Collapse mechanisms and the existence of equilibrium solutions for masonry bodies

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Abstract The equilibrium equations of no–tension (masonry–like) bodies are analyzed. Unlike the existing proofs of the existence of the solution by Anzellotti [2] or Giaquinta & Giusti [6], the present proof does not employ the uniform safe load condition. It is based on the assumption of the absence of a suitably defined collapse mechanisms. The collapse mechanism belongs to a generalized space $BD(\mathrm{cl}\,\Omega)$ of displacements of bounded deformation on the closure $\mathrm{cl}\,\Omega$ of the body Ω. This generalized displacement can have a jump discontinuity on the boundary of the body and the generalized strain is a measure on the closure of the body (instead of the standard interpretation as a measure supported by the interior). The equilibrium solution, however, belongs to the classical space of displacements of bounded deformation $BD(\Omega)$.

Keywords Equilibrium of masonry bodies, collapse mechanism, coercivity

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Contents

1 Introduction

No–tension (masonry–like) materials [2, 6, 3, 5, 8] cannot support all stresses: only negative semidefinite stresses are possible. Therefore, bodies made of no–tension materials cannot support all loads, certain loads lead to the collapse of the body. Therefore, the existence of the solution to the equilibrium equations can be proved only under some restrictions on the loads. The existing proofs of the existence by Anzellotti [2] and Giaquinta & Giusti [6] use a strong version of the safe load condition which amounts to the assumption of the existence of a square integrable stress field that balances the loads and is *uniformly negative definite.* However, the necessary condition for the load in terms of stress is that the loads are balanced by a stress field that is *negative semidefinite.* Indeed, the stress corresponding to the assumed solution is negative semidefinite and balances the loads.

The purpose of this note is to present a different condition, which avoids the strong version of the safe load condition. Apparently, the present condition may be less restrictive in certain situations. Namely, our proof of the existence employs the assumptions (i) that the loads can be balanced by a continuous negative semidefinite stress field on the closure of the body and (ii) that the loads do not admit a suitably defined collapse mechanisms. The above discussion shows that (i) is close to necessary. As for (ii), we mention that collapse mechanisms are used in the engineering *limit analysis* to identify the loads that lead to the collapse of the body. Collapse mechanism is used in cooperation with (i) and for no–tension bodies designates the displacement that amounts to the absence of compression of the body and performs null work on the loads. The choice of the function space for such displacements is subject to debate [4, 9]. Here we employ a novel space $BD(\mathrm{cl}\,\Omega)$ of displacements of bounded deformation on the closure $\mathrm{cl}\,\Omega$ of the body. This space is modeled on a similarly constructed space of bounded variation on the closure of an open set by Souček [10]. A general element of $BD(\mathrm{cl}\,\Omega)$ is a pair $(\boldsymbol{u}, \boldsymbol{\tau})$ where $\boldsymbol{u}$ is in the classical space of displacements of bounded deformation $BD(\Omega)$ [11–12] and $\boldsymbol{\tau}$ is a $\mathbf{R}^n$ valued measure on the boundary $\partial\Omega$ of the body $\Omega \subset \mathbf{R}^n$. Here $\boldsymbol{u}$ represents the displacement field in the interior of the body while $\boldsymbol{\tau}$ represents the deformation of the boundary. The map $\boldsymbol{u}$, being an element of $BD(\Omega)$, has a well defined trace $\boldsymbol{\tau}_{\mathrm{i}}$ on $\partial\Omega$, we call this trace the inner trace; analogously we call $\boldsymbol{\tau}$ the outer trace. The difference $\boldsymbol{j} := \boldsymbol{\tau} - \boldsymbol{\tau}_{\mathrm{i}}$ represents the jump in the deformation on the boundary. It turns out that each element $(\boldsymbol{u}, \boldsymbol{\tau})$ of $BD(\mathrm{cl}\,\Omega)$ has a well defined strain tensor $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ which is a tensor valued measure on the closure of the body. The restriction of $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ to Ω is just the strain tensor of $\boldsymbol{u}$ interpreted as an element of $BD(\Omega)$ while the restriction of $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ to $\partial\Omega$ is the measure of the form $\frac{1}{2}(\boldsymbol{j} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{j})$ where $\boldsymbol{n}$ is the outer normal to $\partial\Omega$.

The internal energy of a no–tension body displays zero growth in the direction of positive semidefinite strain; consequently the total energy functional (internal plus the energy under the loads) is generally not coercive. The strong version of the safe load condition in [2, 6] is used to obtain the coercivity (which is then due to the energy of the loads). Analogously the above conditions (i) and (ii) are used here to prove the coercivity of the total energy functional. The proof under (i) and (ii) involves a limiting procedure (see the proof of Theorem 7.4, below) in which one can obtain a concentration of deformation towards the boundary of the body; hence the measure $\boldsymbol{\tau}$ on the boundary and the possibility of the strain on the boundary.

Under the coercivity, the proof of the existence of the solution goes along the standard lines of the direct method of the calculus of variations: the equilibrium solution belongs to a subset $\mathcal{U}(\mathrm{cl}\,\Omega)$ of the space $BD(\mathrm{cl}\,\Omega)$ of finite internal energy which is obtained as a weak limit of the minimizing sequence. The internal energy is sequentially weakly lower semicontinuous by the results of [2, 6]; Condition (i) is employed once more to prove the continuity of the energy of loads. Moreover, it turns out that for the minimizer the inner and outer traces coincide, i.e., there is no jump of the displacement on the boundary. Thus the solution is actually in $BD(\Omega)$.

We consider only the Neumann problem for simplicity but note that also the Dirichlet problem can be treated by similar methods.

2 Notation

Throughout we use the conventions for vectors and second order tensors identical with those in [7]. Thus Lin denotes the set of all second order tensors on $\mathbf{R}^n$, i.e., linear transformations from $\mathbf{R}^n$ into itself, Sym is the subspace of symmetric tensors, Sym^+ the set of all positive semidefinite elements of Sym; additionally, Sym^- is the set of all negative semidefinite elements of Sym. The scalar product of $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}$ is defined by $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}})$ and $|\cdot|$ denotes the associated euclidean norm on Lin. We denote by $\mathbf{1} \in \mathrm{Lin}$ the unit tensor. If $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}$, we write $\boldsymbol{A} \le \boldsymbol{B}$ if $\boldsymbol{B} - \boldsymbol{A} \in \mathrm{Sym}^+$.

We now introduce some terminology and notation for measures with values in a finite dimensional vector space. We refer to [1; Chapter 1] for further details.

Let V be a finite-dimensional vector space. By a V valued measure in $\mathbf{R}^n$ we mean a map α from a system of all Borel sets in $\mathbf{R}^n$ to V which is countably additive, i.e., if $B_1, B_2, \ldots$ is a disjoint family of Borel sets in $\mathbf{R}^n$ then

$$\alpha\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \alpha(B_i).$$

Below we need the choices $V = \mathrm{Sym}$ and $V = \mathbf{R}^n$. We call the Sym valued measures tensor valued measures; these will be used to model the fracture part of the strain over the body. We call the $\mathbf{R}^n$ valued measures vector valued measures. These will be used to model the value of the displacement on the boundary of the body.

