INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

The transmission problem for the Stokes system

Dagmar Medková

Preprint No. 5-2012

PRAHA 2012

The transmission problem for the Stokes system

D. Medková

Abstract: The transmission problem for the Stokes system is studied: $\Delta \mathbf{u}_+ = \nabla q_+$, $\nabla \cdot \mathbf{u}_+ = 0$ in G_+, $\Delta \mathbf{u}_- = \nabla q_-$, $\nabla \cdot \mathbf{u}_- = 0$ in G_-, $\mathbf{u}_+ - \mathbf{u}_- = \mathbf{g}$, $a_+\{2(\hat\nabla \mathbf{u}_+)\mathbf{n} - q_+\mathbf{n}\} - a_-\{2(\hat\nabla \mathbf{u}_-)\mathbf{n} - q_-\mathbf{n}\} = \mathbf{f}$ on ∂G_+. Here G_+ is a bounded open set with Lipschitz bpundary, $G_- = R^m \setminus \overline{G}_+$, $\mathbf{g} \in H^{1/2}(\partial G_+, R^m)$, $\mathbf{f} \in H^{-1/2}(\partial G_+, R^m)$. Using the integral equation method we show that there exists a unique solution of the transmission problem in the homogeneous Sobolev space. We solve the corresponding boundary integral equation by the successive approximation method. We are able estimate errors. This estimate depends only on a_+ and a_-, not on G_+ and G_-.

Keywords: transmission problem; Stokes system; integral equation method

MSC 2000: 35Q30, 65N38

1 Introduction

Lately the transmission problem for the Stokes system has been studied by the integral equation method (see [5], [6], [1]). The integral equation method is a powerful tool for proving the existence of a solution of the transmission problem. We shall study not only the existence and uniqueness of a solution, we also construct this solution.

Let $G = G_+ \subset R^m$, $m > 2$, be a bounded open set with Lipschitz boundary ∂G. Denote $G_- := R^m \setminus \mathrm{cl}\, G_+$ its complement with $\partial G_- = \partial G$. Here $\mathrm{cl}\, G_+$ denotes the closure of G_+ and ∂G the boundary of G. (Unlike the preceding papers we do not suppose that G_+ or G_- has connected boundary.) Denote by $\mathbf{n} = \mathbf{n}^G$ the outward unit normal of G_+. Let a_+, a_-, b_+, b_- be positive constants. We would like to study the transmission problem

$$a_+\Delta \mathbf{u}_+ = b_+\nabla p_+ \quad \text{in} \quad G_+, \qquad \nabla \cdot \mathbf{u}_+ = 0 \quad \text{in} \quad G_+,$$

$$a_-\Delta \mathbf{u}_- = b_-\nabla p_- \quad \text{in} \quad G_-, \qquad \nabla \cdot \mathbf{u}_- = 0 \quad \text{in} \quad G_-,$$

$$\mathbf{u}_+ - \mathbf{u}_- = \mathbf{g}, \quad \text{on } \partial G.$$

$$\{2a_+(\hat\nabla \mathbf{u}_+)\mathbf{n} - b_+p_+\mathbf{n}\} - \{2a_-(\hat\nabla \mathbf{u}_-)\mathbf{n} - b_-p_-\mathbf{n}\} = \mathbf{f} \quad \text{on } \partial G,$$

where $\hat\nabla \mathbf{u} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$. If we put $q_+ = b_+p_+/a_+$, $q_- = b_-p_-/a_-$, then

$$\Delta \mathbf{u}_+ = \nabla q_+ \quad \text{in} \quad G_+, \qquad \nabla \cdot \mathbf{u}_+ = 0 \quad \text{in} \quad G_+,$$

$$\Delta \mathbf{u}_- = \nabla q_- \quad \text{in} \quad G_-, \qquad \nabla \cdot \mathbf{u}_- = 0 \quad \text{in} \quad G_-,$$

$$\mathbf{u}_+ - \mathbf{u}_- = \mathbf{g}, \quad \text{on } \partial G.$$

$$a_+\{2(\hat\nabla \mathbf{u}_+)\mathbf{n} - q_+\mathbf{n}\} - a_-\{2(\hat\nabla \mathbf{u}_-)\mathbf{n} - q_-\mathbf{n}\} = \mathbf{f} \quad \text{on } \partial G.$$

So, we shall study this problem instead of the original problem. We shall prove the unique solvability of the problem in the homogeneous Sobolev space. We look for a solution of the problem in the form $(D_G\mathbf{g}, \Pi_G\mathbf{g}) + (E_G\mathbf{\Psi}, Q_G\mathbf{\Psi})$, where $(D_G\mathbf{g}, \Pi_G\mathbf{g})$ is the hydrodynamical double layer potential with density $\mathbf{g}$ and $(E_G\mathbf{\Psi}, Q_G\mathbf{\Psi})$ is the hydrodynamical single layer potential with an unknown density $\mathbf{\Psi}$. (For the definition of hydrodynamical potentials see §4.) We obtain an integral equation

$$\mathbf{\Psi} = \frac{2(a_+ - a_-)}{(a_+ + a_-)} K'_G \mathbf{\Psi} + \frac{2}{a_+ + a_-}\mathbf{F} \tag{1}$$

(see §5). (For the definition of the operator K'_G see §4; for the definition of $\mathbf{F}$ see (19).) Fix a constant α such that

$$\frac{|a_+ - a_-|}{a_+ + a_-} < \alpha < 1.$$

We show that there exists an equivalent norm $\| \ \|$ on $H^{-1/2}(\partial\Omega, R^m)$ such that

$$\left\| \frac{2(a_+ - a_-)}{(a_+ + a_-)} K'_G \right\| < \alpha.$$

So, the integral equation (1) has a unique solution which can be obtained by the successive approximation.

Then it is studied the direct integral equation method. The solution of the transmission problem has a representation

$$\mathbf{u}_+ = E_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + D_G\mathbf{g},$$

$$p_+ = Q_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + \Pi_G\mathbf{g},$$

$$\mathbf{u}_- = E_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + D_G\mathbf{g},$$

$$p_- = Q_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + \Pi_G\mathbf{g}.$$

So, it is enough to calculate $[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G]$. But $[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G]$ is a unique solution $\mathbf{\Psi}$ of the equation (1). Hence we can use the successive approximation method and we know how quickly it converges.

2 Formulation of the problem

If $\mathbf{u} = (u_1, \dots, u_m)$ is a velocity field, p is a pressure, denote

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI$$

the corresponding stress tensor. Here I denotes the identity matrix and

$$\hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

is the strain tensor, with $(\nabla \mathbf{u})^T$ as the matrix transposed to $\nabla \mathbf{u} = (\partial_j u_k)$, $(k, j = 1, \dots, m)$. Denote $\nabla \cdot \mathbf{u} = \partial_1 u_1 + \ldots + \partial_m u_m$ the divergence of $\mathbf{u}$.

