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1 Introduction

We study the system of PDEs

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \Delta \boldsymbol{u} + \nabla \pi = \operatorname{div}(\mathbb{F}\mathbb{F}^\top) \tag{1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \tag{1b}$$

$$\partial_t \mathbb{F} + \boldsymbol{u} \cdot \nabla \mathbb{F} = \nabla \boldsymbol{u} \mathbb{F} \tag{1c}$$

describing the unsteady motion of a viscoelastic fluid in a bounded domain $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary. The system is complemented with the no-slip boundary condition

$$\boldsymbol{u}|_{\partial\Omega} = \mathbf{0} \tag{1d}$$

and the initial conditions

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0, \quad \mathbb{F}(0, \cdot) = \mathbb{F}_0. \tag{1e}$$

Our aim is to analyze the error of a suitable approximation of the problem. In particular, we combine the lowest order Taylor-Hood finite element discretization of the flow part (piecewise quadratic velocity and piecewise linear pressure) with either piecewise linear finite elements or finite volumes for the deformation tensor.

The paper is organized as follows. In section 2 we introduce space discretizations of (1). The main result on convergence rates is stated in section 3. Sections

4 and 5 are devoted to the proof of the main result. In the second part of the proof we use a variant of the multiplicative trace inequality, which is proved in section 6. Finally, we present results of a numerical test in section 7.

2 Approximation

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of partitions of Ω into triangles and $h := \|\mathcal{T}_h\|$. For any T and $k = 0, 1, 2, \dots$ we denote by $\mathcal{P}^k(T)$ the space of k-th order polynomials on T. We define the spaces

$$\begin{aligned}
W_h &:= \{\boldsymbol{v} \in W_0^{1,2}(\Omega;\mathbb{R}^2);\ \forall T \in \mathcal{T}_h:\ \boldsymbol{v}_{|T} \in \mathcal{P}^2(T)^2\},\\
L_h &:= \{q \in L_0^2(\Omega) \cap C(\overline{\Omega});\ \forall T \in \mathcal{T}_h:\ q_{|T} \in \mathcal{P}^1(T)\},\\
X_h &:= \{\mathbb{G} \in C(\overline{\Omega};\mathbb{R}^{2\times 2});\ \forall T \in \mathcal{T}_h:\ \mathbb{G}_{|T} \in \mathcal{P}^1(T)^{2\times 2}\},\\
Z_h &:= \{\mathbb{G} \in L^2(\Omega;\mathbb{R}^{2\times 2});\ \forall T \in \mathcal{T}_h:\ \mathbb{G}_{|T} \in \mathcal{P}^0(T)\}.
\end{aligned}$$

Let e be an interior edge shared by elements T_1 and T_2. Define the unit normal vectors $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ on e pointing exterior to T_1 and T_2, respectively. For a function φ, piecewise smooth on $\mathcal{T}_h$, with $\varphi^i = \varphi_{|T_i}$ we define the average $\{\varphi\}$ and the jump $[\varphi]$:

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\varphi] = \varphi^1\boldsymbol{n}^1 + \varphi^2\boldsymbol{n}^2 \quad \text{on } e \in \mathcal{E}_h^0,$$

where $\mathcal{E}_h^0$ is the set of all interior edges e. For a vector-valued function $\boldsymbol{\phi}$, piecewise smooth on $\mathcal{T}_h$, we define the average and the jump analogously

$$\{\boldsymbol{\phi}\} = \frac{1}{2}(\boldsymbol{\phi}^1 + \boldsymbol{\phi}^2), \quad [\boldsymbol{\phi}] = \boldsymbol{\phi}^1 \cdot \boldsymbol{n}^1 + \boldsymbol{\phi}^2 \cdot \boldsymbol{n}^2 \quad \text{on } e \in \mathcal{E}_h^0.$$

Let $\boldsymbol{\beta}$ be a vector-valued function, continuous across e. The upwind value of a quantity $\boldsymbol{\beta}\varphi$ is defined as follows:

$$\{\boldsymbol{\varphi}\}_u = \begin{cases} \boldsymbol{\beta}\varphi^1 & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}^1 > 0,\\ \boldsymbol{\beta}\varphi^2 & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}^1 < 0,\\ \boldsymbol{\beta}\{\varphi\} & \text{if } \boldsymbol{\beta}\cdot\boldsymbol{n}^1 = 0. \end{cases}$$

We introduce the following space semi-discretizations of (1).

A. Finite element approximation. Find $(\boldsymbol{u}_h, \pi_h, \mathbb{F}_h) \in C^1([0,T];W_h) \times C([0,T];L_h) \times C^1([0,T];X_h)$ such that

- for all $(\boldsymbol{v}_h, q_h, \mathbb{G}_h) \in W_h \times L_h \times X_h$ and $t \in (0,T)$ the following integral identities hold:

$$\begin{aligned}
(\partial_t \boldsymbol{u}_h(t), \boldsymbol{v}_h) - (\boldsymbol{u}_h(t) \otimes \boldsymbol{u}_h(t), \nabla \boldsymbol{v}_h) + (\nabla \boldsymbol{u}_h(t), \nabla \boldsymbol{v}_h)&\\
- (\pi_h(t), \operatorname{div} \boldsymbol{v}_h) + (\mathbb{F}_h(t)\mathbb{F}_h^\top(t), \nabla \boldsymbol{v}_h) &= 0, \quad \text{(2a)}
\end{aligned}$$

$$(q_h, \operatorname{div} \boldsymbol{u}_h(t)) = 0, \tag{2b}$$

$$(\partial_t \mathbb{F}_h(t), \mathbb{G}_h) - (\boldsymbol{u}_h(t) \cdot \nabla \mathbb{G}_h, \mathbb{F}_h(t)) - \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h(t))\mathbb{F}_h(t), \mathbb{G}_h) - (\nabla \boldsymbol{u}_h(t)\mathbb{F}_h(t), \mathbb{G}_h) = 0; \tag{2c}$$

- $\boldsymbol{u}_h$ and $\mathbb{F}_h$ satisfy the initial conditions

$$\forall \boldsymbol{v} \in W_h \ (\boldsymbol{u}_h(0,\cdot), \boldsymbol{v}) = (\boldsymbol{u}_0, \boldsymbol{v}), \quad \forall \mathbb{G} \in X_h \ (\mathbb{F}_h(0,\cdot), \mathbb{G}) = (\mathbb{F}_0, \mathbb{G}). \tag{3}$$

B. Finite element-finite volume approximation. Find $(\boldsymbol{u}_h, \pi_h, \mathbb{F}_h) \in C^1([0,T]; W_h) \times C([0,T]; L_h) \times C^1([0,T]; Z_h)$ such that

- for all $(\boldsymbol{v}_h, q_h, \mathbb{G}_h) \in W_h \times L_h \times Z_h$ and $t \in (0,T)$, the integral identities (2a), (2b) hold true and furthermore:

$$(\partial_t \mathbb{F}_h(t), \mathbb{G}_h) + \sum_{e \in \mathcal{E}_h^0} (\{\boldsymbol{u}_h(t)\mathbb{F}_h(t)\}_u, [\mathbb{G}_h])_e - \frac{1}{2}(\operatorname{div} \boldsymbol{u}_h(t)\mathbb{F}_h(t), \mathbb{G}_h) - (\nabla \boldsymbol{u}_h(t)\mathbb{F}_h(t), \mathbb{G}_h) = 0; \tag{4}$$

