ON k-STRONG DISTANCE IN STRONG DIGRAPHS

Ping Zhang, Kalamazoo

(Received January 15, 2001)

Abstract. For a nonempty set S of vertices in a strong digraph D, the strong distance $d(S)$ is the minimum size of a strong subdigraph of D containing the vertices of S. If S contains k vertices, then $d(S)$ is referred to as the k-strong distance of S. For an integer $k \geqslant 2$ and a vertex v of a strong digraph D, the k-strong eccentricity $\mathrm{se}_k(v)$ of v is the maximum k-strong distance $d(S)$ among all sets S of k vertices in D containing v. The minimum k-strong eccentricity among the vertices of D is its k-strong radius $\mathrm{srad}_k\, D$ and the maximum k-strong eccentricity is its k-strong diameter $\mathrm{sdiam}_k\, D$. The k-strong center (k-strong periphery) of D is the subdigraph of D induced by those vertices of k-strong eccentricity $\mathrm{srad}_k(D)$ ($\mathrm{sdiam}_k(D)$). It is shown that, for each integer $k \geqslant 2$, every oriented graph is the k-strong center of some strong oriented graph. A strong oriented graph D is called strongly k-self-centered if D is its own k-strong center. For every integer $r \geqslant 6$, there exist infinitely many strongly 3-self-centered oriented graphs of 3-strong radius r. The problem of determining those oriented graphs that are k-strong peripheries of strong oriented graphs is studied.

Keywords: strong distance, strong eccentricity, strong center, strong periphery

MSC 2000: 05C12, 05C20

1. Introduction

The familiar distance $d(u, v)$ between two vertices u and v in a connected graph is the length of a shortest $u - v$ path in G. Equivalently, this distance is the minimum size of a connected subgraph of G containing u and v. This concept was extended in [2] to connected digraphs, in particular to strongly connected (strong) oriented graphs. We refer to [4] for graph theory notation and terminology not described here.

Research supported in part by the Western Michigan University Arts and Sciences Teaching and Research Award Program.

A digraph D is *strong* if for every pair u, v of distinct vertices of D, there is both a directed $u-v$ path and a directed $v-u$ path in D. A digraph D is an *oriented graph* if D is obtained by assigning a direction to each edge of a graph G. The graph G is referred to as the *underlying graph* of D. In this paper we will be interested in strong oriented graphs. The underlying graph of a strong oriented graph is necessarily 2-edge-connected. Let D be a strong oriented graph of order $n \geqslant 3$ and size m. For two vertices u and v of D, the *strong distance* $\mathrm{sd}(u, v)$ between u and v is defined in [2] as the minimum size of a strong subdigraph of D containing u and v. If $u \neq v$, then $3 \leqslant \mathrm{sd}(u, v) \leqslant m$. In the strong oriented graph D of Figure 1, $\mathrm{sd}(v, w) = 3$, $\mathrm{sd}(u, y) = 4$, and $\mathrm{sd}(u, x) = 5$.

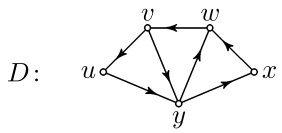

Figure 1. A strong oriented graph

A generalization of distance in graphs was introduced in [5]. For a nonempty set S of vertices in a connected graph G, the *Steiner distance* $d(S)$ of S is the minimum size of a connected subgraph of G containing S. Necessarily, each such subgraph is a tree and is called a *Steiner tree with respect to* S. We now extend this concept to connected strong digraphs. For a nonempty set S of vertices in a strong digraph D, the *strong Steiner distance* $d(S)$ is the minimum size of a strong subdigraph of D containing S. We will refer to such a subgraph as a *Steiner subdigraph with respect to* S, or, simply, S*-subdigraph*. Since D itself is strong, $d(S)$ is defined for every nonempty set S of vertices of D. We denote the size of a digraph D by $m(D)$. If $|S| = k$, then $d(S)$ is referred to as the k*-strong Steiner distance* (or simply k*-strong distance*) of S. Thus $3 \leqslant d(S) \leqslant m(D)$ for each set S of vertices in a strong digraph D with $|S| \geqslant 2$. Then the 2*-strong distance* is the strong distance studied in [2], [3]. For example, in the strong oriented graph D of Figure 1, let $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$, and $S_3 = \{v, w, y\}$. Then the 3-strong distances of S_1, S_2, and S_3 are $d(S_1) = 5$, $d(S_2) = 4$, and $d(S_3) = 3$.

It was shown in [2] that strong distance is a metric on the vertex set of a strong oriented graph D. As such, certain properties are satisfied. Among these are: (1) $\mathrm{sd}(u, v) \geqslant 0$ for vertices u and v of D and $\mathrm{sd}(u, v) = 0$ if and only if $u = v$ and (2) $\mathrm{sd}(u, w) \leqslant \mathrm{sd}(u, v) + \mathrm{sd}(v, w)$ for vertices u, v, w of D. These two properties can be considered in a different setting. Let D be a strong oriented graph and let $S \subseteq V(D)$, where $S \neq \emptyset$. Then $d(S) \geqslant 0$ and $d(S) = 0$ if and only if $|S| = 1$, which is property (1). Let $S_1 = \{u, w\}$, $S_2 = \{u, v\}$, and $S_3 = \{v, w\}$. Then the triangle inequality $\mathrm{sd}(u, w) \leqslant \mathrm{sd}(u, v) + \mathrm{sd}(v, w)$ given in (2) can be restated as $d(S_1) \leqslant d(S_2) + d(S_3)$, where, of course, $|S_i| = 2$ for $1 \leqslant i \leqslant 3$, $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$. We now describe an extension of (2).

