INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

Higher grading affine Toda solitons from quantum algebras

Alexander Zuevsky

Preprint No. 28-2014

PRAHA 2014

Higher grading affine Toda solitons from quantum algebras

Alexander Zuevsky

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

E-mail: zuevsky@yahoo.com

Abstract. We construct special types of quantum soliton solutions using quantum Lie algebras associated to affine Lie algebras.

1. Introduction

The group–theoretical (Lie-algebraic) method [6, 7] of construction and integration of two-dimensional exactly solvable models is one of the most powerful. Using the representation theory for conformal and affine Toda models [9, 7] one is able to reconstruct the general solutions to systems as well as to derive special, in particular, solitonic solutions both in classical and quantum regions. The zero curvature condition applied to elements of connections constructed as sums of products of arbitrary functions with generators of (higher) grading subspaces [5] of a Lie algebra results in systems of equations [3].

Previously, general solution to the higher grading affine Toda equation were obtained in [3]. In [9] group-theoretical specialization for the lower grading subspace were formulated, and classical soliton solutions obtained in frames of algebraic approach to integrable models. In [10] we proposed quantum Lie algebra solutions generating solitons based on further algebraic properties of quantum groups.

In this paper we consider the case of affine Toda models constructed using algebraic elements that belong to grading subspaces up to some number l. The group–theoretical approach allows to extract special solitonic solutions from general ones. The lower grading case for the quantized version of the affine Toda models (the sine–Gordon equation) involving the quantum Lie algebra [1] $(sl_2)_q$ was considered in [10]. Here we extended that construction for a higher grading subspace case. Exactly solvable dynamical systems appearing in this the higher grading formulation are generalizations of lower ($l = 1$) cases, and possess more complicated and deep properties describing more complicated physical situations. It is important to take into account next terms in more general equations in order to understand interrelations of ordinary solutions (to equations associated to lower grading algebraic generators) with solutions corresponding to higher graiding.

2. Higher grading affine Toda system

In this and the next sections we recall [3] the affine Toda system consrtuction. Consider a two dimensional manifold $\mathcal{M}$ with local coordinates $z_{\pm}$. Up to a gauge transformation, $(1, 0)$-component lying in (see subsection 7.1 of Appendix) $\bigoplus_{n=0}^{l} \widehat{\mathcal{G}}_{+n}$ and $(0, 1)$-component in

$\bigoplus_{n=0}^{l} \widehat{\mathcal{G}}_{-n}$ of a flat connection $\mathcal{A}$ in the trivial holomorphic principal fibre bundle $\mathcal{M}\times\widehat{G} \longrightarrow \mathcal{M}$ ($l > 0$ is fixed integer) satisfy the zero curvature condition

$$\partial_+ A_- - \partial_- A_+ + [A_+ \, , \, A_-] = 0. \tag{1}$$

The components $A_\pm$ are the following (we keep notations of [3])

$$A_+ = -B\, F^+ \, B^{-1}\,, \qquad A_- = -\partial_- B\, B^{-1} + F^-. \tag{2}$$

Here B is a mapping $\mathcal{M} \longrightarrow \widehat{G}_0$ ($\widehat{G}_0$ is a group with the Lie algebra $\widehat{\mathcal{G}}_0$) and $F^\pm$ $(1 \le m \le l-1)$ are mappings to $\bigoplus_{n=1}^{l} \widehat{\mathcal{G}}_{\pm n}$

$$F^\pm = E_{\pm l} + \sum_{m=1}^{l-1} F_m^\pm, \tag{3}$$

where $E_{\pm l}$ are some fixed elements of $\widehat{\mathcal{G}}_{\pm l}$ and $F_m^\pm \in \widehat{\mathcal{G}}_{\pm m}$, $(1 \le m \le l-1)$. Substituting (2) into (1) one arrives at the equations of motion

$$\begin{aligned}
\partial_+ \left(\partial_- B\, B^{-1}\right) &= [E_{-l}\, , \, B\, E_l\, B^{-1}] + \sum_{n=1}^{l-1} [F_n^- \, , \, B\, F_n^+\, B^{-1}]\,, & (4)\\
\partial_- F_m^+ &= [E_l\, , \, B^{-1}\, F_{l-m}^-\, B] + \sum_{n=1}^{l-m-1} [F_{n+m}^+\, , \, B^{-1}\, F_n^-\, B]\,, & (5)\\
\partial_+ F_m^- &= -[E_{-l}\, , \, B\, F_{l-m}^+\, B^{-1}] - \sum_{n=1}^{l-m-1} [F_{n+m}^-\, , \, B\, F_n^+\, B^{-1}]\,. & (6)
\end{aligned}$$

Since $Q_{\mathbf{s}}$, $C \in \widehat{\mathcal{G}}_0$ then B can be parameterized as $B = b\, e^{\eta\, Q_{\mathbf{s}}}\, e^{\nu\, C}$ where b is a mapping to G_0, the subgroup of $\widehat{G}_0$ generated by all elements of $\widehat{\mathcal{G}}_0$ other than $Q_{\mathbf{s}}$ and C. Substituting B into the equations of motion (4–6) one has

$$\begin{aligned}
\partial_+ \left(\partial_- b b^{-1}\right) \; + \; \partial_+\partial_-\, \nu\, C &= e^{l\eta} [E_{-l}\, , \, b\, E_l\, b^{-1}] + \sum_{n=1}^{l-1} e^{n\eta}\, [F_n^- \, , \, b\, F_n^+ b^{-1}]\,, & (7)\\
\partial_- F_m^+ &= e^{(l-m)\eta}\, [E_l\, , \, b^{-1}\, F_{l-m}^-\, b] + \sum_{n=1}^{l-m-1} e^{n\eta} [F_{m+n}^+\, , \, b^{-1}\, F_n^-\, b]\,, & (8)\\
\partial_+ F_m^- &= -e^{(l-m)\eta}\, [E_{-l}\, , \, b\, F_{l-m}^+\, b^{-1}] - \sum_{n=1}^{l-m-1} e^{n\eta} [F_{m+n}^-\, , \, b\, F_n^+\, b^{-1}]\,, & (9)\\
\partial_+\partial_-\, \eta\, Q_{\mathbf{s}} &= 0\,. & (10)
\end{aligned}$$

