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LACK OF EXPONENTIAL STABILITY FOR A CLASS OF SECOND-ORDER SYSTEMS WITH MEMORY

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Abstract. We analyze the decay properties of the solution semigroup $S(t)$ generated by the linear integro-differential equation

$$\ddot{u}(t) + Au(t) + \int_0^\infty \mu(s)A^\gamma[u(t) - u(t-s)]\,\mathrm{d}s = 0$$

where the operator A is strictly positive selfadjoint with inverse not necessarily compact. The asymptotic stability is investigated in dependence of the parameter $\gamma \in \mathbb{R}$. In particular, we show that $S(t)$ is not exponentially stable when $\gamma \neq 1$.

1. Introduction

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be an infinite-dimensional separable real Hilbert space, and let

$$A : \mathrm{dom}(A) \subset H \to H$$

be a strictly positive selfadjoint unbounded linear operator, with inverse A^{-1} not necessarily compact. Given a nonincreasing absolutely continuous summable function $\mu : \mathbb{R}^+ = (0,\infty) \to \mathbb{R}^+$ of total mass

$$\int_0^\infty \mu(s)\,\mathrm{d}s = \kappa,$$

we consider the linear integro-differential equation with memory in the unknown $u = u(t)$

$$\ddot{u}(t) + Au(t) + \int_0^\infty \mu(s)A^\gamma[u(t) - u(t-s)]\,\mathrm{d}s = 0, \tag{1.1}$$

where $\gamma \in \mathbb{R}$ is a fixed constant and the *dot* stands for derivative with respect to the time variable t. Here $u(0)$ and $\dot{u}(0)$, as well as the past history $u(-s)_{|s>0}$ of the variable u, are understood to be assigned data of the problem.

Remark 1.1. Equation (1.1) serves as a model for several physical phenomena, choosing $A = -\Delta$ with the appropriate domain. For instance, it rules the evolution of the relative displacement in a linearly viscoelastic solid ($\gamma = 1$) and the electromagnetic field in the ionosphere ($\gamma = 0$). See e.g. [7, 9] for more details.

The asymptotic properties of the solution semigroup $S(t)$ arising from equation (1.1) in the case $\gamma \in [0,1]$ have been widely investigated, rewriting the problem in the so-called history space framework of Dafermos [5]. Roughly speaking, we may summarize the current results as follows: when $\gamma = 1$ the semigroup $S(t)$ is exponentially stable provided that the kernel μ is not completely flat (see [14] for the exact condition), while

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if $\gamma \in [0,1)$ and the embedding $\mathrm{dom}(A) \Subset H$ is compact the semigroup $S(t)$ is not exponentially stable (see [9, 12]).

The purpose of this paper is twofold. First, we study the asymptotic behavior of $S(t)$ when the parameter γ ranges over the whole real line. Secondly, we analyze the decay properties without assuming the compactness of the embedding $\mathrm{dom}(A) \subset H$, which translates into the fact that the spectrum of A is not simply made of an increasing sequence of eigenvalues. In particular, the usual semigroup strategies employed to prove the lack of exponential stability cannot be applied. In the present work, exploiting a recent technique introduced in [6], we prove that $S(t)$ is not exponentially stable when $\gamma \neq 1$. Therefore, roughly speaking, in order to have uniform stability of solutions the operator A^γ acting on the memory has to be "as strong as" the one acting on the variable $u(t)$ outside the integral, but not "stronger". As a complement, in the last part of the paper, we deal with weaker notions of stability, showing that $S(t)$ is stable (i.e. every single trajectory goes to zero) for every $\gamma \in \mathbb{R}$ and semiuniformly stable for $\gamma \in [0,1]$.

Plan of the paper. In §2 we introduce the functional setting and the notation. In §3 we establish the existence of the solution semigroup, and in §4 we discuss the invertibility of its infinitesimal generator. The remaining §5 and §6 are devoted to the main results about the lack of exponential decay and the stability.

2. Functional Setting and Notation

For $r \in \mathbb{R}$, we consider the nested family of Hilbert spaces

$$H^r = \mathrm{dom}(A^{\frac{r}{2}}), \qquad \langle u, v\rangle_r = \langle A^{\frac{r}{2}} u, A^{\frac{r}{2}} v\rangle, \qquad \|u\|_r = \|A^{\frac{r}{2}} u\|.$$

The index r will be always omitted whenever zero. Moreover, when $r > 0$, it is understood that H^{-r} denotes the completion of the domain, so that H^{-r} is the dual space of H^r. Accordingly, the symbol $\langle\cdot,\cdot\rangle$ also stands for duality product between H^r and H^{-r}. Along the paper, we will also encounter the complexifications $H^r_{\mathbb{C}}$ of the spaces H^r, that is, the complex Hilbert spaces

$$H^r_{\mathbb{C}} = H^r \oplus \mathrm{i}H^r = \{z = x + \mathrm{i}y \text{ with } x, y \in H^r\}$$

endowed with the inner product

$$\langle x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2\rangle_r = \langle x_1, x_2\rangle_r + \langle y_1, y_2\rangle_r + \mathrm{i}\langle y_1, x_2\rangle_r - \mathrm{i}\langle x_1, y_2\rangle_r.$$

Analogously, the complexification $\mathbb{A}$ of A is the linear operator on $H_{\mathbb{C}}$ with domain

$$\mathrm{dom}(\mathbb{A}) = \{u + \mathrm{i}v : \, u, v \in \mathrm{dom}(A)\}$$

acting as

$$\mathbb{A}(u + \mathrm{i}v) = Au + \mathrm{i}Av.$$

Since A is strictly positive selfadjoint, so is $\mathbb{A}$, and the two spectra $\sigma(A)$ and $\sigma(\mathbb{A})$ coincide. Besides, setting

$$\alpha_0 = \min\{\alpha : \, \alpha \in \sigma(\mathbb{A})\} > 0,$$

for every $r > 0$ we have the Poincaré type inequality

$$\|z\| \leq \alpha_0^{-\frac{r}{2}} \|z\|_r, \quad \forall z \in H^r_{\mathbb{C}}. \tag{2.1}$$