Proposition 1.1. *For an integer $k \geqslant 2$, let S_1, S_2, S_3 be sets of k vertices in a strong oriented graph with $|S_i| = k$ for $1 \leqslant i \leqslant 3$. If $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$, then*

$$d(S_1) \leqslant d(S_2) + d(S_3).$$

P r o o f. Let D_i be an S_i-digraph of size $d(S_i)$ for $i = 1, 2, 3$. Define a digraph D' to be the subdigraph of D with vertex set $V(D_2) \cup V(D_3)$ and arc set $E(D_2) \cup E(D_3)$. Since $S_2 \cap S_3 \neq \emptyset$ and D_2 and D_3 are strong subdigraphs of D, it follows that D' is also a strong subdigraph of D with $S_1 \subseteq V(D')$. Thus $m(D_1) \leqslant m(D')$. Therefore,

$$d(S_1) = m(D_1) \leqslant m(D') \leqslant m(D_2) + m(D_3) = d(S_2) + d(S_3),$$

as desired. □

As an example, consider the strong oriented graph D of Figure 2. Let $S_1 = \{s, v, x\}$, $S_2 = \{v, x, z\}$, and $S_3 = \{s, x, y\}$. Then $|S_i| = 3$ for $1 \leqslant i \leqslant 3$, where $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$. For each i with $1 \leqslant i \leqslant 3$, let D_i be an S_i-subdigraph of size $d(S_i)$ in D, which is also shown in Figure 2. Hence $d(S_1) = 3$, $d(S_2) = 4$, and $d(S_3) = 5$. Note that the subdigraph D' of D described in the proof of Proposition 1.1 has size 6. Thus $d(S_1) \leqslant m(D') \leqslant d(S_2) + d(S_3)$.

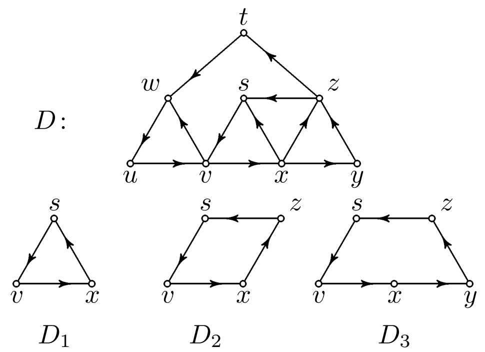

Figure 2. An example of an extension of (2)

The extended triangle inequality $d(S_1) \leqslant d(S_2) + d(S_3)$ stated in Proposition 1.1 suggests a generalization of strong distance in strong oriented graphs, which we introduce in this paper.

2. On k-strong eccentricity, radius, and diameter

Let v be a vertex of a strong oriented graph D of order $n \geqslant 3$ and let k be an integer with $2 \leqslant k \leqslant n$. The *$k$-strong eccentricity* $\mathrm{se}_k(v)$ is defined by

$$\mathrm{se}_k(v) = \max\{d(S);\ S \subseteq V(D), v \in S, |S| = k\}.$$

The *k-strong diameter* $\mathrm{sdiam}_k(D)$ is

$$\mathrm{sdiam}_k(D) = \max\{\mathrm{se}_k(v);\ v \in V(D)\};$$

while the *k-strong radius* $\mathrm{srad}_k(D)$ is defined by

$$\mathrm{srad}_k(D) = \min\{\mathrm{se}_k(v);\ v \in V(D)\}.$$

To illustrate these concepts, consider the strong oriented graph D of Figure 3. The 3-strong eccentricity of each vertex of D is shown in Figure 3. Thus $\mathrm{srad}_3(D) = 8$ and $\mathrm{sdiam}_3(D) = 12$.

D:

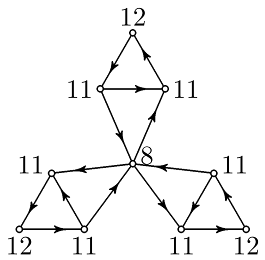

Figure 3. A strong oriented graph D with $\mathrm{srad}_3(D) = 8$ and $\mathrm{sdiam}_3(D) = 12$

For a nontrivial strong oriented graph D of order n, the *radius sequence* $\mathcal{S}_r(D)$ of D is defined as

$$\mathcal{S}_r(D)\colon\ \mathrm{srad}_2(D), \mathrm{srad}_3(D), \mathrm{srad}_4(D), \ldots, \mathrm{srad}_n(D)$$

and the *diameter sequence* $\mathcal{S}_d(D)$ of D is defined as

$$\mathcal{S}_d(D)\colon\ \mathrm{sdiam}_2(D), \mathrm{sdiam}_3(D), \mathrm{sdiam}_4(D), \ldots, \mathrm{sdiam}_n(D).$$

For example, the strong oriented graph D in Figure 4 has order 9. Since $\mathrm{srad}_2(D) = 6$, $\mathrm{srad}_3(D) = 9$, and $\mathrm{srad}_k(D) = 12$ for $4 \leqslant k \leqslant 9$, it follows that $\mathcal{S}_r(D)$: $6, 9, 12, 12, \ldots, 12$. Moreover, $\mathrm{sdiam}_2(D) = 9$ and $\mathrm{sdiam}_k(D) = 12$ for $3 \leqslant k \leqslant 9$.

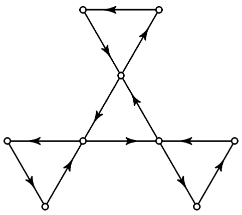

Figure 4. A strong oriented graph

Thus $\mathcal{S}_d(D)$: $9, 12, 12, \ldots, 12$. Note that both $\mathcal{S}_r(D)$ and $\mathcal{S}_d(D)$ are nondecreasing sequences. This is no coincidence, as we now see.

