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ON THE MOTION OF RIGID BODIES IN A COMPRESSIBLE VISCOUS FLUID UNDER THE ACTION OF GRAVITATIONAL FORCES

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Abstract

The global existence of weak solution is proved for the problem of the motion of several rigid bodies in a barotropic compressible fluid, under the influence of gravitational forces.

1. Introduction

The study of motions of rigid or elastic bodies in a fluid has been studied during these last years. The presence of particles affects the flow of the liquid, and the fluid, in turn, affects the motion of particles, so that the problem of determining the flow characteristics strongly couples the fluid and solid motions. Here we are interested in the problem of several rigid bodies embedded into a viscous fluid. The fluid and rigid bodies are contained in a fixed open bounded set of R^3. We will suppose that the underlying continuum is a compressible newtonian fluid. We suppose that both media (fluid and solids) are under the influence of selfgraviting forces. As a physical motivation, one can think to the astrophysical situation of a selfgravitating planetary system moving into a slightly viscous interstellar medium.

Historically, the weak formulation of the problem has been introduced and studied by Judakov see [28] and after that by many authors: Desjardins and Esteban [4, 5], Hoffmann and Starovoitov [26, 27], San Martin, Starovoitov, Tucsnak [33], Serre [34], Galdi [20], among others.

In these problems the chalenging point is the existence of collisions. Let us first mention that in the case of compressible fluids this problem has been clarified by E. Feireisl in [13] who considered a rigid sphere surrounded by a compressible viscous fluid inside a cavity and constructed a "paradoxical" solution to the subsequent problem in which the sphere sticks to the ceiling of the cavity without falling down.

In the incompressible setting Hesla [23] and Hillairet [24] proved a no-collision result when there is only one body in a bounded two dimensional cavity. Later on the result was extended to the three dimensional situation by Hillairet and Takahashi [25].

Let us mention the main results known in the compressible case. In the absence of solids, in dimension $n \geqslant 2$ the proof of the existence of weak solutions for (1) was shown by P. L. Lions for $\gamma \geqslant 9/5$ when $n = 3$ see [29]. This result was extended by E. Feireisl, where the compactness of density was proved for $\gamma > 3/2$ for $n = 3$ in [12] and extension to the gravitational case with non monotone pressure was presented in [7] (for a review of general strategies of approximation schemes in compressible fluids see [11, 15, 32]). Concerning the motion of rigid bodies in a viscous compressible case let us once more mention the work of Feireisl [13], where the existence of global in time weak solution was proved. Comparing with results in the incompressible case, there is no restriction on the existence time, regardless of possible collisions of two or more bodies or contact of a body with the boundary. To be complete, let us mention that strong existence of the motion of rigid bodies in a viscous compressible case was recently investigated by Boulakia and Guerrero [1].

Our modest aim is that those of [13] in the compressible case also extend to the "interacting" gravitational Navier-Stokes-Poisson system with possibly non monotone pressure.

2. The compressible case

We consider the motion of N rigid bodies $\mathbf{B}_i$ for $i = 1, \cdots, N$ with smooth boundaries, in a smooth domain $\Omega \subset \mathbb{R}^3$. The bodies have constant densities ρ_i, $i = 1, \cdots, N$, masses m_i, $i = 1, \cdots, N$ and we denote by $\Omega_f(t) := \Omega \setminus \bigcup_{i=1}^N \overline{\mathbf{B}}_i$ the domain occupied by the fluid, the evolution of which is governed by the compressible Navier-Stokes system

$$\begin{cases} \partial_t \rho_f + \operatorname{div}(\rho_f \vec{v}_f) = 0, \\ \partial_t(\rho_f \vec{v}) + \operatorname{div}(\rho_f \vec{v}_f \otimes \vec{v}_f) + \nabla P(\rho_f) = \operatorname{div} \mathbb{T}(\vec{v}_f) + \rho_f \vec{F}_f + \rho g. \end{cases} \tag{1}$$

Here ρ_f is the density, $\vec{v}_f$ is the velocity, $\vec{F}_f$ is the body-force field (selfgravitation and gravitational action from bodies) given by $F_f(x,t) = 4\pi G \nabla \Phi$, with

$$\Phi(x,t) := \int_\Omega \frac{\rho_f(y,t)}{|x-y|}\, dy + \sum_{i=1}^N \rho_i \int_{B_i(t)} \frac{dy}{|x-y|}, \tag{2}$$

where G is the Newton's constant.

We suppose that the fluid is newtonian with viscous stress tensor $\mathbb{T}$ given by the constitutive law

$$\mathbb{T}(\vec{v}_f) \equiv 2\mu\mathbb{D}(\vec{v}_f) + \lambda\mathbb{I}\ \mathrm{div}\vec{v}_f, \tag{2'}$$

P is the pressure, $\mathbb{D}$ is the strain tensor with $\mathbb{D}(\vec{v}) = 1/2\,(\nabla\vec{v} +^t \nabla\vec{v})$.