- $\boldsymbol{u}_h$ and $\mathbb{F}_h$ satisfy the initial conditions

$$\forall \boldsymbol{v} \in W_h \ (\boldsymbol{u}_h(0,\cdot), \boldsymbol{v}) = (\boldsymbol{u}_0, \boldsymbol{v}), \quad \forall \mathbb{G} \in Z_h \ (\mathbb{F}_h(0,\cdot), \mathbb{G}) = (\mathbb{F}_0, \mathbb{G}). \tag{5}$$

In what follows we shall assume that the family $\{\mathcal{T}_h\}$ is regular. Consequently, the discrete spaces W_h, L_h, X_h and Z_h enjoy the following properties:

- (inf-sup condition) There exists a constant $C > 0$ independent of $h > 0$ such that

$$\forall q_h \in L_h : \qquad \sup_{\boldsymbol{v}_h \in W_h,\ \boldsymbol{v}_h \neq \boldsymbol{0}} \frac{(q_h, \operatorname{div} \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{1,2}} \geq C\|q_h\|_2; \tag{6}$$

- (interpolation into W_h) There exists an operator $\Pi_h^u : W_0^{1,2}(\Omega; \mathbb{R}^2) \to W_h$ such that
  - for all $\boldsymbol{v} \in W_0^{1,2}(\Omega; \mathbb{R}^2) \cap W^{k,q}(\Omega; \mathbb{R}^2)$, $1 \leq q \leq \infty$, $k \in \{1,2,3\}$, $r \in \{0, \dots, k\}$:

$$\|\Pi_h^u \boldsymbol{v} - \boldsymbol{v}\|_{r,q} \leq C h^{k-r} \|\boldsymbol{v}\|_{k,q}, \tag{7}$$

    where $C > 0$ is independent of $h > 0$;
  - for all $\boldsymbol{v} \in W_0^{1,2}(\Omega; \mathbb{R}^2)$ and $q_h \in L_h$:

$$(q_h, \operatorname{div} \Pi_h^u \boldsymbol{v}) = (q_h, \operatorname{div} \boldsymbol{v}); \tag{8}$$

- (interpolation into L_h) There exists an operator $\Pi_h^\pi : L_0^2(\Omega) \to L_h$ and a constant $C > 0$ independent of $h > 0$, such that

$$\|\Pi_h^\pi s - s\|_{r,q} \le C h^{2-r} \|s\|_{2,q} \tag{9}$$

for all $s \in L_0^2(\Omega) \cap W^{2,q}(\Omega)$, $1 \le q \le \infty$, $r \in \{0, 1\}$;

- (interpolation into X_h) There exists an operator $\Pi_h^F : L^2(\Omega; \mathbb{R}^{2\times 2}) \to X_h$ and a constant $C > 0$ independent of $h > 0$, such that

$$\|\Pi_h^F \mathbb{G} - \mathbb{G}\|_{r,q} \le C h^{2-r} \|\mathbb{G}\|_{2,q}, \tag{10}$$

for all $\mathbb{G} \in W^{2,q}(\Omega; \mathbb{R}^{2\times 2})$, $1 \le q \le \infty$, $r \in \{0, 1\}$.

- (interpolation into Z_h) There exists an operator $\Pi_h^0 : L^2(\Omega; \mathbb{R}^{2\times 2}) \to Z_h$ and a constant $C > 0$ independent of $h > 0$, such that

$$\|\Pi_h^0 \mathbb{G} - \mathbb{G}\|_q \le C h \|\mathbb{G}\|_{1,q}, \tag{11}$$

for all $\mathbb{G} \in W^{1,q}(\Omega; \mathbb{R}^{2\times 2})$, $1 \le q \le \infty$.

In the error analysis of the FE/FV scheme we will need the following variant of *multiplicative trace inequality*.

Lemma 1. *Let $F \in W^{2,2}(\Omega)$. Then there exists a constant $c > 0$ independent of h such that*

$$\sum_{e \in \mathcal{E}_h^0} \|F - \Pi_h^0 F\|_{4,e} \le c h^{3/4} \|F\|_{2,2}. \tag{12}$$

The proof will be done in section 6.

The global in time existence of weak solutions to (1) is an open problem. It is only known that (1) has a unique strong solution for small data or on a short time interval [3, 4, 1]. For example Lin et al. [4] proved that if the initial data satisfy

$$\mathbb{F}_0 = \nabla \times \boldsymbol{\Phi}_0, \quad \nabla \boldsymbol{\Phi}_0 \in W^{k,2}(\Omega),\ \boldsymbol{u}_0 \in W^{k,2}(\Omega),\ k \ge 2, \tag{13}$$

then there exists a positive time T, which depends only on $\|\nabla \boldsymbol{\varphi}_0\|_{2,2}$ and $\|\boldsymbol{u}_0\|_{2,2}$, such that the system (1) possesses a unique solution in the time interval $[0, T]$ with

$$\begin{aligned} &\partial_t^j \nabla^\alpha \boldsymbol{u} \in L^\infty(0, T; W^{k-2j-|\alpha|,2}(\Omega)) \cap L^2(0, T; W^{k-2j-|\alpha|+1,2}(\Omega)), \\ &\partial_t^j \nabla^\alpha \mathbb{F} \in L^\infty(0, T; W^{k-2j-|\alpha|,2}(\Omega)), \end{aligned} \tag{14}$$

for all j and α satisfying $2j + |\alpha| \le k$ (see [4], Theorem 2.2).

For the approximate problems A and B we assume the existence of a unique global in time solution that satisfies the inequality:

$$\sup_{\tau \in (0,T)} \|\boldsymbol{u}_h(\tau, \cdot)\|_2^2 + \sup_{\tau \in (0,T)} \|\mathbb{F}_h(\tau, \cdot)\|_2^2 + \int_0^\tau \|\nabla \boldsymbol{u}_h\|_2^2 \le C, \tag{15}$$

where $C > 0$ depends on the data $(\Omega,\ \boldsymbol{u}_0,\ \mathbb{F}_0)$, but is independent of $h > 0$.

3 Main result

We state the main result on the error rates.

Theorem 1. *Let the family* $\{\mathcal{T}_h\}_{h>0}$ *be regular, the initial data* $(\boldsymbol{u}_0, \nabla \times \boldsymbol{\Phi}_0)$ *satisfy* $\boldsymbol{u}_0, \nabla\boldsymbol{\Phi}_0 \in W^{2,2}(\Omega)$ *and* $[0,T]$ *be the maximal time interval in which the strong solution* $(\boldsymbol{u}, \pi, \mathbb{F})$ *to* (1) *exists. Then there exist constants* $h_0 > 0$ *and* $C > 0$ *such that for all* $h \in (0, h_0)$ *it holds:*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(0,T;L^2(\Omega))} + \|\mathbb{F} - \mathbb{F}_h\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch, \quad (16)$$

where $(\boldsymbol{u}_h, \pi_h, \mathbb{F}_h)$ *is the solution to* (2)–(3)*.*

Similarly, there exist constants $h_0 > 0$ *and* $C > 0$ *such that for all* $h \in (0, h_0)$ *it holds:*

$$\|\boldsymbol{u} - \bar{\boldsymbol{u}}_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\boldsymbol{u} - \bar{\boldsymbol{u}}_h)\|_{L^2(0,T;L^2(\Omega))} + \|\mathbb{F} - \bar{\mathbb{F}}_h\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^{3/4}, \quad (17)$$

where $(\bar{\boldsymbol{u}}_h, \bar{\pi}_h, \bar{\mathbb{F}}_h)$ *is the solution to* (2a)*,* (2b)*,* (4) *and* (5)*.*

4 Error estimates for finite element approximation

In this section we prove the first part of Theorem 1.