We shall also employ nonnegative measures ϕ defined on the system of all Borel sets in $\mathbf{R}^n$ with values in $[0, \infty]$ of nonnegative numbers or ∞.

If Δ is a Borel subset of $\mathbf{R}^n$ and α a V valued measure or a nonnegative measure, we say that α is supported by Δ if $\alpha(A) = \mathbf{0}$ for any Borel set A such that $A \cap \Delta = \emptyset$. We denote by $\mathcal{M}(\Delta, V)$ the set of all V valued measures supported by Δ and if $K \subset V$ is any set then $\mathcal{M}(\Delta, K)$ denotes the set of all measures from $\mathcal{M}(\Delta, V)$ which take the values from K. We emphasize that the measures from $\mathcal{M}(\Delta, V)$ or $\mathcal{M}(\Delta, K)$ are defined on all Borel subsets of $\mathbf{R}^n$ but vanish outside Δ.

If α_k is a sequence in $\mathcal{M}(\Delta, V)$ and $\alpha \in \mathcal{M}(\Delta, V)$, we say that α_k weak* converges to α and write $\alpha_k \rightharpoonup \alpha$ if $\int_\Delta \beta \cdot d\alpha_k \to \int_\Delta \beta \cdot d\alpha$ for each continuous function $\beta : \Delta \to V$.

We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbf{R}^n$ [1; Definition 1.52] and we denote by $\mathcal{H}^{n-1}$ the $n-1$-dimensional Hausdorff measure in $\mathbf{R}^n$ [1; Section 2.8]. If ϕ is a nonnegative measure or a V valued measure, we denote by $\phi \llcorner A$ the restriction of ϕ to a Borel set $A \subset \mathbf{R}^n$ defined by

$$\phi \llcorner A(B) = \phi(A \cap B)$$

for any Borel subset B of $\mathbf{R}^n$. Thus if $\mathcal{S}$ is an $n-1$ dimensional surface in $\mathbf{R}^n$ then $\mathcal{H}^{n-1} \llcorner \mathcal{S}$ is the area measure on $\mathcal{S}$.

If ϕ is a nonnegative measure, we denote by $f\phi$ the product of the measure ϕ by a ϕ integrable V valued function f on $\mathbf{R}^n$; one has

$$(f\phi)(A) = \int_A f \, d\phi$$

for any Borel subset A of $\mathbf{R}^n$.

The polar decomposition of measures says that if $\alpha \in \mathcal{M}(\Delta, V)$, there exists a pair $(r, |\alpha|)$ consisting of a Borel function $r : \Delta \to V$ and of a nonnegative measure $|\alpha|$ on Δ such that

$$\alpha = r|\alpha|$$

and

$$|r(\boldsymbol{x})| = 1 \quad \text{for } |\alpha| \text{ a.e. } \boldsymbol{x} \in \Delta.$$

The measure $|\alpha|$ is unique and the function r is unique up to a change on a $|\alpha|$ null set. The measure $|\alpha|$ is called the total variation measure of α, and r the amplitude. We denote by $\mathrm{M}(\alpha)$ the mass of α, defined by $\mathrm{M}(\alpha) = |\alpha|(\mathbf{R}^n)$.

If Ω is an open subset of $\mathbf{R}^n$ then $C^0(\mathrm{cl}\,\Omega, V)$ denotes the space of all continuous V valued functions on the closure $\mathrm{cl}\,\Omega$ of Ω, $C^1(\mathrm{cl}\,\Omega, V)$ the space of all class 1 V valued functions on Ω such that both the function and its gradient have continuous extensions to $\mathrm{cl}\,\Omega$. Finally, $C_0^1(\Omega, V)$ denotes the space of all class 1 V valued functions on $\mathbf{R}^n$ such that their supports are compact and contained in Ω.

Throughout the paper c denotes a general constant that changes from line to line and that is independent of the local variables in the surrounding text.

3 No–tension materials

We here outline briefly the constitutive theory of no–tension materials. The response of a no–tension material is completely determined by the tensor of elastic constants C. Here and below $\mathsf{C} : \mathrm{Sym} \to \mathrm{Sym}$ is a given linear transformation, such that

$$\left.\begin{aligned} &\boldsymbol{E} \cdot \mathsf{C}\boldsymbol{E} > 0 \quad \text{for all } \boldsymbol{E} \in \mathrm{Sym}, \boldsymbol{E} \neq \boldsymbol{0}, \\ &\boldsymbol{E}_1 \cdot \mathsf{C}\boldsymbol{E}_2 = \boldsymbol{E}_2 \cdot \mathsf{C}\boldsymbol{E}_1 \quad \text{for all } \boldsymbol{E}_1, \boldsymbol{E}_2 \in \mathrm{Sym}. \end{aligned}\right\} \quad (3.1)$$

We introduce the energetic scalar product $\langle \cdot, \cdot \rangle$ and the energetic norm $\|\cdot\|$ on Sym by

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \boldsymbol{A} \cdot \mathsf{C}\boldsymbol{B}, \quad \|\boldsymbol{A}\| = \sqrt{\langle \boldsymbol{A}, \boldsymbol{A} \rangle}$$

for each $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}$.

Proposition 3.1. *If* $\boldsymbol{E} \in \mathrm{Sym}$, *there exists a unique triplet* $(\boldsymbol{T}, \boldsymbol{E}^{\mathrm{e}}, \boldsymbol{E}^{\mathrm{f}})$ *of elements of* Sym *such that*

$$\left.\begin{aligned} &\boldsymbol{E} = \boldsymbol{E}^{\mathrm{e}} + \boldsymbol{E}^{\mathrm{f}}, \\ &\boldsymbol{T} = \mathsf{C}\boldsymbol{E}^{\mathrm{e}}, \\ &\boldsymbol{T} \in \mathrm{Sym}^-, \quad \boldsymbol{E}^{\mathrm{f}} \in \mathrm{Sym}^+, \\ &\boldsymbol{T} \cdot \boldsymbol{E}^{\mathrm{f}} = 0. \end{aligned}\right\} \quad (3.2)$$

Equivalently, the triplet $(\boldsymbol{T}, \boldsymbol{E}^{\mathrm{e}}, \boldsymbol{E}^{\mathrm{f}})$ *is characterized by* $(3.2)_{1,2}$ *and*

$$\left.\begin{aligned} &\boldsymbol{E}^{\mathrm{e}} \text{ is the orthogonal projection of } \boldsymbol{E} \text{ onto } \mathsf{C}^{-1}\mathrm{Sym}^- \text{ with respect to } \langle \cdot, \cdot \rangle, \\ &\boldsymbol{E}^{\mathrm{f}} \text{ is the orthogonal projection of } \boldsymbol{E} \text{ onto } \mathrm{Sym}^+ \text{ with respect to } \langle \cdot, \cdot \rangle. \end{aligned}\right\} \quad (3.3)$$