The two Lamé viscosity coefficients λ and μ are real constants and satisfy the stability requirements

$$\mu > 0, \qquad 2\mu + 3\lambda \geqslant 0, \tag{3}$$

and the pressure $P(\rho_f)$ is related to the density ρ_f by a general barotropic constitutive law (see [7] for motivations) satisfying

$$\begin{cases} P \in C^1(\mathbb{R}_+), \quad P(0) = 0, \\ \dfrac{1}{a}\, z^{\gamma-1} - b \leqslant P'(z) \leqslant a z^{\gamma-1} + b \quad \text{for all } z \geqslant 0, \end{cases} \tag{4}$$

for two constants $a > 0$ and $b \geqslant 0$.

The evolution of the rigid body $\mathbf{B}_i$ is characterized by the motion of its center of mass

$$X_i(t) = \frac{1}{m_i}\int_{B_i(t)} \rho_i \vec{x}\, dx,$$

and its inertia tensor $\mathcal{J}_i$ given for any pair of vectors $\vec{a}, \vec{b}$ by

$$\mathcal{J}(t)\vec{a}\cdot\vec{b} = \int_{\mathbf{B}_i(t)} \rho_i(\vec{a}\times(\vec{x}-\vec{X}_i))\cdot(\vec{b}\times(\vec{x}-\vec{X}_i))\, dx.$$

The velocity of $x \in \mathbf{B}_i$ in the Eulerian coordinate system can be written as

$$\vec{u}_i(t,x) = \vec{U}_i(t) + \mathcal{O}_i(t)(\vec{x} - \vec{X}_i(t)), \quad \vec{U}_i(t)\frac{d}{dt}\vec{X}_i(t).$$

where $\mathcal{O}_i(t)$ is the angular velocity matrix of $\mathbf{B}_i$.

The matrix $\mathcal{O}_i(t)$ and is skew-symmetric then there is a vector $\vec{\omega}_i$ such that

$$\mathcal{Q}_i(t)(\vec{x}-\vec{X}_i) = \vec{\omega}_i(t)\times(\vec{x}-\vec{X}_i).$$

Each solid $\mathbf{B}_i$ is submitted to exterior forces (gravitation) $\vec{F}_i$ together with the contact forces at the various interfaces $\partial\mathbf{B}_i(t)$. Assuming continuity of velocity, we set

$$\lim_{y\to x}\vec{v}(t,y) = \vec{u}_i(t,x) \quad \text{for any } x \in \partial\mathbf{B}_i(t), \quad i = 1,\cdot\cdot\cdot, N. \tag{5}$$

Whenever Ω is bounded, we also consider the no-slip boundary condition

$$\lim_{y\to x}\vec{v}(t,y) = \vec{0} \quad \text{for any } x \in \partial\Omega. \tag{6}$$

So assuming continuity of stress, the balance laws for linear and angular momenta for $B_i(t), \quad i = 1, \cdots, N$ read

$$m_i \frac{d}{dt}\, \vec{U}_i(t) = \int_{\mathbf{B}_i(t)} \rho_i \vec{F}_i \; dx + \int_{\partial \mathbf{B}_i(t)} (\mathbb{T} - P\ \mathbb{I})\, \vec{n}\; d\sigma, \tag{7}$$

and

$$\mathcal{J}_i(t) \frac{d}{dt}\, \vec{\omega}_i(t) \int_{\partial \mathbf{B}_i(t)} \rho_i \left(\vec{x} - \vec{X}_i\right) \times (\mathbb{T} - P\ \mathbb{I})\, \vec{n}\; d\sigma + \int_{\mathbf{B}_i(t)} \rho_i (\vec{x} - \vec{X}_i) \times \vec{F}_i \; dx, \tag{8}$$

with $F_i(x,t) = 4\pi G \nabla \Phi_i$, with

$$\Phi_i(x,t) := \int_{\Omega} \frac{\rho_f(y,t)}{|x-y|}\; dy + \sum_{j \neq i} \rho_j \int_{\mathbf{B}_j(t)} \frac{dy}{|x-y|}. \tag{9}$$

2.1. Variational formulation

Let us assume that

$$\text{“The sets } \overline{\mathbf{B}}_i(t) \text{ are connected and compact, with } \mathbf{B}_i(t) \neq \emptyset,\ \left|\partial \overline{\mathbf{B}}_i(t)\right| = 0,\text{”} \tag{10}$$

for any $t \in [0,T]$ and each $i = 1, \cdots, N$.

We denote the time-dependent fluid region by $Q_f \equiv ((0,T) \times \Omega) \setminus Q_s$, where $Q_s \equiv \{(t,x) \mid t \in [0,T],\ x \in \cup_{i=1}^N \overline{\mathbf{B}}_i(t)\}$ is the solid region, we also set $(Q_s)_i \equiv \{(t,x) \mid t \in [0,T],\ x \in \overline{\mathbf{B}}_i(t)\}$

$$\rho = \rho_f + \sum_{i=1}^N \rho_i \chi_i, \quad \vec{v} = \left| \begin{array}{l} \vec{v}_f \ \text{ in } Q_f, \\ \vec{u}_i \ \text{ in } (Q_s)_i. \end{array} \right. , \quad \vec{F} = \vec{F}_s + \sum_{i=1}^N \vec{F}_i \chi_i, \tag{11}$$

where χ_i is the characteristic function of $\mathbf{B}_i$.