Next, we introduce the so-called memory spaces

$$\mathcal{M}^r = L^2_\mu(\mathbb{R}^+; H^r)$$

endowed with the weighted L^2-inner products

$$\langle \eta, \xi \rangle_{\mathcal{M}^r} = \int_0^\infty \mu(s) \langle \eta(s), \xi(s) \rangle_r \, \mathrm{d}s,$$

and we consider the infinitesimal generator of the right-translation semigroup on $\mathcal{M}^\gamma$, that is, the linear operator

$$T\eta = -\eta' \qquad \text{with domain} \qquad \mathrm{dom}(T) = \big\{\eta \in \mathcal{M}^\gamma : \eta' \in \mathcal{M}^\gamma,\ \lim_{s\to 0} \eta(s) = 0 \ \text{ in } \ H^\gamma\big\},$$

the *prime* standing for weak derivative, along with its complexification $\mathbb{T}$ acting on the space $\mathcal{M}^\gamma_{\mathbb{C}} = L^2_\mu(\mathbb{R}^+; H^\gamma_{\mathbb{C}})$. The phase space of our problem will be

$$\mathcal{H}^\gamma = H^1 \times H \times \mathcal{M}^\gamma.$$

3. The Contraction Semigroup

We translate the problem in the history space framework of Dafermos [5]. To this end, defining the auxiliary variable

$$\eta^t(s) = u(t) - u(t-s),$$

system (1.1) can be given the form

$$\begin{cases} \ddot{u} + Au + \displaystyle\int_0^\infty \mu(s) A^\gamma \eta(s) \, \mathrm{d}s = 0, \\ \dot{\eta} = T\eta + \dot{u}. \end{cases} \tag{3.1}$$

Then, introducing the 3-component vector

$$Z(t) = (u(t), v(t), \eta^t),$$

we rewrite system (1.1) as the ODE in $\mathcal{H}^\gamma$

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = LZ(t),$$

where the linear operator L is given by

$$L\begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -A\big(u + \displaystyle\int_0^\infty \mu(s) A^{\gamma-1} \eta(s) \, \mathrm{d}s\big) \\ T\eta + v \end{pmatrix},$$

with domain

$$\mathrm{dom}(L) = \left\{ z \in \mathcal{H}^\gamma \,\middle|\, \begin{array}{c} v \in H^{\max\{1,\gamma\}} \\ u + \displaystyle\int_0^\infty \mu(s) A^{\gamma-1} \eta(s) \, \mathrm{d}s \in H^2 \\ \eta \in \mathrm{dom}(T) \end{array} \right\}.$$

Theorem 3.1. *For every fixed $\gamma \in \mathbb{R}$, the operator L is the infinitesimal generator of a contraction semigroup*

$$S(t) = \mathrm{e}^{tL} : \mathcal{H}^\gamma \to \mathcal{H}^\gamma.$$

The proof is based on an application of the classical Lumer-Phillips Theorem (see [16]).

Theorem 3.2 (Lumer-Phillips)**.** *The operator L is the infinitesimal generator of a contraction semigroup $S(t) = \mathrm{e}^{tL}$ on $\mathcal{H}^\gamma$ if and only if*

(i) *L is dissipative; and*
(ii) $\mathrm{ran}(1 - L) = \mathcal{H}^\gamma$.

In the next proposition, we prove condition (i).

Proposition 3.3. *The operator L is dissipative for every $\gamma \in \mathbb{R}$.*

Proof. This amounts to show that

$$\langle Lz, z\rangle_{\mathcal{H}^\gamma} \leq 0, \quad \forall z \in \mathrm{dom}(L).$$

Indeed, given $z = (u, v, \eta) \in \mathrm{dom}(L)$, direct computations yield

$$\langle Lz, z\rangle_{\mathcal{H}^\gamma} = \langle T\eta, \eta\rangle_{\mathcal{M}^\gamma}. \tag{3.2}$$

Moreover, integrating by parts,

$$\langle T\eta, \eta\rangle_{\mathcal{M}^\gamma} = \lim_{y\to 0} \frac{1}{2}\Big(- \mu(1/y)\|\eta(1/y)\|_\gamma^2 + \mu(y)\|\eta(y)\|_\gamma^2 + \int_y^{1/y} \mu'(s)\|\eta(s)\|_\gamma^2 \,\mathrm{d}s\Big).$$

Exploiting the monotonicity of μ and the Hölder inequality, we now infer that

$$\begin{aligned}\lim_{y\to 0} \mu(y)\|\eta(y)\|_\gamma^2 &\leq \limsup_{y\to 0} \mu(y) \left(\int_0^y \|\eta'(r)\|_\gamma \,\mathrm{d}r\right)^2 \\ &\leq \limsup_{y\to 0} y \int_0^y \mu(r)\|\eta'(r)\|_\gamma^2 \,\mathrm{d}r = 0,\end{aligned}$$

and thus

$$\langle T\eta, \eta\rangle_{\mathcal{M}^\gamma} = \lim_{y\to 0} \frac{1}{2}\Big(- \mu(1/y)\|\eta(1/y)\|_\gamma^2 + \int_y^{1/y} \mu'(s)\|\eta(s)\|_\gamma^2 \,\mathrm{d}s\Big).$$

Since the left-hand side is bounded and the two terms in the right-hand side are negative, we conclude that both the limit and the integral exist and are finite. In particular, this forces the convergence

$$\lim_{y\to 0} \mu(1/y)\|\eta(1/y)\|_\gamma^2 = 0,$$

and therefore

$$\langle T\eta, \eta\rangle_{\mathcal{M}^\gamma} = \frac{1}{2}\int_0^\infty \mu'(s)\|\eta(s)\|_\gamma^2 \,\mathrm{d}s \leq 0. \tag{3.3}$$

The proof is finished. $\square$

In order to prove condition (ii) of Theorem 3.2, we need the following well known measure-theoretical result (see e.g. [11]).