Proposition 2.1. *For a nontrivial strong oriented graph D of order n and every integer k with $2 \leqslant k \leqslant n-1$,*

$$\text{(a) } \mathrm{srad}_k(D) \leqslant \mathrm{srad}_{k+1}(D) \text{ and (b) } \mathrm{sdiam}_k(D) \leqslant \mathrm{sdiam}_{k+1}(D).$$

P r o o f. To verify (a), let u and v be two vertices of D with $\mathrm{se}_k(u) = \mathrm{srad}_k(D)$ and $\mathrm{se}_{k+1}(v) = \mathrm{srad}_{k+1}(D)$. Let S be a set of k vertices of D such that $\mathrm{se}_k(u) = d(S) = \mathrm{srad}_k(D)$. Now let x be a vertex of D such that $x = v$ if $v \notin S$ and $x \in V(D) - S$ if $v \in S$. Let $S' = \{x\} \cup S$. Since $S \subseteq S'$, it follows that $d(S) \leqslant d(S')$. Moreover, S' is a set of $k+1$ vertices of D containing v and so $d(S') \leqslant \mathrm{se}_{k+1}(v)$. Thus

$$\mathrm{srad}_k(D) = d(S) \leqslant d(S') \leqslant \mathrm{se}_{k+1}(v) = \mathrm{srad}_{k+1}(D)$$

and so (a) holds. To verify (b), let S be a set of k vertices of D with $d(S) = \mathrm{sdiam}_k(D)$. If S' is any set of $k+1$ vertices of D with $S \subseteq S'$, then

$$\mathrm{sdiam}_k(D) = d(S) \leqslant d(S') \leqslant \mathrm{sdiam}_{k+1}(D)$$

and so (b) holds. $\square$

Equalities in (a) and (b) of Proposition 2.1 hold for certain strong oriented graphs, for example, the directed n-cycle $\overrightarrow{C_n}$ for $n \geqslant 3$. In fact, $\mathrm{srad}_k(\overrightarrow{C_n}) = \mathrm{sdiam}_k(\overrightarrow{C_n}) = n$ for all k with $2 \leqslant k \leqslant n$. As another example, let D be the strong oriented graph of order $n \geqslant 3$ with $V(D) = \{v_1, v_2, \ldots, v_n\}$ such that for $1 \leqslant i < j \leqslant n$, $(v_i, v_j) \in E(D)$, except when $i = 1$ and $j = n$, and $(v_n, v_1) \in E(D)$ (see Figure 5). Then $\mathrm{srad}_k(D) = \mathrm{sdiam}_k(D) = n$ for all k with $2 \leqslant k \leqslant n$. In fact, there are many other strong oriented graphs D with the property that $\mathrm{srad}_k(D) = \mathrm{sdiam}_k(D)$.

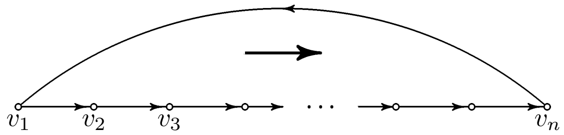

Figure 5. A strong oriented graph D of order n with $\mathrm{srad}_k(D) = \mathrm{sdiam}_k(D)$ for $2 \leqslant k \leqslant n$

On the other hand, for a strong oriented graph D, the difference between $\mathrm{srad}_{k+1}(D)$ and $\mathrm{srad}_k(D)$ (or $\mathrm{sdiam}_{k+1}(D)$ and $\mathrm{sdiam}_k(D)$) can be arbitrarily large for some k.

Proposition 2.2. *For every integer $N \geqslant 3$, there exist a strong oriented graph D and an integer k such that*

$$\mathrm{srad}_{k+1}(D) - \mathrm{srad}_k(D) \geqslant N \quad \textit{and} \quad \mathrm{sdiam}_{k+1}(D) - \mathrm{sdiam}_k(D) \geqslant N.$$

Proof. Let $\ell \geqslant 3$ be an integer. For each i with $1 \leqslant i \leqslant \ell$, let D_i be a copy of the directed N-cycle $\overrightarrow{C_N}$ and let $v_i \in V(D_i)$. Now let D be the strong oriented graph obtained from the digraphs D_i $(1 \leqslant i \leqslant \ell)$ by identifying the ℓ vertices $v_1, v_2, \ldots, v_\ell$. It can be verified that $\mathrm{srad}_{k+1}(D) - \mathrm{srad}_k(D) = N$ and $\mathrm{sdiam}_{k+1}(D) - \mathrm{sdiam}_k(D) = N$ for all k with $2 \leqslant k \leqslant \ell - 1$. □

For an integer $k \geqslant 2$, the k-strong radius and k-strong diameter of a strong oriented graph satisfy familiar inequalities, which are verified with familiar arguments.

Proposition 2.3. *Let $k \geqslant 2$ be an integer. For every strong oriented graph D,*

$$\mathrm{srad}_k(D) \leqslant \mathrm{sdiam}_k(D) \leqslant 2\mathrm{srad}_k(D).$$

Proof. The inequality $\mathrm{srad}_k(D) \leqslant \mathrm{sdiam}_k(D)$ follows directly from the definitions. It was shown in [2] that result is true for $k = 2$. So we may assume that $k \geqslant 3$. Let $S_1 = \{w_1, w_2, \ldots, w_k\}$ be a set of vertices of D with $d(S) = \mathrm{sdiam}_k(D)$ and let v be a vertex of D with $\mathrm{se}_k(v) = \mathrm{srad}_k(D)$. Define $S_2 = \{v, w_1, w_2, \ldots, w_{k-1}\}$ and $S_3 = \{v, w_2, w_3, \ldots, w_k\}$. Thus $S_1 \subseteq S_2 \cup S_3$ and $S_2 \cap S_3 \neq \emptyset$. It then follows from Proposition 1.1 that

$$\mathrm{sdiam}_k(D) = d(S_1) \leqslant d(S_2) + d(S_3) \leqslant 2\mathrm{srad}_k(D),$$

producing the desired result. □

3. On k-strong centers and peripherals

A vertex v in a strong digraph D is a *k-strong central vertex* if $se_k(v) = \mathrm{srad}_k(G)$, while the *$k$-strong center* $SC_k(D)$ of D is the subgraph induced by the k-strong central vertices of D. These concepts were first introduced in [3] for $k = 2$. For example, consider the strong digraph D of Figure 4, which is also shown in Figure 6. Each vertex of D is labeled with its 3-strong eccentricity. Thus the vertices x, y, z are the 3-strong central vertices of D. The 3-strong center $SC_3(D)$ of D is a 3-cycle as shown in Figure 6.