The reader is referred to [2, 6] or [3] for proofs. We define the stored energy $\hat{w}$: $\mathrm{Sym} \to \mathbf{R}$ by

$$\hat{w}(\boldsymbol{E}) = \tfrac{1}{2}\hat{\boldsymbol{T}}(\boldsymbol{E}) \cdot \boldsymbol{E} \equiv \tfrac{1}{2}\|\Pi \boldsymbol{E}\|^2$$

for any $\boldsymbol{E} \in \mathrm{Sym}$ where $\Pi : \mathrm{Sym} \to \mathsf{C}^{-1}\mathrm{Sym}^-$ is the orthogonal projection onto $\mathsf{C}^{-1}\mathrm{Sym}^-$ with respect to $\langle \cdot, \cdot \rangle$; $\boldsymbol{E}^{\mathrm{e}}$ and $\boldsymbol{E}^{\mathrm{f}}$ are called the elastic and fracture parts of the deformation $\boldsymbol{E}$.

4 Projections of measures

Let Δ be a Borel subset of $\mathbf{R}^n$. Later in this work, Δ will be the closure of and open set with class 1 boundary. The following two results are proved by Anzellotti [2] in the case Δ is an open set, but the relevant proofs hold verbatim if Δ is a Borel set.

Theorem 4.1 (Cf. Anzellotti [2; Definition immediately preceding Lemma 2.2; Lemma 2.6])**.** *Let K be a closed convex cone in an finite dimensional space V with inner product $\langle \cdot, \cdot \rangle$. Denote by $K^\perp$ the conjugate cone,*

$$K^\perp = \{\boldsymbol{A} \in \mathrm{Sym} : \langle \boldsymbol{A}, \boldsymbol{B} \rangle \le 0 \text{ for all } \boldsymbol{B} \in K\}.$$

If $\alpha \in \mathcal{M}(\Delta, V)$ then there exists a unique pair α_1, α_2 of measures such that $\alpha_1 \in \mathcal{M}(\Delta, K)$, $\alpha_2 \in \mathcal{M}(\Delta, K^\perp)$,

$$\alpha = \alpha_1 + \alpha_2$$

and for any pair β_1, β_2 of measures such that $\beta_1 \in \mathcal{M}(\Delta, K)$, $\beta_2 \in \mathcal{M}(\Delta, K^\perp)$, and

$$\alpha = \beta_1 + \beta_2,$$

we have

$$|\beta_1| \ge |\alpha_1|, \quad |\beta_2| \ge |\alpha_2|$$

where $|\cdot|$ denotes the total variation measure with respect to the norm $\|\cdot\|$ derived from $\langle \cdot, \cdot \rangle$.

The measure $\mathcal{P}_K \alpha := \alpha_1$ (respectively, $\mathcal{P}_{K^\perp} \alpha := \alpha_2$) is called the projection of α onto K (respectively, $K^\perp$).

Remark 4.2 (Cf. Anzellotti [2; Lemma 2.5])**.** *Let K be a closed convex cone in an inner product space V. Then the set $\mathcal{M}(\Delta, K)$ is weak* sequentially closed in the sense that if $\alpha_k \in \mathcal{M}(\Delta, K)$ and $\alpha_k \rightharpoonup \alpha$ for some $\alpha \in \mathcal{M}(\Delta, V)$ then $\alpha \in \mathcal{M}(\Delta, K)$.*

5 The space $BD(\mathrm{cl}\,\Omega)$

Throughout the rest of the paper, let Ω be a bounded open connected subset of $\mathbf{R}^n$ with class 1 boundary. We denote by $BD(\mathrm{cl}\,\Omega)$ the set of all pairs $(\boldsymbol{u}, \boldsymbol{\tau})$ where $\boldsymbol{u} \in L^1(\Omega, \mathbf{R}^n)$ and $\boldsymbol{\tau} \in \mathcal{M}(\partial\Omega, \mathbf{R}^n)$ such that there exists a measure $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym})$ satisfying

$$\int_\Omega \operatorname{div} \boldsymbol{T} \cdot \boldsymbol{u}\, d\mathcal{L}^n + \int_{\mathrm{cl}\,\Omega} \boldsymbol{T} \cdot d\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) = \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n} \cdot d\boldsymbol{\tau} \tag{5.1}$$

for every $\boldsymbol{T} \in C^1(\mathrm{cl}\,\Omega, \mathrm{Sym})$. Here $\boldsymbol{n}$ is the (continuous) outer normal to Ω. We call $\boldsymbol{\tau}$ the outer trace of $(\boldsymbol{u}, \boldsymbol{\tau})$. It will be shown that the measure $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ is uniquely determined by $(\boldsymbol{u}, \boldsymbol{\tau})$. We call the elements of $BD(\mathrm{cl}\,\Omega)$ generalized displacements.

Let $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$. By taking $\boldsymbol{T} \in C_0^1(\Omega, \mathrm{Sym})$ in (5.1) we learn that $\boldsymbol{u} \in BD(\Omega)$, and $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \llcorner \Omega$ is the small strain tensor corresponding to $\boldsymbol{u}$. Here $BD(\Omega)$ is the classical space of displacements of bounded deformation [11–12]. We thus see that $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \llcorner \Omega$ is uniquely determined by $\boldsymbol{u}$. We denote the trace of $\boldsymbol{u}$ in the sense of $BD(\Omega)$ by $\mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau}) \in L^1(\partial\Omega, \mathbf{R}^n)$ so that we have

$$\int_\Omega \operatorname{div} \boldsymbol{T} \cdot \boldsymbol{u}\, d\mathcal{L}^n + \int_\Omega \boldsymbol{T} \cdot d\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) = \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n} \cdot \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\, d\mathcal{H}^{n-1} \tag{5.2}$$

for every $\boldsymbol{T} \in C^1(\mathrm{cl}\,\Omega, \mathrm{Sym})$. Subtracting (5.2) from (5.1) we obtain

$$\int_{\partial\Omega} \boldsymbol{T} \cdot d\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) = \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n} \cdot d(\boldsymbol{\tau} - \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1})$$

for every $\boldsymbol{T} \in C^1(\mathrm{cl}\,\Omega, \mathrm{Sym})$. The arbitrariness of $\boldsymbol{T}|\partial\Omega$ then implies that

$$\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \llcorner \partial\Omega = (\boldsymbol{\tau} - \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}) \odot \boldsymbol{n}. \tag{5.3}$$

Here $\boldsymbol{a} \odot \boldsymbol{b} = \frac{1}{2}(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})$. Combining the uniqueness of $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \llcorner \Omega$ with (5.3), we see that $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ is uniquely determined.