Now consider the case $l = 1$. Let us parameterize the element B in principal grading of $\widehat{\mathcal{G}}$ and take $b = e^{(\phi(x_+,x_-),H^0)}$, $\gamma_0^\pm(x_\pm) = e^{(\phi^\pm(x_\pm),H^0)}$, where $(\phi, H^0) \equiv \sum_{i=1}^{r} \phi_i H_i^0$. From the equations (7–10) for an infinite dimensional Lie algebra $\widehat{\mathcal{G}}$ in the principal grading we obtain the affine Toda field theory systems of equations

$$\partial_+\partial_-\phi + \tfrac{4\mu}{\beta} \sum_{i=1}^{r} \left(m_i \tfrac{\alpha_i}{\alpha_i^2} \exp(\beta\alpha_i\cdot\phi) - \tfrac{\psi}{2}\exp(-\beta\cdot\phi) \right) = 0. \tag{11}$$

The formal general solution to the above equation was introduced in [9]:

$$e^{-\beta\lambda_i\cdot\phi} = e^{-\beta\lambda_i\cdot\phi_0} \frac{{}^{(1)}\langle\lambda_i|(\gamma_0^+)^{-1}\mu_+^{-1}(z_+)\mu_-(z_-)(\gamma_0^-)|\lambda_i\rangle^{(1)}}{\left({}^{(1)}\langle\lambda_0|(\gamma_0^+)^{-1}\mu_+^{-1}(z_+)\mu_-(z_-)(\gamma_0^-)|\lambda_0\rangle^{(1)}\right)^{m_j}} = e^{-\beta\lambda_i\cdot\phi_0} \frac{{}^{(1)}\langle\lambda_i|B^{-1}|\lambda_i\rangle^{(1)}}{\left({}^{(1)}\langle\lambda_0|B^{-1}|\lambda_0\rangle^{(1)}\right)^{m_j}}, \tag{12}$$

The fact that (12) is indeed a solution to (11) may be checked by using the representation theory of $\widehat{\mathcal{G}}$. A map $g : \mathcal{M} \longrightarrow G$ appearing in the gradient form of the flat connection $A_\pm = g^{-1}\partial_\pm g$, may be factorized (according to the Lie algebra decomposition $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+$) by the modified Gauss decomposition $g = \mu_-\nu_+\gamma_{0-}$ or $g = \mu_+\nu_-\gamma_{0+}$ with maps $\gamma_{0\pm} : \mathcal{M} \longrightarrow G_0$, $\mu_\pm, \nu_\pm : \mathcal{M} \longrightarrow G_\pm$. The grading condition provides the holomorphic property of $\mu_\pm$, i.e., they satisfy the initial value problem

$$\partial_\pm \mu_\pm(z_\pm) = \mu_\pm(z_\pm)\tilde{\mathcal{E}}_\pm(z_\pm), \tag{13}$$

$$\tilde{\mathcal{E}}_\pm(z_\pm) = \sum_{m=1}^{M} \tilde{\mathcal{E}}^\pm_m(\Phi^\pm), \qquad \tilde{\mathcal{E}}^\pm_m(\Phi^\pm) = \sum_{\alpha\in\Delta^+_m} \Phi^{\pm m}_\alpha(z_\pm) X_{\pm\alpha}, \tag{14}$$

with arbitrary functions $\Phi^{\pm m}_\alpha(z_\pm)$ determining the general solution to the system. Note that the summations in (14) are performed over the set of positive roots Δ^+_m of $\mathcal{G} = \sum_{m\in\mathbb{Z}} \mathcal{G}_m$ in the subspace $\mathcal{G}_m$. Using notations of [3], $\widehat{g}_0 \equiv \gamma_{0+}\gamma_{0-}^{-1}$, and $\gamma_0^\pm(x_\pm) : \mathcal{M} \longrightarrow G_0$ are arbitrary mappings. The element B is parameterized as $B^{-1} = (\gamma_0^+(x_+))^{-1}\widehat{g}_0\gamma_0^-(x_-)$.

3. Generalization of the affine Toda equations

Consider for example the case of $\widehat{sl}_2$ in principal grading (see subsection 7.1 of Appendix). The mapping B is parameterised as

$$B = e^{\varphi\, H^0}\, e^{\tilde{\nu}\, C}\, e^{\eta\, Q} = e^{\varphi\, \tilde{H}^0}\, e^{\nu\, C}\, e^{\eta\, Q}, \tag{15}$$

where $\tilde{H}^0 = H^0 - \frac{1}{2}C$ is Cartan element, and $\tilde{\nu} = \nu - \frac{1}{2}\varphi$. The case $l = 3$ delivers a generalization of (11). In this case $E_3 = a_+E^1_+ + a_-E^2_-$, $E_{-3} = a_+E^{-2}_+ + a_-E^{-1}_-$, [3] and therefore $[E_3\,,\, E_{-3}] = 3a_+a_-C \equiv \beta C$. Introduce the notations

$$E_1 = a_+E^0_+ + a_-E^1_-,\; E_{-1} = a_+E^{-1}_+ + a_-E^0_-,\; f_1 = a_+E^0_+ - a_-E^1_-,\; f_{-1} = a_+E^{-1}_+ - a_-E^0_-,$$
$$f_2 = -\sqrt{a_+a_-}\, H^1, \quad f_{-2} = -\sqrt{a_+a_-}\, H^{-1}.$$