Let $G = G_+ \subset R^m$, $m > 2$, be a bounded open set with Lipschitz boundary. Denote $G_- = R^m \setminus \operatorname{cl} G_+$, where $\operatorname{cl} G_+$ is the closure of G_+. Denote by $\mathbf{n} = \mathbf{n}^G$ the outward unit normal of G_+. We shall study the transmission problem

$$\Delta \mathbf{u}_+ = \nabla p_+ \quad \text{in} \quad G_+, \qquad \nabla \cdot \mathbf{u}_+ = 0 \quad \text{in} \quad G_+, \tag{2}$$

$$\Delta \mathbf{u}_- = \nabla p_- \quad \text{in} \quad G_-, \qquad \nabla \cdot \mathbf{u}_- = 0 \quad \text{in} \quad G_-, \tag{3}$$

$$\mathbf{u}_+ - \mathbf{u}_- = \mathbf{g}, \quad a_+ T(\mathbf{u}_+, p_+)\mathbf{n} - a_- T(\mathbf{u}_-, p_-)\mathbf{n} = \mathbf{f} \quad \text{on } \partial G. \tag{4}$$

Here $\mathbf{g} \in H^{1/2}(\partial G_+, R^m)$, $\mathbf{f} \in H^{-1/2}(\partial G_+, R^m)$ and a_+, a_- are fixed positive constants.

Denote by $W^{1,2}(G)$ the space of all functions $u \in L^2(G)$ such that $\partial_j u \in L^2(G)$ in the sense of distributions for each $j = 1, \dots, m$ equipped with the norm

$$\|u\|_{W^{1,2}(G)} = \sqrt{\int_G [|u|^2 + |\nabla u|^2] \, d\mathcal{H}_m}.$$

(Here $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in R^k.) Denote by $H^{1/2}(\partial G)$ the space of traces of $W^{1,2}(G)$ endowed with the norm

$$\|v\|_{H^{1/2}(\partial G)} = \inf\{\|u\|_{W^{1,2}(G)}; u \in W^{1,2}(G), v = u|\partial G\}$$

and by $H^{-1/2}(\partial G)$ the dual space of $H^{1/2}(\partial G)$.

If $X(M)$ is a vector space of real functions (or distributions) on a set M denote by $X(M, C)$ its complexification, i.e. $X(M, C) = \{v_1 + iv_2; v_1 \in X(M, R) = X(M), v_2 \in X(M)\}$. If $K = R$ or $K = C$ and $k \in N$, we denote $X(M, K^k) = \{\mathbf{u} = (u_1, \dots, u_k); u_j \in X(M, K) \text{ for } j = 1, \dots, k\}$.

If $\mathbf{h} \in H^{-1/2}(\partial G, R^m)$ then the Neumann problem for the Stokes system

$$\Delta \mathbf{u} = \nabla p \quad \text{in} \quad G, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad G, \tag{5}$$

$$T(\mathbf{u}, p)\mathbf{n}^G = \mathbf{h} \quad \text{on} \quad \partial G \tag{6}$$

has a weak formulation (compare [10]): We say that $\mathbf{u} \in W^{1,2}(G, R^m)$, $p \in L^2(G, R^1)$ is a weak solution of the problem (5), (6) if $\nabla \cdot \mathbf{u} = 0$ and

$$2 \int_G \hat{\nabla} \mathbf{u} : \hat{\nabla} \mathbf{v} \, d\mathcal{H}_m - \int_G p(\nabla \cdot \mathbf{v}) \, d\mathcal{H}_m = \langle \mathbf{h}, \mathbf{v} \rangle$$

for each $\mathbf{v} \in W^{1,2}(G, R^m)$.

We need a weak characterization of Neumann problem for the Stokes system also for G_-.

Denote by $L^{1,2}(R^m)$ the space of all functions $u \in L^2_{loc}(R^m)$ such that $\partial_j u \in L^2(R^m)$ in the sense of distributions for each $j = 1, \dots, m$. Then $L^{1,2}(R^m)$ is a Banach space with the norm

$$\|u\|_{L^{1,2}(R^m)} = \sqrt{\int\limits_G |u|^2 \, d\mathcal{H}_m + \int\limits_{R^m} |\nabla u|^2 \, d\mathcal{H}_m}$$

(see [9], § 1.5.3). Denote by $\mathcal{C}^\infty_c(R^m)$ the space of all infinitely differentiable functions in R^m with compact support. Denote by $\tilde W^{1,2}(R^m)$ the closure of $\mathcal{C}^\infty_c(R^m)$ in $L^{1,2}(R^m)$. The space $L^{1,2}(R^m)$ is the direct sum of $\tilde W^{1,2}(R^m)$ and the space of constant functions (see [2], p. 155). If we put

$$\|u\|_{\tilde W^{1,2}(R^m)} = \|\nabla u\|_{L^2(R^m)},$$

then this norm is in $\tilde W^{1,2}(R^m)$ equivalent with the norm induced from $L^{1,2}(R^m)$ (see [9], §1.5.2 and [9], §1.5.3). According to [7], Lemma 2.2 we have $\tilde W^{1,2}(R^m) = \{u \in L^{2m/(m-2)}(R^m); \nabla u \in L^2(R^m; R^m)\}$. For an open set Ω denote by $\tilde W^{1,2}(\Omega)$ the space of restrictions of functions from $\tilde W^{1,2}(R^m)$ onto Ω. Denote

$$\|u\|_{\tilde W^{1,2}(\Omega)} = \inf\{\|v\|_{\tilde W^{1,2}(R^m)}; v = u \text{ on } \Omega\}.$$

Then $\tilde W^{1,2}(\Omega)$ is a Banach space. If $u \in \tilde W^{1,2}(\Omega)$ then $u \in W^{1,2}(V)$ for every bounded open subset V of Ω. If Ω is a bounded open set with Lipschitz boundary then $\tilde W^{1,2}(\Omega) = W^{1,2}(\Omega)$ and both norms are equivalent. If Ω is an unbounded domain with compact Lipschitz boundary then $\|\nabla u\|_{L^2(\Omega)}$ is an equivalent norm in $\tilde W^{1,2}(\Omega)$.