Regularity of the solution to (1)**.** In accordance with (13)–(14), the assumptions of Theorem 1 imply that the exact solution $(\boldsymbol{u}, \pi, \mathbb{F})$ satisfies:

$$\begin{aligned} \partial_t \boldsymbol{u} &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)), \\ \boldsymbol{u} &\in L^\infty(0,T;W^{2,2}(\Omega)) \cap L^2(0,T;W^{3,2}(\Omega)), \\ \partial_t \mathbb{F} &\in L^\infty(0,T;L^2(\Omega)), \\ \mathbb{F} &\in L^\infty(0,T;W^{2,2}(\Omega)). \end{aligned} \quad (18)$$

From (1b) and (1c) we furthermore obtain that

$$\begin{aligned} \nabla\pi &= \operatorname{div}(\mathbb{F}\mathbb{F}^\top) - \partial_t \boldsymbol{u} - \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \Delta\boldsymbol{u} \in L^2(0,T;W^{1,2}(\Omega)), \\ \partial_t \mathbb{F} &= \nabla\boldsymbol{u}\mathbb{F} - \boldsymbol{u} \cdot \nabla\mathbb{F} \in L^\infty(0,T;W^{1,2}(\Omega)). \end{aligned} \quad (19)$$

Equations satisfied by the errors. Multiplying (1) with $\boldsymbol{v}_h \in W_h$, $q_h \in L_h$ and $\mathbb{G}_h \in X_h$, respectively, and integrating over Ω, we obtain for a.a. $t \in (0,T)$ the following identities:

$$(\partial_t \boldsymbol{u}(t), \boldsymbol{v}_h) - (\boldsymbol{u}(t) \otimes \boldsymbol{u}_h(t), \nabla\boldsymbol{v}_h) + (\nabla\boldsymbol{u}(t), \nabla\boldsymbol{v}_h) - (\pi(t), \operatorname{div}\boldsymbol{v}_h) + (\mathbb{F}(t)\mathbb{F}^\top(t), \nabla\boldsymbol{v}_h) = 0, \quad (20a)$$

$$(q_h, \operatorname{div} \boldsymbol{u}(t)) = 0, \tag{20b}$$

$$(\partial_t \mathbb{F}(t), \mathbb{G}_h) - (\boldsymbol{u}(t) \cdot \nabla \mathbb{G}_h, \mathbb{F}(t)) - (\nabla \boldsymbol{u}(t) \mathbb{F}(t), \mathbb{G}_h) = 0; \tag{20c}$$

Let us denote $e_u := \boldsymbol{u} - \boldsymbol{u}_h$, $e_\pi := \pi - \pi_h$ and $e_F := \mathbb{F} - \mathbb{F}_h$. Taking the difference of (20) and (2) we obtain:

$$\int_0^T \big[(\partial_t e_u, \boldsymbol{v}_h) - (e_u \otimes \boldsymbol{u} + \boldsymbol{u}_h \otimes e_u, \nabla \boldsymbol{v}_h) + (\nabla e_u, \nabla \boldsymbol{v}_h) - (e_\pi, \operatorname{div} \boldsymbol{v}_h) + (\mathbb{F}\mathbb{F}^\top - \mathbb{F}_h \mathbb{F}_h^\top, \nabla \boldsymbol{v}_h)\big] = 0, \tag{21a}$$

$$\int_0^T (q_h, \operatorname{div} e_u) = 0, \tag{21b}$$

$$\int_0^T (\partial_t e_F, \mathbb{G}_h) - (u_i \mathbb{F} - u_{hi} \mathbb{F}_h, \partial_i \mathbb{G}_h) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \mathbb{G}_h) - (\nabla \boldsymbol{u} \mathbb{F} - \nabla \boldsymbol{u}_h \mathbb{F}_h, \mathbb{G}_h) = 0, \tag{21c}$$

for any $(\boldsymbol{v}_h, q_h, \mathbb{G}_h) \in L^2(0,T; W_h) \times L^2(0,T; L_h) \times L^2(0,T; X_h)$.

Estimates of the errors. In order to derive estimates of e_u, e_π, e_F in suitable norms, we decompose

$$e_u = (\boldsymbol{u} - \Pi_h^u \boldsymbol{u}) + (\Pi_h^u \boldsymbol{u} - \boldsymbol{u}_h) =: \eta_u + \delta_u.$$

Similarly we introduce

$$\eta_\pi := \pi - \Pi_h^\pi \pi, \quad \delta_\pi := \Pi_h^\pi \pi - \pi_h, \quad \eta_F := \mathbb{F} - \Pi_h^F \mathbb{F}, \quad \delta_F := \Pi_h^F \mathbb{F} - \mathbb{F}_h.$$

The properties of the interpolation operators imply the following estimates:

$$\sup_{\tau \in (0,T)} \|\eta_u(\tau, \cdot)\|_2^2 \le Ch^4 \sup_{\tau \in (0,T)} \|\boldsymbol{u}(\tau, \cdot)\|_{2,2}^2,$$

$$\int_0^T \|\nabla \eta_u\|_2^2 \le Ch^4 \int_0^T \|\boldsymbol{u}\|_{3,2}^2,$$

$$\sup_{\tau \in (0,T)} \|\eta_F(\tau, \cdot)\|_2^2 \le Ch^4 \sup_{\tau \in (0,T)} \|\mathbb{F}(\tau, \cdot)\|_{2,2}^2.$$

Hence, with the help of (18) we obtain:

$$\sup_{\tau \in (0,T)} \big(\|e_u(\tau, \cdot)\|_2^2 + \|e_F(\tau, \cdot)\|_2^2\big) + \int_0^T \|\nabla e_u\|_2^2 \le Ch^4 + \sup_{\tau \in (0,T)} \big(\|\delta_u(\tau, \cdot)\|_2^2 + \|\delta_F(\tau, \cdot)\|_2^2\big) + \int_0^T \|\nabla \delta_u\|_2^2. \tag{22}$$

The proof of Theorem 1 will be complete as soon as we estimate the δ-terms in the previous inequality by Ch^2. Due to the initial conditions (3) it holds that $\|\delta_u(0)\|_2 = \|\delta_F(0)\|_2 = 0$. Hence, for any $\tau \in (0, T)$ we have:

$$\begin{aligned}
&\frac{1}{2}\left(\|\delta_u(\tau,\cdot)\|_2^2 + \|\delta_F(\tau,\cdot)\|_2^2\right) + \int_0^\tau \|\nabla\delta_u\|_2^2 \\
&\quad= \frac{1}{2}\left(\|\delta_u(\tau,\cdot)\|_2^2 - \|\delta_u(0,\cdot)\|_2^2 + \|\delta_F(\tau,\cdot)\|_2^2 - \|\delta_F(0,\cdot)\|_2^2\right) + \int_0^\tau \|\nabla\delta_u\|_2^2 \\
&\quad= \int_0^\tau [(\partial_t\delta_u, \delta_u) + (\partial_t\delta_F, \delta_F) + (\nabla\delta_u, \nabla\delta_u)] \\
&\quad= \int_0^\tau [(\partial_t e_u, \delta_u) + (\partial_t e_F, \delta_F) + (\nabla e_u, \nabla\delta_u)] \\
&\quad\quad - \int_0^\tau [(\partial_t\eta_u, \delta_u) + (\partial_t\eta_F, \delta_F) + (\nabla\eta_u, \nabla\delta_u)] = J_1 + J_2. \qquad (23)
\end{aligned}$$