The fields of the model are in (15) as well as matter fields $\psi^i_{R/L}$, $i = 1,\, 2$, $\chi_\pm$ defined as

$$F^+_1 = \psi^2_R f_1 + \chi_+E_1, \qquad F^+_2 = \psi^1_R f_2, \quad F^-_1 = \psi^1_L f_{-1} + \chi_-E_{-1}, \qquad F^-_2 = \psi^2_L f_{-2},$$

$\psi^i_{R/L}$, $i = 1, 2$, are components of two the Dirac fields ψ^i. It follows then that

$$[E_3, [E_{-3}, E_{\pm1}]] = 0,\; [E_3, [E_{-3}, \tilde{H}^0]] = 4a_+a_-\tilde{H}^0,\; [E_3, [E_{-3}, f_i]] = 4a_+a_-f_i,\; i = \pm1, \pm2.$$

Using (7–10), the equations of motion are

$$
\begin{aligned}
\partial_+\partial_-\,\varphi &= -a_+a_-\left(\left(e^{2\varphi}-e^{-2\varphi}\right)\left(e^{3\eta}-e^{\eta}\left(\psi_L^1\psi_R^2-\chi_+\chi_-\right)\right)\right.\\
&\quad - \left. e^{\eta}\left(e^{2\varphi}+e^{-2\varphi}\right)\left(\psi_L^1\chi_+-\psi_R^2\chi_-\right)\right), \qquad (16)\\
\partial_+\partial_-\,\nu &= -a_+a_-\left(\left(e^{2\varphi}+e^{-2\varphi}\right)\left(\frac{3}{2}e^{3\eta}-\frac{1}{2}e^{\eta}\left(\psi_L^1\psi_R^2-\chi_+\chi_-\right)\right)\right.\\
&\quad - \left.\frac{1}{2}e^{\eta}\left(e^{2\varphi}-e^{-2\varphi}\right)\left(\psi_L^1\chi_+-\psi_R^2\chi_-\right)+2e^{2\eta}\psi_L^2\psi_R^1\right), \qquad (17)\\
\partial_-\,\psi_R^1 &= \sqrt{a_+a_-}e^{\eta}\left(\psi_L^1\left(e^{2\varphi}+e^{-2\varphi}\right)-\chi_-\left(e^{2\varphi}-e^{-2\varphi}\right)\right), \qquad (18)\\
\partial_+\,\psi_L^1 &= 2\sqrt{a_+a_-}\left(-e^{2\eta}\psi_R^1+\frac{1}{2}e^{\eta}\psi_L^2\left(\psi_R^2\left(e^{2\varphi}-e^{-2\varphi}\right)+\chi_+\left(e^{2\varphi}+e^{-2\varphi}\right)\right)\right), \qquad (19)\\
\partial_-\,\psi_R^2 &= 2\sqrt{a_+a_-}\left(e^{2\eta}\psi_L^2+\frac{1}{2}e^{\eta}\psi_R^1\left(\psi_L^1\left(e^{2\varphi}-e^{-2\varphi}\right)-\chi_-\left(e^{2\varphi}+e^{-2\varphi}\right)\right)\right), \qquad (20)\\
\partial_+\,\psi_L^2 &= \sqrt{a_+a_-}e^{\eta}\left(-\psi_R^2\left(e^{2\varphi}+e^{-2\varphi}\right)-\chi_+\left(e^{2\varphi}-e^{-2\varphi}\right)\right), \qquad (21)\\
\partial_-\,\chi_+ &= \sqrt{a_+a_-}e^{\eta}\psi_R^1\left(\chi_-\left(e^{2\varphi}-e^{-2\varphi}\right)-\psi_L^1\left(e^{2\varphi}+e^{-2\varphi}\right)\right), \qquad (22)\\
\partial_+\,\chi_- &= \sqrt{a_+a_-}e^{\eta}\psi_L^2\left(\chi_+\left(e^{2\varphi}-e^{-2\varphi}\right)+\psi_R^2\left(e^{2\varphi}+e^{-2\varphi}\right)\right), \qquad (23)\\
\partial_+\partial_-\,\eta &= 0. \qquad (24)
\end{aligned}
$$

The general solutions to the matter fields $F_i^{\pm}$ may be written in the following form. For $m=1$ in (7–10) one has [3]

$$\langle i|F_1^+|i;i\rangle = f_i^+ = e^{\sum\limits_{l=0}^{r} k_{il}(\phi^- - \phi)_l} e^{\nu_0}\partial_+\left(\langle i|\mu_+^{-1}\mu_-|i;i\rangle\frac{\langle 0|\mu_+^{-1}\mu_-|0\rangle^{m_i}}{\langle i|\mu_+^{-1}\mu_-|i\rangle}\right). \qquad (25)$$

Here $|i;i\rangle$ denotes an element of the Verma module which is result of the action of the lowering generator on the highest state vector.

