# On bounded solutions to the compressible isentropic Euler system

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## Setting of the problem

We study the compressible isentropic Euler system in the whole 2D space

$$
\begin{cases}
\partial_t \rho + \operatorname{div}_x(\rho v) = 0 \\
\partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] = 0 \\
\rho(\cdot, 0) = \rho^0 \\
v(\cdot, 0) = v^0 .
\end{cases}
\tag{1}
$$

Unknowns:

- $\rho(x, t)$ ... density
- $v(x, t)$ ... velocity

The pressure $p(\rho)$ is given.

## Introduction

- It is a hyperbolic system of conservation laws
- The theory of hyperbolic conservation laws is far frome being completely understood
- Solutions develop singularities in finite time even for smooth initial data
- Admissibility comes into play due to the entropy inequality ("selector" of physical solutions in case of existence of many solutions)
- There are satisfactory results in the case of scalar conservation laws (in 1D as well as in multi-D), there is a lot of entropies: $\Rightarrow$ Kruzkov, 1970: Well–posedness theory in $BV$.
- There are also satisfactory results in the case of systems of conservation laws in 1D: Lax, Glimm, Bianchini, Bressan, ...

## Introduction II

Back to our case, the Euler system:

- In more than 1D there is only one (entropy, entropy flux) pair, which is

$$
\left( \rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}, \left( \rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho) \right) v \right)
$$

with the internal energy $\varepsilon(\rho)$ given through

$$
p(\rho) = \rho^2 \varepsilon'(\rho).
$$

- Local existence of strong (and therefore admissible) solutions is proved
- On the other hand global existence of weak solutions in general (it is a system in multi D!) is still an open problem, there are only partial results
- The weak–strong uniqueness property holds for this system

# Weak solution

### Definition 1

By a *weak solution* of Euler system on $\mathbb{R}^2 \times [0, \infty)$ we mean a pair $(\rho, v) \in L^\infty(\mathbb{R}^2 \times [0, \infty))$ such that the following identities hold for every test functions $\psi \in C_c^\infty(\mathbb{R}^2 \times [0, \infty))$, $\phi \in C_c^\infty(\mathbb{R}^2 \times [0, \infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho \partial_t \psi + \rho v \cdot \nabla_x \psi] \, dx dt + \int_{\mathbb{R}^2} \rho^0(x) \psi(x, 0) dx = 0$$

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho v \cdot \partial_t \phi + \rho v \otimes v : \nabla_x \phi + p(\rho) \operatorname{div}_x \phi] \, dx dt$$
$$+ \int_{\mathbb{R}^2} \rho^0(x) v^0(x) \cdot \phi(x, 0) dx = 0.$$

# Admissible weak solution

### Definition 2

A bounded weak solution $(\rho, v)$ of Euler system is *admissible* if it satisfies the following inequality for every nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0, \infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} \left[ \left( \rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} \right) \partial_t \varphi \right.$$

$$+ \left. \left( \rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right] dxdt$$

$$+ \int_{\mathbb{R}^2} \left( \rho^0(x)\varepsilon(\rho^0(x)) + \rho^0(x)\frac{|v^0(x)|^2}{2} \right) \varphi(x, 0) \, dx \geq 0.$$

# First results

- De Lellis, Székelyhidi: There exist initial data $(\rho^0, v^0) \in L^\infty$ such that there are infinitely many bounded admissible weak solutions
- Chiodaroli: ill-posedness of bounded admissible weak solutions for initial data with regular density ($\rho^0 \in C^1$) and irregular velocity ($v^0 \in L^\infty$)

## Previous result

Motivated by the result of László Székelyhidi about vortex sheet initial data for incompressible Euler equations we studied similar problem for the compressible isentropic Euler system and proved the following

### Theorem 3 (Chiodaroli, De Lellis, K.)

*Let $p(\rho) = \rho^2$. There exist Lipschitz initial data $(\rho^0, v^0)$ for which there are infinitely many bounded admissible weak solutions $(\rho, v)$ of Euler system on $\mathbb{R}^2 \times [0, \infty)$ with $\inf \rho > 0$. These solutions are all locally Lipschitz on a finite interval of time where they therefore all coincide with the unique classical solution.*

The proof is based on analysis of the Riemann problem and a suitable application of the theory of De Lellis and Székelyhidi for incompressible Euler equations.

Denote $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \tag{2}$$

where $\rho_\pm, v_\pm$ are constants.

In particular the initial data are "1D" and there is a classical theory about self-similar solutions to the Riemann problem in 1D (they are unique in the class of $BV$ functions).

In the case of system (1), the initial singularity can resolve to at most 3 structures (rarefaction wave, admissible shock or contact discontinuity) connected by constant states.