Lemma 3.4. *Given $\delta > 0$ and $h \in L^2(\mathbb{R}^+)$, consider the function $E_\delta[h]$ defined as*

$$E_\delta[h; s] = \int_0^s \mathrm{e}^{-\frac{\delta}{2}(s-r)} h(r)\,\mathrm{d}r.$$

Then, $E_\delta[h] \in L^2(\mathbb{R}^+)$ and

$$\|E_\delta[h]\|_{L^2(\mathbb{R}^+)} \le \tfrac{2}{\delta}\|h\|_{L^2(\mathbb{R}^+)}.$$

Proposition 3.5. *The operator $1 - L : \mathrm{dom}(L) \subset \mathcal{H}^\gamma \to \mathcal{H}^\gamma$ is onto for every $\gamma \in \mathbb{R}$.*

Proof. Let $\bar{z} = (\bar{u}, \bar{v}, \bar{\eta}) \in \mathcal{H}^\gamma$ be arbitrarily chosen. We look for a solution $z = (u, v, \eta) \in \mathrm{dom}(L)$ to the equation

$$z - Lz = \bar{z},$$

which, written componentwise, reads

$$\begin{aligned}
&u - v = \bar{u},\\
&v + A\big(u + \int_0^\infty \mu(s) A^{\gamma-1}\eta(s)\,\mathrm{d}s\big) = \bar{v},\\
&\eta - T\eta - v = \bar{\eta}.
\end{aligned}$$

Integrating the third equation with $\eta(0) = 0$ we find

$$\eta(s) = (1 - \mathrm{e}^{-s})v + (E * \bar{\eta})(s),$$

where $E(s) = \mathrm{e}^{-s}$ and $*$ denotes the convolution product on $(0, s)$. Then, substituting this expression and the first equation of the system above into the second equation, we obtain

$$v = (1 + A + \varkappa A^\gamma)^{-1} w, \tag{3.4}$$

having set

$$\varkappa = \int_0^\infty \mu(s)(1 - \mathrm{e}^{-s})\,\mathrm{d}s > 0$$

and

$$w = \bar{v} - A\big(\bar{u} + \int_0^\infty \mu(s) A^{\gamma-1}(E * \bar{\eta})(s)\,\mathrm{d}s\big).$$

Our next step is to prove that $v \in H^{\max\{1,\gamma\}}$. To this aim, we consider two cases separately.

Case 1: $\gamma \le 1$. Owing to (3.4) it is sufficient to show that $w \in H^{-1}$. Appealing to (2.1), we begin by estimating

$$\begin{aligned}
\|w\|_{-1} &\le \|\bar{v}\|_{-1} + \|\bar{u}\|_1 + \Big\| \int_0^\infty \mu(s) A^\gamma (E * \bar{\eta})(s)\,\mathrm{d}s \Big\|_{-1}\\
&\le c\|\bar{v}\| + \|\bar{u}\|_1 + \int_0^\infty \mu(s) \|(E * \bar{\eta})(s)\|_{2\gamma-1}\,\mathrm{d}s,
\end{aligned}$$

for some $c > 0$. Since $\gamma \le 1$, we have

$$\int_0^\infty \mu(s)\|(E * \bar{\eta})(s)\|_{2\gamma-1}\,\mathrm{d}s \le c\int_0^\infty \mu(s)\|(E * \bar{\eta})(s)\|_\gamma\,\mathrm{d}s \le c\sqrt{\kappa}\|E_2[h]\|_{L^2(\mathbb{R}^+)}$$

where we set

$$h(r)=\sqrt{\mu(r)}\|\bar\eta(r)\|_\gamma \qquad \text{and} \qquad E_2[h;s]=\int_0^s \mathrm{e}^{-(s-r)}h(r)\,\mathrm{d}r.$$

At this point an exploitation of Lemma 3.4 yields

$$c\sqrt{\kappa}\|E_2[h]\|_{L^2(\mathbb{R}^+)}\le c\sqrt{\kappa}\|h\|_{L^2(\mathbb{R}^+)}=c\sqrt{\kappa}\|\bar\eta\|_{\mathcal{M}^\gamma},$$

and the conclusion follows.

Case 2: $\gamma>1$. In light of (3.4) we need to prove that $w\in H^{-\gamma}$. With the aid of (2.1), it is sufficient to estimate

$$\begin{aligned}\|w\|_{-\gamma}&\le\|\bar v\|_{-\gamma}+\|\bar u\|_{2-\gamma}+\left\|\int_0^\infty\mu(s)A^\gamma(E*\bar\eta)(s)\,\mathrm{d}s\right\|_{-\gamma}\\&\le c\|\bar v\|+c\|\bar u\|_1+\int_0^\infty\mu(s)\|(E*\bar\eta)(s)\|_\gamma\,\mathrm{d}s,\end{aligned}$$

for some $c>0$, and then the argument is analogous to the previous case.

In order to finish the proof, we are left to show that

$$u+\int_0^\infty\mu(s)A^{\gamma-1}\eta(s)\,\mathrm{d}s\in H^2 \qquad \text{and} \qquad \eta\in\mathrm{dom}(T).$$

To this aim, using the notation above and appealing once more to Lemma 3.4, we estimate

$$\begin{aligned}\|\eta\|^2_{\mathcal{M}^\gamma}&\le 2\int_0^\infty\mu(s)\|v\|^2_\gamma\,\mathrm{d}s+2\int_0^\infty\mu(s)\|(E*\bar\eta)(s)\|^2_\gamma\mathrm{d}s\\&\le 2\kappa\|v\|^2_\gamma+2\|E_2[h]\|^2_{L^2(\mathbb{R}^+)}\\&\le 2\kappa\|v\|^2_\gamma+2\|\bar\eta\|^2_{\mathcal{M}^\gamma},\end{aligned}$$

and thus $\eta\in\mathcal{M}^\gamma$. As a consequence, $T\eta=\eta-v-\bar\eta\in\mathcal{M}^\gamma$. It is also apparent to see that

$$\lim_{s\to0}\eta(s)=0 \ \text{ in } \ H^\gamma,$$

yielding $\eta\in\mathrm{dom}(T)$. Finally,

$$u+\int_0^\infty\mu(s)A^{\gamma-1}\eta(s)\,\mathrm{d}s=A^{-1}(\bar v-v)\in H^2,$$

and the conclusion follows. □

Remark 3.6. For every initial datum $z_0=(u_0,v_0,\eta_0)\in\mathcal{H}^\gamma$, the third component of the solution $S(t)z_0=(u(t),\dot u(t),\eta^t)$ admits the explicit representation formula (see [10])

$$\eta^t(s)=\begin{cases}u(t)-u(t-s), & 0\le s\le t,\\ \eta_0(s-t)+u(t)-u_0, & s>t.\end{cases} \tag{3.5}$$

4. Invertibility of the Operator L

In this section we discuss the invertibility of the infinitesimal generator L. To this end, introducing the nonnegative function $k(s,r) : \mathbb{R}^+ \times \mathbb{R}^+ \to [0,1]$ defined as

$$k(s,r) = \begin{cases} \sqrt{\frac{\mu(s)}{\mu(r)}} & \text{if } r \le s, \\ 0 & \text{otherwise,} \end{cases}$$

we consider the integral operator K with kernel k

$$Kf(s) = \int_0^\infty k(s,r) f(r)\,\mathrm{d}r, \quad f \in L^2(\mathbb{R}^+).$$

The result reads as follows.