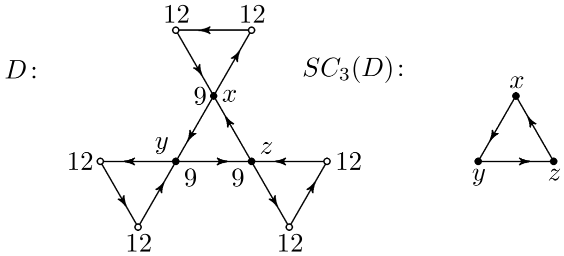

Figure 6. The 3-strong center of a strong digraph D

It was shown in [3] that every 2-strong center of every strong oriented graph D lies in a block of the underlying graph of D. However, it is not true in general for $k \geqslant 3$. For example, although the 3-strong center of the strong oriented graph D in Figure 6 lies in a block of the underlying graph of D, the 4-strong center of D is D itself and D is not a block. On the other hand, as Hedetniemi (see [1]) showed that every graph is the center of some connected graph, it was also shown in [3] that every oriented graph is the 2-strong center of some strong digraph. We now extend this result by showing that, for each integer $k \geqslant 2$, every oriented graph is the k-strong center of some strong digraph.

Theorem 3.1. *Let $k \geqslant 2$ be an integer. Then every oriented graph is the k-strong center of some strong digraph.*

Proof. For an oriented graph D, we construct a strong oriented graph D^* from D by adding the $3k$ new vertices u_i, v_i, w_i $(1 \leqslant i \leqslant k)$ and arcs (1) $(w_i, v_i), (v_i, u_i)$, and (u_i, w_i) for all i with $1 \leqslant i \leqslant k$ and (2) (u_i, x) and (x, v_i) for all $x \in V(D)$ and for all i with $1 \leqslant i \leqslant k$. The oriented graph D^* is shown in Figure 7. Certainly, D^* is strong. Next, we show that D is the k-strong center of D^*.

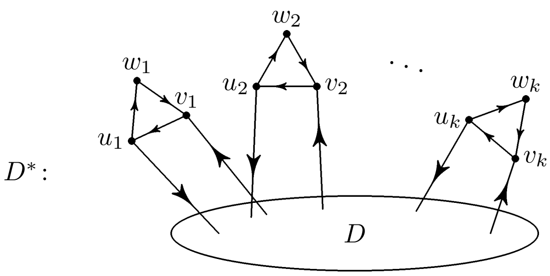

Figure 7. A strong oriented graph D^* containing D as its k-strong center

Let $U = \{u_1, u_2, \ldots, u_k\}$, $V = \{v_1, v_2, \ldots, v_k\}$, and $W = \{w_1, w_2, \ldots, w_k\}$. For each $x \in V(D)$, let $S(x) = \{x\} \cup (W - \{w_k\})$. Then $\mathrm{se}_k(x) = d(S) = 6(k-1)$. For each $u_i \in U$, where $1 \leqslant i \leqslant k$, let $S(u_i) = \{u_i\} \cup (W - \{w_i\})$. Then $\mathrm{se}_k(u_i) =$

$d(S) = 6(k-1)+3$ for $1 \leqslant i \leqslant k$. For each $v_i \in V$, where $1 \leqslant i \leqslant k$, let $S(v_i) = \{v_i\} \cup (W - \{w_i\})$. Then $\mathrm{se}_k(v_i) = d(S) = 6(k-1)+3$ for $1 \leqslant i \leqslant k$. For each $w_i \in W$, where $1 \leqslant i \leqslant k$, let $S = W$. Then $\mathrm{se}_k(w_i) = d(S) = 6k$ for $1 \leqslant i \leqslant k$. Since $\mathrm{se}_k(x) = 6(k-1)$ for all $x \in V(D)$ and $\mathrm{se}_k(v) > 6(k-1)$ for all $v \in V(D^*) - V(D)$, it follows that D is the k-strong center of D^*, as desired. $\square$

Independently, V. Castellana and M. Raines also discovered Theorem 3.1 (personal communication). A vertex v in a strong digraph D is called a *k-strong peripheral vertex* if $\mathrm{se}_k(v) = \mathrm{sdiam}_k(D)$, while the subgraph induced by the k-strong peripheral vertices of D is the *k-strong periphery $SP_k(D)$* of D. Also, these concepts were first introduced in [3] for $k = 2$. A strong digraph D and its 3-strong periphery are shown in Figure 8. The following result appeared in [3].

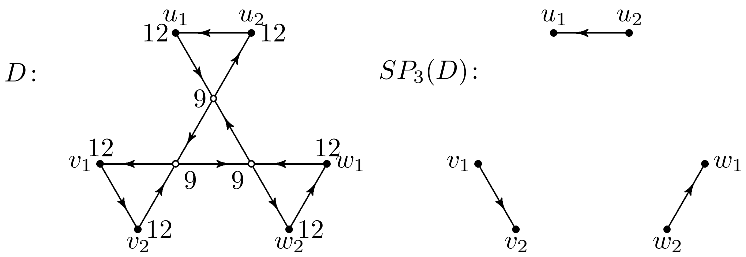

Figure 8. The 3-strong periphery of a strong digraph

Theorem A. *If D is an oriented graph with* $\mathrm{srad}_2(D) = 3$ *and* $\mathrm{sdiam}_2(D) > 3$, *then D is not the 2-strong periphery of any oriented graph.*

We now extend Theorem A to the k-strong periphery of a strong oriented graph for $k \geqslant 3$ and show that not all oriented graphs are the k-strong peripheries of strong oriented graphs.