We introduce a norm $|\cdot|$ on $BD(\mathrm{cl}\,\Omega)$ by setting

$$|(\boldsymbol{u}, \boldsymbol{\tau})| = |\boldsymbol{u}|_{L^1(\Omega, \mathbf{R}^n)} + \mathrm{M}(\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})).$$

It is clear that the just defined $|\cdot|$ is a seminorm. Let us show that it is a norm, i.e., $|(\boldsymbol{u}, \boldsymbol{\tau})| = 0$ implies $\boldsymbol{u} = \boldsymbol{0}$, $\boldsymbol{\tau} = \boldsymbol{0}$. Thus let $|(\boldsymbol{u}, \boldsymbol{\tau})| = 0$, so that $\boldsymbol{u} = \boldsymbol{0}$ and $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) = \boldsymbol{0}$. Then the left hand side of (5.1) vanishes and hence also the right hand side vanishes which by the arbitrariness of $\boldsymbol{T}$ gives $\boldsymbol{\tau} = \boldsymbol{0}$. Thus $(\boldsymbol{u}, \boldsymbol{\tau})$ is the null element of $BD(\mathrm{cl}\,\Omega)$.

Remark 5.1. If $\boldsymbol{\sigma} \in \mathcal{M}(\partial\Omega, \mathbf{R}^n)$ then $(\boldsymbol{0}, \boldsymbol{\sigma}) \in BD(\mathrm{cl}\,\Omega)$ with $\tilde{\boldsymbol{E}}(\boldsymbol{0}, \boldsymbol{\sigma}) = \boldsymbol{\sigma} \odot \boldsymbol{n}$ as one easily finds. Considering $(\boldsymbol{u}, \boldsymbol{\tau}) + (\boldsymbol{0}, \boldsymbol{\sigma})$ with $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ fixed and $\boldsymbol{\sigma}$ varying over $\mathcal{M}(\partial\Omega, \mathbf{R}^n)$, we see that the two components $\boldsymbol{u}$ and $\boldsymbol{\tau}$ of any element of $BD(\mathrm{cl}\,\Omega)$ are independent, with $\boldsymbol{u}$ restricted to belong to $BD(\Omega)$. Thus

$$BD(\mathrm{cl}\,\Omega) = BD(\Omega) \times \mathcal{M}(\partial\Omega).$$

Proposition 5.2. *There exists a $c \in \mathbf{R}$ such that*

$$\mathrm{M}(\boldsymbol{\tau}) \le c|(\boldsymbol{u}, \boldsymbol{\tau})| \tag{5.4}$$

for every $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$.

Proof (Cf. Souček [10; Proof of Theorem 2(i)].) For every $\boldsymbol{x} \in \partial\Omega$ there exists an orthogonal frame such that in this frame, $\boldsymbol{n}(\boldsymbol{x}) = (1, \ldots, 1)/\sqrt{n}$. Then there exists an open ball $B(\boldsymbol{x}, r_{\boldsymbol{x}})$ such that $\boldsymbol{n}(\boldsymbol{y}) = (n_1(\boldsymbol{y}), \ldots, n_n(\boldsymbol{y}))$ with $n_i(\boldsymbol{y}) \ge 1/(2\sqrt{n})$ for every $\boldsymbol{y} \in \partial\Omega \cap B(\boldsymbol{x}, r_{\boldsymbol{x}})$. We can then find a finite number of such balls $B_k := B(\boldsymbol{x}_k, r_{\boldsymbol{x}_k})$, $k = 1, \ldots, K$, which covers $\partial\Omega$. Let φ^k be a partition of unity on $\partial\Omega$ subordinated to the covering B_k, $k = 1, \ldots, K$. Then

$$\mathrm{M}(\boldsymbol{\tau}) \le \sum_{k=1}^{K} \mathrm{M}(\boldsymbol{\sigma}^k) \tag{5.5}$$

where $\boldsymbol{\sigma}^k = \varphi^k \boldsymbol{\tau}$. Further,

$$\mathrm{M}(\boldsymbol{\sigma}^k) \le \sum_{i=1}^{n} \mathrm{M}(\sigma_i^k) \tag{5.6}$$

where $\boldsymbol{\sigma}^k = (\sigma_1^k, \ldots, \sigma_n^k)$. Let i and k be fixed and let

$$C = \{\eta \in C^1(\partial\Omega \cap B_k), |\eta|_{C^0(\partial\Omega\cap B_k)} \le 1\}.$$

Next let us extend, without changing notation, an arbitrary $\eta \in C$ as a constant on lines parallel to the x_i coordinate axis (in the new coordinates). Let η be such an extended function. We now apply (5.1) to $\boldsymbol{T}$ such that $T_{ii} = \eta\varphi^k$ and all other components of $\boldsymbol{T}$ vanish (no summation convention throughout the proof). We obtain, writing $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$, and denoting by E_{ij} the components of $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$,

$$\begin{aligned}\int_{\partial\Omega\cap B_k} \eta n_i \, d\sigma_i^k &= \int_{\partial\Omega\cap B_k} \eta n_i \varphi^k \, d\tau_i \\ &= \int_{\Omega\cap B_k} (\eta\varphi^k)_{,i} u_i \, d\mathcal{L}^n + \int_{\mathrm{cl}\,\Omega\cap\mathrm{cl}\,B_k} \eta\varphi^k \, dE_{ii} \\ &= \int_{\Omega\cap B_k} \eta\varphi^k_{,i} u_i \, d\mathcal{L}^n + \int_{\mathrm{cl}\,\Omega\cap\mathrm{cl}\,B_k} \eta\varphi^k \, dE_{ii}\end{aligned}$$

Since $n_i \ge 1/(2\sqrt{n})$, we have

$$\begin{aligned}|\sigma_i^k| &= \sup\left\{\int \eta \, d\sigma_i^k : \eta \in C\right\} \\ &\le c \sup\left\{\int_{\Omega\cap B_k} \eta\varphi^k_{,i} u_i \, d\mathcal{L}^n : \eta \in C\right\} + c \sup\left\{\int_{\mathrm{cl}\,\Omega\cap\mathrm{cl}\,B_k} \eta\varphi^k \, dE_{ii} : \eta \in C\right\} \\ &\le c \int_{\Omega\cap B_k} |u_i| \, d\mathcal{L}^n + c\mathrm{M}(E_{ii}) \\ &\le |(\boldsymbol{u}, \boldsymbol{\tau})|.\end{aligned}$$

Summing over i and k and using (5.5) and (5.6) we obtain (5.4). □

6 No–tension bodies

We now apply the results of Section 4 to bodies made of no–tension materials. We put

$$\left.\begin{aligned} V &= \mathrm{Sym}, \\ \langle \boldsymbol{A}, \boldsymbol{B} \rangle &= \mathsf{C}\boldsymbol{A} \cdot \boldsymbol{B} \quad \text{for every } \boldsymbol{A}, \boldsymbol{B} \in \mathrm{Sym}, \\ K &= \mathsf{C}^{-1}\mathrm{Sym}^-, \\ K^\perp &= \mathrm{Sym}^+. \end{aligned}\right\} \tag{6.1}$$

Furthermore, we denote by $\mathcal{P}_K : \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym}) \to \mathcal{M}(\mathrm{cl}\,\Omega, K)$ and $\mathcal{P}_{K^\perp} : \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym}) \to \mathcal{M}(\mathrm{cl}\,\Omega, K^\perp)$ the orthogonal projections of measures onto the cones $\mathcal{M}(\mathrm{cl}\,\Omega, K)$ and $\mathcal{M}(\mathrm{cl}\,\Omega, K^\perp)$ with respect to the scalar product $\langle\cdot,\cdot\rangle$.