If $\mathbf{h} \in H^{-1/2}(\partial G, R^m)$ we say that $\mathbf{u} \in \tilde W^{1,2}(G_-, R^m)$, $p \in L^2(G_-, R^1)$ is a weak solution of the problem

$$\Delta \mathbf{u} = \nabla p \quad \text{in} \quad G_-, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad G_-,$$

$$T(\mathbf{u}, p)\mathbf{n}^{G_-} = \mathbf{h} \quad \text{on} \quad \partial G$$

if $\nabla \cdot \mathbf{u} = 0$ and

$$2 \int\limits_{G_-} \hat\nabla \mathbf{u} \cdot \hat\nabla \mathbf{v} \, d\mathcal{H}_m - \int\limits_{G_-} p(\nabla \cdot \mathbf{v}) \, d\mathcal{H}_m = \langle \mathbf{h}, \mathbf{v} \rangle$$

for each $\mathbf{v} \in \tilde W^{1,2}(G_-, R^m)$.

Using weak characterizations of the Neumann boundary condition for the Stokes system in G_+ and in G_- and the fact that $\mathbf{n}^{G_-} = -\mathbf{n}^{G_+}$ we give a weak formulation of the transmission problem for the Stokes system (2), (3), (4):

We say that $\mathbf{u}_+ \in \tilde W^{1,2}(G_+, R^m)$, $p_+ \in L^2(G_+, R^1)$, $\mathbf{u}_- \in \tilde W^{1,2}(G_-, R^m)$, $p_- \in L^2(G_-, R^1)$ *is a weak solution of the transmission problem for the Stokes* *system (2), (3), (4) if* $\nabla \cdot \mathbf{u}_+ = 0$, $\nabla \cdot \mathbf{u}_- = 0$, $\mathbf{u}_+ - \mathbf{u}_- = \mathbf{g}$ *on* ∂G_+ *in the sense of traces and*

$$a_+ \int\limits_{G_+} (2\hat{\nabla}\mathbf{u}_+ \cdot \hat{\nabla}\mathbf{v} - p_+(\nabla \cdot \mathbf{v}))\, d\mathcal{H}_m + a_- \int\limits_{G_-} (2\hat{\nabla}\mathbf{u}_- \cdot \hat{\nabla}\mathbf{v} - p_-(\nabla \cdot \mathbf{v}))\, d\mathcal{H}_m = \langle \mathbf{f}, \mathbf{v} \rangle$$

for all $\mathbf{v} \in \tilde{W}^{1,2}(R^m, R^m)$.

If $\mathbf{u}_+$, p_+, $\mathbf{u}_-$, p_- is a classical solution of the transmission problem for the Stokes system (2), (3), (4) and $\mathbf{u}_-$, $\nabla \mathbf{u}_-$ and p_- go to 0 at infinity sufficiently quickly, then the Green formula gives that $\mathbf{u}_+$, p_+, $\mathbf{u}_-$, p_- is also a weak solution of the transmission problem for the Stokes system (2), (3), (4).

3 Uniqueness

Proposition 3.1. *Suppose that* $\mathbf{u}_+ \in \tilde{W}^{1,2}(G_+, R^m)$, $p_+ \in L^2(G_+, R^1)$, $\mathbf{u}_- \in \tilde{W}^{1,2}(G_-, R^m)$, $p_- \in L^2(G_-, R^1)$ *is a weak solution of the transmission problem for the Stokes system (2), (3), (4). If* $\mathbf{g} = 0$, $\mathbf{f} = 0$ *then* $\mathbf{u}_+ = 0$, $\mathbf{u}_- = 0$, $p_+ = 0$, $p_- = 0$.

Proof. Put $\mathbf{v} = \mathbf{u}_+$ on G_+, $\mathbf{v} = \mathbf{u}_-$ on G_-. Since $\mathbf{u}_+ - \mathbf{u}_- = 0$ on ∂G_+, we have $\mathbf{v} \in \tilde{W}^{1,2}(R^m, R^m)$. Since $\nabla \cdot \mathbf{v} = 0$, we obtain

$$0 = \langle \mathbf{f}, \mathbf{v} \rangle = a_+ \int\limits_{G_+} 2|\hat{\nabla}\mathbf{u}_+|^2\, d\mathcal{H}_m + a_- \int\limits_{G_-} 2|\hat{\nabla}\mathbf{u}_-|^2\, d\mathcal{H}_m.$$

Thus $\hat{\nabla}\mathbf{v} = 0$ on $R^m \setminus \partial G$. If V is a component of $R^m \setminus \partial G$ then there exists a skew symmetric matrix A (i.e. $(A^T = -A)$) and a vector $\mathbf{b}$ such that $\mathbf{v}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in V (see for example [10], Lemma 3.1). Suppose first that V is unbounded. Since $\mathbf{v} \in L^{2m/(m-2)}(V, R^m)$, we infer that $A = 0$, $\mathbf{b} = 0$. Let now $V_1, \ldots, V_k$ are all component of $R^m \setminus \partial\Omega$ on which $\mathbf{v} = 0$. Denote by D the closure of $V_1 \cup \ldots \cup V_k$. Suppose that $D \neq R^m$. Let S be a component of ∂D. Then $\mathbf{v} = 0$ on S. Choose a component V of $R^m \setminus \partial G$ such that $S \subset \partial V$ and $D \cap V = \emptyset$. Suppose that $\mathbf{v}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in V. Let W be a bounded component of $R^m \setminus S$. Put $\mathbf{w}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in W. Then $\Delta \mathbf{w} = 0$ in W, $\mathbf{w} = 0$ on $\partial W = S$. The maximum principle for harmonic functions gives that $\mathbf{w} = 0$ in W. Since $V \subset W$, we have $\mathbf{v} = 0$ in V, what is a contradiction. Thus $\mathbf{v} = 0$ in R^m.

The Stokes system gives $\nabla p_+ = \Delta \mathbf{u}_+ = 0$, $\nabla p_- = \Delta \mathbf{u}_- = 0$. Thus p_+ is constant on each component of G_+, p_- is constant on each component of G_-. Since $p_- \in L^2(G_-, R^1)$, we deduce $p_- = 0$ on the unbounded component of G_-. Let S be a component of ∂G. Then there are a component V_+ of G_+ and a component V_- of G_- such that $S = \partial V_+ \cap \partial V_-$. Let c_+, c_- are such constants that $p_+ = c_+$ on V_+, $p_- = c_-$ on V_-. Since $\mathbf{u}_+$, p_+, $\mathbf{u}_-$, p_- is a classical solution of the transmission problem for the Stokes system (2), (3), (4), we have $0 = a_+T(\mathbf{u}_+, p_+)\mathbf{n} - a_-T(\mathbf{u}_-, p_-)\mathbf{n} = (a_-c_- - a_+c_+)\mathbf{n}$ on S. Therefore $a_-c_- = a_+c_+$. Since $p_- = 0$ on an unbounded component of G_-, we deduce that $p_+ = 0$, $p_- = 0$.

4 The surface potentials

The aim of this section is to assemble some basic facts on hydrodynamical potentials.