Then, we have the weak form of the continuity equation $(1)_1$

$$\int_0^T \int_{\Omega} \rho \partial_t \phi + \rho \vec{v} \cdot \nabla \phi \; dx\; dt = 0 \tag{12}$$

for any test function $\phi \in \mathcal{D}'((0,T) \times \Omega)$.

Similarly the momentum equation $(1)_2$ with (2) and (9) reads

$$\int_0^T \int_{\Omega} (\rho \vec{v}) \cdot \partial_t \phi + [\rho \vec{v} \otimes \vec{v}] : \mathbb{D}(\phi) + P \mathrm{div} \phi \; dx\; dt = \int_0^T \int_{\Omega} \mathbb{T} : \mathbb{D}(\phi) - \rho \vec{F} \cdot \phi \; dx\; dt \tag{13}$$

for any test function $\phi \in \mathcal{T}(\overline{Q}_s)$, where

$$\mathcal{T}(\overline{Q}_s) \equiv \{\phi \in \mathcal{D}((0,T) \times \Omega) \mid \mathbb{D}(\phi) = 0 \ \text{ on an open neighbourhood of } \overline{Q}_s\}.$$

The motion of the solids is described through a family of isometries of $\mathbb{R}^3$ by

$$\vec{\eta}_i(t,s) : \mathbb{R}^3 \to \mathbb{R}^3, \quad \overline{\mathbf{B}}_i(t) = \vec{\eta}_i(t,s)[\overline{\mathbf{B}}_i(s)] \text{ for } 0 \leqslant s \leqslant t \leqslant T,$$

or equivalently

$$\vec{\eta}_i(t,s) = \vec{\eta}_i(t,0)\,(\vec{\eta}_i(s,0))^{-1},$$

where the mappings $\vec{\eta}_i$ satisfy

$$\vec{\eta}_i(t,0)(\vec{x}) \equiv \vec{\eta}_i[t](\vec{x}) = \vec{X}_i(t) + \mathcal{O}_i(t)\vec{x}, \quad \mathcal{O}_i(t) \in SO(3).$$

We say that the velocity $\vec{v}$ is compatible with the family $\{\mathbf{B}_i, \vec{\eta}_i, \quad i = 1,\cdots,N\}$ if the functions $t \to \vec{\eta}_i[t](\vec{x})$ are absolutely continuous on $(0,T)$ for any $x \in \mathbb{R}^3$ and if

$$\left(\frac{\partial}{\partial}\,\vec{\eta}_i[t]\right)\left((\vec{\eta}_i[t])^{-1}(\vec{x})\right) = \vec{v}(t,x) \text{ for } i = 1,\cdots,N, \text{ for } x \in \overline{\mathbf{B}}_i(t) \text{ and a.e. } t \in (0,T). \tag{14}$$

In other words if

$$\vec{v}(t,x) = \vec{u}_i(t,x) \equiv \vec{U}_i(t) + \mathcal{O}_i(t)(\vec{x} - \vec{X}_i(t)) \text{ for } x \in \overline{\mathbf{B}}_i(t) \text{ and a.e. } t \in (0,T),$$

where

$$\vec{U}_i(t) = \frac{d}{dt}\vec{X}_i(t), \quad \frac{d}{dt}\,\mathcal{O}_i(t)\mathcal{O}_i^T(t) = \mathcal{Q}_i(t).$$

We say that relations (12) (13) and (14) represent a weak formulation of (1)(5)(6).

In the spirit of [15], just mention two facts. The first one is that any classical solution ρ of $(1)_1$ is also a renormalized solution i.e. for any $b \in C^1(\mathbb{R})$ such that $b'(z) \equiv 0$ for z large enough

$$\partial_t b(\rho) + \mathrm{div}(b(\rho)\vec{v}) + (\rho b'(\rho) - (b(\rho))\,\mathrm{div}\vec{v}) = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega), \tag{15}$$

and secondly, the following dissipative condition implied by the energy inequality

$$\int_{\mathbb{R}^3}\left[\frac{1}{2}\,\rho|\vec{v}|^2 + \Pi(\rho) - \frac{1}{2}\,G\rho\Phi(\rho)\right]\,dx + \int_0^T\int_{\mathbb{R}^3} 2\mu(\rho)|\mathbb{D}(\vec{v})|^2 + \lambda(\rho)(\mathrm{div}\,\vec{v})^2 dx\,dt \leqslant E_0, \tag{16}$$

where $\Pi(\rho) = \rho\int_1^\rho \frac{P(z)}{z^2}\,dz$.

