On isentropic solutions to the Riemann problem for the Euler system

Eduard Feireisl
joint work with O.Kreml (Praha) and A.Vasseur (Austin, TX)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

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Euler system

Mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho \vartheta = 0$$

Energy balance

$$\partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta \right] + \operatorname{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta + \varrho \vartheta \right) \mathbf{u} \right] = 0$$

Entropy inequality

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) \geq 0, \ \ s = s(\varrho, \vartheta) \equiv \log \left(\frac{\vartheta^{c_v}}{\varrho} \right)$$

Riemann problem

Geometry

$$\Omega = R^1 \times \mathcal{T}^1, \text{ where } \mathcal{T}^1 \equiv [0,1]_{\{0,1\}} \text{ is the "flat" sphere}$$

Initial data

$$\varrho(0,x_1,x_2) = R_0(x_1),\ R_0 = \begin{cases} R_L \text{ for } x_1 \leq 0 \\ R_R \text{ for } x_1 > 0 \end{cases}$$

$$\vartheta(0,x_1,x_2) = \Theta_0(x_1),\ \Theta_0 = \begin{cases} \Theta_L \text{ for } x_1 \leq 0 \\ \Theta_R \text{ for } x_1 > 0 \end{cases}$$

$$u^1(0,x_1,x_2) = U_0(x_1),\ U_0 = \begin{cases} U_L \text{ for } x_1 \leq 0, \\ U_R \text{ for } x_1 > 0 \end{cases} \quad u^2(0,x_1,x_2) = 0.$$

Shock free Riemann solutions

Solution class

$$0 < \varrho \leq \overline{\varrho},\ 0 < \vartheta \leq \overline{\vartheta},\ |s(\varrho, \vartheta)| < \overline{s},\ |\mathbf{u}| < \overline{u}$$

Isentropic solutions

- the entropy S is *constant* in $[0, T] \times \Omega$
- $\Theta = R^{\frac{1}{c_v}} \exp\left(\frac{1}{c_v} S\right)$
- $R = R(t, x_1)$ and $U = U(t, x_1)$ represent a rarefaction wave solution of the 1-D *isentropic* system

$$\partial_t R + \partial_{x_1}(RU) = 0,\ R\left[\partial_t U + U \partial_{x_1} U\right] + \exp\left(\frac{1}{c_v} S\right) \partial_{x_1} R^{\frac{c_v+1}{c_v}} = 0$$

Main result

Theorem, EF, O.Kreml, A.Vasseur [2014]

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Euler system in $(0, T) \times \Omega$ originating from the Riemann data. Suppose in addition that the Riemann data give rise to the shock-free solution $[R, \Theta, U]$ of the 1-D Riemann problem.
Then

$$\varrho = R, \ \vartheta = \Theta, \ \mathbf{u} = [U, 0] \text{ a.a. in } (0, T) \times \Omega$$

Relative energy

Relative energy (entropy) functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + H_{\tilde{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) \right] \ \mathrm{d}x$$

Ballistic free energy

$$H_{\tilde{\vartheta}}(\varrho, \vartheta) = \varrho \left(c_v \vartheta - \tilde{\vartheta} s(\varrho, \vartheta) \right).$$

$$\varrho \mapsto H_{\tilde{\vartheta}}(\varrho, \tilde{\vartheta}) \text{ convex}$$

$$\vartheta \mapsto H_{\tilde{\vartheta}}(\varrho, \vartheta) \begin{cases} \text{decreasing for } \vartheta < \tilde{\vartheta} \\ \\ \text{increasing for } \vartheta > \tilde{\vartheta} \end{cases}$$

Relative energy inequality, dissipative solutions

Relative energy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \right]_{t=0}^{t=\tau} \leq \int_0^\tau \mathcal{R} \left(\varrho, \vartheta, \mathbf{u}, \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \ \mathrm{d}t$$

Test functions

$$\tilde{\varrho} > 0, \ \tilde{\vartheta} > 0, \ \left\{ \begin{array}{c} \tilde{\varrho} = R_L, \tilde{\vartheta} = \Theta_L, \ \tilde{u}^1 = U_L, \ \tilde{u}^2 = 0 \text{ if } x_1 < -A, \\ \\ \tilde{\varrho} = R_R, \tilde{\vartheta} = \Theta_R, \ \tilde{u}^1 = U_R, \ \tilde{u}^2 = 0 \text{ if } x_1 > A \end{array} \right\}$$

Remainder

Remainder in the relative energy inequality

$$\mathcal{R}\left(\varrho, \vartheta, \mathbf{u}, \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$

$$= \int_\Omega \left[\varrho(\tilde{\mathbf{u}} - \mathbf{u}) \cdot \partial_t \tilde{\mathbf{u}} + \varrho(\tilde{\mathbf{u}} - \mathbf{u}) \otimes \mathbf{u} : \nabla_x \tilde{\mathbf{u}} + (\tilde{\varrho}\tilde{\vartheta} - \varrho\vartheta)\mathrm{div}_x \tilde{\mathbf{u}} \right] \ \mathrm{d}x$$

$$- \int_\Omega \left[\varrho \Big(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \Big) \partial_t \tilde{\vartheta} + \varrho \Big(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \Big) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} \right] \ \mathrm{d}x$$

$$+ \int_\Omega \left[\left(1 - \frac{\varrho}{\tilde{\varrho}} \right) \partial_t (\tilde{\varrho}\tilde{\vartheta}) + \left(\tilde{u} - \frac{\varrho}{\tilde{\varrho}} \right) \mathbf{u} \cdot \nabla_x (\tilde{\varrho}\tilde{\vartheta}) \right] \ \mathrm{d}x$$

Ansatz for the Riemann problem

Relative energy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| R, \Theta, [U, 0] \right) \right]_{t=0}^{t=\tau}$$

$$\leq \int_0^\tau \int_\Omega \left[\varrho(U - u^1)\partial_t U + \varrho(U - u^1)u^1 \partial_{x_1} U + (R\Theta - \varrho\vartheta)\, \partial_{x_1} U \right] \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \left[\varrho \Big(s(\varrho, \vartheta) - S \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - S \Big) u^1 \partial_{x_1} \Theta \right] \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{R} \right) \partial_t (R\Theta) - \left(U - \frac{\varrho}{R} u^1 \right) \partial_{x_1} (R\Theta) \right] \mathrm{d}x \, \mathrm{d}t$$

Ideas of the proof

Source term

Show that the production term in the relative energy inequality is non-positive!

Some observations

$$\left| \frac{\partial_x \Theta}{\partial_x U} \right|^2 = \frac{1}{c_v(c_v+1)} \Theta, \ \partial_{x_1} U \geq 0$$

Final goal

$$[\varrho, s] \mapsto \left[\Theta - \vartheta(\varrho, s) - \frac{1}{c_v} \left(\frac{R}{\varrho} - 1 \right) \Theta + \frac{1}{c_v} (s - S) \Theta \right]$$

$$+ \frac{\Theta}{4 c_v (c_v + 1)} (s - S)^2$$

non-positive for $\varrho > 0, \ s \geq S$