Denote by ω_m the surface of the unit sphere in R^m. For $\mathbf{x} \in R^m$, $m > 2$, and $j, k = 1, \ldots, m$ define

$$E_{jk}(\mathbf{x}) = \frac{1}{2\omega_m}\left[\delta_{jk}\frac{|\mathbf{x}|^{2-m}}{m-2} + \frac{x_j x_k}{|\mathbf{x}|^m}\right]$$

$$Q_k(\mathbf{x}) = \frac{x_k}{\omega_m|\mathbf{x}|^m}.$$

For $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ define the hydrodynamical single layer potential with density $\mathbf{\Psi}$ in $R^m \setminus \partial G$ by

$$(E_G\mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} E(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}) \tag{7}$$

and the corresponding pressure

$$(Q_G\mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} Q(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}). \tag{8}$$

Then $E_G\mathbf{\Psi} \in C^\infty(R^m\setminus\partial G, R^m)$, $Q_G\mathbf{\Psi} \in C^\infty(R^m\setminus\partial G, R^1)$, $\nabla Q_G\mathbf{\Psi} - \Delta E_G\mathbf{\Psi} = 0$, $\nabla\cdot E_G\mathbf{\Psi} = 0$ in $R^m\setminus\partial G$. Moreover, $E_G\mathbf{\Psi} \in W^{1,2}(G, R^m)$, $Q_G\mathbf{\Psi} \in L^2(G, R^1)$ (see [8], Theorem 4.4). We have the following decay behavior as $|\mathbf{x}| \to \infty$:

$$E_G\mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{2-m}),$$

$$Q_G\mathbf{\Psi}(\mathbf{x}),\ |(\nabla E_G\mathbf{\Psi})(\mathbf{x})| = O(|\mathbf{x}|^{1-m}).$$

This gives that $E_G\mathbf{\Psi} \in \tilde{W}^{1,2}(G_-, R^m)$, $Q_G\mathbf{\Psi} \in L^2(G_-, R^1)$.

If $\mathbf{\Psi} \in L^2(\partial G, R^m)$ then $E_G\mathbf{\Psi}$ can be defined by (7) for almost all $\mathbf{x} \in \partial G$ and $E_G\mathbf{\Psi} \in H^{1/2}(\partial G, R^m)$. The operator $E_G : \mathbf{\Psi} \mapsto E_G\mathbf{\Psi}$ can be extended as a bounded linear operator from $H^{-1/2}(\partial G; R^m)$ to $H^{1/2}(\partial G; R^m)$ (see [8], Proposition 4.5). Moreover, $E_G\mathbf{\Psi}$ is the trace of $E_G\mathbf{\Psi}$ on ∂G (see for example [10]). But these means that $E_G\mathbf{\Psi} \in \tilde{W}^{1,2}(R^m, R^m)$.

Remark that

$$E_G\mathbf{n}^G \equiv 0, \quad Q_G\mathbf{n}^G = -1 \text{ in } G_+, \quad Q_G\mathbf{n}^G = 0 \text{ in } G_- \tag{9}$$

(see for example [10]).

Now we define a hydrodynamical double layer potential. Fix $\mathbf{y} \in \partial G$ such that there is the unit outward normal $\mathbf{n}^G(\mathbf{y})$ of G at $\mathbf{y}$. For $\mathbf{x} \in R^m \setminus \{\mathbf{y}\}$, $j, k \in \{1, \ldots, m\}$ put

$$K^G_{jk}(\mathbf{x}, \mathbf{y}) = \frac{m}{\omega_m} \frac{(y_j - x_j)(y_k - x_k)(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}^G(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{m+2}},$$

$$\Pi^G_k(\mathbf{x}, \mathbf{y}) = \frac{2}{\omega_m} \left\{ -m \frac{(y_k - x_k)(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}^G(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^{m+2}} + \frac{n^G_k(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^m} \right\}.$$

For $\mathbf{\Psi} = [\Psi_1, \ldots, \Psi_m] \in H^{1/2}(\partial G, R^m)$ define the hydrodynamical double layer potential with density $\mathbf{\Psi}$ by

$$(D_G \mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} K^G(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}) \tag{10}$$

and the corresponding pressure

$$(\Pi_G \mathbf{\Psi})(\mathbf{x}) = \int\limits_{\partial G} \Pi^G(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}) \tag{11}$$

in $R^m \setminus \partial G$. Then $D_G \mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^m)$, $\Pi_G \mathbf{\Psi} \in C^\infty(R^m \setminus \partial G, R^1)$ and $\nabla \Pi_G \mathbf{\Psi} - \Delta D_G \mathbf{\Psi} = 0$, $\nabla \cdot D_G \mathbf{\Psi} = 0$ in $R^m \setminus \partial G$. Moreover, $D_G \mathbf{\Psi} \in W^{1,2}(G; R^m)$ (see [8], Theorem 4.4). We have the following decay behavior as $|\mathbf{x}| \to \infty$:

$$(D_G \mathbf{\Psi})(\mathbf{x}) = O(|\mathbf{x}|^{1-m}),$$

$$|(\nabla D_G \mathbf{\Psi})(\mathbf{x})|, \; \Pi_G \mathbf{\Psi}(\mathbf{x}) = O(|\mathbf{x}|^{-m}).$$

This gives that $D_G \mathbf{\Psi} \in \tilde{W}^{1,2}(G_-; R^m)$.

Define

$$K_G \mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \searrow 0} \int\limits_{\partial G \setminus B(\mathbf{x};\epsilon)} K^G(\mathbf{x}, \mathbf{y}) \mathbf{\Psi} \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y})$$

on ∂G, where $B(\mathbf{x}; \epsilon) = \{\mathbf{y}; |\mathbf{x} - \mathbf{y}| < \epsilon\}$. Then K_G is a bounded linear operator on $H^{1/2}(\partial G; R^m)$ (compare [8], Proposition 4.5). If we denote by $[D_G \mathbf{\Psi}]_+$ the trace of $D_G \mathbf{\Psi}$ as a function on G_+ and by $[D_G \mathbf{\Psi}]_-$ the trace of $D_G \mathbf{\Psi}$ as a function on G_-, then

$$[D_G \mathbf{\Psi}]_+(\mathbf{x}) = \frac{1}{2} \mathbf{\Psi}(\mathbf{z}) + K_G \mathbf{\Psi}(\mathbf{z}), \quad [D_G \mathbf{\Psi}]_-(\mathbf{x}) = -\frac{1}{2} \mathbf{\Psi}(\mathbf{z}) + K_G \mathbf{\Psi}(\mathbf{z}) \tag{12}$$

(see [10]).