2.2. Main result

Using the previous global quantities $(\rho(t,x), \vec{v}(t,x))$ for the fluid-solid mixture leads formally to the integro-differential system equivalent to (1), (5), (6), (7), (8)

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\vec{v}) = 0, \\ \partial_t(\rho\vec{v}) + \mathrm{div}(\rho\vec{v}\otimes\vec{v}) - \mathrm{div}\,(2\mu(\rho)\mathbb{D}(\vec{v})) - \nabla\,(\lambda(\rho)\mathrm{div}\vec{v}) + \nabla P = \rho\vec{F}, \end{cases} \tag{17}$$

for $t > 0$, with initial conditions

$$\rho|_{t=0} = \rho^0(x), \quad \rho\vec{v}|_{t=0} = \vec{m}_0(x), \qquad \text{on } \mathbb{R}^3. \tag{18}$$

Theorem 1. Suppose that $\Omega \subset \mathbb{R}^3$ and let initial data ρ_0, $\vec{m}_0$ be such that

$$\rho_0 \geqslant 0, \quad \rho_0 \in L^\gamma(\Omega), \tag{19}$$

$$\vec{m}_0 = 0 \ \text{ a.e. on the set } \{x \in \Omega \mid \rho_0 = 0\}, \quad \frac{\vec{m}_0^2}{\rho_0} \in L^1(\Omega), \tag{20}$$

and $\{(\overline{B}_i)_0\}_{i=1,\cdots,N}$ be given satisfying (10).

Suppose that $\gamma > 3/2$ and that λ and μ are two positive constants.

Then the problem (1), (5), (6), (7), (8) admits a variational solution ρ, $\vec{v}$ and $\{\overline{\mathbf{B}}_i, \vec{\eta}_i\}_{i=1,\cdots,N}$ on $Q := ((0,T) \times \Omega$ satisfying

$$\rho(0) = \rho_0, \quad (\rho\vec{v})(0) = \vec{m}_0, \quad \overline{\mathbf{B}}_i(0) = (\overline{\mathbf{B}}_i)_0 \ \ i = 1, \cdots, N,$$

where the second equality is understood in the sense

$$\int_\Omega \vec{m}_0 \cdot \phi \, dx = \lim_{t \to 0+} \int_\Omega (\rho\vec{v})(t) \cdot \phi \, dx, \tag{20'}$$

for any test function $\phi \in \mathcal{D}(\Omega)$ such that $\mathbb{D}(\phi) = 0$ in a neighbourhood of $\bigcup_{i=1}^N \overline{\mathbf{B}}_i(0)$.

The proof of Theorem 1 will rely on the weak-convergence technique developed in [13] with the gravitational ingredient of [7] (see also [11, 15, 32]): assuming that we have constructed a sequence of approximate solutions

$$\left(\rho_n, \vec{v}_n, \{(\overline{\mathbf{B}}_i)_n, (\vec{\eta}_i)_n\}_{i=1,\cdots,N}\right),$$

one has to prove the convergence of the density

$$\rho_n \to \rho \quad \text{strongly in } \ C([0,T]; L^1(\Omega)), \tag{21}$$

and of the convective term

$$\rho_n \vec{v}_n \otimes \vec{v}_n \to \rho\vec{v} \otimes \vec{v} \quad \text{weakly in } \ L^1(Q_f), \tag{22}$$

where ρ and $\vec{v}$ are the respective weak limits of the sequences ρ_n and $\vec{v}_n$.

Let us first quote a result from [13].

Let K be a compact non void domain in $\mathbb{R}^3$. We denote by $d_K(x)$ the distance from x to K and we define for any subset $B \subset \mathbb{R}^3$ $\mathbf{db}_B(x)$, the signed distance from x to ∂B, by

$$\mathbf{db}_B(x) = d_{\overline{\mathbb{R}^3 \setminus B}}(x) - d_{\overline{B}}(x) \quad x \in \mathbb{R}^3.$$

The sequence of sets $B_n \subset \mathbb{R}^3$ is said to converge to $B \subset \mathbb{R}^3$ in the sense of boundaries $B_n \xrightarrow{b} B$ if $\mathbf{db}_{B_n} \to \mathbf{db}_B$ in $C_{loc}(\mathbb{R}^3)$.

Proposition 1. Let $\vec{v}_n(t,x)$ a family of functions such that $t \to \vec{v}_n(t,\cdot)$ is continuous from $[0,T]$ to $\mathbb{R}^3$ and $x \to \vec{v}_n(\cdot,x)$ is measurable from $\mathbb{R}^3$ to $\mathbb{R}^3$ (Caratheodory), and such that

$$t \to \|\vec{v}_n(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla\vec{v}_n(t,\cdot)\|_{L^\infty(\mathbb{R}^3)},$$

is bounded in $L^2(0,T)$.

Let $\vec{\eta}_n[t] \,:\, \mathbb{R}^3 \to \mathbb{R}^3$ the solution of the problem

$$\frac{d}{dt}\,\vec{\eta}_n[t](x) = \vec{v}_n(t,\vec{\eta}_n[t](x)), \quad \eta_n[0](x) = x \;\; x \in \mathbb{R}^3.$$

Let also $\mathbf{B}_n \subset \mathbb{R}^3$ be a sequence such that $\mathbf{B}_n \xrightarrow{b} \mathbf{B}$, and denote by $\mathbf{B}_n(t) = \eta_n[t](\mathbf{B}_n)$ the image of B_n by the flow $\vec{v}_n$.