Theorem 4.1. *The infinitesimal generator L is invertible if and only if*

$$\gamma \le 1 \qquad \textit{and} \qquad K \in \mathfrak{L}(L^2(\mathbb{R}^+)).^1$$

Proof. The operator L is invertible if and only if, for any $\bar z = (\bar u, \bar v, \bar\eta) \in \mathcal{H}^\gamma$, the equation

$$Lz = \bar z$$

admits a unique solution $z = (u, v, \eta) \in \mathrm{dom}(L)$. Componentwise, this translates into

$$v = \bar u, \tag{4.1}$$

$$A\big(u + \int_0^\infty \mu(s) A^{\gamma-1}\eta(s)\,\mathrm{d}s\big) = -\bar v, \tag{4.2}$$

$$T\eta + v = \bar\eta. \tag{4.3}$$

If $\gamma > 1$ we see from (4.1) that, choosing $\bar u \in H^1$ but not more regular, $v \notin H^\gamma$ and therefore L is not invertible. Let us now prove that, when $\gamma \le 1$,

$$L \text{ invertible} \quad \Leftrightarrow \quad K \in \mathfrak{L}(L^2(\mathbb{R}^+)).$$

First we show sufficiency. Assuming $K \in \mathfrak{L}(L^2(\mathbb{R}^+))$, we claim that the (unique) solution of system (4.1)-(4.3) is given by

$$\begin{aligned} u &= -A^{-1}\bar v - A^{\gamma-1}\bar u \int_0^\infty s\mu(s)\,\mathrm{d}s + \int_0^\infty \mu(s) \Big(\int_0^s A^{\gamma-1}\bar\eta(r)\,\mathrm{d}r \Big)\,\mathrm{d}s, \\ v &= \bar u, \\ \eta(s) &= s\bar u - \int_0^s \bar\eta(r)\,\mathrm{d}r. \end{aligned}$$

Indeed, setting

$$f_1(r) = \sqrt{\mu(r)}\|\bar\eta(r)\|_\gamma \in L^2(\mathbb{R}^+),$$

we infer that

$$\begin{aligned} \int_0^\infty \mu(s) \Big\| \int_0^s \bar\eta(r)\,\mathrm{d}r \Big\|_\gamma^2 \mathrm{d}s &\le \int_0^\infty \mu(s) \Big(\int_0^s \|\bar\eta(r)\|_\gamma\,\mathrm{d}r \Big)^2 \mathrm{d}s \\ &= \|Kf_1\|^2_{L^2(\mathbb{R}^+)} < \infty. \end{aligned}$$

[1] With standard notation, $\mathfrak{L}(L^2(\mathbb{R}^+))$ denotes the space of bounded linear operators on $L^2(\mathbb{R}^+)$.

Moreover, calling

$$f_2(r) = \sqrt{\mu(r)} \in L^2(\mathbb{R}^+),$$

and exploiting the condition $\gamma \leq 1$, we obtain

$$\int_0^\infty \mu(s)\|s\bar{u}\|_\gamma^2 \,\mathrm{d}s = \|\bar{u}\|_\gamma^2 \int_0^\infty s^2\mu(s)\,\mathrm{d}s = \|\bar{u}\|_\gamma^2 \|Kf_2\|^2_{L^2(\mathbb{R}^+)} < \infty.$$

Collecting the two inequalities above and observing that

$$\lim_{s\to 0} \eta(s) = 0 \;\text{ in }\; H^\gamma \qquad \text{and} \qquad \eta' \in \mathcal{M}^\gamma$$

we conclude that $\eta \in \mathrm{dom}(T)$. At this point, it is easy to see that the whole vector $(u, v, \eta) \in \mathrm{dom}(L)$. In order to prove necessity we begin to show that, when L is invertible, the operator K maps $L^2(\mathbb{R}^+)$ into $L^2(\mathbb{R}^+)$. Indeed, taking $\bar{u} = v = 0$ and integrating equation (4.3) on $(0, s)$, we learn that

$$\int_0^s \bar{\eta}(r)\,\mathrm{d}r \in \mathcal{M}^\gamma, \quad \forall \bar{\eta} \in \mathcal{M}^\gamma.$$

Hence, for every $f \in L^2(\mathbb{R}^+)$, choosing

$$\bar{\eta}(r) = \frac{f(r)}{\sqrt{\mu(r)}}\, w$$

for some unit $w \in H^\gamma$, we have

$$\|Kf\|^2_{L^2(\mathbb{R}^+)} = \int_0^\infty \mu(s) \left(\int_0^s \frac{f(r)}{\sqrt{\mu(r)}}\,\mathrm{d}r \right)^2 \mathrm{d}s = \int_0^\infty \mu(s) \left\| \int_0^s \bar{\eta}(r)\,\mathrm{d}r \right\|_\gamma^2 \mathrm{d}s < \infty.$$

To complete the argument it is sufficient to prove that $K : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ is closed; then the conclusion will follow applying the Closed Graph theorem. To this end, introducing the further integral operator

$$K^\star f(s) = \int_0^\infty k(r,s) f(r)\,\mathrm{d}r, \quad f \in L^2(\mathbb{R}^+),$$

and exploiting the Fubini-Tonelli theorem, for every $\xi_1, \xi_2 \in L^2(\mathbb{R}^+)$ we draw the inequality

$$\int_0^\infty |\xi_1(s) K^\star \xi_2(s)|\,\mathrm{d}s \leq \int_0^\infty |\xi_1(s)| \int_0^\infty k(r,s)|\xi_2(r)|\,\mathrm{d}r\,\mathrm{d}s = \langle K|\xi_1|, |\xi_2|\rangle_{L^2(\mathbb{R}^+)} < \infty,$$

which in turn implies $K^\star \xi_2 \in L^2(\mathbb{R}^+)$ (see e.g. [11, p. 232]). Next, taking a sequence $\varphi_n \in L^2(\mathbb{R}^+)$ such that

$$\varphi_n \to \varphi \qquad \text{and} \qquad K\varphi_n \to \psi \quad \text{in } L^2(\mathbb{R}^+)$$

for some $\varphi, \psi \in L^2(\mathbb{R}^+)$, we have

$$\langle \psi, \zeta \rangle = \lim_{n\to\infty} \langle K\varphi_n, \zeta \rangle = \lim_{n\to\infty} \langle \varphi_n, K^\star \zeta \rangle = \langle K\varphi, \zeta \rangle$$

for every smooth compactly supported $\zeta \in C_{\mathrm{c}}^\infty(\mathbb{R}^+)$.[2] By density, we reach the desired equality $K\varphi = \psi$. ☐

[2]Again, the order of integration in the iterated integral can be changed due to the Fubini-Tonelli theorem.