Denote by K'_G the adjoint operator of K_G. Then K'_G is a bounded linear operator on $H^{-1/2}(G, R^m)$. Remark that

$$K'_G \mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \searrow 0} \int\limits_{\partial G \setminus B(\mathbf{x};\epsilon)} K^G(\mathbf{y}, \mathbf{x}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y})$$

for $\mathbf{\Psi} \in L^2(\partial G, R^m)$.

According to [10], Proposition 4.2

$$[T(E_G \mathbf{\Psi}, Q_G \mathbf{\Psi})]_+ \mathbf{n}^G = \frac{1}{2} \mathbf{\Psi} - K'_G \mathbf{\Psi}, \tag{13}$$

i.e.

$$\int\limits_{G_+} [(2\hat{\nabla} E_G \mathbf{\Psi}) \cdot (\hat{\nabla} \mathbf{v}) - (Q_G \mathbf{\Psi})(\nabla \cdot \mathbf{v})] \, d\mathcal{H}_m = \left\langle \frac{1}{2} \mathbf{\Psi} - K'_G \mathbf{\Psi}, \mathbf{v} \right\rangle \tag{14}$$

for all $\mathbf{v} \in \tilde{W}^{1,2}(R^m, R^m)$. If $\mathbf{v} \in \mathcal{C}_c^\infty(R^m, R^m)$ choose $r > 0$ such that $\mathbf{v}$ is supported in $B(0; r)$ and $\partial G \subset B(0; r)$. Put $V = G_- \cap B(0; r)$. Then

$$\int\limits_{G_-} [(2\hat{\nabla} E_G \mathbf{\Psi}) \cdot (\hat{\nabla} \mathbf{v}) - (Q_G \mathbf{\Psi})(\nabla \cdot \mathbf{v})] \, d\mathcal{H}_m = \int\limits_{V} [(2\hat{\nabla} E_G \mathbf{\Psi}) \cdot (\hat{\nabla} \mathbf{v}) - (Q_G \mathbf{\Psi})(\nabla \cdot \mathbf{v})] \, d\mathcal{H}_m$$

$$= \left\langle \frac{1}{2} \mathbf{\Psi} - K'_V \mathbf{\Psi}, \mathbf{v} \right\rangle = \left\langle \frac{1}{2} \mathbf{\Psi} + K'_G \mathbf{\Psi}, \mathbf{v} \right\rangle$$

Since $\mathcal{C}_c^\infty(R^m, R^m)$ is a dense subset of $\tilde{W}^{1,2}(R^m, R^m)$ we have

$$\int\limits_{G_-} [(2\hat{\nabla} E_G \mathbf{\Psi}) \cdot (\hat{\nabla} \mathbf{v}) - (Q_G \mathbf{\Psi})(\nabla \cdot \mathbf{v})] \, d\mathcal{H}_m = \left\langle \frac{1}{2} \mathbf{\Psi} + K'_G \mathbf{\Psi}, \mathbf{v} \right\rangle \tag{15}$$

for all $\mathbf{v} \in \tilde{W}^{1,2}(R^m, R^m)$, i.e.

$$[T(E_G \mathbf{\Psi}, Q_G \mathbf{\Psi})]_- \mathbf{n}^G = -\frac{1}{2} \mathbf{\Psi} - K'_G \mathbf{\Psi}. \tag{16}$$

Lemma 4.1. *For $\mathbf{\Psi} \in H^{1/2}(\partial G, R^m)$, $\mathbf{v} \in \tilde{W}^{1,2}(R^m, R^m)$ denote*

$$\langle [T(D_G \mathbf{\Psi}, \Pi_G \mathbf{\Psi})]_+ \mathbf{n}^G, \mathbf{v} \rangle = \int\limits_{G_+} [(2\hat{\nabla} D_G \mathbf{\Psi}) \cdot (\hat{\nabla} \mathbf{v}) - (\Pi_G \mathbf{\Psi})(\nabla \cdot \mathbf{v})] \, d\mathcal{H}_m,$$

$$\langle [T(D_G \mathbf{\Psi}, \Pi_G \mathbf{\Psi})]_- \mathbf{n}^G, \mathbf{v} \rangle = -\int\limits_{G_-} [(2\hat{\nabla} D_G \mathbf{\Psi}) \cdot (\hat{\nabla} \mathbf{v}) - (\Pi_G \mathbf{\Psi})(\nabla \cdot \mathbf{v})] \, d\mathcal{H}_m.$$

Then $T(D_G\mathbf{\Psi},\Pi_G\mathbf{\Psi})]_+\mathbf{n}^G \in H^{-1/2}(\partial G,R^m), T(D_G\mathbf{\Psi},\Pi_G\mathbf{\Psi})]_-\mathbf{n}^G \in H^{-1/2}(\partial G,R^m)$.

Proof. D_G is a bounded linear operator from $H^{1/2}(\partial G,R^m)$ to $W^{1,2}(G,R^m)$ (see [8], Theorem 4.4). Similarly, Π_G is a bounded linear operator from $H^{1/2}(\partial G,R^m)$ to $L^2(G,R^m)$. Thus $\mathbf{\Psi}\mapsto T(D_G\mathbf{\Psi},\Pi_G\mathbf{\Psi})]_+\mathbf{n}^G$ is a bounded linear operator from $H^{1/2}(\partial G,R^m)$ to $[W^{1,2}(G,R^m)]'$ (the dual space of $W^{1,2}(G,R^m)$). This and the behavior of $D_G\mathbf{\Psi}$, $\Pi_G\mathbf{\Psi}$ at infinity gives that $\mathbf{\Psi}\mapsto T(D_G\mathbf{\Psi},\Pi_G\mathbf{\Psi})]_-\mathbf{n}^G$ is a bounded linear operator from $H^{1/2}(\partial G,R^m)$ to $[\tilde W^{1,2}(G_-,R^m)]'$. If $\mathbf{v}\in\mathcal{C}_c^\infty(G,R^m)$ then Green's formula gives

$$\int\limits_{G_+}[(2\hat\nabla D_G\mathbf{\Psi})\cdot(\hat\nabla\mathbf{v})-(\Pi_G\mathbf{\Psi})(\nabla\cdot\mathbf{v})]\,d\mathcal{H}_m=\int\limits_{G_+}\mathbf{v}[\Delta D_G\mathbf{\Psi}+\nabla\Pi_G\mathbf{\Psi}]\,d\mathcal{H}_m=0.$$

Thus $[T(D_G\mathbf{\Psi},\Pi_G\mathbf{\Psi})]_+\mathbf{n}^G$ is supported on ∂G. But $H^{-1/2}(\partial G,R^m)$ is the set of distributions from $[W^{1,2}(\partial G,R^m)]'$ supported on ∂G. By the same way we show that $[T(D_G\mathbf{\Psi},\Pi_G\mathbf{\Psi})]_-\mathbf{n}^G\in H^{-1/2}(\partial G,R^m)$.