# First observation

If $v_{-1} = v_{+1}$, then any self-similar solution to (1), (2) has to satisfy $v_1(t, x) = v_{-1} = v_{+1}$ and in particular there is no contact discontinuity in the self-similar solution.

The initial singularity then resolves into at most 2 structures (rarefaction waves or admissible shocks) connected by constant states.

# Classification of self-similar solutions I

1) If

$$v_{+2} - v_{-2} \geq \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a $1-$rarefaction wave and a $3-$rarefaction wave. The intermediate state is vacuum, i.e. $\rho_m = 0$.

2) If

$$\left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau \right| < v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a $1-$rarefaction wave and a $3-$rarefaction wave. The intermediate state has $\rho_m > 0$.

3) If $\rho_- > \rho_+$ and

$$-\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_- \rho_+}} < v_{+2} - v_{-2} < \int_{\rho_+}^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a $1-$rarefaction wave and an admissible $3-$shock.

4) If $\rho_- < \rho_+$ and

$$-\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+ \rho_-}} < v_{+2} - v_{-2} < \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of an admissible $1-$shock and a $3-$rarefaction wave.

5) If

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+ \rho_-}} \tag{3}$$

then the self-similar solution consists of an admissible
$1-$shock and an admissible $3-$shock.

## Main Theorem

### Theorem 4 (Chiodaroli, K.)

*Let $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$. For every Riemann data (2) such that the self-similar solution to the Riemann problem (1), (2) consists of an admissible $1-$shock and an admissible $3-$shock, i.e. $v_{-1} = v_{+1}$ and*

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_- \rho_+}}, \tag{4}$$

*there exist infinitely many admissible solutions to (1), (2).*

- Compared to Theorem 3, the new Theorem 4 widely extends the set of initial data for which there exist infinitely many admissible solutions to the Riemann problem.

- Moreover Theorem 4 gives this result for any pressure law $p(\rho) = \rho^{\gamma}$, instead of the specific case $\gamma = 2$ in Theorem 3

## Entropy rate admissibility I

Define the *total energy* of the solutions $(\rho, v)$ to (1) as

$$E[\rho, v](t) = \int_{\mathbb{R}^2} \left( \rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} \right) \mathrm{d}x \qquad (5)$$

and the *energy dissipation rate* of $(\rho, v)$ at time $t$:

$$D[\rho, v](t) = \frac{\mathrm{d}_+ E[\rho, v](t)}{\mathrm{d}t}. \qquad (6)$$

In our case the energy is always infinite, so we restrict the integrals to a finite box:

$$E_L[\rho, v](t) = \int_{(-L,L)^2} \left( \rho\varepsilon(\rho) + \rho\frac{|v|^2}{2} \right) \mathrm{d}x \qquad (7)$$

$$D_L[\rho, v](t) = \frac{\mathrm{d}_+ E_L[\rho, v](t)}{\mathrm{d}t}. \qquad (8)$$

### Definition 5 (Entropy rate admissible solution)

A weak solution $(\rho, v)$ of (1) is called *entropy rate admissible* if there exists $L^* > 0$ such that there is no other weak solution $(\overline{\rho}, \overline{v})$ with the property that for some $\tau \geq 0$, $(\overline{\rho}, \overline{v})(x, t) = (\rho, v)(x, t)$ on $\mathbb{R}^2 \times [0, \tau]$ and $D_L[\overline{\rho}, \overline{v}](\tau) < D_L[\rho, v](\tau)$ for all $L \geq L^*$.

This definition is motivated by Dafermos. He proved that for a single equation the entropy rate criterion is equivalent to the viscosity criterion in the class of piecewise smooth solutions. Following the approach of Dafermos, Hsiao proved, in the class of piecewise smooth solutions, the equivalence of the entropy rate criterion and the viscosity criterion for the one-dimensional system of equations of nonisentropic gas dynamics in lagrangian formulation with pressure laws $p(\rho) = \rho^\gamma$ for $\gamma \geq 5/3$ while the same equivalence is disproved for $\gamma < 5/3$.

# Main Theorem II

### Theorem 6 (Chiodaroli, K.)

*Let $p(\rho) = \rho^\gamma$, $1 \leq \gamma < 3$. There exist Riemann data (2) for which the self-similar solution to (1) emanating from these data is not entropy rate admissible.*

Theorem 6 ensures that for $1 \leq \gamma < 3$ there exist initial Riemann data (2) for which some of the infinitely many nonstandard solutions constructed as in Theorem 4 dissipate more total energy than the self-similar solution, suggesting in particular that the Dafermos entropy rate admissibility criterion would not pick the self-similar solution as the admissible one.

# References

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Thank you for your attention.