We now discuss two consequences of Theorem 4.1.

Corollary 4.2. *Assume that* $\gamma \leq 1$ *and*

$$\liminf_{t\to\infty} \left[\sup_{s\in\mathbb{R}^+} \frac{\mu(t+s)}{\mu(s)} \right] < 1. \tag{4.4}$$

Then the operator L *is invertible.*

Proof. We begin by showing that (4.4) is equivalent to

$$\mu(t+s) \leq C\mathrm{e}^{-\delta t}\mu(s), \tag{4.5}$$

for every $t \geq 0$ and $s > 0$, and some $C \geq 1$ and $\delta > 0$. Indeed, let $\varrho < 1$ and $r > 0$ such that

$$\mu(r+s) \leq \varrho\mu(s), \quad \forall s > 0.$$

Then, for every $t \geq 0$, writing

$$t = nr + \tau, \quad n \in \mathbb{N},\ \tau \in [0, r),$$

and exploiting the monotonicity of μ, we get

$$\mu(s+t) \leq \mu(s+nr) \leq \varrho^n\mu(s) = \mathrm{e}^{n\log\varrho}\mu(s) \leq C\mathrm{e}^{-\delta t}\mu(s),$$

with $C = \frac{1}{\varrho}$ and $\delta = -\frac{1}{r}\log\varrho$. Let now $f \in L^2(\mathbb{R}^+)$ be arbitrarily fixed. In light of (4.5) we have

$$\|Kf\|^2_{L^2(\mathbb{R}^+)} \leq C\|E_\delta[f]\|^2_{L^2(\mathbb{R}^+)}$$

where

$$E_\delta[f;s] = \int_0^s \mathrm{e}^{-\frac{\delta}{2}(s-r)}|f(r)|\,\mathrm{d}r.$$

Appealing to Lemma 3.4, the right-hand side is controlled by

$$C\|E_\delta[f]\|^2_{L^2(\mathbb{R}^+)} \leq \frac{4C}{\delta^2}\|f\|^2_{L^2(\mathbb{R}^+)},$$

and thus

$$\|K\|_{\mathfrak{L}(L^2(\mathbb{R}^+))} \leq \frac{2\sqrt{C}}{\delta}.$$

The proof is finished. ☐

From the proof above we learn that if (4.4) holds (and thus L is invertible for $\gamma \leq 1$), then μ has an exponential decay at infinity. The situation is completely different when the kernel decays polynomially.

Corollary 4.3. *Let* $p > 1$ *be fixed, and let*

$$\mu(s) = \frac{1}{(1+s)^p}.$$

Then, the operator L *is not invertible.*

Proof. Considering the function
$$f(s) = s^{(p-2)/2}\sqrt{\mu(s)} \in L^2(\mathbb{R}^+),$$
it is immediate to see that
$$\|Kf\|^2_{L^2(\mathbb{R}^+)} = \int_0^\infty \mu(s)\bigg(\int_0^s r^{(p-2)/2}\,\mathrm{d}r\bigg)^2 \mathrm{d}s = \frac{4}{p^2}\int_0^\infty \Big(\frac{s}{1+s}\Big)^p \mathrm{d}s = \infty,$$
and hence $K \notin \mathfrak{L}(L^2(\mathbb{R}^+))$. □

5. Lack of Exponential Stability

We now analyze the exponential stability of $S(t)$. Recall that $S(t)$ is said to be *exponentially stable* if there exist $M \geq 1$ and $\beta > 0$ such that
$$\|S(t)\|_{\mathfrak{L}(\mathcal{H}^\gamma)} \leq M\mathrm{e}^{-\beta t}, \quad \forall t \geq 0.$$
The main result of the paper reads as follows.

Theorem 5.1. *The semigroup $S(t)$ is not exponentially stable if*

(i) $\gamma > 1$*; or*

(ii) $\gamma < 0$*; or*

(iii) $\gamma \in [0,1)$ *and the kernel μ satisfies the condition*[3]
$$\lim_{s\to 0} s^{1-\gamma}\mu(s) = 0.$$

The proof is based on the next abstract criterion from [17] (see also [4, 8] for the statement used here).

Lemma 5.2. *The contraction semigroup $S(t) = \mathrm{e}^{tL}$ on $\mathcal{H}^\gamma$ is exponentially stable if and only if there exists $\varepsilon > 0$ such that*
$$\inf_{\lambda\in\mathbb{R}} \|\mathrm{i}\lambda z - \mathbb{L}z\|_{\mathcal{H}^\gamma_\mathbb{C}} \geq \varepsilon\|z\|_{\mathcal{H}^\gamma_\mathbb{C}}, \quad \forall z \in \mathrm{dom}(\mathbb{L}).^4 \tag{5.1}$$

We also need two technical lemmas. The first can be found in [15]. For the second one, we address the reader to [6].

Lemma 5.3. *Given $\theta \in [0,1)$, let us denote with $F(\lambda)$ the Fourier transform of μ*
$$F(\lambda) = \int_0^\infty \mu(s)\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s.$$
Then, the following implication holds
$$\lim_{s\to 0} s^{1-\theta}\mu(s) = 0 \quad \Rightarrow \quad \lim_{\lambda\to\infty} \lambda^\theta F(\lambda) = 0.$$

Lemma 5.4. *Let $\alpha \in \sigma(\mathbb{A})$ be fixed, and let $\mathcal{Q} \subset \mathbb{R}$ be a given bounded set. Then, for every $\varepsilon > 0$ small enough, there exists a unit vector $w_\varepsilon \in H_\mathbb{C}$ such that the vector*
$$\xi_{q,\varepsilon} = \mathbb{A}^q w_\varepsilon - \alpha^q w_\varepsilon$$
satisfies the relation
$$\|\xi_{q,\varepsilon}\| \leq \varepsilon, \quad \forall q \in \mathcal{Q}.$$

[3]When $\gamma = 0$ the condition is automatically satisfied.