We denote by $\mathcal{U}(\mathrm{cl}\,\Omega)$ the set of all $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ such that $\mathcal{P}_K\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ is absolutely continuous with respect to $\mathcal{L}^n$ and the density, still denoted by $\mathcal{P}_K\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$, satisfies

$$\mathsf{E}(\boldsymbol{u}, \boldsymbol{\tau}) := \tfrac{1}{2}\int_{\Omega} \|\mathcal{P}_K\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})\|^2 \, d\mathcal{L}^n < \infty$$

where $\|\cdot\|$ is the norm corresponding to the scalar product $\langle\cdot,\cdot\rangle$. We call $\mathsf{E}(\boldsymbol{u}, \boldsymbol{\tau})$ the internal energy of the displacement $(\boldsymbol{u}, \boldsymbol{\tau})$. Furthermore in view of (3.3), we call

$\mathcal{P}_K \tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ the elastic strain and $\mathcal{P}_{K^\perp} \tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ the fracture strain corresponding to the generalized displacement $(\boldsymbol{u}, \boldsymbol{\tau})$.

Theorem 6.1. *Let* $(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \in \mathcal{U}(\mathrm{cl}\,\Omega)$ *and let* $\boldsymbol{u} \in L^{n/(n-1)}(\Omega, \mathbf{R}^n)$, $\boldsymbol{F} \in \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym})$ *and* $\boldsymbol{\tau} \in \mathcal{M}(\partial\Omega, \mathbf{R}^n)$ *be such that*

$$\left.\begin{aligned} \boldsymbol{u}_k &\rightharpoonup \boldsymbol{u} \quad \text{in } L^{n/(n-1)}(\Omega, \mathbf{R}^n), \\ \boldsymbol{u}_k &\to \boldsymbol{u} \quad \text{in } L^1(\Omega, \mathbf{R}^n), \\ \hat{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) &\rightharpoonup \boldsymbol{F} \quad \text{in } \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym}), \\ \boldsymbol{\tau}_k &\rightharpoonup \boldsymbol{\tau} \quad \text{in } \mathcal{M}(\partial\Omega, \mathbf{R}^n), \end{aligned}\right\} \tag{6.2}$$

and

$$\mathsf{E}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \le c \tag{6.3}$$

for all k and some $c \in \mathbf{R}$. *Then* $(\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{U}(\mathrm{cl}\,\Omega)$, $\boldsymbol{F} = \hat{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$, *and*

$$\liminf_{k\to\infty} \mathsf{E}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \ge \mathsf{E}(\boldsymbol{u}, \boldsymbol{\tau}). \tag{6.4}$$

Proof We have

$$\int_\Omega \operatorname{div} \boldsymbol{T} \cdot \boldsymbol{u}_k \, d\mathcal{L}^n + \int_{\mathrm{cl}\,\Omega} \boldsymbol{T} \cdot d\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) = \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n} \cdot d\boldsymbol{\tau}_k$$

for every $\boldsymbol{T} \in C^1(\mathrm{cl}\,\Omega, \mathrm{Sym})$ and every k; the limit using (6.2) provides

$$\int_\Omega \operatorname{div} \boldsymbol{T} \cdot \boldsymbol{u} \, d\mathcal{L}^n + \int_{\mathrm{cl}\,\Omega} \boldsymbol{T} \cdot d\boldsymbol{F} = \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n} \cdot d\boldsymbol{\tau}$$

which shows that $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ and $\boldsymbol{F} = \tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$. Furthermore, since (6.3) holds, [2; Theorem 3.3] gives (6.4) and completes the proof. □

7 Collapse mechanisms and the coercivity of energy

We consider loads which consist of the body force $\boldsymbol{b} \in L^n(\Omega, \mathbf{R}^n)$ and the surface traction $\boldsymbol{s} \in C^0(\partial\Omega, \mathbf{R}^n)$. The energy of a displacement $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ under the loads is given by

$$\mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{b} \cdot \boldsymbol{u} \, d\mathcal{L}^n + \int_{\partial\Omega} \boldsymbol{s} \cdot d\boldsymbol{\tau}.$$

If $(\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{U}(\mathrm{cl}\,\Omega)$, we define the total energy $\mathsf{F}(\boldsymbol{u}, \boldsymbol{\tau})$ by

$$\mathsf{F}(\boldsymbol{u}, \boldsymbol{\tau}) = \mathsf{E}(\boldsymbol{u}, \boldsymbol{\tau}) - \mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau}).$$

Definition 7.1. We say that $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ is a collapse mechanism if $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym}^+)$, $\mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau}) = 0$, and $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \ne \mathbf{0}$.

Remark 7.2. Let us say that the loads satisfy the uniform safe load condition if there exists a map $\boldsymbol{T} \in C^0(\mathrm{cl}\,\Omega, \mathrm{Sym})$ such that

$$\int_{\mathrm{cl}\,\Omega} \boldsymbol{T} \cdot d\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) = \mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau})$$

for all $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ and

$$-\boldsymbol{T}(\boldsymbol{x}) \cdot \boldsymbol{A} \geq \varepsilon_0 |\boldsymbol{A}|$$

for all $\boldsymbol{A} \in \mathrm{Sym}^+$, all $\boldsymbol{x} \in \mathrm{cl}\,\Omega$, and some $\varepsilon_0 > 0$. Under the uniform safe load condition there is no collapse mechanism. Indeed, assuming that $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ is a collapse mechanism we obtain

$$0 = -\mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau}) = -\int_{\mathrm{cl}\,\Omega} \boldsymbol{T} \cdot d\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \geq \varepsilon_0 \mathrm{M}(\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}))$$

and hence $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) = \boldsymbol{0}$ contrary to the assumption on $(\boldsymbol{u}, \boldsymbol{\tau})$. Under the uniform safe load condition one can actually modify the proofs in [2, 6] to prove that the total energy is coercive in the sense that

$$\mathsf{F}(\boldsymbol{u}, \boldsymbol{\tau}) \geq c_1\big(\mathsf{E}(\boldsymbol{u}, \boldsymbol{\tau}) + \mathrm{M}(\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}))\big) - c_2$$

for all $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$ some $c_1 > 0$ and some $c_2 \in \mathbf{R}$. (We here note that the proofs in [2, 6] allow for a slightly more general condition that $\boldsymbol{T}$ is actually only square integrable, defined almost everywhere with respect to $\mathcal{L}^n$.) The goal of this paper is to relax the uniform safe load condition and to prove (a weaker, but still sufficient, version of) the coercivity of the energy functional under the weaker assumption of the absence of collapse mechanism.