4. Solitonic solutions from general solutions

In [9] it was shown how to extract solitonic solutions from the formal general solutions of the affine Toda field equations. Let's take $\gamma_0^{\pm}=1$ in (12) to be a constant function. Then the mappings $\mu_\pm$ are $\mu_\pm=\mu_\pm^0 e^{z_\pm\mathcal{E}_\pm}$ with $\mu_\pm^0$ being some fixed mappings independent of $z_\pm$. Next take $\tilde{\mathcal{E}}_\pm$ in (14) as $\mathcal{E}_\pm \equiv E_{\pm l}+\sum_{N=1}^{l-1}c_N^{\pm}E_{\pm N}$ where $E_\pm$ are elements of a Heisenberg subalgebra of $\widehat{\mathcal{G}}$, namely $[\mathcal{E}_+,\mathcal{E}_-]=\Omega C$. One can consider principal of homogenious Heisenberg subalgebras for that purpose. In this paper we only deel with the principal case while the homogeneous case will be discussed elsewhere. Thus, we arrive at a special solution to (11)

$$e^{-\beta\lambda_i\cdot\phi} = e^{-\beta\lambda_i\cdot\phi_0}\frac{{}^{(1)}\langle\lambda_i|e^{x_\pm\mathcal{E}_\pm}\mu^0 e^{x_\pm\mathcal{E}_\pm}|\lambda_i\rangle^{(1)}}{\left({}^{(1)}\langle\lambda_0|e^{x_\pm\mathcal{E}_\pm}\mu^0 e^{x_\pm\mathcal{E}_\pm}|\lambda_0\rangle^{(1)}\right)^{m_j}.} \qquad (26)$$

In order to compute these solutions explicitly we have to remove $\mathcal{E}_\pm$-dependence from (26) moving $\mathcal{E}_+$ to the right and $\mathcal{E}_-$ to the left . Then we should find such a $\mu_0=\prod_{i=1}^N e^{\mathcal{V}_i}$ so that $\mathcal{V}_i$ would be eigenvectors with respect to the adjoint action of $\mathcal{E}_\pm$, i.e., $[\mathcal{E}_\pm,\mathcal{V}_i]=\omega_\pm^{(i)}\mathcal{V}_i$. Then it turns out [9] that resulting expressions provide us with solitonic solutions to the equations under considerations while parameters $\omega_\pm^{(i)}$ characterize solitons.

In the next sections we will explain the affinization $(sl_2)_q^t$ of the quantum Lie algebra $(sl_2)_q$ [1] and then apply those formulations to the construction of q-deformed solutions to the quantised [8] affine Toda equations in the specific example of the higher grading the sine–Gordon equation.

5. Quantum Lie algebra

5.1. Universal enveloping algebra $U_q(sl_2)$

The complex Lie algebra sl_2 is generated by three elements X_+, X_- and H satisfying

$$HX^{\pm} - X^{\pm}H = \pm 2X^{\pm}, \qquad X^+X^- - X^-X^+ = H.$$

The Lie bracket is skew-symmetric and satisfies Jacobi identity. The quantized enveloping algebra $U_q(sl_2)$ [2], [4] is an associative algebra generated by X^+, X^-, h with q-deformed commutator relations

$$X^+X^- - X^-X^+ = \frac{q^H - q^{-H}}{q - q^{-1}}, \qquad HX^{\pm} - X^{\pm}H = \pm 2X^{\pm},$$

It possesses a Hopf algebra structure with deformed adjoint action

$$(ad_{X^{\pm}})_q a = X^{\pm} a q^{H/2} - q^{\mp 1} q^{H/2} a X^{\pm}, \qquad (ad_H)_q a = Ha - aH, \tag{27}$$

(for all $a \in U_q(sl_2)$).

5.2. Quantum algebra $(sl_2)_q$

In [1] new generators in the quantized universal enveloping algebra $U_q(\widehat{sl}_2)$ were introduced as

$$X_h^{\pm} = \sqrt{\frac{2}{q+q^{-1}}}\, q^{-H/2} X^{\pm}, \qquad H_h = \frac{2}{q+q^{-1}}\left(qX^+X^- - q^{-1}X^-X^+\right). \tag{28}$$

It forms a three-dimensional subspace $(sl_2)_q = X_h^+, X_h^-, H_h$ in $U_q(sl_2)$, closed under the quantum Lie bracket

$$[a, b]_h = (ad_a)_q b, \tag{29}$$

a, $b \in (sl_2)_q$. These generators possesses the following commutation relations

$$\begin{aligned} &[H_h, X_h^{\pm}]_h = \pm 2q^{\pm 1} X_h^{\pm},\ [X_h^{\pm}, H_h]_h = \mp 2q^{\mp 1} X_h^{\pm},\ [X_h^+, X_h^-]_h = H_h, \\ &\left[X_h^-, X_h^+\right]_h = -H_h\ [H_h, H_h]_h = 2(q - q^{-1})H_h,\ [X_h^{\pm}, X_h^{\pm}]_h = 0. \end{aligned} \tag{30}$$

We should mention that $(sl_2)_q$ is not a Lie algebra in the standard sense. The generators (28) do not satisfy Jacobi identity and q-deformed Lie bracket is not skew-symmetric. Although q-analog of Jacobi identity for $(sl_2)_q$ is still absent, nevertheless (29) is q-skew-symmetric, [1]. Under q-skew-symmetry we mean a symmetry under q-conjugation, (that we will denote by tilde) an automophism of $(sl_2)_q$ defined by $q \mapsto \frac{1}{q}$. Then for any element of $(sl_2)_q$ $(aX_h^+ + bX_h^- + cH_h)\tilde{} = \tilde{a}X_h^+ + \tilde{b}X_h^- + \tilde{c}H_h$, ($a$, b, $c \in C$) and q-deformed Lie bracket satisfies $\widetilde{[a, b]_h} = [\tilde{b}, \tilde{a}]_h$.