[4]Here and in the sequel, $\mathbb{L}$ denotes the complexification of the infinitesimal generator L.

We are now in a position to prove Theorem 5.1. Along the proof, $C \geq 0$ will denote a *generic* constant depending only on the structural parameters of the problem. Case (i) follows directly from the fact that, as we saw in the previous section, when $\gamma > 1$ the infinitesimal generator L is not invertible (and thus $S(t)$ cannot be exponentially stable). In order to prove cases (ii) and (iii), choose $\alpha_n \in \sigma(\mathbb{A})$ with $\alpha_n \to \infty$ (this is possible since $\mathbb{A}$ is unbounded). By Lemma 5.4, given a positive sequence $\nu_n \to 0$, there exist unitary $w_n \in H_{\mathbb{C}}$ such that the vectors

$$\xi_{q,n} = \mathbb{A}^q w_n - \alpha_n^q w_n$$

fulfill the inequality

$$\|\xi_{q,n}\| \leq \nu_n, \quad \text{for } q = \tfrac{\gamma}{2}, \gamma, 1. \tag{5.2}$$

Next, setting

$$\zeta_n = c_n w_n$$

where

$$c_n = \begin{cases} 1 & \text{if } \gamma < 0, \\ \alpha_n^{-\frac{\gamma}{2}} & \text{if } \gamma \in [0,1), \end{cases}$$

we consider the sequence

$$\hat{z}_n = (0, 0, \zeta_n) \in \mathcal{H}^\gamma_{\mathbb{C}}.$$

Exploiting (2.1) and (5.2), it is apparent to see that

$$\|\hat{z}_n\|_{\mathcal{H}^\gamma_{\mathbb{C}}} \leq C,$$

that is, $\hat{z}_n$ is bounded. Suppose now by contradiction that the semigroup $S(t)$ is exponentially stable. Then, for any given $\lambda_n \in \mathbb{R}$ to be chosen later, the resolvent equation

$$\mathrm{i}\lambda_n z_n - \mathbb{L} z_n = \hat{z}_n$$

admits a unique solution

$$z_n = (u_n, v_n, \eta_n) \in \mathrm{dom}(\mathbb{L}).$$

Moreover, in light of Lemma 5.2, there exists $\varepsilon > 0$ such that

$$\|z_n\|_{\mathcal{H}^\gamma_{\mathbb{C}}} \leq \frac{1}{\varepsilon}\|\hat{z}_n\|_{\mathcal{H}^\gamma_{\mathbb{C}}} \leq \frac{C}{\varepsilon}, \tag{5.3}$$

namely, the sequence z_n is bounded as well. We will reach a contradiction by showing it is not so. To this aim, we first write the resolvent equation above componentwise

$$\begin{aligned}
&\mathrm{i}\lambda_n u_n - v_n = 0, \\
&\mathrm{i}\lambda_n v_n + \mathbb{A}\Big(u_n + \int_0^\infty \mu(s)\mathbb{A}^{\gamma-1}\eta_n(s)\,\mathrm{d}s\Big) = 0, \\
&\mathrm{i}\lambda_n \eta_n - \mathbb{T}\eta_n - v_n = \zeta_n.
\end{aligned}$$

An integration of the third equation with $\eta_n(0) = 0$ entails

$$\eta_n(s) = \frac{1}{\mathrm{i}\lambda_n}(v_n + \zeta_n)(1 - \mathrm{e}^{-\mathrm{i}\lambda_n s}).$$

Therefore, substituting this expression and the first equation of the system above into the second one, we obtain

$$\lambda_n^2 v_n - \mathbb{A} v_n - (\kappa - F(\lambda_n))\mathbb{A}^\gamma(\zeta_n + v_n) = 0. \tag{5.4}$$

At this point, for every n, the solution v_n can be written as

$$v_n = p_n w_n + r_n,$$

for some $p_n \in \mathbb{C}$ and some vector $r_n \perp w_n$. It is apparent from (5.3) that

$$\|r_n\| \leq C \qquad \text{and} \qquad |p_n| \leq C. \tag{5.5}$$

Taking the inner product in $H_{\mathbb{C}}$ of (5.4) with w_n, we obtain the identity

$$(\lambda_n^2 - \alpha_n - \kappa\alpha_n^\gamma + F(\lambda_n)\alpha_n^\gamma)p_n = (\kappa - F(\lambda_n))c_n\alpha_n^\gamma + f_n, \tag{5.6}$$

having set

$$f_n = p_n\langle w_n, \xi_{1,n}\rangle + \langle r_n, \xi_{1,n}\rangle + (\kappa - F(\lambda_n))[(c_n + p_n)\langle w_n, \xi_{\gamma,n}\rangle + \langle r_n, \xi_{\gamma,n}\rangle].$$

Next, choosing

$$\lambda_n = \sqrt{\alpha_n + \kappa\alpha_n^\gamma} \sim \sqrt{\alpha_n},$$

equation (5.6) yields[5]

$$p_n = \frac{(\kappa - F(\lambda_n))c_n}{F(\lambda_n)} + \frac{f_n}{F(\lambda_n)\alpha_n^\gamma}. \tag{5.7}$$

Owing to (5.2), (5.5) and the Riemann-Lebesgue lemma, it is clear that

$$|f_n| \leq C\nu_n.$$

Hence, if $\gamma < 0$, selecting $\nu_n = \mathrm{o}(\alpha_n^\gamma)$ we infer from (5.7) that

$$|p_n| \geq \left|\frac{\kappa - F(\lambda_n)}{F(\lambda_n)}\right| - \left|\frac{f_n}{F(\lambda_n)\alpha_n^\gamma}\right| \sim \left|\frac{\kappa}{F(\lambda_n)}\right| \to \infty.$$

If otherwise $\gamma \in [0, 1)$, exploiting Lemma 5.3 we still learn from (5.7) that

$$|p_n| \geq \left|\frac{\kappa - F(\lambda_n)}{F(\lambda_n)\sqrt{\alpha_n^\gamma}}\right| - \left|\frac{f_n}{F(\lambda_n)\alpha_n^\gamma}\right| \sim \left|\frac{\kappa}{F(\lambda_n)\lambda_n^\gamma}\right| \to \infty.$$

In both cases, we end up with

$$\|z_n\|_{\mathcal{H}_{\mathbb{C}}^\gamma} \geq \|v_n\| \geq |p_n| \to \infty,$$

contradicting (5.3). □

[5]Since μ is nonincreasing, absolutely continuous and positive, $F(\lambda_n) \neq 0$ for every n.