Remark 7.3. (i) *We have*

$$\mathrm{M}(\boldsymbol{G}) \leq \mathrm{tr}(\boldsymbol{G}(\mathrm{cl}\,\Omega)) \tag{7.1}$$

for every $\boldsymbol{G} \in \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym}^+)$, *and* (ii) $\boldsymbol{G} \mapsto \mathrm{tr}(\boldsymbol{G}(\mathrm{cl}\,\Omega))$ *is a continuous linear functional on* $\mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym})$ (*with respect to weak** *convergence*).

Proof (i): Let $\boldsymbol{G} \in \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym}^+)$ and let A be a Borel set. Denote by λ_i, $i = 1, \dots, n$, the eigenvalues of the tensor $\boldsymbol{G}(A)$ respecting the multiplicities. Then

$$\|\boldsymbol{G}(A)\| = \Big(\sum_{i=1}^{n} \lambda_i^2\Big)^{1/2} \leq \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} \lambda_i = \mathrm{tr}(\boldsymbol{G}(A))$$

Consequently,

$$\begin{aligned} \mathrm{M}(\boldsymbol{G}) &= \sup\Big\{\sum_{k=1}^{\infty} \|\boldsymbol{G}(A_i)\| : \bigcup_{k=1}^{\infty} A_i = \mathrm{cl}\,\Omega,\ A_k \cap A_l = \emptyset \ \text{ if } \ k \neq l\Big\} \\ &\leq \sup\Big\{\sum_{k=1}^{\infty} \mathrm{tr}(\boldsymbol{G}(A_k)) : \bigcup_{k=1}^{\infty} A_i = \mathrm{cl}\,\Omega,\ A_k \cap A_l = \emptyset \ \text{ if } \ k \neq l\Big\} \\ &= \mathrm{tr}(\boldsymbol{G}(\mathrm{cl}\,\Omega)) \end{aligned}$$

i.e., we have (7.1). (ii): This follows immediately by noting that $\mathrm{tr}(\boldsymbol{G}(\mathrm{cl}\,\Omega)) = \int_{\mathrm{cl}\,\Omega} \mathbf{1} \cdot d\boldsymbol{G}$ where $\mathbf{1}$ stands for the function on $\mathrm{cl}\,\Omega$ that is identically equal to the unit tensor $\mathbf{1}$. □

Theorem 7.4 (Coercivity). *Assume that the following two conditions hold:*

(i) *the loads* $(\boldsymbol{b}, \boldsymbol{s})$ *have an admissible equilibrating stress field in the sense that there exists a* $\boldsymbol{T} \in C(\mathrm{cl}\,\Omega, \mathrm{Sym}^-)$ *such that*

$$\int_{\mathrm{cl}\,\Omega} \boldsymbol{T} \cdot d\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) = \mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau}) \tag{7.2}$$

for every $(\boldsymbol{u}, \boldsymbol{\tau}) \in BD(\mathrm{cl}\,\Omega)$;

(ii) *the loads do not admit a collapse mechanism.*

Then the total energy is coercive in the sense that for every sequence $(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \in \mathcal{U}(\mathrm{cl}\,\Omega)$ *such that the sequence* $\mathsf{F}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)$ *is bounded from above we have that the sequences* $\mathrm{M}(|\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)|)$ *and* $\mathsf{E}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)$ *are bounded.*

Proof Consider the sequence $(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \in \mathcal{U}(\mathrm{cl}\,\Omega)$ such that the sequence $\mathsf{F}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)$ is bounded. We have

$$\mathsf{F}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) = \tfrac{1}{2} \int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n - \int_\Omega \boldsymbol{T} \cdot \boldsymbol{E}_k^{\mathrm{e}} \, d\mathcal{L}^n - \int_{\mathrm{cl}\,\Omega} \boldsymbol{T} \cdot d\boldsymbol{E}_k^{\mathrm{f}} \le c < \infty \tag{7.3}$$

where

$$\boldsymbol{E}_k^{\mathrm{e}} := \mathcal{P}_K \tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k), \quad \boldsymbol{E}_k^{\mathrm{f}} := \mathcal{P}_{K^\perp} \tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k).$$

We note that the third integral in (7.3) is nonpositive since $\boldsymbol{T} \le \boldsymbol{0}$ and $\boldsymbol{E}_k^{\mathrm{f}} \ge \boldsymbol{0}$. Thus we have

$$\tfrac{1}{2} \int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n - \int_\Omega \boldsymbol{T} \cdot \boldsymbol{E}_k^{\mathrm{e}} \, d\mathcal{L}^n \le c$$

and using

$$-\int_\Omega \boldsymbol{T} \cdot \boldsymbol{E}_k^{\mathrm{e}} \, d\mathcal{L}^n \ge -|\boldsymbol{T}|_{L^2(\Omega, \mathrm{Sym})} \Big(\int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n \Big)^{1/2}$$

we obtain

$$\tfrac{1}{2} \int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n - |\boldsymbol{T}|_{L^2(\Omega, \mathrm{Sym})} \Big(\int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n \Big)^{1/2} \le c.$$

This implies that the sequence $\mathsf{E}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) = \frac{1}{2} \int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n$ is bounded.

To prove that the sequence $|\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)|$ is bounded, assume on the contrary that the mass $\mathrm{M}(|\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)|)$ of the sequence $|\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)|$ satisfies

$$\mathrm{M}(|\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)|) \to \infty. \tag{7.4}$$

We have

$$\begin{aligned} \mathrm{M}(|\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)|) &\le \mathrm{M}(|\boldsymbol{E}_k^{\mathrm{e}}|) + \mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|) \\ &= \int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\| \, d\mathcal{L}^n + \mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|) \\ &\le \mathcal{L}^n(\Omega)^{1/2} \Big(\int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n \Big)^{1/2} + \mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|). \end{aligned}$$

The boundedness of the sequence $\int_\Omega \|\boldsymbol{E}_k^{\mathrm{e}}\|^2 \, d\mathcal{L}^n$ proved above and (7.4) then imply

$$\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|) \to \infty.$$

The sequence $\boldsymbol{F}_k := (\boldsymbol{E}_k^{\mathrm{e}} + \boldsymbol{E}_k^{\mathrm{f}})/\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|)$ is bounded in mass, i.e., $\sup\{\mathrm{M}(\boldsymbol{F}_k), k = 1, \ldots\} < \infty$. Indeed, $\mathrm{M}(\boldsymbol{E}_k^{\mathrm{e}})/\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|) \to 0$ while $\boldsymbol{E}_k^{\mathrm{f}}/\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|)$ has mass equal to 1. The sequence $\boldsymbol{F}_k$ is the sequence of strains of the sequence of displacements $(\boldsymbol{v}_k, \boldsymbol{\sigma}_k)$ where $\boldsymbol{v}_k = \boldsymbol{u}_k/\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|)$ and $\boldsymbol{\sigma}_k = \boldsymbol{\tau}_k/\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|)$.