5.3. Affinization of the $(sl_2)_q$

In [10] an affinized version, the principal Heisenberg subalgebra, an eigenvalue vertex operator, and representations of the quantum algebra $(sl_2)_q$ were considered. In this section and subsection 7.2 of Appendix we recall these constructions and then apply them in section 6 to the construction of solutions for the higher grading the sine–Gordon equation. Let $\mathcal{G} = (sl_2)_q$ and $\mathcal{L} = C[t, t^{-1}]$ be the algebra of Lourant polynomials in t and let $\mathcal{L}(\mathcal{G}) = \mathcal{L} \otimes_C \mathcal{G}$. Introduce a complex vector space $(sl_2)_q^t$: $\widetilde{\mathcal{L}}(\mathcal{G}) = \mathcal{L}(\mathcal{G}) \oplus CK \oplus Cd$. This is a loop algebra $\mathcal{L}(\mathcal{G})$ completed with the derivation d (acting as $t\frac{d}{dt}$ in $\mathcal{L}$ and trivially on K) extended by a one-dimensional center K corresponding to C-valued q-deformed 2-cocycle on $\mathcal{L}(\mathcal{G})$ $\Psi_q(a, b) = (x|y)_h \Phi(P, Q)$,

$\Phi(P,Q) = Res_t \frac{dP}{dt} Q$. Here $(x|y)_h$ is a quantum bilinear form on $(sl_2)_q$ [1] and P, Q are polynoms in t. We define q-deformed Lie bracket in this algebra as

$$[t^m \otimes x \oplus \omega K \oplus \nu d, \quad t^n \otimes y \oplus \omega_1 K \oplus \nu_1 d]_h = \\ (t^{m+n} \otimes [x,y]_h + \nu n t^n \otimes y - n\nu_1 m t^m \otimes x) \oplus m\delta_{m+n,0}(x|y)_h K \tag{31}$$

where $x,\, y \in \mathcal{G},\, \nu,\, \omega,\, \nu_1,\, \omega_1 \in C$.

Now we introduce generators that constitute an affinization of quantum algebra $(sl_2)_q$

$$H_1 = 1 \otimes H_h,\ H_0 = 1 \otimes (K - H_h),\ e_1 = 1 \otimes X_h^+,\ e_0 = t \otimes X_h^-,\ f_1 = 1 \otimes X_h^-,\ f_0 = t^{-1} \otimes X_h^+.$$

Then one can calculate the adjoint action

$$[H_0, e_0]_h = 2e_0 q^{-1}, [H_0, f_0]_h = -2f_0 q^{-1}, [H_0, e_1]_h = -2e_1 q, [H_0, f_1]_h = 2f_1 q^{-1},$$
$$[H_1, e_1]_h = 2e_1 q, [H_1, f_1]_h = -2f_1 q^{-1}, [H_1, e_0]_h = 2e_0 q^{-1}, [H_1, f_0]_h = 2q f_0,$$
$$[H_0, H_1]_h = -2(q - q^{-1})H_1, [H_0, H_0]_h = 2(q - q^{-1})H_1, [H_1, H_1]_h = 2(q - q^{-1})H_1,$$
$$[H_1, H_0]_h = -2(q - q^{-1})H_1, [e_0, f_0]_h = H_0, [e_1, f_1]_h = H_1, [f_0, e_0]_h = -H_0, [f_1, e_1]_h = -H_1,$$
$$[e_0, H_0]_h = -2qe_0, [e_0, H_1]_h = 2qe_0, [e_1, H_0]_h = 2e_1 q, [e_1, H_1]_h = -2e_1 q,$$
$$[f_0, H_0]_h = 2q^{-1} f_0, [f_0, H_1]_h = -2q^{-1} f_0, [f_1, H_1]_h = 2q f_1, [f_1, H_0]_h = -2q f_1.$$

5.4. Heisenberg subalgebra and eigenvector of the q-deformed adjoint action

The key point in the construction of solitonic solution to the affine Toda equation is the existence of eigenvectors with respect to elements of the Heisenberg subalgebra of the underlying affine algebra. Let

$${}_qE_{+1} = 1 \otimes X_h^+ + t \otimes X_h^-, \qquad {}_qE_{-1} = 1 \otimes X_h^- + t^{-1} \otimes X_h^+$$

so that the following series

$$A_{2m} = (q+q^{-1})^m t^m \otimes (-H_h),\ A_{2m+1} = (q+q^{-1})^m t^m \otimes q^{-1} X_h^+ - t^{m+1} \otimes q X_h^- + \left(X_h^- | X_h^+\right)_h, \\ \mathcal{F}_q = \sum_{k=-\infty}^{+\infty} \zeta^k A_k, \tag{32}$$

is an eigenvector of ${}_qE_{\pm 1}$ with respect to the quantum affine Lie bracket (31). Using $\mathcal{F}_q$ we can find q-deformation of the solitonic solution (26) corresponding to the affinization of the quantum algebra $(sl_2)_q$.

6. Solution to the affine Toda equations corresponding to $(sl_2)_q^t$

In this section we will use results of the section 5. We propose q-deformed solitonic solutions to the affine Toda equations associated with the affine Lie algebra $\widehat{sl}_2$, i.e., the sine–Gordon equation. Those solutions will be constructed by means of the Heisenberg subalgebra operators and the eigenvector $\mathcal{F}$ involved in the exponent in the mapping μ_0. Using properties of the highest weight representation of the affinized quantum algebra $(sl_2)_q$ one can check (similar to [9], [8]) that (12) is a solution to the affine Toda equation (11) when we replace its ingredients by their quantum group counterparts. A consideration in the case of the sine–Gordon equation is given in [9].