Theorem 3.2. *Let $k \geqslant 3$ be an integer. If D is an oriented graph with* $\mathrm{sdiam}_k(D) > \mathrm{srad}_k(D)$, *then D is not the k-strong periphery of any oriented graph.*

P r o o f. Let D satisfy the conditions of the theorem. Assume, to the contrary, that D is the k-strong periphery of some oriented graph D'. Assume that $\mathrm{srad}_k(D) = r$ and $\mathrm{sdiam}_k(D) = d$. So $d > r \geqslant 3$. Let u be a k-strong central vertex of D. Since $\mathrm{sdiam}_k(D) = d > r$, we have $\mathrm{sdiam}_k(D') = d' \geqslant d > r$. Moreover, since D is the k-strong periphery of D' and $u \in V(D)$, it follows that D' contains a set $S = \{u, v_1, v_2, \ldots, v_{k-1}\}$ such that $d(S) = \mathrm{sdiam}_k(D') = d'$. Because u is a k-strong central vertex of D, that is, u has k-strong eccentricity r in D, and $r < d'$, at least one vertex from $\{v_1, v_2, \ldots, v_{k-1}\}$ does not belong to $V(D)$. Assume, without loss of generality, that $v_1 \notin V(D)$. Then the k-strong eccentricity $\mathrm{se}_k(v_1)$ of v_1 in D' is at least $d(S)$ and so $\mathrm{se}_k(v_1) \geqslant d(S) = d'$. Thus $\mathrm{se}_k(v_1) = d'$, which implies that v_1 is a k-strong peripheral vertex of D'. Since $v_1 \notin V(D)$, it follows that D is not the k-strong periphery of D', which is a contradiction. $\square$

In [3], a sufficient condition was established for an oriented graph D to be the 2-strong periphery of some oriented graph D', which we state next.

Theorem B. *Let D be an oriented graph of order n with strong diameter at least 4. If $\mathrm{id}\, v + \mathrm{od}\, v < n - 1$ for every vertex v of D, then D is the 2-strong periphery of some oriented graph D'.*

Observe that if v is a vertex of an oriented graph D of order n such that $\mathrm{id}\, v + \mathrm{od}\, v < n - 1$, then there is a vertex $u \in V(D)$ such that v and u are nonadjacent vertices of D, that is, v belongs to an independent set, namely $\{u, v\}$, of cardinality 2 in D. Thus the sufficient condition given in Theorem B is equivalent to that every vertex in D belongs to an independent set of cardinality 2 in D. We now extend Theorem B to obtain a sufficient condition for an oriented digraph D to be the k-strong periphery of some oriented graph D' for all integers $k \geqslant 2$.

Theorem 3.3. *Let $k \geqslant 2$ be an integer and let D be a connected oriented graph. If every vertex of D belongs to an independent set of cardinality k in D, then D is the k-strong periphery of some oriented graph D'.*

P r o o f. By Theorem B the result holds for $k = 2$. So we assume that $k \geqslant 3$. Let D be an oriented graph of order n which satisfies the conditions of the theorem and let $V(D) = \{u_1, u_2, \ldots, u_n\}$. We construct a new oriented graph D' of order $2n + 2$ with $V(D') = V(D) \cup \{v_1, v_2, \ldots, v_n, x, y\}$ such that the arc set of D' consists of $E(D)$ together with arcs (1) (u_i, v_i) and (v_i, u_j) for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, (2) (v_i, v_j) for $1 \leqslant i < j \leqslant n$, and (3) $(y, x), (v_i, x), (x, u_i), (u_i, y), (y, v_i)$ for $1 \leqslant i \leqslant n$. The oriented graph D' is shown in Figure 9. We claim that D is the k-strong periphery of D'. We will show it only for $k = 3$ since the argument for $k \geqslant 4$ is similar.

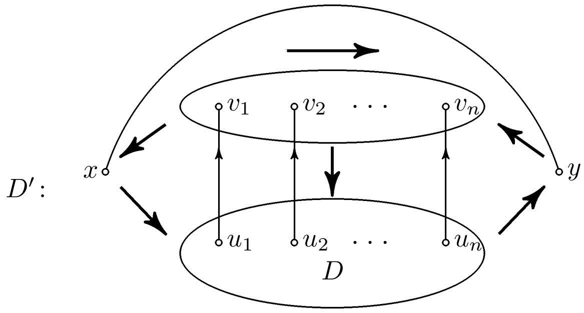

Figure 9. An oriented graph D' containing D as its k-strong periphery

We first show that $\mathrm{se}_3(u_i) = 6$ in D' for all i with $1 \leqslant i \leqslant n$. Without loss of generality, we consider only $u_1 \in V(D)$ and show that $\mathrm{se}_3(u_1) = 6$. Let $S_0 = \{u_1, u_p, u_q\}$ be an independent set of three vertices in D', where $2 \leqslant p < q \leqslant n$. Then the size of a strong subdigraph containing S_0 is at least 6. On the other hand, the directed 6-cycle C shown in Figure 10 contains S_0. Thus $d(S_0) = 6$ and so $\mathrm{se}_3(u_1) \geqslant 6$.