Then passing to sub-sequences

$$\eta_n[t] \to \eta[t] \;\text{ in } C_{loc}(\mathbb{R}^3) \text{ as } n \to \infty \;\text{ uniformly in } [0,T],$$

where $\eta[t]$ solves

$$\frac{d}{dt}\,\vec{\eta}[t](x) = \vec{v}(t,\vec{\eta}[t](x)), \quad \eta[0](x) = x \;\; x \in \mathbb{R}^3,$$

and $\vec{v}_n \to \vec{v}$ in $L^2(0,T;W^{1,\infty}(\mathbb{R}^3))$ weak-star.

Moreover $\mathbf{B}_n(t) \xrightarrow{b} \mathbf{B}(t)$ uniformly in $[0,T]$, where $\mathbf{B}(t) = \eta[t](\mathbf{B})$.

2.3. The approximation scheme

As in [13] we use the penalization method of San Martin, Starovoitov and Tucsnak [33] which amounts to replace the solids by a fluid with very high viscosity and consider the regularized system

$$\left.\begin{aligned} \partial_t\rho + \operatorname{div}(\rho\vec{v}) &= \varepsilon\Delta\rho, \\ \partial_t(\rho\vec{v}) + \operatorname{div}(\rho\vec{v}\otimes\vec{v}) - \operatorname{div}\left(2\mu(\chi)\mathbb{D}(\vec{v})\right) - \nabla\left(\lambda(\chi)\operatorname{div}\vec{v}\right) + \nabla p + \varepsilon\nabla\vec{v}\cdot\nabla\rho &= \rho\vec{F}, \end{aligned}\right\} \tag{23}$$

where $\varepsilon > 0$, p is a regularized pressure defined by the constitutive relation

$$p = p(\rho) = P(\rho) + A\rho^\beta \;\text{ with } K > 0 \text{ and } \beta > 4, \tag{24}$$

and the viscosity coefficients are supposed to depend on an extra variable χ related to velocity as follows. The system is completed by the Neumann boundary conditions

$$\nabla\rho\cdot n|_{\partial\Omega} = 0. \tag{25}$$

Defining a regularized velocity field by the convolution $\mathcal{R}_\delta[\vec{v}](t,x) = \vec{v} * \vartheta_\delta$ where $\{\vartheta_\delta\}_{\delta>0}$ is a sequence of regularizing kernels (radially symmetric and radially non increasing functions on $\mathbb{R}^3$ supported in the ball $B(0,\delta)$ with $\int \vartheta_\delta \, dx = 1$), and the corresponding characteristic curves $\vec{\eta}$ are such that

$$\frac{d}{dt}\,\vec{\eta}[t](x) = \mathcal{R}_\delta[\vec{v}]\,(t,\vec{\eta}[t](x))\,, \quad \eta[0](x) = x \text{ for } x \in \mathbb{R}^3.$$

Then, for a bounded open set $O \subset \mathbb{R}^3$ we denote $O(t) = \eta[t]O$ and we define

$$\chi(t,x) \equiv \mathbf{db}_{O(t)}(x) \ \text{ for any } t \in [0,T],\ x \in \mathbb{R}^3. \tag{26}$$

We supplement the system (23) with the initial data

$$\rho(0) = \rho_0 \in C^{2+\nu}(\mathbb{R}^3), \quad 0 < \underline{\rho} \leqslant \rho_0(x) \leqslant \overline{\rho}, \tag{27}$$

$$(\rho\vec{v})(0) = \vec{m}_0, \quad \vec{m}_0 \in C^2(\mathbb{R}^3) \tag{28}$$

Proposition 2. Let $O \in \mathbb{R}^3$ be a bounded open set, the initial data $(\rho_0, \vec{m}_0)$ satisfy (27) and (28) and let the pressure satisfy (24) with $\beta > 4$ and $\gamma > 3/2$. Finally, let the viscosity coefficients μ, λ be smooth functions of χ satisfying

$$\mu(\chi) \geqslant \mu_0 > 0, \quad \lambda(\chi) + \mu(\chi) \geqslant 0 \ \text{ for any } \chi \in \mathbb{R}.$$

Then the problem (23) admits a weak solution $\rho \geqslant 0$, such that

- Equation $(23)_1$ holds a.e. on $(0,T)\times\mathbb{R}^3$ and equation $(23)_2$ holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.
- The solution satisfies the energy inequality

$$\int_{\mathbb{R}^3}\left[\frac{1}{2}\,\rho|\vec{v}|^2 + \Pi(\rho) - \frac{1}{2}\,G\rho\Phi(\rho)\right]\,dx + \int_0^\tau\int_{\mathbb{R}^3} 2\mu|\mathbb{D}(\vec{v})|^2 dx\,dt + \lambda(\operatorname{div}\,\vec{v})^2 dx\,dt \leqslant E_0, \tag{29}$$

  where $\Pi(\rho) = \rho\int_1^\rho \frac{P(z)}{z^2}\,dz$ and $E_0 = \int_{\mathbb{R}^3}\left[\frac{1}{2}\,\frac{\vec{m}^2}{\rho_0} + \Pi(\rho_0) - \frac{1}{2}\,G\rho_0\Phi(\rho_0)\right]\,dx.$