First consider the case of $l = 1$. Starting from the general solution (12) to the affine Toda equations for $\widehat{sl}_2$ we take $\gamma_0^\pm = 1$ and substitute the vectors of the fundamental highest weight representation of $\widehat{sl}_2$ by the fundamental highest weight representation vectors of $(sl_2)_q^t$

$$e^{-\beta\lambda_i \cdot \phi} = e^{-\beta\lambda_i \cdot \phi_0} \frac{{}_q^{(1)}\langle \lambda_i | \mu_+^{-1}(z_+) \mu_-(z_-) | \lambda_i \rangle_q^{(1)}}{\left({}_q^{(1)}\langle \lambda_0 | \mu_+^{-1}(z_+) \mu_-(z_-) | \lambda_0 \rangle_q^{(1)}\right)^{m_j}}, \tag{33}$$

where $|i\rangle_q^{(1)} = |\lambda_i\rangle_q^{(1)} = v_0^{(i)}$ denotes the highest vector in the $i-$th fundamental representation of $(sl_2)_q^t$. Taking $\mathcal{E}_\pm = {}_qE_{\pm 1}$ in the solitonic solution and inserting exponential of the eigen operator $\mathcal{F}_q$ as a group element, we obtain [10]

$$e^{-\beta\lambda_j\cdot\phi} = e^{-\beta\lambda_j\cdot\phi_0}\frac{{}_q^{(1)}\langle j|e^{-\mathcal{E}_+z_+}e^{Q\mathcal{F}_q}e^{\mathcal{E}_-z_-}|j\rangle_q^{(1)}}{\left({}_q^{(1)}\langle 0|e^{\mathcal{E}_+z_+}e^{Q\mathcal{F}_q}e^{\mathcal{E}_-z_-}|0\rangle_q^{(1)}\right)^{m_j}}. \tag{34}$$

Then it follows that

$$e^{-\beta\lambda_j\cdot\phi} = e^{-\beta\lambda_j\cdot\phi_0}\frac{{}_q^{(1)}\langle j|\exp\left(Qe^{-2z_+\zeta-2\frac{1}{t}z_-\zeta^{-1}}\mathcal{F}_q\right)\exp(-z_+z_-)|j\rangle_q^{(1)}}{\left({}_q^{(1)}\langle 0|\exp\left(Qe^{-2z_+\zeta-2\frac{1}{t}z_-\zeta^{-1}}\mathcal{F}_q\right)\exp(-z_+z_-)|0\rangle_q^{(1)}\right)^{m_j}}. \tag{35}$$

So for $j = 0$

$$e^{-\beta\lambda_1\cdot\phi} = e^{-\beta\lambda_1\cdot\phi_0}, \tag{36}$$

and for $j = 1$ this gives

$$e^{-\beta\lambda_1\cdot\phi} = e^{-\beta\lambda_1\cdot\phi_0}\exp\left(-Qe^{-2z_+\zeta-2\frac{1}{t}z_-\zeta^{-1}}\sum_{m=-\infty}^{+\infty}\zeta^{2m}t^m(q+q^{-1})^m\right). \tag{37}$$

6.1. The case $l = 3$

The affine Toda equation system (11) is a particular example of more general construction described in the first section. Namely equations (11) correspond to $l = 1$ grading subspace. The case $l = 3$ leads us to a generalization of (11). Using the general form of solution to the affine Toda equation for $l > 1$ (12) we can form a class of q-deformed solutions constructed with by means of the eigenvector $\mathcal{F}_q$ (32). We take in (12)

$$\mathcal{E}_\pm = {}_qE_{\pm 1} + {}_qE_{\pm 3}, \tag{38}$$

where ${}_qE_{+3} = t\otimes X_h^+ + t^2\otimes X_h^-$, ${}_qE_{-3} = t^{-1}\otimes X_h^- + t^{-2}\otimes X_h^+$. Then we arrive at a solution for Toda type field ϕ

$$e^{-\beta\lambda_1\cdot\phi} = e^{-\beta\lambda_1\cdot\phi_0}\exp\left(-Qe^{-2z_+(1+t)\zeta-2\frac{1}{t}z_-(1+\frac{1}{t})\zeta^{-1}}\sum_{m=-\infty}^{+\infty}\zeta^{2m}t^m(q+q^{-1})^m\right). \tag{39}$$

Now we want to find q-deformed solitonic solutions for matter fields $F_m^\pm$ just in the same way as we did it for Toda type fields. We substitute elements (38) to (25) and take the state vectors from (42) from the basis of the fundamental highest weight representation of $(sl_2)_q^t$. Finally the q-solitonic solutions are

$$\begin{aligned}
f_1^+ = \exp\left(\sum_{l=0}^{1}k_{1l}(\phi^- - \phi)_l\right)e^{\nu_0}\partial_+\Bigg({}_q\langle 1|\exp\left(Qe^{-2z_+(1+t)\zeta-2\frac{1}{t}z_-(1+\frac{1}{t})\zeta^{-1}}\sum_{m=-\infty}^{+\infty}\zeta^{2m}t^m(q+q^{-1})^m\right) \\
\times\sum_{n=0}^{+\infty}\frac{1}{n!}(Qe^{-2z_+(1+t)\zeta-2\frac{1}{t}z_-(1+\frac{1}{t})\zeta^{-1}})^n\, q^{-1}q^{-\lambda^{(1)}} \\
\times\left(\sum_{m=-\infty}^{+\infty}\zeta^{2m+1}t^m(q+q^{-1})^m\right)^n\left((1-2qq^{-(\lambda^{(1)}-2)})^n|1;1\rangle_q + \sum_{k=0}^{n-1}(1-2qq^{-(\lambda^{(1)}-2)})^k|1\rangle_q\right)\Bigg) \\
= e^{\sum_{l=0}^{1}k_{1l}(\phi^- - \phi)_l}e^{\nu_0}\partial_+\Bigg(\exp\left(2Qe^{-2z_+(1+t)\zeta-2\frac{1}{t}z_-(1+\frac{1}{t})\zeta^{-1}}\sum_{m=-\infty}^{+\infty}\zeta^{2m}t^m(q+q^{-1})^m\right)q^{-1}q^{-\lambda^{(1)}} \\
\times\left(\exp(1-2qq^{-(\lambda^{(1)}-2)})|1;1\rangle_q + \frac{\exp\left((1-2qq^{-(\lambda^{(1)}-2)})\right)-1}{(1-2qq^{-(\lambda^{(1)}-2)})-1}|1\rangle_q\right)\Bigg),
\end{aligned}$$