The last integral can be estimated using the Hölder and the Young inequality and (7)–(10):

$$\begin{aligned}
J_2 &\leq \left(\int_0^T \|\partial_t\eta_u\|_2^2\right)^{1/2}\left(\int_0^\tau \|\delta_u\|_2^2\right)^{1/2} + \left(\int_0^T \|\partial_t\eta_F\|_2^2\right)^{1/2}\left(\int_0^\tau \|\delta_F\|_2^2\right)^{1/2} \\
&\quad + \left(\int_0^T \|\nabla\eta_u\|_2^2\right)^{1/2}\left(\int_0^\tau \|\nabla\delta_u\|_2^2\right)^{1/2} \\
&\leq \frac{1}{2}\int_0^\tau \|\delta_u\|_2^2 + \frac{1}{2}\int_0^T \|\partial_t\eta_u\|_2^2 + \frac{1}{2}\int_0^\tau \|\delta_F\|_2^2 + \frac{1}{2}\int_0^T \|\partial_t\eta_F\|_2^2 \\
&\quad + \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha}\int_0^T \|\nabla\eta_u\|_2^2 \\
&\leq \frac{1}{2}\int_0^\tau \|\delta_u\|_2^2 + Ch^2\int_0^T \|\partial_t\boldsymbol{u}\|_{1,2}^2 + \frac{1}{2}\int_0^\tau \|\delta_F\|_2^2 + Ch^2\int_0^T \|\partial_t\mathbb{F}\|_{1,2}^2 \\
&\quad + \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^4}{\alpha}\int_0^T \|\boldsymbol{u}\|_{3,2}^2. \qquad (24)
\end{aligned}$$

Here $\alpha > 0$ is arbitrary number to be specified later. The regularity of the solution (18), (19) implies that

$$\boxed{J_2 \leq \frac{1}{2}\int_0^\tau \|\delta_u\|_2^2 + \frac{1}{2}\int_0^\tau \|\delta_F\|_2^2 + \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + Ch^2 + \frac{Ch^4}{\alpha}.} \qquad (25)$$

The first integral on the r.h.s. of (23) can be substituted from (21) as follows:

$$J_1 = \int_0^\tau \big[(e_u \otimes \boldsymbol{u} + \boldsymbol{u}_h \otimes e_u, \nabla\delta_u) + (e_\pi, \operatorname{div}\delta_u) - (\mathbb{F}\mathbb{F}^\top - \mathbb{F}_h\mathbb{F}_h^\top, \nabla\delta_u) + (u_i\mathbb{F} - u_{hi}\mathbb{F}_h, \partial_i\delta_F) + \frac{1}{2}((\operatorname{div}\boldsymbol{u}_h)\mathbb{F}_h, \delta_F) + (\nabla\boldsymbol{u}\mathbb{F} - \nabla\boldsymbol{u}_h\mathbb{F}_h, \delta_F)\big] =: \int_0^\tau \sum_{j=1}^6 T_j.$$

We shall estimate the resulting terms $T_1, \ldots, T_6$ subsequently.

Term T_1 can be decomposed as follows:

$$T_1 = (\eta_u \otimes \boldsymbol{u}, \nabla\delta_u) + \underbrace{(\delta_u \otimes \boldsymbol{u}, \nabla\delta_u)}_{=0} + (\boldsymbol{u}_h \otimes \eta_u, \nabla\delta_u) + (\boldsymbol{u}_h \otimes \delta_u, \nabla\delta_u),$$

where the second term vanishes since $\operatorname{div}\boldsymbol{u} = 0$. The remaining terms can be estimated with help of the Hölder, Sobolev and Young inequality and the properties of the interpolation operators:

$$\int_0^\tau |(\eta_u \otimes \boldsymbol{u}, \nabla\delta_u)| \le C\int_0^\tau \|\eta_u\|_2^{1/2}\|\nabla\eta_u\|_2^{1/2}\|\boldsymbol{u}\|_2^{1/2}\|\nabla\boldsymbol{u}\|_2^{1/2}\|\nabla\delta_u\|_2$$

$$\le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha}\left(\sup_{(0,T)}\|\eta_u\|_2\right)\left(\sup_{(0,T)}\|\boldsymbol{u}\|_2\right)\left(\int_0^T \|\nabla\boldsymbol{u}\|_2^2\right)^{1/2}\left(\int_0^T \|\nabla\eta_u\|_2^2\right)^{1/2}$$

$$\le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^4}{\alpha}\left(\sup_{(0,T)}\|\boldsymbol{u}\|_{2,2}\right)^2\left(\int_0^T \|\boldsymbol{u}\|_{3,2}^2\right).$$

Similarly we get:

$$\int_0^\tau |(\boldsymbol{u}_h \otimes \eta_u, \nabla\delta_u)| \le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^4}{\alpha}\left(\sup_{(0,T)}\|\boldsymbol{u}_h\|_2\right)\left(\sup_{(0,T)}\|\boldsymbol{u}\|_{2,2}\right)\left(\int_0^T \|\boldsymbol{u}_h\|_{1,2}^2\right)^{1/2}\left(\int_0^T \|\boldsymbol{u}\|_{3,2}^2\right)^{1/2}.$$

Finally, using the same inequalities we obtain:

$$\int_0^\tau |(\boldsymbol{u}_h \otimes \delta_u, \nabla\delta_u)| \le \int_0^\tau \|\boldsymbol{u}_h\|_2^{1/2}\|\nabla\boldsymbol{u}_h\|_2^{1/2}\|\delta_u\|_2^{1/2}\|\nabla\delta_u\|_2^{3/2}$$

$$\le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha^3}\left(\sup_{(0,T)}\|\boldsymbol{u}_h\|_2\right)^2\int_0^\tau \|\boldsymbol{u}_h\|_{1,2}^2\|\delta_u\|_2^2.$$

Due to (18), (19) and (15), the previous estimates altogether yield:

$$\boxed{\int_0^\tau T_1 \le 3\alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha^3}\int_0^\tau \|\boldsymbol{u}_h\|_{1,2}^2\|\delta_u\|_2^2 + \frac{Ch^4}{\alpha}.} \tag{26}$$

Term T_2. Thanks to (21b) and (8), we can write:

$$T_2 = (\eta_\pi, \operatorname{div}\delta_u) + \underbrace{(\delta_\pi, \operatorname{div} e_u)}_{=0} - \underbrace{(\delta_\pi, \operatorname{div}\eta_u)}_{=0} \le \|\eta_\pi\|_2 \|\nabla\delta_u\|_2 \le \frac{Ch^2}{\alpha}\|\pi\|_{1,2}^2 + \alpha\|\nabla\delta_u\|_2.$$