5 The indirect integral equation method

Put $\mathbf{u}_+=\mathbf{v}_++D_G\mathbf{g}$, $\mathbf{u}_-=\mathbf{v}_-+D_G\mathbf{g}$, $p_+=q_++\Pi_G\mathbf{g}$, $p_-=q_-+\Pi_G\mathbf{g}$. Then $\mathbf{u}_+$, $\mathbf{u}_-$, p_+, p_- is a weak solution of the problem (2), (3), (4) if and only if $\mathbf{v}_+$, $\mathbf{v}_-$, q_+, q_- is a weak solution of the problem

$$\Delta\mathbf{v}_+=\nabla q_+,\ \nabla\cdot\mathbf{v}_+=0\text{ in }G_+,\quad \Delta\mathbf{v}_-=\nabla q_-,\ \nabla\cdot\mathbf{v}_-=0\text{ in }G_-,\tag{17}$$

$$\mathbf{v}_+-\mathbf{v}_-=0,\quad a_+T(\mathbf{v}_+,q_+)\mathbf{n}-a_-T(\mathbf{v}_-,v_-)\mathbf{n}=\mathbf{F}\quad\text{on }\partial G,\tag{18}$$

where

$$\mathbf{F}=\mathbf{f}-a_+[T(D_G\mathbf{g},\Pi_G\mathbf{g})]_+\mathbf{n}+a_-[T(D_G\mathbf{g},\Pi_G\mathbf{g})]_-\mathbf{n}.\tag{19}$$

Since $\mathbf{v}_+=\mathbf{v}_-$ on ∂G, the function $\mathbf{v}=\mathbf{v}_+$ on G_+, $\mathbf{v}=\mathbf{v}_-$ on G_- must be in $\tilde W^{1,2}(R^m,R^m)$. We shall look for a solution of the problem (17), (18) in the form of a hydrodynamical single layer potential $\mathbf{v}=E_G\mathbf{\Psi}$, $q=Q_G\mathbf{\Psi}$ with an unknown density $\mathbf{\Psi}\in H^{-1/2}(\partial G,R^m)$. Boundary behavior of hydrodynamic single layer potentials gives that $\mathbf{v}=E_G\mathbf{\Psi}$, $q=Q_G\mathbf{\Psi}$ is a weak solution of the problem if

$$a_+\left(\frac{1}{2}\mathbf{\Psi}-K'_G\mathbf{\Psi}\right)+a_-\left(\frac{1}{2}\mathbf{\Psi}+K'_G\mathbf{\Psi}\right)=\mathbf{F}.\tag{20}$$

We would like to solve this equation by the successive approximation method. For this aim we rewrite this equation as

$$\mathbf{\Psi}=\frac{2(a_+-a_-)}{(a_++a_-)}K'_G\mathbf{\Psi}+\frac{2}{a_++a_-}\mathbf{F}.\tag{21}$$

Definition 5.1. *Lex X be a Banach space. Denote by I the identity operator on X. If M is a subspace of X denote by* $\dim M$ *the dimension of M. If Y*

is a subspace of X such that $X = M \bigoplus Y$, i.e. X is the direct sum of M and Y, denote by $\operatorname{codim} Y = \dim M$ *the codimension of Y. If T is a bounded linear operator in X, denote by* $\operatorname{Ker} T = \{x \in X; Tx = 0\}$ *the kernel of T,* $\alpha(T) = \dim \operatorname{Ker} T$, $\beta(T) = \operatorname{codim} T(X)$. *We say that T is Fredholm if $T(X)$ is a closed subset of X and $\alpha(T) < \infty$, $\beta(T) < \infty$. For a Fredholm operator T denote $i(T) = \alpha(T) - \beta(T)$ the index of T. If X is a complex Banach space denote by $\sigma(T)$ the spectrum of T and by* $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$ *the spectral radius of T.*

Lemma 5.2. *Let $\mathbf{\Psi} \in H^{-1/2}(\partial G, C^m)$. Then*

$$\langle \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}} \rangle = 2 \int\limits_{R^m \setminus \partial G} |\hat{\nabla} E_G \mathbf{\Psi}|^2 \, d\mathbf{y} \geq 0. \tag{22}$$

If $\langle \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}} \rangle = 0$ then $E_G \mathbf{\Psi} = 0$ in R^m and for each component S of ∂G there exists a constant c_S such that $\mathbf{\Psi} = c_s \mathbf{n}^G$ on S.

Proof. For (22) see [10], Corollary 4.4. Let now $\langle \mathbf{\Psi}, E_G \overline{\mathbf{\Psi}} \rangle = 0$. Then $\hat{\nabla} E_G \mathbf{\Psi} = 0$ in $R^m \setminus \partial G$. If V is a component of $R^m \setminus \partial G$ then there is a matrix A and a vector $\mathbf{b}$ such that $E_G \mathbf{\Psi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in V (see [10], Lemma 3.1). Denote by $V_1, \ldots, V_k$ all components of $R^m \setminus \partial G$ and suppose that V_1 is unbounded. Since $E_G \mathbf{\Psi}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, we infer that $E_G \mathbf{\Psi} = 0$ in V_1. Denote $D = \cup\{V_j; E_G \mathbf{\Psi} \equiv 0 \text{ in } V_j\}$. The boundary behavior of a hydrodynamical single layer potential gives $E_G \mathbf{\Psi} = 0$ on $\operatorname{cl} D$. Suppose now that $\operatorname{cl} D \neq R^m$. Fix a component S of ∂D. Choose a component V_j of $R^m \setminus \partial G$ such that $S \subset \partial V_j$ and $D \cap V_j = \emptyset$. Then there is a matrix A and a vector $\mathbf{b}$ such that $E_G \mathbf{\Psi}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ in V_j. Put $\mathbf{u}(\mathbf{x}) A\mathbf{x} + \mathbf{b}$. Denote by U the component of $R^m \setminus S$ such that $V_j \subset U$. Then $\mathbf{u}$ is a solution of the problem $\Delta \mathbf{u} = 0$ in U, $\mathbf{u} = 0$ on ∂U. The maximum principle for harmonic functions gives that $\mathbf{u} = 0$ in U. Thus $E_G \mathbf{\Psi} = 0$ in V_j, what is a contradiction. So, $\operatorname{cl} D = R^m$.