$$f_0^+ = \exp\left(\sum_{l=0}^{1} k_{1l}(\phi^- - \phi)_l\right) e^{\nu_0}\partial_+ \left(\exp\left(Qe^{-2z_+(1+t)\zeta - 2\frac{1}{t}z_-(1+\frac{1}{t})\zeta^{-1}} \sum_{m=-\infty}^{+\infty} \zeta^{2m} t^m (q+q^{-1})^m\right)\right.$$

$$\left. \times q^{-1} q^{-\lambda^{(0)}} \left(\exp(1-2q^{-1}q^{(2+\lambda_0^{(0)})})|0;0\rangle_q + \frac{\exp\left((1-2q^{-1}q^{(2+\lambda_0^{(0)})})\right)-1}{(1-2q^{-1}q^{(2+\lambda_0^{(0)})})-1}|0\rangle_q\right)\right).$$

Similar relations can be obtain all higher cases of l elsewhere.

7. Appendix

7.1. Affine Kac–Moody algebras

Here we recall facts about affine Kac–Moody algebras [5], [3]. Consider an untwisted affine Kac-Moody algebra $\widehat{\mathcal{G}}$ endowed with an integral grading $\widehat{\mathcal{G}} = \bigoplus_{n\in\mathbb{Z}} \widehat{\mathcal{G}}_n$, and denote $\widehat{\mathcal{G}}_\pm = \bigoplus_{n>0} \widehat{\mathcal{G}}_{\pm n}$. By an affine Lie algebra we mean a loop algebra corresponding to a finite dimensional simple Lie algebra $\mathcal{G}$ of rank r, extended by the center C and the derivation D. According to [5], integral gradings of $\widehat{\mathcal{G}}$ are labelled by a set of co-prime integers $\mathbf{s} = (s_0, s_1, \dots s_r)$, and the grading operators are given by

$$Q_{\mathbf{s}} \equiv H_{\mathbf{s}} + N_{\mathbf{s}}\, D - \frac{1}{2N_{\mathbf{s}}}\,\mathrm{Tr}\,(H_{\mathbf{s}})^2\; C\,. \tag{40}$$

Here $H_{\mathbf{s}} \equiv \sum_{a=1}^r s_a \lambda_a^v \cdot H^0$, $N_{\mathbf{s}} \equiv \sum_{i=0}^r s_i m_i^\psi$, $\psi = \sum_{a=1}^r m_a^\psi \alpha_a\,,\quad m_0^\psi = 1$. H^0 is an element of Cartan subalgebra of $\mathcal{G}$; α_a, $a = 1, 2, \dots r$, are its simple roots; ψ is its maximal root; m_a^ψ the integers in expansion $\psi = \sum_{a=1}^r m_a^\psi \alpha_a$; and λ_a^v are the fundamental co–weights satisfying the relation $\alpha_a \cdot \lambda_b^v = \delta_{ab}$.

The principal grading operator Q_{ppal} is given by (40) where $N_{\mathbf{s}} = h$ is Coxeter number. Therefore $\widehat{\mathcal{G}}_0 = \{H_a^0\,, a = 1, 2, \dots r\,; C; Q_{\mathrm{ppal}}\}$, $\widehat{\mathcal{G}}_m = \{E^0_{\alpha^{(m)}}, E^1_{-\alpha^{(h-m)}}\}$, $\widehat{\mathcal{G}}_{-m} = \{E^0_{-\alpha^{(m)}}, E^{-1}_{\alpha^{(h-m)}}\}$ where $0 < m < h$, and $\alpha^{(m)}$ are positive roots of height m. The element B is parameterized as $B = e^{\varphi\cdot\tilde{H}^0}\, e^{\nu\, C}\, e^{\eta Q_{\mathrm{ppal}}} = e^{\varphi\cdot H^0}\, e^{\tilde{\nu}\, C}\, e^{\eta Q_{\mathrm{ppal}}}$, where $\tilde{H}^0$ was defined in [3] as $\widetilde{H}_a^0 = H_a^0 - \frac{1}{N_{\mathbf{s}}}\,\mathrm{Tr}\left(H_{\mathbf{s}}\, H_a^0\right) C = H_a^0 - \frac{2}{\alpha_a^2}\frac{s_a}{N_{\mathbf{s}}}\, C$, and $\tilde{\nu} = \nu - \frac{2}{h}\widehat{\delta}\cdot\varphi$, with $\widehat{\delta} = \sum_{a=1}^r \frac{\lambda_a}{\alpha_a^2}$, and λ_a being the fundamental weights of $\mathcal{G}$. Let us denote by H^n, $E^n_\pm$, D, C the Chevalley basis generators of $\widehat{sl}_2$. The commutation relations are

$$\begin{gathered}[H^m\,,\, H^n] = 2\,m\,C\,\delta_{m+n,0}, \quad [H^m\,,\, E^n_\pm] = \pm 2\,E^{m+n}_\pm, \\ [E^m_+\,,\, E^n_-] = H^{m+n} + m\,C\,\delta_{m+n,0}, \quad [D\,,\, T^m] = m\,T^m\,, \quad T^m \equiv H^m, E^m_\pm.\end{gathered} \tag{41}$$

The grading operator for the principal grading ($\mathbf{s} = (1,1)$) is $Q \equiv \frac{1}{2}H^0 + 2D$. Then the eigensubspaces are $\widehat{\mathcal{G}}_0 = \{H^0, C, Q\}$, $\widehat{\mathcal{G}}_{2n+1} = \{E^n_+, E^{n+1}_-\}$, $n \in \mathbb{Z}$, $\widehat{\mathcal{G}}_{2n} = \{H^n\}, n \in \{\mathbb{Z} - 0\}$.