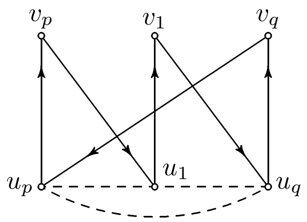

Figure 10. A directed 6-cycle C in D' containing S_0

To show that $\mathrm{se}_3(u_1) \leqslant 6$. Let S be a set of three vertices of D containing u_1. Then the only possible choices for S are $S_1 = \{u_1, u_i, u_j\}$, where $2 \leqslant i < j \leqslant n$, $S_2 = \{u_1, v_i, v_j\}$, where $1 \leqslant i < j \leqslant n$, $S_3 = \{u_1, u_i, v_j\}$, where $i \geqslant 2$ and $1 \leqslant j \leqslant n$, $S_4 = \{u_1, x, y\}$, $S_5 = \{u_1, u_i, y\}$, where $2 \leqslant i \leqslant n$, $S_6 = \{u_1, u_i, x\}$, where $2 \leqslant i \leqslant n$, $S_7 = \{u_1, v_i, y\}$, and $S_8 = \{u_1, v_i, x\}$, where $1 \leqslant i \leqslant n$. If $S = S_1$, then the directed 6-cycle u_1, v_1, u_i, v_i, u_j, v_j, u_1 is a strong subdigraph of D' containing S and so $d(S) \leqslant 6$. Let $S = S_2 = \{u_1, v_i, v_j\}$, where $1 \leqslant i < j \leqslant n$. If $i = 1$, then the directed 4-cycle u_1, v_1, u_j, v_j, u_1 is a strong subdigraph of D' containing S and so $d(S) \leqslant 4$. If $i \geqslant 2$, then the directed 4-cycle u_1, y, v_i, v_j, u_1 is a strong subdigraph of D' containing S and so $d(S) \leqslant 4$. Let $S = S_3 = \{u_1, u_i, v_j\}$, where $i \geqslant 2$ and $1 \leqslant j \leqslant n$. If $j = 1$ or $j = i$, say $j = 1$, then the directed 4-cycle u_1, v_1, u_1, v_i, u_1 is a strong subdigraph of D' containing S and so $d(S) \leqslant 4$; Otherwise, the directed 5-cycle $u_1, y, v_j, u_i, v_i, u_1$ is a strong subdigraph of D' containing S and so $d(S) \leqslant 5$. If $S = S_4$, then the directed 3-cycle u_1, y, x, u_1 is a strong subdigraph of D' containing S and so $d(S) \leqslant 3$. If $S = S_5$ (or $S = S_6$), then the directed 5-cycle $u_1, v_1, u_i, y, v_i, u_1$ contains S (or the directed 5-cycle $u_1, v_1, x, u_i, v_i, u_1$ contains S). Thus $d(S) \leqslant 5$. Let $S = S_7 = \{u_1, v_i, y\}$ or $S = S_8 = \{u_1, v_i, x\}$, where $1 \leqslant i \leqslant n$. If $i = 1$, then directed 4-cycle u_1, y, v_1, x, u_1 contains S and $d(S) \leqslant 4$. If $i \geqslant 2$, then either the directed 5-cycle u_1, v_1, u_i, y, v_i, u_1 contains S or the directed 5-cycle u_1, v_1, x, u_i, v_i, u_1 contains S. Thus $d(S) \leqslant 5$. Hence $d(S) \leqslant 6$ for all possible choices for S and so $\mathrm{se}_3(u_1) \leqslant 6$. Therefore, $\mathrm{se}_3(u_1) = 6$. Similarly, $\mathrm{se}_3(u_i) = 6$ for all i with $2 \leqslant i \leqslant n$.

Next we show that $\mathrm{se}(x) \leqslant 5$ and $\mathrm{se}(y) \leqslant 5$ in D'. Let S be a set of three vertices in D' containing x. Then the only possible choices for S are $S_1 = \{x, u_i, u_j\}$, where

$1 \leqslant i < j \leqslant n$, $S_2 = \{x, v_i, v_j\}$, where $1 \leqslant i < j \leqslant n$, $S_3 = \{x, u_i, v_j\}$, where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, $S_4 = \{x, y, u_i\}$, where $1 \leqslant i \leqslant n$, and $S_5 = \{x, y, v_i\}$, where $1 \leqslant i \leqslant n$. For $S = S_1, S_2, S_3$, the directed 5-cycle u_i, v_i, x, u_j, v_j, u_i contains S and so $d(S) \leqslant 5$. For $S = S_4$, the directed 3-cycle x, u_i, y, x contains S and so $d(S) \leqslant 3$. For $S = S_5$, the directed 4-cycle u_1, y, v_i, x, v_1 contains S and so $d(S) \leqslant 4$. Therefore, $\mathrm{se}(x) \leqslant 5$. Similarly, $\mathrm{se}(y) \leqslant 5$.

Finally, we show that $\mathrm{se}(v_i) \leqslant 5$ in D' for all i with $1 \leqslant i \leqslant n$. Without loss of generality, let $v_i = v_1$ and let S be a set of three vertices in D' containing v_1. Then the only possible choices for S are $S_1 = \{v_1, u_i, u_j\}$, where $1 \leqslant i < j \leqslant n$, $S_2 = \{v_1, v_i, v_j\}$, where $2 \leqslant i < j \leqslant n$, $S_3 = \{v_1, u_i, v_j\}$, where $1 \leqslant i \leqslant n$ and $j \geqslant 2$, $S_4 = \{v_1, u_i, x\}$, where $1 \leqslant i \leqslant n$, $S_5 = \{v_1, v_i, x\}$, where $2 \leqslant i \leqslant n$, $S_6 = \{v_1, u_i, y\}$, where $1 \leqslant i \leqslant n$, and $S_7 = \{v_1, v_i, y\}$, where $2 \leqslant i \leqslant n$. An argument similar to the one above shows that $d(S) \leqslant 5$ for each choice of S and so $\mathrm{se}_3(v_1) \leqslant 5$.