Since $\mathbf{u} = E_G \mathbf{\Psi}$, $p = Q_G \mathbf{\Psi}$ is a solution of the Stokes system (5) in $R^m \setminus \partial G$, we have $\nabla Q_G \mathbf{\Psi} = \Delta E_G \mathbf{\Psi} = 0$ in $R^m \setminus \partial G$. So, there are constants c_j such that $Q_G \mathbf{\Psi} = c_j$ in V_j. Let now S be a component of ∂G. Choose j and k such that $V_j \subset G_+$, $V_k \subset G_-$ and $S = \partial V_j \cap V_k$. According to boundary behavior of a hydrodynamical potential we have on S

$$\mathbf{\Psi} = \left[\frac{1}{2} \mathbf{\Psi} - K'_G \mathbf{\Psi}\right] - \left[-\frac{1}{2} \mathbf{\Psi} - K'_G \mathbf{\Psi}\right] = [T(E_G \mathbf{\Psi}, Q_G \mathbf{\Psi})]_+ \mathbf{n}^G$$

$$-[T(E_G \mathbf{\Psi}, Q_G \mathbf{\Psi})]_- \mathbf{n}^G = T(0, c_j) \mathbf{n}^G - T(0, c_k) \mathbf{n}^G = -c_j \mathbf{n}^G + c_k \mathbf{n}^G.$$

Lemma 5.3. *If $\lambda \in C$ is an eigenvalue of $(1/2)I - K'_G$ in $H^{-1/2}(\partial G, C^m)$ then $0 \leq \lambda \leq 1$.*

Proof. Let $\mathbf{\Psi}$ be an eigenfunction corresponding to an eigenvalue λ. According to Lemma 5.2 and [10], Proposition 4.3

$$2\int\limits_G |\hat{\nabla} E_G \mathbf{\Psi}(x)|^2 \, d\mathbf{x} = \left\langle \frac{1}{2}\mathbf{\Psi} - K'_G\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}} \right\rangle$$

$$= \langle \lambda\mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = 2\lambda \int\limits_{R^m\setminus\partial G} |\hat{\nabla} E_G\mathbf{\Psi}|^2 \, d\mathbf{x}.$$

If $\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle \neq 0$ then

$$\lambda = \frac{\int\limits_G |\hat{\nabla} E_G\mathbf{\Psi}|^2 \, d\mathbf{x}}{\int\limits_{R^m\setminus\partial G} |\hat{\nabla} E_G\mathbf{\Psi}|^2 \, d\mathbf{x}}$$

and $0 \leq \lambda \leq 1$. Let now $\langle \mathbf{\Psi}, E_G\overline{\mathbf{\Psi}}\rangle = 0$. Then for each component S of ∂G there exists a constant c_S such that $\mathbf{\Psi} = c_S\mathbf{n}^G$ on S. By virtue of (9) we obtain that $\lambda = 1$ or $\lambda = 0$.

Proposition 5.4. In $H^{-1/2}(\partial G, C^m)$ we have

$$\sigma\left(\frac{2(a_+ - a_-)}{a_+ + a_-}K'_G\right) \subset \left\langle -\frac{|a_+ - a_-|}{a_+ + a_-}, \frac{|a_+ - a_-|}{a_+ + a_-}\right\rangle. \tag{23}$$

If

$$\frac{|a_+ - a_-|}{a_+ + a_-} < \alpha < 1 \tag{24}$$

then there exists an equivalent norm $\| \ \|$ on $H^{-1/2}(\partial G, C^m)$ such that

$$\left\| \frac{2(a_+ - a_-)}{a_+ + a_-}K'_G \right\| < \alpha. \tag{25}$$

Proof. If $\lambda \in C \setminus \langle 0, 1\rangle$ then $(1/2)I - K'_G - \lambda I$ is a Fredholm operator with index 0 by [10], Theorem 4.12 and [11],§ 16, Theorem 16. Since λ is not an eigenvalue of $(1/2)I - K'_G$ by Lemma 5.3, we deduce that $\lambda \notin \sigma((1/2)I - K'_G)$. Easy calculation gives (23). Let now α satisfy (24). Then there exists an equivalent norm $\| \ \|$ on $H^{-1/2}(\partial G, C^m)$ such that (25) holds true by [3].

Theorem 5.5. *Let* $\mathbf{f} \in H^{-1/2}(\partial G, R^m)$, $\mathbf{g} \in H^{1/2}(\partial G, R^m)$. *Then there exists unique weak solution* $\mathbf{u}_+ \in \tilde{W}^{1,2}(G_+, R^m)$, $p_+ \in L^2(G_+, R^1)$, $\mathbf{u}_- \in \tilde{W}^{1,2}(G_-, R^m)$, $p_- \in L^2(G_-, R^1)$ *of the transmission problem for the Stokes system (2), (3), (4). Let* $\mathbf{F}$ *be given by (19). Then there exists unique solution* $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ *of the equation (20). Fix* $\mathbf{\Psi}_0 \in H^{-1/2}(\partial G, R^m)$. *Set*

$$\mathbf{\Psi}_k = \frac{2(a_+ - a_-)}{(a_+ + a_-)}K'_G\mathbf{\Psi}_{k-1} + \frac{2}{a_+ + a_-}\mathbf{F}.$$

Then $\mathbf{\Psi}_k \to \mathbf{\Psi}$ in $H^{-1/2}(\partial G, R^m)$. Fix a constant α satisfying (24). Then there exists an equivalent norm $\| \ \|$ on $H^{-1/2}(\partial G, C^m)$ (dependent only on G, a_+, a_- and α) such that

$$\|\mathbf{\Psi}_k - \mathbf{\Psi}\| \leq \frac{\alpha^k}{1-\alpha}\Big(\|\mathbf{\Psi}_0\| + \frac{2}{a_+ + a_-}\|F\|\Big).$$

Moreover, $\mathbf{u}_+ = E_G\mathbf{\Psi} + D_G\mathbf{g}$, $p_+ = Q_G\mathbf{\Psi} + \Pi_G\mathbf{g}$ in G_+, $\mathbf{u}_- = E_G\mathbf{\Psi} + D_G\mathbf{g}$, $p_- = Q_G\mathbf{\Psi} + \Pi_G\mathbf{g}$ in G_-.

Proof. The theorem is an easy consequence of Proposition 3.1 and Proposition 5.4.