Let $\mathcal{R}$ be any linear operator from $BD(\Omega)$ to the subspace of rigid body displacements such that $\mathcal{R}\boldsymbol{v} = \boldsymbol{v}$ for any rigid body displacement [12; Remark 1.1, Chapter II, Section 1]. By passing from $\boldsymbol{v}_k$ to $\boldsymbol{v}_k - \mathcal{R}\boldsymbol{v}_k$ we can assume that $\mathcal{R}\boldsymbol{v}_k = \boldsymbol{0}$ without affecting the energy and the strain tensor. By the Sobolev inequality for $BD(\Omega)$ [12; Remark 2.5, Chapter II, Section 2] we have

$$|\boldsymbol{v}_k|_{L^{n/(n-1)}(\Omega, \mathbf{R}^n)} \le c\mathrm{M}(\boldsymbol{F}_k) \to c$$

and thus the sequences

$$|\boldsymbol{v}_k|_{L^{n/(n-1)}(\Omega,\mathbf{R}^n)} \qquad \text{and} \qquad |\boldsymbol{v}_k|_{L^1(\Omega,\mathbf{R}^n)}$$

are bounded. As also $\mathrm{M}(\boldsymbol{F}_k)$ is bounded, it follows from Proposition 5.2 that also the sequence $\mathrm{M}(\boldsymbol{\sigma}_k)$ is bounded. Hence we can find a subsequence, still denoted by $\boldsymbol{v}_k$, and an element $\boldsymbol{v} \in L^{n/(n-1)}(\Omega,\mathbf{R}^n)$ and measures $\boldsymbol{F} \in \mathcal{M}(\mathrm{cl}\,\Omega,\mathrm{Sym})$ and $\boldsymbol{\sigma} \in \mathcal{M}(\partial\Omega,\mathbf{R}^n)$ such that

$$\left.\begin{aligned} \boldsymbol{v}_k &\rightharpoonup \boldsymbol{v} && \text{in } L^{n/(n-1)}(\Omega,\mathbf{R}^n), \\ \boldsymbol{v}_k &\to \boldsymbol{v} && \text{in } L^1(\Omega,\mathbf{R}^n), \\ \boldsymbol{F}_k &\rightharpoonup \boldsymbol{F} && \text{in } \mathcal{M}(\mathrm{cl}\,\Omega,\mathrm{Sym}), \\ \boldsymbol{\sigma}_k &\rightharpoonup \boldsymbol{\sigma} && \text{in } \mathcal{M}(\partial\Omega,\mathbf{R}^n). \end{aligned}\right\} \tag{7.5}$$

By Theorem 6.1, $(\boldsymbol{v},\boldsymbol{\sigma}) \in BD(\mathrm{cl}\,\Omega)$ and $\tilde{\boldsymbol{E}}(\boldsymbol{v},\boldsymbol{\sigma}) = \boldsymbol{F}$.

The rest of the proof shows that $(\boldsymbol{v},\boldsymbol{\sigma})$ is a collapse mechanism.

Dividing (7.3) by $\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|)$ we obtain

$$\tfrac{1}{2}\int_{\Omega} \|\boldsymbol{F}_k^{\mathrm{e}}\|^2\, d\mathcal{L}^n - \int_{\mathrm{cl}\,\Omega} \boldsymbol{T}\cdot d\boldsymbol{F}_k \le c/\mathrm{M}(|\boldsymbol{E}_k^{\mathrm{f}}|)$$

where we have put $\boldsymbol{F}_k^{\mathrm{e}} = \mathcal{P}_K \boldsymbol{F}_k$, $\boldsymbol{F}_k^{\mathrm{f}} = \mathcal{P}_{K^\perp}\boldsymbol{F}_k$. The limit using $\boldsymbol{F}_k^{\mathrm{e}} \to \boldsymbol{0}$ in $L^2(\Omega,\mathrm{Sym})$ gives

$$\mathsf{W}(\boldsymbol{v},\boldsymbol{\sigma}) \equiv \int_{\mathrm{cl}\,\Omega} \boldsymbol{T}\cdot d\boldsymbol{F} = 0. \tag{7.6}$$

Furthermore, by the above, $\boldsymbol{F}_k^{\mathrm{e}}\mathcal{L}^n \rightharpoonup \boldsymbol{0}$ in $\mathcal{M}(\mathrm{cl}\,\Omega,\mathrm{Sym})$ and hence

$$\boldsymbol{F}_k^{\mathrm{f}} = \boldsymbol{F}_k - \boldsymbol{F}_k^{\mathrm{e}}\mathcal{L}^n \rightharpoonup \boldsymbol{F} \qquad \text{in } \mathcal{M}(\mathrm{cl}\,\Omega,\mathrm{Sym}). \tag{7.7}$$

Since $\boldsymbol{F}_k^{\mathrm{f}} \in \mathcal{M}(\mathrm{cl}\,\Omega,\mathrm{Sym}^+)$, we also have

$$\boldsymbol{F} \equiv \tilde{\boldsymbol{E}}(\boldsymbol{v},\boldsymbol{\sigma}) \in \mathcal{M}(\mathrm{cl}\,\Omega,\mathrm{Sym}^+) \tag{7.8}$$

by Remark 4.2. In view of (7.6) and (7.8), to prove that $(\boldsymbol{v},\boldsymbol{\sigma})$ is a collapse mechanism, it now remains to be showed that $\tilde{\boldsymbol{E}}(\boldsymbol{v},\boldsymbol{\sigma}) \neq \boldsymbol{0}$. But (7.1) gives

$$1 = \mathrm{M}(\boldsymbol{F}_k^{\mathrm{f}}) \le \mathrm{tr}(\boldsymbol{F}_k^{\mathrm{f}}(\mathrm{cl}\,\Omega)) \to \mathrm{tr}(\boldsymbol{F}(\mathrm{cl}\,\Omega))$$

by (7.7). Thus $\mathrm{tr}\,\boldsymbol{F} \ge 1$ and hence $\boldsymbol{F} \equiv \tilde{\boldsymbol{E}}(\boldsymbol{v},\boldsymbol{\sigma}) \neq \boldsymbol{0}$. □

Remark 7.5. *Assume that the loads satisfy Conditions* (i) *and* (ii) *of Theorem* 7.4. *Then* $\boldsymbol{s}\cdot\boldsymbol{n} < 0$ *everywhere on* $\partial\Omega$.