Since $\mathrm{se}_3(v) = 6$ for all $v \in V(D)$ and $\mathrm{se}_3(v) \leqslant 5$ for all $v \in V(D') - V(D)$, it follows that D is the 3-strong periphery of the oriented graph D'. In general, for $k \geqslant 3$, we have $\mathrm{se}_k(v) = 2k$ for all $v \in V(D)$ and $\mathrm{se}_k(v) \leqslant 2k - 1$ for all $v \in V(D') - V(D)$. Therefore, D is the k-strong periphery of the oriented graph D'. □

4. On strongly k-self-centered oriented graphs

Let D be a nontrivial strong digraph of order n and let k be an integer with $2 \leqslant k \leqslant n$. Then D is called strongly *k-self-centered* if $\mathrm{srad}_k D = \mathrm{sdiam}_k D$, that is, if D is its own k-strong center. For example, the directed n-cycle $\overrightarrow{C_n}$ and the strong digraph D in Figure 5 are k-self-centered for all k with $2 \leqslant k \leqslant n$. The 2-self-centered digraph was studied in [3]. The following result was established in [3].

Theorem C. *For every integer $r \geqslant 3$, there exist infinitely many strongly 2-self-centered oriented graphs of strong radius r.*

We now extend Theorem C to strongly 3-*self-centered* oriented graphs.

Theorem 4.1. *For every integer $r \geqslant 6$, there exist infinitely many strongly 3-self-centered oriented graphs of strong radius r.*

P r o o f. For each integer $r \geqslant 6$, we construct an infinite sequence $\{D_n\}$ of strongly 3-self-centered oriented graphs of strong radius r. We consider two cases, according to whether r is even or r is odd.

C a s e 1. *r is even.* Let $r = 2p$, where $p \geqslant 3$. Let D_1 be the digraph obtained from the directed p-cycle $C_p : w_1, w_2, \ldots, w_p$ by adding the $2(p-1)$ new vertices u_1, u_2, …,

u_{p-1} and $v_1, v_2, \ldots, v_{p-1}$ and the new arcs (1) (u_i, u_{i+1}), (v_i, v_{i+1}) for $1 \leqslant i \leqslant p-2$ and (2) (v, u_1), (u_{p-1}, v), (v, v_1), and (v_{p-1}, v) for all $v \in V(C_p)$. The digraph D_1 is shown in Figure 11 for $r = 6$. Let $U = \{u_1, u_2, \ldots, u_{p-1}\}$, $V = \{v_1, v_2, \ldots, v_{p-1}\}$, and $W = \{w_1, w_2, \ldots, w_p\}$. We show that D_1 is a strongly 3-self-centered digraph with 3-strong radius r.

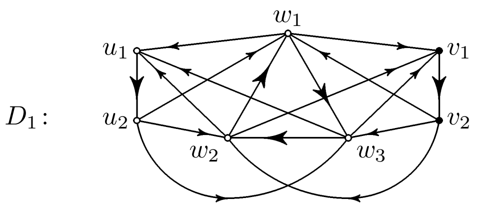

Figure 11. The digraph D_1 in Case 1 for $r = 6$

First, we make an observation. If $S = \{u, v, w\}$, where $u \in U$, $v \in V$, and $w \in W$, then $d(S) \geqslant r$ by the construction of D_1. On the other hand, let D_S be the strong subdigraph in D_1 consisting of two p-cycles $w, v_1, v_2, \ldots, v_{p-1}, w$ and $w, u_1, u_2, \ldots, u_{p-1}, w$. Since D_S contains S and has size $2p = r$, it follows that $d(S) = r$. Therefore, for every vertex x of $V(D_1)$, there is a set S of three vertices of D_1 such that S contains x and $d(S) = r$. This implies that $\mathrm{se}_3(x) \geqslant r$ for all $x \in V(D_1)$. So it remains to show that $\mathrm{se}_3(x) \leqslant r$ for all $x \in V(D_1)$. There are two subcases.

Subcase 1.1. $x \in U$ *or* $x \in V$. Without loss of generality, assume that $x \in U$. We will only consider $x = u_1 \in U$ since the proofs for other vertices are similar. Let S be a set of three vertices in D_1 containing u_1. If $S \cap V \neq \emptyset$ and $S \cap W \neq \emptyset$, then $d(S) = r$ by the observation above. So we may assume that S is one of the following sets: $S_1 = \{u_1, u_i, u_j\}$, where $2 \leqslant i < j \leqslant p-1$, $S_2 = \{u_1, u_i, w_j\}$, where $2 \leqslant i \leqslant p-1$ and $1 \leqslant j \leqslant p$, $S_3 = \{u_1, u_i, v_j\}$, where $2 \leqslant i \leqslant p-1$ and $1 \leqslant j \leqslant p-1$, $S_4 = \{u_1, v_i, v_j\}$, where $1 \leqslant i < j \leqslant p-1$, and $S_5 = \{u_1, w_i, w_j\}$, where $1 \leqslant i < j \leqslant p$. If $S = S_1, S_2$, then the directed p-cycle $w_j, u_1, u_2, \ldots, u_{p-1}, w_j$ is a strong subdigraph in D_1 containing S and so $d(S) \leqslant p$. If $S = S_3, S_4$, then the strong subdigraph D_S in D_1 consisting of two p-cycles $w_1, v_1, v_2, \ldots, v_{p-1}, w_1$ and $w_1, u_1, u_2, \ldots, u_{p-1}, w_1$ contains S and so $d(S) \leqslant 2p = r$. If $S = S_5$, then the strong subdigraph consisting of two p-cycles $w_i, v_1, v_2, \ldots, v_{p-1}, w_i$ and $w_j, u_1, u_2, \ldots, u_{p-1}, w_j$ contains S and so $d(S) \leqslant 2p = r$.