6. Stability and Semiuniform Stability

6.1. **Stability.** We analyze the stability of $S(t)$ within the assumption $\mathrm{dom}(A) \Subset H$. Recall that, for a fixed $\gamma \in \mathbb{R}$, the semigroup $S(t)$ is said to be *stable* if

$$\lim_{t\to\infty} \|S(t)z\|_{\mathcal{H}^\gamma} = 0, \quad \forall z \in \mathcal{H}^\gamma.$$

Theorem 6.1. *If A^{-1} is a compact, then the semigroup $S(t)$ is stable for every $\gamma \in \mathbb{R}$.*

The proof is based on the following result, yielding a sufficient condition for the stability of $S(t)$.

Lemma 6.2. *Let A^{-1} be a compact operator. Assume that, for every $z_0 = (u_0, v_0, \eta_0) \in \mathrm{dom}(L)$, the condition*

$$\int_0^\infty \mu'(s)\|\eta^t(s)\|_\gamma^2 \,\mathrm{d}s = 0, \quad \forall t \geq 0, \tag{6.1}$$

implies

$$u(t) = u_0, \quad \forall t \geq 0.$$

Then, $S(t)$ is stable.

The above lemma can be proved in the very same way of [13, Lemma 4.7], where the case $\gamma = 1$ is treated. Thus, we limit ourselves to sketch the argument.

Sketch of the proof of Lemma 6.2. Setting

$$\beta = \beta(\gamma) = \max\{1, \gamma\},$$

we define the subspace of $\mathcal{H}^\gamma$

$$\mathcal{V}^\gamma = H^{1+\beta} \times H^\beta \times [\mathcal{M}^{\gamma+\beta} \cap \mathrm{dom}(T)] \subset \mathrm{dom}(L).$$

Introducing the norm

$$\|(u, v, \eta)\|_{\mathcal{V}^\gamma}^2 = \|u\|_{1+\beta}^2 + \|v\|_\beta^2 + \|\eta\|_{\mathcal{M}^{\gamma+\beta}}^2 + \|T\eta\|_{\mathcal{M}^\gamma}^2,$$

$\mathcal{V}^\gamma$ turns out to be a reflexive Banach space (actually, a Hilbert space) continuously and densely embedded into $\mathcal{H}^\gamma$. Following the proofs of [13, Lemmas 4.5 and 4.6] one can show that, for every $z \in \mathcal{V}^\gamma$, the set

$$\mathcal{K}_z = \bigcup_{t\geq 1} S(t)z$$

is bounded in $\mathcal{V}^\gamma$ and precompact in $\mathcal{H}^\gamma$. Thanks to the reflexivity of $\mathcal{V}^\gamma$, the inclusion $\mathcal{V}^\gamma \subset \mathrm{dom}(L)$ and the precompactness of $\mathcal{K}_z$, the argument devised in [13, Lemma 4.7] can be now repeated word by word, simply changing the spaces accordingly. $\square$

Proof of Theorem 6.1. Let $z_0 = (u_0, v_0, \eta_0) \in \mathrm{dom}(L)$ satisfying (6.1). Due to Lemma 6.2, in order to reach the desired conclusion it is sufficient to show that u is constant. To this aim, introducing the set

$$\mathcal{B}_{z_0} = \{s \in \mathbb{R}^+ : \eta^t(s) = 0,\ \forall t \geq 0\}$$

and owing to (3.5), it is readily seen that, if $\sigma \in \mathcal{B}_{z_0}$, then u is σ-periodic. Therefore, if $\mathcal{B}_{z_0}$ contains two rationally independent numbers, then u must be constant. Since μ is absolutely continuous, the set

$$\mathfrak{D} = \{s \in \mathbb{R}^+ :\ \mu'(s) < 0\}$$

has positive Lebesgue measure. In light of (6.1) the same holds for $\mathcal{B}_{z_0}$, and thus it certainly contains two rationally independent numbers. ☐

6.2. **Semiuniform stability.** The semigroup $S(t)$ is said to be *semiuniformly stable* if there is a function $h : [0,\infty) \to [0,\infty)$ vanishing at infinity such that

$$\|S(t)z\|_{\mathcal{H}^\gamma} \leq h(t)\|Lz\|_{\mathcal{H}^\gamma}, \quad \forall z \in \mathrm{dom}(L).$$

In order to analyze the semiuniform stability of $S(t)$, we need the following well-known criterion [1, 2, 3].

Lemma 6.3. *The contraction semigroup $S(t) = \mathrm{e}^{tL}$ on $\mathcal{H}^\gamma$ is semiuniformly stable if and only if the imaginary axis $\mathrm{i}\mathbb{R}$ belongs to the resolvent set $\rho(\mathbb{L})$.*

Our result reads as follows.

Theorem 6.4. *Assume that the kernel μ satisfies the additional condition*

$$\mu'(s) + \nu\mu(s) \leq 0 \tag{6.2}$$

for every $s \in \mathbb{R}^+$ and some $\nu > 0$. If $\gamma \in [0,1]$, then the semigroup $S(t)$ is semiuniformly stable. Moreover, it is not semiuniformly stable when $\gamma > 1$.

Remark 6.5. Observe that, analogously to the previous Section 5, we are not assuming the compactness of the embedding $\mathrm{dom}(A) \subset H$.

Proof. Along the proof, $C \geq 0$ will denote a *generic* constant depending only on the structural parameters of the problem. Moreover, the Poincaré inequality (2.1) will be used several times without explicit mention.