Proof Let $\boldsymbol{x} \in \partial\Omega$ be fixed and consider a displacement $(\boldsymbol{0}, \boldsymbol{a}\delta_{\boldsymbol{x}}) \in BD(\mathrm{cl}\,\Omega)$ where $\boldsymbol{a} \in \mathbf{R}^n$ is arbitrary and $\delta_{\boldsymbol{x}}$ is the Dirac measure supported by $\boldsymbol{x}$. Then $\tilde{\boldsymbol{E}}(\boldsymbol{0},\boldsymbol{a}\delta_{\boldsymbol{x}}) = \boldsymbol{a}\odot\boldsymbol{n}(\boldsymbol{x})\delta_{\boldsymbol{x}}$ and applying (7.2) to this generalized displacement we obtain

$$\boldsymbol{T}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{a} = \boldsymbol{s}(\boldsymbol{x})\cdot\boldsymbol{a};$$

hence $\boldsymbol{T}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{s}(\boldsymbol{x})$ by the arbitrariness of $\boldsymbol{a}$. Consequently, $\boldsymbol{s}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{T}(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) \le 0$ i.e., $\boldsymbol{s}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) \le 0$. Furthermore, if $\boldsymbol{s}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) = 0$ then the generalized displacement $(\boldsymbol{0},\boldsymbol{n}(\boldsymbol{x})\delta_{\boldsymbol{x}})$ has positive definite strain tensor, viz., $\boldsymbol{n}(\boldsymbol{x})\otimes\boldsymbol{n}(\boldsymbol{x})\delta_{\boldsymbol{x}}$ and $\mathsf{W}(\boldsymbol{0},\boldsymbol{n}(\boldsymbol{x})\delta(\boldsymbol{x})) = 0$. Thus $(\boldsymbol{0},\boldsymbol{n}(\boldsymbol{x})\delta_{\boldsymbol{x}})$ is a collapse mechanism, a contradiction. Hence $\boldsymbol{s}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) < 0$. □

8 The existence of solution to the equilibrium problem

We now state the main result of this note.

Theorem 8.1. *Assume that Conditions* (i) *and* (ii) *of Theorem* 7.4 *hold. Then there exists a* $(\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{U}(\mathrm{cl}\,\Omega)$ *which minimizes the total energy* F *on* $\mathcal{U}(\mathrm{cl}\,\Omega)$*. For this solution, the inner and outer traces coincide,* $\boldsymbol{\tau} = \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}$.

Proof Let

$$I := \inf \{ \mathsf{F}(\boldsymbol{u}, \boldsymbol{\tau}) : (\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{U}(\mathrm{cl}\,\Omega) \}$$

we have $I \in [-\infty, \infty)$. Let $(\boldsymbol{u}_k, \boldsymbol{\tau}_k)$ be a minimizing sequence, i.e., a sequence such that

$$F(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \to I.$$

Let $\mathcal{R}$ be the linear transformation as in the proof of Theorem 7.4. As in that proof, we can assume that $\mathcal{R}\boldsymbol{u}_k = \mathbf{0}$ for all k. By Theorem 7.4, the sequences $|\tilde{\boldsymbol{E}}((\boldsymbol{u}_k, \boldsymbol{\tau}_k)|$ and $\mathsf{E}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)$ are bounded. The boundedness of $|\tilde{\boldsymbol{E}}(\boldsymbol{u}_k, \boldsymbol{\tau}_k)|$ and Proposition 5.2 imply that also the sequence $\mathrm{M}(\sigma_k)$ is bounded and by the Sobolev inequality also $|\boldsymbol{u}_k|_{L^{n/(n-1)}(\Omega, \mathbf{R}^n)}$ is bounded. Then there exist $\boldsymbol{u} \in L^{n/(n-1)}(\Omega, \mathbf{R}^n)$, $\boldsymbol{F} \in \mathcal{M}(\mathrm{cl}\,\Omega, \mathrm{Sym})$ and $\boldsymbol{\tau} \in \mathcal{M}(\partial\Omega, \mathbf{R}^n)$ such that (6.2) hold for some subsequence, still denoted $(\boldsymbol{u}_k, \boldsymbol{\tau}_k)$. Hence by Theorem 6.1 we have $(\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{U}(\mathrm{cl}\,\Omega)$ and (6.4). By the assumptions on the loads we have

$$\mathsf{W}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \to \mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau})$$

and thus

$$I = \liminf_{k \to \infty} \mathsf{F}(\boldsymbol{u}_k, \boldsymbol{\tau}_k) \ge \mathsf{F}(\boldsymbol{u}, \boldsymbol{\tau}) \ge I.$$

Thus $(\boldsymbol{u}, \boldsymbol{\tau}) \in \mathcal{U}(\mathrm{cl}\,\Omega)$ which minimizes the total energy F on $\mathcal{U}(\mathrm{cl}\,\Omega)$.

To prove the second part of the assertion, assume that $\boldsymbol{\tau} \neq \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}$ and consider the generalized displacement $(\boldsymbol{u}, \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1})$. Prove that

$$\mathsf{F}(\boldsymbol{u}, \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}) < \mathsf{F}(\boldsymbol{u}, \boldsymbol{\tau}).$$

Indeed, since the projection of the measure $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau})$ onto $\mathsf{C}^{-1}\mathrm{Sym}^-$ is absolutely continuous with respect to the measure $\mathcal{L}^n$, we see that the singular measure $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \llcorner \partial\Omega = (\boldsymbol{\tau} - \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}) \odot \boldsymbol{n}$ takes its values from Sym^+. This occurs if and only if $\boldsymbol{\tau} - \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1} = \phi\boldsymbol{n}$ where ϕ is a scalar positive valued finite measure. Moreover, ϕ is not identically equal to 0 since the measure $\tilde{\boldsymbol{E}}(\boldsymbol{u}, \boldsymbol{\tau}) \llcorner \partial\Omega$ is different from $\mathbf{0}$. Then

$$\mathsf{W}(\boldsymbol{u}, \boldsymbol{\tau}) - \mathsf{W}(\boldsymbol{u}, \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}) = \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{n}\, d\phi < 0$$

because $\boldsymbol{s} \cdot \boldsymbol{n} < 0$ everywhere on $\partial\Omega$, see Remark 7.5. As clearly

$$\mathsf{E}(\boldsymbol{u}, \boldsymbol{\tau}) = \mathsf{E}(\boldsymbol{u}, \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}),$$

we have

$$\mathsf{F}(\boldsymbol{u}, \boldsymbol{\tau}) > \mathsf{F}(\boldsymbol{u}, \mathrm{T_i}(\boldsymbol{u}, \boldsymbol{\tau})\mathcal{H}^{n-1}).$$

But this is a contradiction with $(\boldsymbol{u}, \boldsymbol{\tau})$ being a minimizer of the total energy. □

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