- Moreover, the following estimate holds independently of χ

$$\varepsilon\|\nabla\rho\|^2_{L^2((0,T)\times\mathbb{R}^3)} \leqslant C(\rho_0,\vec{m},K,\beta,\mu_0,G) \tag{30}$$

  and

$$\|\nabla\rho\|_{L^{r+1}(0,T;L^2(\mathbb{R}^3))} + \|\rho\|_{L^{\beta+1}((0,T)\times\mathbb{R}^3))} + \|\partial_t\rho\|_{L^r((0,T)\times\mathbb{R}^3))} + \|\Delta\rho\|_{L^r((0,T)\times\mathbb{R}^3))} \leqslant C(\rho_0,\vec{m},K,\beta,\mu_0,G) \tag{31}$$

  for a certain $r > 1$.

Proof: The proof follows from the results of [16] and [7] (the presence of the variable χ does not modify essentially the arguments). □

2.4. The high viscosity limit

We now study the problem (23)–(28) where ε, $A > 0$ are fixed and we only consider the viscosity limit (i.e. the limit of viscosity coefficients $\mu = \mu_n$, $\lambda = \lambda_n$). We set

$$\lambda_n(X) = \lambda + nH(X+\delta),\ \mu_n(X) = \mu + \mu H(X+\delta) \tag{32}$$

with $H = H(z)$ a smooth convex function, $H(z) = 0$, $z \le 0$, $H(z) > 0$. Otherwise, the positive constants μ and λ satisfy

$$\mu > 0,\ \lambda + \mu \ge 0. \tag{33}$$

From Proposition 1, the problem (23)–(28) has a weak solution ρ_n, v_n for $n = 1, 2$. Since (29)–(31) are independent of n then it implies

$$\rho_n \to \rho \text{ in } L^\beta((0,T)\times\Omega) \tag{34}$$

$$v_n \to v \text{ weakly in } L^2(0,T,W_0^{1,2}(\Omega)) \tag{35}$$

$$(\rho_n v_n) \to \rho v \text{ weakly in } L^2((0,T)\times\Omega) \tag{36}$$

$$\rho_n v_n \otimes v_n \to Q \text{ weakly in } L^{6/5}((0,T)\times\Omega), \tag{37}$$

where ρ, v satisfy $(23)_1$ a.e. on $(0,T)\times\Omega$, the boundary condition (24) in the sense of traces and the estimates (29)–(31) are satisfied.

From (34)–(37), (29), (32) we see that the limit functions ρ, u satisfy the energy inequality (29) with μ and λ constant as in (33).

Moreover, from the previous limits, we obtain

$$\rho_n, \rho \in C([0,T]; L^2(\Omega)),$$

$$\rho_n(0) = \rho(0) = \rho_0.$$

Further

$$\|\nabla\rho_n\|^2_{L^2((0,T)\times\Omega)} \to \|\nabla\rho\|^2_{L^2((0,T)\times\Omega)} \tag{38}$$

and

$$\nabla v_n \nabla\rho_n \to \nabla v\nabla\rho \text{ in } \mathcal{D}'((0,T)\times\Omega). \tag{39}$$

Similarly as in [F] and applying Proposition 2.4 we get that $v_n = R_\delta[v_n]$, $n = 1, 2, \ldots$ satisfy the hypotheses of Proposition 2.4 which implies

$$X_n(t) = db_{\mathcal{O}_n(t)} \to db_{\mathcal{O}(t)} \text{ in } C_{loc}(\mathbb{R}^3) \text{ uniformly in } t \in [0,T) \tag{40}$$

where $\mathcal{O}(t) = \mu[t](\mathcal{O})$,

$$\frac{d}{dt}\eta(t)(x) = R_0[u](t, \mu[t](x)),$$

$$\eta[0](x) = x.$$

Denoting

$$\begin{array}{rcl} Q^f & = & \{(t,x)|t \in (0,T), db_{\mathcal{O}(t)}(x) + \delta < 0\} \\ Q^s & = & \{(t,x)|t \in (0,T), db_{\mathcal{O}(t)}(x) + \delta > 0\}, \end{array} \tag{41}$$

then from (40) we get

$$\mu_n(X_n) = \mu,\ \lambda(X_n) = \lambda \text{ on any compact } K^f \subset Q^f \tag{42}$$

for all $n \geq n\ (K^f)$.