6 The direct integral equation method

Let $\mathbf{f} \in H^{-1/2}(\partial G, R^m)$, $\mathbf{g} \in H^{1/2}(\partial G, R^m)$, $\mathbf{u}_+$, p_+, $\mathbf{u}_-$, p_- be a weak solution of the transmission problem (2), (3), (4). Then

$$\mathbf{u}_+(\mathbf{x}) = [E_G T(\mathbf{u}_+, p_+)\mathbf{n}^G](\mathbf{x}) + D_G\mathbf{u}_+(\mathbf{x}), \qquad \mathbf{x} \in G_+, \tag{26}$$

$$p_+(\mathbf{x}) = [Q_G T(\mathbf{u}_+, p_+)\mathbf{n}^G](\mathbf{x}) + \Pi_G\mathbf{u}_+(\mathbf{x}), \quad \mathbf{x} \in G_+ \tag{27}$$

(see for example [4]). Using Green's formula we obtain

$$[E_G T(\mathbf{u}_+, p_+)\mathbf{n}^G](\mathbf{x}) + D_G\mathbf{u}_+(\mathbf{x}) = 0, \quad \mathbf{x} \in G_-, \tag{28}$$

$$[Q_G T(\mathbf{u}_+, p_+)](\mathbf{x}) + \Pi_G\mathbf{u}_+(\mathbf{x}) = 0, \qquad \mathbf{x} \in G_-. \tag{29}$$

Since $\mathbf{u}_-$, p_- is a sum of a hydrodynamical single layer potential and a hydrodynamical double layer potential (see Theorem 5.5), we have $\mathbf{u}_-(\mathbf{x}) = O(|\mathbf{x}|^{2-m})$, $|\nabla\mathbf{u}_-(\mathbf{x})| + |p_-(\mathbf{x})| O(|\mathbf{x}|^{1-m})$ as $|\mathbf{x}| \to \infty$. We now use (26), (27), (28), (29) for $G_- \cap B(0;r)$. Letting $r \to \infty$ we obtain

$$\mathbf{u}_-(\mathbf{x}) = [E_{G_-} T(\mathbf{u}_-, p_-)\mathbf{n}^{G_-}](\mathbf{x}) + D_{G_-}\mathbf{u}_-(\mathbf{x}), \qquad \mathbf{x} \in G_-, \tag{30}$$

$$p_-(\mathbf{x}) = [Q_{G_-} T(\mathbf{u}_-, p_-)\mathbf{n}^{G_-}](\mathbf{x}) + \Pi_{G_-}\mathbf{u}_-(\mathbf{x}), \quad \mathbf{x} \in G_- \tag{31}$$

$$[E_{G_-} T(\mathbf{u}_-, p_-)\mathbf{n}^{G_-}](\mathbf{x}) + D_{G_-}\mathbf{u}_-(\mathbf{x}) = 0, \quad \mathbf{x} \in G_+, \tag{32}$$

$$[Q_{G_-} T(\mathbf{u}_-, p_-)](\mathbf{x}) + \Pi_{G_-}\mathbf{u}_-(\mathbf{x}) = 0, \qquad \mathbf{x} \in G_+. \tag{33}$$

Adding

$$\mathbf{u}_+ = E_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + D_G\mathbf{g}, \tag{34}$$

$$p_+ = Q_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + \Pi_G\mathbf{g}, \tag{35}$$

$$\mathbf{u}_- = E_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + D_G\mathbf{g}, \tag{36}$$

$$p_- = Q_G[T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G] + \Pi_G\mathbf{g}. \tag{37}$$

So, it is enough to calculate $T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G$. We have proved in the preceding paragraph that there exists unique $\mathbf{\Psi} \in H^{-1/2}(\partial G, R^m)$ such that $\mathbf{u}_+ = E_G\mathbf{\Psi} + D_G\mathbf{g}$, $p_+ = Q_G\mathbf{\Psi} + \Pi_G\mathbf{g}$, $\mathbf{u}_- = E_G\mathbf{\Psi} + D_G\mathbf{g}$, $p_- = Q_G\mathbf{\Psi} + \Pi_G\mathbf{g}$. This $\mathbf{\Psi}$ is given by Theorem 5.5. So, we have proved the following

Theorem 6.1. *Let* $\mathbf{f} \in H^{-1/2}(\partial G, R^m)$*,* $\mathbf{g} \in H^{1/2}(\partial G, R^m)$*. Let* $\mathbf{F}$ *be given by (19). Fix* $\mathbf{\Psi}_0 \in H^{-1/2}(\partial G, R^m)$*. Set*

$$\mathbf{\Psi}_k = \frac{2(a_+ - a_-)}{(a_+ + a_-)} K'_G \mathbf{\Psi}_{k-1} + \frac{2}{a_+ + a_-}\mathbf{F}.$$

Then $\mathbf{\Psi}_k \to T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G$ *in* $H^{-1/2}(\partial G, R^m)$*. Fix a constant* α *satisfying (24). Then there exists an equivalent norm* $\| \ \|$ *on* $H^{-1/2}(\partial G, C^m)$ *(dependent only on* G*,* a_+*,* a_- *and* α*) such that*

$$\|\mathbf{\Psi}_k - [T(\mathbf{u}_+, p_+)\mathbf{n}^G - T(\mathbf{u}_-, p_-)\mathbf{n}^G]\| \le \frac{\alpha^k}{1-\alpha}\Big(\|\mathbf{\Psi}_0\| + \frac{2}{a_+ + a_-}\|F\|\Big).$$

Acknowledgment: Supported by GA ČR Grant P201/11/1304 and RVO: 67985840.

References

[1] T. K. Chang, D. H. Pahk, Spectral properties for layer potentials associated to the Stokes equation in Lipschitz domains, Manuscripta Math. 130 (2009) 359–373.

[2] J. Deny, Les potentiels d'énergie finie, Acta Math. 82 (1950) 107–183.

[3] I. Gohberg, A. Marcus, Some remarks on topologically equivalent norms, Izvestija Mold. Fil. Akad. Nauk SSSR 10(1960) 91–95. (in Russian)

[4] G. C. Hsiao, W. L. Wendland, Boundary Integral Equations, Springer, Heidelberg, 2008.

[5] M. Kohr, C. Pintea, W. L. Wendland, Brinkman-type operators on Riemannian manifolds: Transmission problems in Lipschitz and C^1 domains, Potential Anal. 32 (2010) 229–273.

[6] M. Kohr, C. Pintea, W. L. Wendland, Stokes-Brinkman transmission problems on Lipschitz and C^1 domains in Riemannian manifolds, Commun. Pure Appl. Anal. 9 (2010), 493–537.

[7] H. Kozono, H. Sohr, On a new class of generalized solutions for the Stokes equations in exterior domains, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 4 ser. 19 (1998) 155–181.

[8] V. Maz'ya, M. Mitrea, T. Shaposhnikova, The inhomogenous Dirichlet problem for the Stokes system in Lipschitz domains with unit normal close to VMO^*, Funct. Anal. Appl. 43 (2009) 217–235.

[9] V. G. Maz'ya, S. V. Poborchi, Differentiable Functions on Bad Domains, World Scientific, Singapore, 1997.

[10] D. Medková, Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system, Acta Appl. Math. 116 (2011) 281–304.

[11] V. Müller, Spectral Theorey of Linear Operators and Spectral Systems in Banach Algebras, Birkhäuser, Basel, 2007.

Dagmar Medkova
Mathematical Institute,
Academy of Sciences of the Czech Republic,
Žitná 25,
115 67 Praha 1, Czech Republic, medkova@math.cas.cz