Thus

$$\boxed{\int_0^\tau T_2 \le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^2}{\alpha}.} \tag{27}$$

Terms T_3 and T_6 can be rewritten in the following way:

$$T_3 + T_6 = -(\eta_F \mathbb{F}^\top, \nabla\delta_u) - (\Pi_h^F \mathbb{F}\eta_F^\top, \nabla\delta_u) + (\delta_F\eta_F^\top, \nabla\boldsymbol{u}) + (\delta_F \Pi_h^F \mathbb{F}^\top, \nabla\eta_u) + (\delta_F\delta_F^\top, \nabla\Pi_h^u\boldsymbol{u}) - (\Pi_h^F\mathbb{F}\delta_F^\top, \nabla\delta_u).$$

We estimate the resulting terms as follows:

$$\int_0^\tau |(\eta_F\mathbb{F}^\top, \nabla\delta_u)| \le \int_0^\tau \|\mathbb{F}\|_\infty \|\eta_F\|_2 \|\nabla\delta_u\|_2 \le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^4}{\alpha}\left(\sup_{(0,T)}\|\mathbb{F}\|_{2,2}\right)^4,$$

$$\int_0^\tau |(\Pi_h^F\mathbb{F}\eta_F^\top, \nabla\delta_u)| \le \int_0^\tau \|\Pi_h^F\mathbb{F}\|_\infty \|\eta_F\|_2 \|\nabla\delta_u\|_2 \le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{Ch^4}{\alpha}\left(\sup_{(0,T)}\|\mathbb{F}\|_{2,2}\right)^4,$$

$$\int_0^\tau |(\delta_F\eta_F^\top, \nabla\boldsymbol{u})| \le \int_0^\tau \|\delta_F\|_2\|\eta_F\|_3\|\nabla\boldsymbol{u}\|_6 \le \int_0^\tau \|\delta_F\|_2^2\|\boldsymbol{u}\|_{2,2}^2 + Ch^2\int_0^T \|\mathbb{F}\|_{2,2}^2,$$

$$\int_0^\tau |(\delta_F\Pi_h^F\mathbb{F}^\top, \nabla\eta_u)| \le \int_0^\tau \|\delta_F\|_2\|\Pi_h^F\mathbb{F}\|_\infty\|\nabla\eta_u\|_2 \le \left(\sup_{(0,T)}\|\mathbb{F}\|_{2,2}\right)^2\int_0^\tau \|\delta_F\|_2^2 + Ch^4\int_0^T \|\boldsymbol{u}\|_{3,2}^2,$$

$$\int_0^\tau |(\delta_F\delta_F^\top, \nabla\Pi_h^u\boldsymbol{u})| \le C\int_0^\tau \|\delta_F\|_2^2\|\boldsymbol{u}\|_{3,2},$$

$$\int_0^\tau |(\Pi_h^F\mathbb{F}\delta_F^\top, \nabla\delta_u)| \le \int_0^\tau \|\Pi_h^F\mathbb{F}\|_\infty\|\delta_F\|_2\|\nabla\delta_u\|_2 \le \alpha\int_0^\tau \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha}\left(\sup_{(0,T)}\|\mathbb{F}\|_{2,2}\right)^2\int_0^\tau \|\delta_F\|_2^2.$$

In summary we have:

$$\boxed{\int_0^\tau T_3 + T_6 \le 3\alpha \int_0^\tau \|\nabla \delta_u\|_2^2 + C\int_0^\tau \left(\frac{1}{\alpha} + \|\boldsymbol{u}\|_{3,2}\right) \|\delta_F\|_2^2 + Ch^2 + \frac{Ch^4}{\alpha}.} \tag{28}$$

Before proceeding to the remaining two terms we recall some basic identities related to the advective term $\boldsymbol{u} \cdot \nabla \mathbb{F}$.

Lemma 2. *Let $\boldsymbol{v} \in W^{1,2}(\Omega; \mathbb{R}^2)$ be such that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. Then*

$$\forall \mathbb{G} \in W^{1,2}(\Omega; \mathbb{R}^{2\times 2}): \ (v_i \mathbb{G}, \partial_i \mathbb{G}) = \frac{1}{2}((\operatorname{div} \boldsymbol{v})\mathbb{G}, \mathbb{G}); \tag{29}$$

$$\forall \mathbb{G}, \mathbb{H} \in W^{1,2}(\Omega; \mathbb{R}^{2\times 2}): (v_i \mathbb{G}, \partial_i \mathbb{H}) = -((\operatorname{div} \boldsymbol{v})\mathbb{G}, \mathbb{H}) - (v_i \mathbb{H}, \partial_i \mathbb{G}). \tag{30}$$

The proof is done by integrating by parts.

Terms T_4 and T_5. We rearrange these terms as follows:

$$T_4 + T_5 = (u_i \eta_F, \partial_i \delta_F) + (e_{ui} \Pi_h^F \mathbb{F}, \partial_i \delta_F) + (u_{hi} \delta_F, \partial_i \delta_F) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \delta_F). \tag{31}$$

Using (30) we obtain:

$$(u_i \eta_F, \partial_i \delta_F) = -(u_i \delta_F, \partial_i \eta_F), \tag{32}$$

$$(e_{ui} \Pi_h^F \mathbb{F}, \partial_i \delta_F) = ((\operatorname{div} \boldsymbol{u}_h) \Pi_h^F \mathbb{F}, \delta_F) - (e_{ui} \delta_F, \partial_i \Pi_h^F \mathbb{F}). \tag{33}$$

Due to (29), the last two terms in (31) can be rewritten as

$$\begin{aligned}(u_{hi} \delta_F, \partial_i \delta_F) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \delta_F) &= \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\delta_F, \delta_F) + \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\mathbb{F}_h, \delta_F) \\ &= \frac{1}{2}((\operatorname{div} \boldsymbol{u}_h)\Pi_h^F \mathbb{F}, \delta_F). \end{aligned} \tag{34}$$

Equations (31)–(34) and the fact that $\operatorname{div} \boldsymbol{u} = 0$ yield:

$$T_4 + T_5 = -(u_i \delta_F, \partial_i \eta_F) - \frac{3}{2}((\operatorname{div} e_u)\Pi_h^F \mathbb{F}, \delta_F) - (e_{ui} \delta_F, \partial_i \Pi_h^F \mathbb{F}). \tag{35}$$