Applying the energy inequality (29) we obtain

$$\mathbb{D}(v_n) \to 0 \text{ in } L^2(K^s) \text{ for any compact } K^s \subset Q^s. \tag{43}$$

Then as in [F] we can find an open time interval $J \subset (0,T)$ and a neighbourhood $U = U_\varepsilon(x)$ such that $\overline{J \times U} \subset Q^f$, which together with (42), $(23)_2$, (39)–(31) we get

$$\partial_t \rho_n \in L^q(J, W^{-k,q}(U)) \text{ for a certain, } q > 1,\ k \geq 1$$

which gives us

$$\rho_n v_n \to \rho v \in C(\overline{J}, L^{\frac{2s}{s+1}}_{\text{weak}}(U)). \tag{44}$$

This together with the imbedding

$$L^{\frac{2\beta}{\beta+1}}(U) \subset W^{-1,2}(U) \tag{45}$$

implies that

$$\rho_n v_n \otimes v_n \to \rho v \otimes v \text{ weakly in } \ L^{6/5}((0,T) \times U), \tag{46}$$

and shows that $Q = \rho v \otimes v$ a.e. on the set Q^f.

Lemma 1. The following equivalence holds:

$$\overline{Q}^s \equiv d(Q^s) = \{(t,x),\ t \in [0,T],\ db_{\mathcal{O}(t)}(x) + \delta \geq 0\}$$

Proof: see [F].

Lemma 1 together with (44) implies that $Q : \mathbb{D}(\varphi) = \rho u \otimes u : \mathbb{D}(\varphi)$ for any test function $\varphi \in \mathcal{T}(\overline{Q}^s)$. Then we can pass to the limit in the momentum equation to obtain the integral identity

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} (\rho v) \cdot \partial_t \varphi + [\rho v \otimes v] : \mathbb{D}(\varphi) + p(p) \operatorname{div} \varphi + d \nabla v \nabla \rho \varphi dx dt \\ &= \int_0^t \int_{\mathbb{R}^N} T(u) \mathbb{D}(\varphi) - \rho F \cdot \varphi dx dt \end{aligned} \tag{47}$$

for any test function $\varphi \in \mathcal{T}(\overline{Q}^s)$, where $\mathcal{T}$ is given by $(2)'$.

Similarly as in [F] we can consider the open set $S \subset \mathbb{R}^3$, which can be written as a union of a finite number of balls

$$S = \bigcup_{i=1}^{k} U_{\varepsilon_i}(x_i).$$

Taking $0 < \delta < \min;\ (\varepsilon_i)$

$$S = \{x\,|db_{\mathcal{O}}(x) + \delta > 0\},\ \mathcal{O} = \bigcup_{i=1}^{k} U_{\varepsilon_i - \delta}(x_i).$$

We define $U^i(t) = \eta[t](U_{\varepsilon_i-\delta}(x_i))$. Because (43) it implies $\mathbb{D}(v(t)) = 0$ a.e. on the set

$$U^i_\delta(t) = \{x \in \mathbb{R}^3 | db_{U^i(t)}(x) + \delta > 0\} \text{ for a.e. } t \in [0,T].$$

It implies $v(t)$ is a rigid velocity on U^i_δ and $R_\delta[v(t)] = v(t)$ a.e. on $U^i(t)$ for a.e. $t \in [0,T)$. This implies that there is a system of isometries $\eta^i[t]$ such that the system of compacts $\{\overline{U}^i_\delta, \eta^i\}_{i=1}^k$ is compatible with the velocity v and denoting $S = \bigcup_{i=1}^N \mathbf{B}_i$ ($\mathbf{B}^i$ are open connected) we find that v is compatible with the system $\{\overline{\mathbf{B}}^i, \eta^i\}_{i=1}^N$, where the isometries $\eta^i[t]$ coincide with $\eta[t]$ on any of the balls $U^j_{\varepsilon_j-\delta}$ contained in $\mathbf{B}^i$.

We have then proved.

Proposition 1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary $\partial\Omega$ of class $C^{2+\nu}$, $\nu > 0$. Let the pressure p be given by the constitutive law (24) where

$$\beta > \max\{4, \gamma\},\ \gamma > \frac{4}{3}.$$

Let the initial data ρ_0, $\vec{m}_0$ have the properties required by (27)–(28), $q \in \mathcal{D}((0,T) \times \Omega)$. Finally, let $S \subset \mathbb{R}^3$ be an open set which can be written as a finite union of balls. Let $\mathbf{S} = \bigcup_{i=1}^N \mathbf{B}^i$, where $\mathbf{B}^i$ are open connected. Then there exists functions $\rho \geq 0, v$ such that

$$\rho \in L^\infty((0,T);\ L^\beta(\Omega)),\ v \in L^2(0,T,W_0^{1,2}(\Omega)),$$

and a system of isometries $\eta^i(t) : \mathbb{R}^3 \to \mathbb{R}^3$ having the following properties.

- The functions ρ, v satisfy the regularized continuity equation $(23)_1$ a.e. on $(0,T) \times \Omega$ while the boundary condition (25) hold in the sense of traces.