7.2. Highest weight representations of $(sl_2)_q$

The highest weight vector v_0 of the fundamental representation of the quantum Lie algebra $(sl_2)_q$ [10] satisfies the relations $H_h v_0 = v_0$, $X_h^+ v_0 = 0$, $X_h^- v_0 = v_1$. Note that because of the definition of generators $(sl_2)_h$ (28) we have

$$[H, X_h^+]_h = [H, X_h^+] = 2X_h^+,\ [H, X_h^-]_h = [H, X_h^-] = -2X_h^-,\ [H, H_h]_h = [H, H_h] = 0.$$

The element H lies in Cartan subalgebra of $(sl_2)_q$. Therefore $H_h(Hv_0) = (Hv_0)$, $Hv_0 = \lambda v_0$. Using the definitions (28) we find that $H_h = \frac{2}{q+q^{-1}}\left(qX^+X^- - q^{-1}X^-X^+\right) = q^H(X_h^+X_h^- - X_h^-X_h^+)$, and the action of H_h on v_0 in such a form gives $q^H(X_h^+X_h^- - X_h^-X_h^+)v_0 = v_0$, and finally $X_h^\pm v_1 = q^{-\lambda}v_0$, where λ can be found from $\frac{2}{q+q^{-1}}\left(qX^+X^- - q^{-1}X^-X^+\right)v_0 = \frac{2}{q+q^{-1}}[H]v_0 = v_0$, with $\frac{2q(q^{\lambda}-q^{-\lambda})}{(q+q^{-1})(q-q^{-1})}=1$. This procedure can be recurrently continued for all v_n in the basis of the representation. In section 7 we introduced a quantum affine algebra $(sl_2)_q^t$ as an affinization of $(sl_2)_q$. The highest weight vector of the i-th ($i=1,2$) fundamental representation of $(sl_2)_q^t$ possesses the properties similar to the properties of the highest weight vector of fundamental representation of $(sl_2)_q$. The action of $h_{1,2}$, $e_{1,2}$ and $f_{1,2}$ generators on highest weight vectors $v_0^{(1)}$ and $v_0^{(0)}$ is given by

$$h_1 v_0^{(1)} = v_0^{(1)},\ h_1 v_0^{(0)} = 0,\ h_0 v_0^{(1)} = 0,\ h_0 v_0^{(0)} = v_0^{(0)}, e_1 v_0^{(0)} = e_1 v_0^{(1)} = e_0 v_0^{(0)} = e_0 v_0^{(1)} = 0,$$
$$f_0 v_0^{(1)} = f_1 v_0^{(0)} = 0, f_0 v_0^{(0)} = v_1^{(0)}, f_1 v_0^{(0)} = v_1^{(0)}, \tag{42}$$

where superscripts correspond to representation and subscripts are vector basis numbers. In the same way as for $(sl_2)_q$

$$e_1 v_1^{(1)} = q^{-H} v_0^{(1)},\ e_1 v_1^{(1)} = q^{-\lambda^{(1)}} v_0^{(1)},\ h v_0^{(1)} = \lambda^{(1)} v_0^{(1)},\ e_1 v_1^{(0)} = 0\ h = 1 \otimes H, \tag{43}$$

$$e_0 v_1^{(0)} = q^{-H} v_0^{(0)},\ e_0 v_1^{(0)} = q^{-\lambda^{(0)}} v_0^{(0)},\ h v_0^{(0)} = \lambda^{(0)} v_0^{(0)},\ e_0 v_1^{(1)} = 0, \tag{44}$$

$$H_1 v_1^{(1)} = (1 - 2qq^{-(\lambda^{(1)}-2)}) v_1^{(1)},\ H_1 v_1^{(0)} = 2q^{-1} q^{(2+\lambda_0^{(0)})} v_1^{(0)}. \tag{45}$$

8. Conclusions

In this paper we discussed affine Toda models formulated with the inclusion of higher then one grading subspaces for affine Lie algebras. The zero curevature condition applied to elements of connection involving higher gradign Lie algebra generators lead us to more suffisticated and interesting systems of equations describing dynamical systems that are closer (with respect to ordinary affine Toda models) to phenomena in nature. In particular, these systems of equations take into accound interractions of original Toda fields with matter fields. The group–theoretical way to treat higher grading affine Toda models allows us to find special solitonic solutions. In the quantum situation, corresponding methods using properties of quatnum groups and quantum Lie algebras [1] make possible constructions of special quantum soliton solutions.

9. References

[1] Delius G W and Hueffmann A 1996 J. Phys.A29
[2] Drinfeld V G 1985 Sov. Math. Dokl. 32
[3] Ferreira L A, Gervais J-L, Guillen, J S and Saveliev M V 1996 Nuclear Phys. B 470 1-2
[4] Jimbo M 1985 Lett. Math. Phys. 10, 63
[5] Kac V G 1990 *Infinite dimensional Lie algebras.* Third edition (Cambridge university Press, Cambridge)
[6] Leznov A N and Saveliev M V 1979 Lett. Math. Phys. 3
[7] Leznov A N and Saveliev M V 1992 *Group-Theoretical Methods for Integration of Non–Linear Dynamical Systems.* Progress in Physics Series, v. 15, (Birkhauser–Verlag, Basel)
[8] Mansfield P 1983 Nucl. Phys. B 222
[9] Olive D I Turok N and Underwood J W R 1993 Nucl. Phys. B401
[10] Zuevsky A. Quantum Lie algebra solitons. J. of Physics: Conf. Series (JPCS) 2014