Decomposing $e_u = \eta_u + \delta_u$ and using similar arguments as in the previous paragraphs we can estimate terms on the r.h.s. of (35) as follows:

$$\begin{aligned} -\int_0^\tau (u_i \delta_F, \partial_i \eta_F) &\le \int_0^\tau \|\boldsymbol{u}\|_\infty \|\delta_F\|_2 \|\nabla \eta_F\|_2 \\ &\le \frac{1}{2}\int_0^\tau \|\boldsymbol{u}\|_{2,2}^2 \|\delta_F\|_2^2 + \frac{1}{2}\int_0^T \|\nabla \eta_F\|_2^2 \\ &\le \left(\sup_{(0,T)} \|\boldsymbol{u}\|_{2,2}\right)^2 \int_0^T \|\delta_F\|_2^2 + Ch^2 \left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2}\right)^2, \end{aligned} \tag{36}$$

$$-\frac{3}{2}\int_0^\tau ((\operatorname{div} e_u)\Pi_h^F \mathbb{F}, \delta_F) \leq \frac{3}{2}\int_0^\tau (\|\nabla \eta_u\|_2 + \|\nabla \delta_u\|_2)\,\|\Pi_h^F \mathbb{F}\|_\infty \|\delta_F\|_2$$
$$\leq Ch^4 \int_0^T \|\boldsymbol{u}\|_{3,2}^2 + \alpha \int_0^\tau \|\nabla \delta_u\|_2^2 + \frac{C}{\alpha}\left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2}\right)^2 \int_0^\tau \|\delta_F\|_2^2,$$

$$-\int_0^\tau (e_{ui}\delta_F, \partial_i \Pi_h^F \mathbb{F}) \leq \int_0^\tau (\|\eta_u\|_4 + \|\delta_u\|_4)\,\|\delta_F\|_2 \|\nabla \Pi_h^F \mathbb{F}\|_4$$
$$\leq Ch^4 \int_0^T \|\boldsymbol{u}\|_{3,2}^2 + \alpha \int_0^T \|\nabla \delta_u\|_2^2 + \frac{C}{\alpha}\left(\sup_{(0,T)} \|\mathbb{F}\|_{2,2}\right)^2 \int_0^T \|\delta_F\|_2^2. \quad (37)$$

Altogether we have:

$$\boxed{\int_0^\tau T_4 + T_5 \leq 2\alpha \int_0^\tau \|\nabla \delta_u\|_2^2 + \frac{C}{\alpha}\int_0^\tau \|\delta_F\|_2^2 + Ch^2 + Ch^4.} \quad (38)$$

Gronwall's inequality for the errors and the end of the proof. Collecting (25), (26), (27), (28), (38) and inserting the result to (23), we obtain:

$$\frac{1}{2}\left(\|\delta_u(\tau)\|_2^2 + \|\delta_F(\tau)\|_2^2\right) + (1 - C\alpha)\int_0^\tau \|\nabla \delta_u\|_2^2$$
$$\leq C\int_0^\tau \left(1 + \frac{\|\boldsymbol{u}_h\|_{1,2}^2}{\alpha^3}\right)\|\delta_u\|_2^2 + C\int_0^\tau \left(\frac{1}{\alpha} + \|\boldsymbol{u}\|_{3,2}\right)\|\delta_F\|_2^2 + \frac{C}{\alpha}\left(h^2 + h^4\right).$$

Choosing α sufficiently small and using the Gronwall inequality we obtain for $h \in (0, h_0)$:

$$\sup_{(0,T)} \|\delta_u\|_2^2 + \sup_{(0,T)} \|\delta_F\|_2^2 + \int_0^T \|\delta_u\|_{1,2}^2 \leq Ch^2.$$

This in accordance with (22) completes the proof of Theorem 1.

Remark 1. *The question arises, whether the derived error estimate can be improved provided the exact solution is more regular. Essentially, the first order with respect to h arises due to (36), where $\|\nabla \eta_F\|_2$ can only be bounded by $O(h)$ due to the piecewise linear approximation of $\mathbb{F}$. In view of this, it seems that the result cannot be improved.*

Remark 2. *One can easily incorporate a forcing term $\boldsymbol{f} \in L^2(0,T;L^2(\Omega;\mathbb{R}^2))$ on the right hand side of the momentum equation. This would yield the same error estimates.*

5 Error estimates for finite element - finite volume approximation

The essential difference in the proof of error estimates for the solution of problem (2a), (2b), (4) and (5) is in the term "$\boldsymbol{u} \cdot \nabla \mathbb{F}$". It is therefore sufficient to estimate the error of

$$\int_0^T \left((\boldsymbol{u} \cdot \nabla \mathbb{F}, \delta_F) - \sum_{e \in \mathcal{E}_h^0} (\{\boldsymbol{u}_h \mathbb{F}_h\}_u, [\delta_F])_e + \frac{1}{2} (\operatorname{div} \boldsymbol{u}_h \mathbb{F}_h, \delta_F) \right). \tag{39}$$

Proof. (Error estimate for term "$\boldsymbol{u} \cdot \nabla \mathbb{F}$")
Let us first denote by F one component of the tensor $\mathbb{F}$, similarly by G_h one component of $\mathbb{G}_h$. Then we can write:

$$(\boldsymbol{u} \cdot \nabla F, G_h) = \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(\boldsymbol{u}\mathrm{F}) \mathrm{G}_h \; d\boldsymbol{x} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{u} \cdot \boldsymbol{n}) \mathrm{F} \; [\mathrm{G}_h] \; dS$$
$$= \sum_{e \in \mathcal{E}_h^0} \int_e \{\boldsymbol{u} F\} [\mathrm{G}_h] \; dS.$$

For the approximate solution it holds:

$$\int_e \{\boldsymbol{u}_h \mathrm{F}_h\}_u \; [\mathrm{G}_h] \; dS = \int_e \{\boldsymbol{u}_h \mathrm{F}_h\} [\mathrm{G}_h] \; dS + \int_e \frac{|\boldsymbol{u}_h \cdot \boldsymbol{n}|}{2} [\mathrm{F}_h][\mathrm{G}_h] \; dS \quad \forall \; \mathrm{G}_h \in Z_h.$$

Thus, we can rewrite (39) as follows:

$$\ldots = \int_0^T \sum_{e \in \mathcal{E}_h^0} \int_e \{(\boldsymbol{u} - \boldsymbol{u}_h)\mathrm{F}\} [\delta_F] \; dS \; dt + \int_0^T \sum_{e \in \mathcal{E}_h^0} \int_e \{\boldsymbol{u}_h (\mathrm{F} - \mathrm{F}_h)\} [\delta_F] \; dS \; dt +$$
$$+ \int_0^T \sum_{e \in \mathcal{E}_h^0} \int_e \frac{|\boldsymbol{u}_h \cdot \boldsymbol{n}|}{2} [\mathrm{F} - \mathrm{F}_h][\delta_F] \; dS \; dt + \frac{1}{2} \int_0^T \int_\Omega \operatorname{div} \boldsymbol{u}_h \mathrm{F}_h \mathrm{G}_h =$$
$$= \sum_{i=1}^4 I_i.$$

In what follows, we will estimate these terms separately.

Term I_1.