- The variational form of the momentum equation $(23)_2$ holds for any test function $\varphi \in T(\overline{Q}^s)$ where

$$\overline{Q}^s = \{(t,x)\,|t \in [0,T],\ x \in \bigcup_{i=1}^{m} \eta^i[t](\mathbf{B}_i)\}$$

- the density ρ belongs to the class $C([0,T];L^1(\Omega))$ and satisfies the initial condition (27). (The momentum (ρv) satisfies the initial condition (28) in the sense of $(20)'$.

- the functions ρ, v satisfy the energy inequality (29) with the constant viscosity coefficients μ, ν as well as (30)–(31) independently of $\mathbf{B}$.

- the velocity v is compatible with the system $\{\overline{\mathbf{B}}^i, \eta^i\}_{i=1}^N$, $\mathbf{B}^i(0) = \mathbf{B}^i$, $i = 1, \dots, N$.

2.5. The vanishing viscosity limit

Proposition 4. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with the boundary $\partial\Omega$ of class $C^{2+\nu}$, $\nu > 0$. Let the pressure p be given by the constitute law (24) with

$$\beta > \max\{7, \gamma\},\ \gamma > 4/3.$$

Let the initial data φ_0, m_0 have properties (27), (28). Finally, let $S_0 \subset R^3$ be an open set which can be written as a finite union of holds,

$$\mathbf{B}_0 = \bigcup_{i=1}^{N} \mathbf{B}_0^i, \text{ where } \mathbf{B}_0^i \text{ are open and connected.}$$

Then the problem (1)–(2) admits a variational solution ρ,v and $\{\overline{\mathbf{B}}^i,\eta^i\}_{i=1}^N, n(0,T)\times\Omega$ satisfying the initial conditions (27)–(28) and $\mathbf{B}^i(0) = \mathbf{B}_0^i$.

Proof: See [9].

It remains to show the energy inequality (29).

2.6. Sequential stability

The following stability results holds

Theorem 2. Let $\Omega_n, \Omega \subset \mathbb{R}^3$, be bounded domains such that $\Omega_n \subset \Omega_{n+1}$, $\Omega_n \xrightarrow{b} \Omega$ as $n \to \infty$. Assume that the pressure $p = p_n$ is given by the constitutive relation

$$p_n(p) = a\rho^\gamma + b_n\rho^\beta$$

with $\gamma > 3/2$, $\beta > 1$, $b_n \to 0$ as $n \to \infty$.

Finally, let $\rho_n, v_n, \{\bar{\mathbf{B}}_n^i, \eta_n^i\}_{i=1}^N$ be a sequence of variational solutions to the problem (1)–(9) on the sets $(0,T) \times \Omega_n$ such that

$$\begin{aligned} \rho_n(0) &= \rho_{0,n} \to \rho_0 \text{ in } L^\gamma(\mathbb{R}^3) \\ (\rho_n v_n)(0) &= q_n \to q \text{ in } L^1(\mathbb{R}^3) \end{aligned} \tag{48}$$

where $\rho_0, \vec{m}_0$ satisfy the compatibility conditions (20). Moreover, let

$$E_{0,n} \to E_0 = \int_\Omega \frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{a}{\gamma - 1}\rho^\gamma dx;$$

and

$$\overline{\mathbf{B}}_{0,n}^{i} \equiv \overline{\mathbf{B}}_{n}^{i}(0) \xrightarrow{b} \overline{\mathbf{B}}_{0}^{i} \text{ for any } i = 1, \ldots, N,$$

where $\overline{\mathbf{B}}_{0}^{i} \subset \mathbb{R}^3$ satisfy (10) and

$$\overline{\mathbf{B}}_{0,n}^{i} = cl(\mathbf{B}_{0,n}^{i}),\ \mathbf{B}_{0,n}^{i} \text{ open},\ \overline{\mathbf{B}}^{i} \subset \mathbf{B}_{0,n}^{i} \text{ for all } n = 1, 2, \ldots \tag{49}$$

Then there is a subsequence such that

$$\begin{aligned} \rho_n &\to \rho \text{ in } C([0,T];\mathbb{R}^3) \\ v_n &\to v \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega)), \\ \eta_n^i[t] &\to \eta^i[t] \text{ in } C_{\text{loc}}(\mathbb{R}^3) \text{ uniformly in } t \in [0,T], \end{aligned} \tag{50}$$

and

$$\overline{\mathbf{B}}_{n}^{i}(t) \xrightarrow{b} \overline{\mathbf{B}}^{i}(t) \text{ for all } t \in [0,T]$$

where $\rho, v, \{\overline{\mathbf{B}}^{i}, \eta^i\}_{i=1}^{N}$ is a variational solution of the Problem (1)–(9) on $(0,T) \times \Omega$ satisfying the initial conditions

$$\rho(0) = \rho_0,\ (\rho u)(0) = \vec{m}_0,\ \overline{\mathbf{B}}^{i}(0) = \overline{\mathbf{B}}_{0}^{i} \text{ for } i = 1, \ldots, N.$$

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