Subcase 1.2. $x \in W$. We may assume that $x = w_1 \in W$ and let S be a set of three vertices in D_1 containing w_1. Again, if $S \cap V \neq \emptyset$ and $S \cap U \neq \emptyset$, then $d(S) = r$ by the observation above. So we may assume that S is one of the following sets $S_1 = \{w_1, w_i, w_j\}$, where $2 \leqslant i < j \leqslant p$, $S_2 = \{w_1, w_i, u_j\}$, where $2 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant p-1$, $S_3 = \{w_1, w_i, v_j\}$, where $2 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant p-1$, $S_4 = \{w_1, u_i, u_j\}$, where $1 \leqslant i < j \leqslant p-1$, and $S_5 = \{w_1, v_i, v_j\}$, where $1 \leqslant i < j \leqslant p-1$. An argument similar to the one in Subcase 1.1 shows that $d(S) \leqslant r$ for all possible choices for S.

Therefore, $\mathrm{se}_3(x) = r$ for all $x \in V(D_1)$ and so D_1 is a strongly 3-self-centered digraph with 3-strong radius r.

For $n \geqslant 1$, we define the strong digraph D_{n+1} recursively from D_n by adding the $2(p-1)$ new vertices $x_1, x_2, \ldots, x_{p-1}$ and $y_1, y_2, \ldots, y_{p-1}$ and the new arcs (1) (x_i, x_{i+1}), (y_i, y_{i+1}) for $1 \leqslant i \leqslant p-2$ and (2) (v, x_1), (x_{p-1}, v), (v, y_1), and (y_{p-1}, v) for all $v \in V(D_n)$. The digraph D_{n+1} is shown in Figure 12. We assume that D_n is a strongly 3-self-centered oriented graph of 3-strong radius r for some integer $n \geqslant 1$ and show that D_{n+1} is also a strongly 3-self-centered oriented graph of 3-strong radius r.

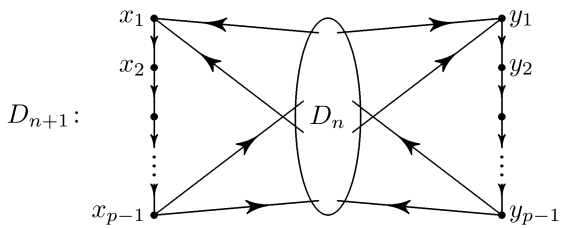

Figure 12. The digraph D_{n+1} in Case 1

Let $X = \{x_1, x_2, \ldots, x_{p-1}\}$ and $Y = \{y_1, y_2, \ldots, y_{p-1}\}$. For $v \in V(D_{n+1})$, let S be a set of three vertices in D_{n+1} containing v. If $v \in V(D_n)$ and $S = \{v, x_1, y_1\}$, then $\mathrm{se}_3(v) = d(S) = r$. So we may assume that $v \in X \cup Y$, say $v = x_1$. Let $S = \{v, y_1, z\}$, where $z \in V(D_n)$. Then $d(S) = \mathrm{se}_3(v) = r$. Therefore, $\mathrm{se}_3(v) = r$ for all $v \in V(D_{n+1})$ and so D_{n+1} is also a strongly 3-self-centered oriented graph of 3-strong radius r.

C a s e 2. *r is odd.* Let $r = 2p+1$, where $p \geqslant 3$. Let D_1 be the digraph obtained from the directed $(p+1)$-cycle $C_{p+1}\colon w_1, w_2, w_3, w_4, w_1$ by adding the $p-1$ new vertices $u_1, u_2, \ldots, u_{p-1}$ and the new arcs (1) (u_i, u_{i+1}) for $1 \leqslant i \leqslant p-2$ and (2) (v, u_1) and (u_{p-1}, v) for all $v \in V(C_{p+1})$. The digraph D_1 is shown in Figure 13 for $r = 7$.

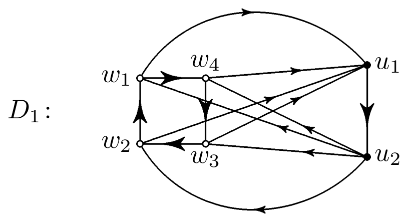

Figure 13. The digraph D_1 in Case 2 for $r = 7$

For $n \geqslant 1$, we define D_{n+1} recursively from D_n by adding the $p-1$ new vertices x_1, x_2, …, x_{p-1} and the new arcs (1) (x_i, x_{i+1}), for $1 \leqslant i \leqslant p-2$ and (2) (v, x_1) and (x_{p-1}, v) for all $v \in V(D_n)$. The digraph D_{n+1} is shown in Figure 14.

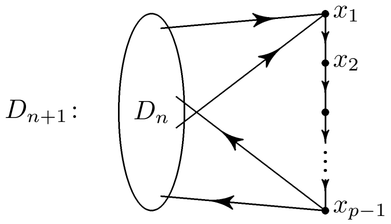

Figure 14. The digraph D_{n+1} in Case 2

An argument similar to the one used in Case 1 shows that each strong digraph D_n is a strongly 3-self-centered oriented graph of strong radius r for all $n \geqslant 1$. □

Acknowledgments. The author is grateful to Professor Gary Chartrand for suggesting the concept of strong Steiner distance and kindly providing useful information on this topic.

References

[1] *F. Buckley, Z. Miller, P. J. Slater*: On graphs containing a given graph as center. J. Graph Theory *5* (1981), 427–434.
[2] *G. Chartrand, D. Erwin, M. Raines, P. Zhang*: Strong distance in strong digraphs. J. Combin. Math. Combin. Comput. *31* (1999), 33–44.
[3] *G. Chartrand, D. Erwin, M. Raines, P. Zhang*: On strong distance in strong oriented graphs. Congr. Numer. *141* (1999), 49–63.
[4] *G. Chartrand, L. Lesniak*: Graphs & Digraphs, third edition. Chapman & Hall, New York, 1996.
[5] *G. Chartrand, O. R. Oellermann, S. Tian, H. B. Zou*: Steiner distance in graphs. Čas. Pěst. Mat. *114* (1989), 399–410.

Author's address: *Ping Zhang*, Department of Mathematics and Statistics, Western Michigan University, Kalamazoo, MI 49008, USA, e-mail: `ping.zhang@wmich.edu`.