$$I_1 = \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{(\boldsymbol{u}-\boldsymbol{u}_h)\mathrm{F}\}[\delta_F]\ dS\ dt = \int_0^T \sum_{T\in\mathcal{T}_h} \int_{\partial T} (\boldsymbol{u}-\boldsymbol{u}_h)\cdot\boldsymbol{n}\ \mathrm{F}\ \delta_F\ dS\ dt =$$

$$= \int_0^T \sum_{T\in\mathcal{T}_h} \int_T \operatorname{div}\big((\boldsymbol{u}-\boldsymbol{u}_h)\mathrm{F}\big)\delta_F\ d\boldsymbol{x}\ dt =$$

$$= \int_0^T \int_\Omega \operatorname{div}\delta_u\ \mathrm{F}\delta_F\ d\boldsymbol{x}\ dt + \int_0^T \int_\Omega \operatorname{div}\eta_u\ \mathrm{F}\delta_F\ d\boldsymbol{x}\ dt +$$

$$+ \int_0^T \int_\Omega \delta_u\cdot\nabla\mathrm{F}\ \delta_F\ d\boldsymbol{x}\ dt + \int_0^T \int_\Omega \eta_u\cdot\nabla\mathrm{F}\ \delta_F\ d\boldsymbol{x}\ dt \le$$

$$\le \int_0^T c_1\|\nabla\delta_u\|_{2,\Omega}\|\mathrm{F}\|_{2,2,\Omega}\|\delta_F\|_{2,\Omega} + c_2\|\nabla\eta_u\|_{2,\Omega}\|\mathrm{F}\|_{2,2,\Omega}\|\delta_F\|_{2,\Omega} +$$

$$+ \int_0^T c_3\|\delta_u\|_{4,\Omega}\|\nabla\mathrm{F}\|_{4,\Omega}\|\delta_F\|_{2,\Omega} + c_4\|\eta_u\|_{4,\Omega}\|\nabla\mathrm{F}\|_{4,\Omega}\|\delta_F\|_{2,\Omega} \le$$

$$\le \alpha\int_0^T \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha}\Big(\sup_{(0,T)}\|\mathrm{F}\|_{2,2}\Big)^2 \int_0^T \|\delta_F\|_2^2 +$$

$$+ C\int_0^T \|\delta_F\|_2^2 + Ch^4\Big(\sup_{(0,T)}\|\mathrm{F}\|_{2,2}\Big)^2 \int_0^T \|\boldsymbol{u}\|_{3,2}^2.$$

Terms $I_2 + I_4$.

$$I_2 = \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{\boldsymbol{u}_h(\mathrm{F}-\mathrm{F}_h)\}[\delta_F]\ dS\ dt =$$

$$= \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{\boldsymbol{u}_h\delta_F\}[\delta_F]\ dS\ dt + \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \{\boldsymbol{u}_h\eta_F\}[\delta_F]\ dS\ dt = A_1 + A_2$$

$$\begin{aligned} A_1 &= \int_0^T \int_\Omega \frac{1}{2}\operatorname{div}\boldsymbol{u}_h\delta_F^2\ d\boldsymbol{x}\ dt = -\int_0^T \int_\Omega \frac{1}{2}\operatorname{div} e_u\Pi_h^0\mathrm{F}\delta_F\ d\boldsymbol{x}\ dt - I_4 \\ &= -\int_0^T \int_\Omega \frac{1}{2}\operatorname{div}\eta_u\Pi_h^0\mathrm{F}\delta_F\ d\boldsymbol{x}\ dt - \int_0^T \int_\Omega \frac{1}{2}\operatorname{div}\delta_u\Pi_h^0\mathrm{F}\delta_F\ d\boldsymbol{x}\ dt - I_4 \\ &\le C\int_0^T \|\eta_u\|_{1,2}\|\delta_F\|_2 + C\int_0^T \|\nabla\delta_u\|_2\|\delta_F\|_2 - I_4 \\ &\le Ch^2 + \alpha\int_0^T \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha}\int_0^T \|\delta_F\|_2^2 - I_4. \end{aligned}$$

$$
\begin{aligned}
A_2 &\le \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}\{|\eta_F|\}|[\delta_F]|\ dS\ dt \\
&\le \int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e}\|c_e^{1/2}\|_{4,e}\|\{|\eta_F|\}\|_{4,e} \\
&\le \alpha \int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e} + \frac{C}{\alpha}h^{3/2}\left(\sup_{(0,T)}\|\mathbb{F}\|_{2,2}\right)^2 \int_0^T \|\nabla \boldsymbol{u}_h\|_2^2\ dt,
\end{aligned}
$$

where the trace inequalities for η_F and c_e were used and $c_e = \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}$. Further we have:

$$
\begin{aligned}
I_3 &= \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}[\eta_F][\delta_F]\ dS\ dt + \int_0^T \sum_{e\in\mathcal{E}_h^0} \int_e \frac{|\boldsymbol{u}_h\cdot\boldsymbol{n}|}{2}[\delta_F][\delta_F]\ dS\ dt = \\
&= B_1 + B_2,
\end{aligned}
$$

where B_1 can be estimated in the same way as A_2 and

$$
B_2 = \int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e}^2.
$$

Finally, we can conclude:

$$
\boxed{(1-\alpha)\int_0^T \sum_{e\in\mathcal{E}_h^0} \|c_e^{1/2}[\delta_F]\|_{2,e}^2 \le C\alpha \int_0^T \|\nabla\delta_u\|_2^2 + \frac{C}{\alpha}\int_0^T \|\delta_F\|_2^2 + Ch^{3/2} + Ch^2 + Ch^4}. \tag{40}
$$

□

6 Multiplicative trace inequality

This section is devoted to the proof of Lemma 1. We first state an auxiliary result.

Lemma 3. *There exists a constant $c > 0$ independent of h and T, such that for $T \in \tau_h$, $v \in H^1(T)$, $h \in (0, h_0)$ we have*

$$
\|v\|_{L^4(\partial T)}^4 \le c\left[4\|v\|_{L^6(T)}^3 |v|_{H^1(T)} + \frac{d}{h_T}\|v\|_{L^4(T)}^4\right]. \tag{41}
$$

Proof of Lemma 1. Assume that (41) holds, take $v = F - \Pi_h^F F$, where

$$
\begin{aligned}
&F \in W^{2,2}(\Omega) \\
&\Pi_h^F F : W^{2,2}(\Omega) \to \mathcal{P}_0, \quad \mathcal{P}_0 = \{p_{|_T} = const.\}.
\end{aligned}
$$

Then it holds

$$\|F - \Pi_h^F F\|^4_{L^4(\partial T)} = \|\eta_F\|^4_{L^4(\partial T)} \le c_1 \left[4h^3 \|F\|^3_{W^{1,6}(T)} |F|_{H^1(T)} + \frac{2}{h} h^4 \|F\|^4_{W^{1,4}(T)} \right] \le$$
$$\le c_2 \left[h^3 \|F\|^4_{W^{2,2}(T)} + h^3 \|F\|^4_{W^{2,2}(T)} \right] \le$$
$$\le c_3 h^3 \|F\|^4_{W^{2,2}(T)}.$$

This implies

$$\|F - \Pi_h^F F\|_{L^4(e)} \le c h^{3/4} \|F\|_{W^{2,2}(T)}.$$

□

The proof of Lemma 3 is analogous as the proof of Lemma 3.1. in [2].

Proof. Let us denote by $\tilde{\boldsymbol{x}}_i$ the center of the largest d-dimensional ball inscribed in $T = T_i$. W.l.o.g. put $\tilde{\boldsymbol{x}}_i$ to the origin of the coordinate system.
We start from the relation

$$\int_{\partial T} v^4 \; \boldsymbol{x} \cdot \boldsymbol{n} \; dS = \int_T \nabla \cdot (v^4 \; \boldsymbol{x}) \; d\boldsymbol{x} \quad \forall \, v \in H^1(T).$$

Let us denote $\boldsymbol{n}_{ij}$ the outer normal to T_i on $\partial T_{ij} = \partial T_i \cap \partial T_j$, where T_j is a neighbor of T_i, i.e.

$$j \in S(i) = \{j; T_j \cap T_i \neq 0\}.$$

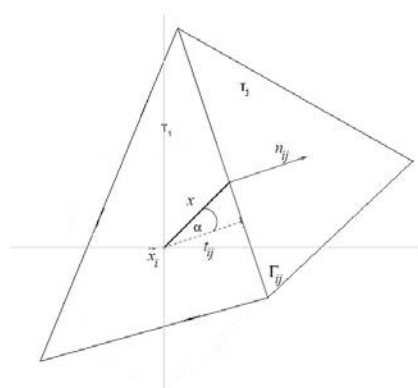
We have

$$\boldsymbol{x} \cdot \boldsymbol{n}_{ij} = |\boldsymbol{x}||\boldsymbol{n}_{ij}| \cos\alpha = |\boldsymbol{x}| \cos\alpha = t_{ij}, \quad \boldsymbol{x} \in \partial T_{ij}, \; j \in S(i),$$

where t_{ij} is the distance of $\tilde{\boldsymbol{x}}_i$ from ∂T_{ij}. We know

$$t_{ij} \ge \rho_{T_i} \quad \forall \, j \in S(i),$$

where ρ_{T_i} is the radius of the inscribed ball.



Now, the following holds:

$$\int_{\partial T_i} v^4 \, \boldsymbol{x} \cdot \boldsymbol{n} \, dS = \sum_{j \in S(i)} \int_{\partial T_i \cap \partial T_j} v^4 \, \boldsymbol{x} \cdot \boldsymbol{n}_{ij} \, dS = \sum_{j \in S(i)} t_{ij} \int_{\partial T_i \cap \partial T_j} v^4 \, dS \geq$$
$$\geq \rho_{T_i} \sum_{j \in S(i)} \int_{\partial T_i \cap \partial T_j} v^4 \, dS = \rho_{T_i} \|v\|^4_{L^4(\partial T_i)}. \tag{42}$$

Further it holds

$$\int_{T_i} \nabla \cdot (v^4 \boldsymbol{x}) \, d\boldsymbol{x} = \int_{T_i} v^4 \, \nabla \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \nabla v^4 \, d\boldsymbol{x} = d \int_{T_i} v^4 \, d\boldsymbol{x} + 4 \int_{T_i} v^3 \, \boldsymbol{x} \cdot \nabla v \, d\boldsymbol{x} \leq$$
$$\leq d\|v\|^4_{L^4(T_i)} + 4 \int_{T_i} |v^3 \, \boldsymbol{x} \cdot \nabla v| \, d\boldsymbol{x} \leq$$
$$\leq d\|v\|^4_{L^4(T_i)} + 4 \sup_{\boldsymbol{x} \in T_i} |\boldsymbol{x}| \int_{T_i} |v|^3 . |\nabla v| \, d\boldsymbol{x} \leq$$
$$\leq d\|v\|^4_{L^4(T_i)} + 4 h_{T_i} \|v\|^3_{L^6(T_i)} |v|_{H^1(T_i)}. \tag{43}$$

From (42) and (43) we have

$$\rho_{T_i} \|v\|^4_{L^4(\partial T_i)} \leq \int_{\partial T_i} v^4 \, \boldsymbol{x} \cdot \boldsymbol{n} \, dS = \int_{T_i} \nabla \cdot (v^4 \, \boldsymbol{x}) \, d\boldsymbol{x} \leq$$
$$\leq d\|v\|^4_{L^4(T_i)} + 4 h_{T_i} \|v\|^3_{L^6(T_i)} |v|_{H^1(T_i)},$$

which yields

$$\|v\|^4_{L^4(\partial T_i)} \leq \frac{d}{\rho_{T_i}} \|v\|^4_{L^4(T_i)} + 4 \frac{h_{T_i}}{\rho_{T_i}} \|v\|^3_{L^6(T_i)} |v|_{H^1(T_i)} \leq$$
$$\leq c \left(4 \|v\|_{L^6(T_i)^3} |v|_{H^1(T_i)} + \frac{d}{h_{T_i}} \|v\|^4_{L^4(\partial T_i)} \right).$$

□

7 Numerical experiments

We consider the flow in a rectangular domain $\Omega = (0,1)^2$ driven by the boundary condition

$$\boldsymbol{u} = \begin{cases} (4x(1-x), 0)^\top & \text{if } y = 1, \\ (0,0)^\top & \text{otherwise,} \end{cases}$$

with the initial conditions $\boldsymbol{u}_0 = (0,0)^\top$, $\mathbb{F}_0 = \mathbb{I}$. The calculation was run on the time interval $(0, 0.2)$ with fixed timestep 0.01. The problem was solved on a series of regular triangular meshes consisting of 8 to 256 elements in each direction. We have compared 3 methods:

a) finite element method (FEM) for velocity, pressure and stress,

	e_u		∇e_u		e_π		e_F	
h	$L^\infty(L^2)$	EOC	$L^2(L^2)$	EOC	$L^2(L^2)$	EOC	$L^\infty(L^2)$	EOC
1/8	2.08e-02	2.11	1.77e+02	1.00	4.19e+02	0.96	1.37e-01	0.73
1/16	4.80e-03	2.02	8.85e+01	1.00	2.14e+02	0.99	8.21e-02	0.89
1/32	1.18e-03	2.00	4.42e+01	0.99	1.07e+02	1.00	4.41e-02	0.94
1/64	2.95e-04	2.00	2.21e+01	0.99	5.36e+01	1.00	2.29e-02	0.95
1/128	7.37e-05		1.10e+01		2.68e+01		1.17e-02	

(a) FEM

	e_u		∇e_u		e_π		e_F	
h	$L^\infty(L^2)$	EOC	$L^2(L^2)$	EOC	$L^2(L^2)$	EOC	$L^\infty(L^2)$	EOC
1/8	2.07e-02	2.00	1.77e+02	1.00	4.73e+02	0.88	1.94e-01	0.73
1/16	5.16e-03	1.40	8.87e+01	0.98	2.55e+02	0.94	1.16e-01	0.88
1/32	1.94e-03	1.34	4.47e+01	0.98	1.33e+02	0.96	6.33e-02	0.90
1/64	7.67e-04	1.35	2.25e+01	0.98	6.83e+01	0.97	3.37e-02	0.91
1/128	3.00e-04		1.13e+01		3.47e+01		1.78e-02	

(b) FEM/dual FVM

	e_u		∇e_u		e_π		e_F	
h	$L^\infty(L^2)$	EOC	$L^2(L^2)$	EOC	$L^2(L^2)$	EOC	$L^\infty(L^2)$	EOC
1/16	0.0111	1.3033	0.2031	1.0027	2.62e-4	1.4997	4.29e-2	0.9633
1/32	0.0045	0.8264	0.1013	1.0338	9.26e-5	1.4215	2.20e-2	1.0086
1/64	0.0025	0.5759	0.0495	1.0216	3.46e-5	1.2854	1.09e-2	1.0099
1/128	0.0017	0.5159	0.0244	0.9968	1.42e-5	1.1783	5.4e-3	0.9985
1/256	0.0012		0.0122		6.3e-6		2.7e-3	

(c) FDM+FVM

Table 1: Error norms and experimental order of convergence for driven cavity problem.

b) FEM for velocity and pressure, dual finite volume method (FVM) for stress,

c) finite difference method (FDM) for velocity and pressure, FVM for stress.

In the case b) the stress was approximated by piecewise constants on dual elements, that arise by connecting the barycenters of primary elements with the edge midpoints. In the latter case the so-called staggered discretization of velocity was used.

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