When $\gamma > 1$, we already know from Theorem 4.1 that L is not invertible, and hence $S(t)$ cannot be semiuniformly stable. Thus, we restrict our attention to the case $\gamma \in [0,1]$. We preliminarily observe that, since $S(t)$ is a contraction semigroup,

$$\sigma(\mathbb{L}) \cap \mathrm{i}\mathbb{R} = \sigma_{\mathrm{ap}}(\mathbb{L}) \cap \mathrm{i}\mathbb{R},$$

where $\sigma_{\mathrm{ap}}(\mathbb{L})$ denotes the set of the approximate eigenvalues of the operator $\mathbb{L}$ (see [1, Proposition 2.2]). Hence, due to Lemma 6.3, it is sufficient to show that no approximate eigenvalues of $\mathbb{L}$ lie on the imaginary axis. By contradiction, suppose that there exists $\lambda \in \mathbb{R}$ with $\mathrm{i}\lambda \in \sigma_{\mathrm{ap}}(\mathbb{L})$. Then, there is a sequence $z_n = (u_n, v_n, \eta_n) \in \mathrm{dom}(\mathbb{L})$ with

$$\|z_n\|^2_{\mathcal{H}^\gamma_{\mathbb{C}}} = \|u_n\|_1^2 + \|v_n\|^2 + \|\eta_n\|^2_{\mathcal{M}^\gamma_{\mathbb{C}}} = 1, \tag{6.3}$$

such that

$$\mathrm{i}\lambda z_n - \mathbb{L} z_n \to 0 \quad \text{in } \mathcal{H}^\gamma_{\mathbb{C}} \tag{6.4}$$

which, written componentwise, reads

$$\mathrm{i}\lambda u_n - v_n \to 0 \quad \text{in } H^1_{\mathbb{C}}, \tag{6.5}$$

$$\mathrm{i}\lambda v_n + \mathbb{A}\Big(u_n + \int_0^\infty \mu(s)\mathbb{A}^{\gamma-1}\eta_n(s)\mathrm{d}s\Big) \to 0 \quad \text{in } H_{\mathbb{C}}, \tag{6.6}$$

$$\mathrm{i}\lambda \eta_n - \mathbb{T}\eta_n - v_n \to 0 \quad \text{in } \mathcal{M}^\gamma_{\mathbb{C}}. \tag{6.7}$$

In light of (6.2) and Corollary 4.2 the generator $\mathbb{L}$ is invertible, and thus $\lambda \neq 0$. In addition, exploiting (3.2) and (3.3), together with (6.2)-(6.4), we infer that

$$\|\eta_n\|^2_{\mathcal{M}^\gamma_{\mathbb{C}}} \leq -\frac{1}{\nu}\int_0^\infty \mu'(s)\|\eta_n(s)\|^2_\gamma \, \mathrm{d}s = \frac{2}{\nu}\,\mathfrak{Re}\langle \mathrm{i}\lambda z_n - \mathbb{L}z_n, z_n\rangle_{\mathcal{H}^\gamma_{\mathbb{C}}} \to 0. \tag{6.8}$$

Moreover, due to (6.7),

$$\mathrm{i}\lambda\eta_n - \mathbb{T}\eta_n - v_n = \varepsilon_n$$

for some vanishing sequence $\varepsilon_n \in \mathcal{M}^\gamma_{\mathbb{C}}$. Hence, an integration over $(0,s)$ yields

$$(1-\mathrm{e}^{-\mathrm{i}\lambda s})v_n = \mathrm{i}\lambda\Big[\eta_n(s) - \int_0^s \mathrm{e}^{-\mathrm{i}\lambda(s-y)}\varepsilon_n(y)\mathrm{d}y\Big].$$

Taking the inner product in $\mathcal{M}_{\mathbb{C}}$ with v_n, we get

$$\|v_n\|^2\int_0^\infty \mu(s)(1-\mathrm{e}^{-\mathrm{i}\lambda s})\,\mathrm{d}s = \mathrm{i}\lambda\Big[\langle \eta_n, v_n\rangle_{\mathcal{M}_{\mathbb{C}}} - \int_0^\infty \mu(s)\int_0^s \mathrm{e}^{-\mathrm{i}\lambda(s-y)}\langle \varepsilon_n(y), v_n\rangle\,\mathrm{d}y\,\mathrm{d}s\Big].$$

Appealing to (6.3) and (6.8) it is apparent to see that

$$|\mathrm{i}\lambda\langle \eta_n, v_n\rangle_{\mathcal{M}_{\mathbb{C}}}| \leq C|\lambda|\|v_n\|\|\eta_n\|_{\mathcal{M}^\gamma_{\mathbb{C}}} \to 0$$

while, setting $h(y) = \sqrt{\mu(y)}\|\varepsilon_n(y)\|_\gamma$ and exploiting (6.2), (6.3) and Lemma 3.4,

$$\begin{aligned}\Big|\mathrm{i}\lambda\int_0^\infty \mu(s)\int_0^s \mathrm{e}^{-\mathrm{i}\lambda(s-y)}\langle\varepsilon_n(y), v_n\rangle\,\mathrm{d}y\,\mathrm{d}s\Big| &\leq C|\lambda|\|v_n\|\int_0^\infty \mu(s)\int_0^s \|\varepsilon_n(y)\|_\gamma\,\mathrm{d}y\,\mathrm{d}s\\ &\leq C|\lambda|\|v_n\|\int_0^\infty \sqrt{\mu(s)}E_\nu[h;s]\,\mathrm{d}s,\\ &\leq C|\lambda|\|v_n\|\|\varepsilon_n\|_{\mathcal{M}^\gamma_{\mathbb{C}}} \to 0.\end{aligned}$$

In conclusion,

$$\|v_n\|^2\Big|\int_0^\infty \mu(s)(1-\mathrm{e}^{-\mathrm{i}\lambda s})\,\mathrm{d}s\Big| \to 0,$$

and since

$$\mathfrak{Re}\int_0^\infty \mu(s)(1-\mathrm{e}^{-\mathrm{i}\lambda s})\,\mathrm{d}s = \int_0^\infty \mu(s)(1-\cos\lambda s)\,\mathrm{d}s > 0$$

it is easy to see that, in turn,

$$v_n \to 0 \quad \text{in } H_{\mathbb{C}}.$$

At this point (6.6) reduces to

$$\mathbb{A}\Big(u_n + \int_0^\infty \mu(s)\mathbb{A}^{\gamma-1}\eta_n(s)\mathrm{d}s\Big) \to 0 \quad \text{in } H_{\mathbb{C}}.$$

Taking the inner product of the above relation with u_n in $H_{\mathbb{C}}$, and appealing again to (6.3) and (6.8), we conclude that

$$u_n \to 0 \quad \text{in } H^1_{\mathbb{C}}.$$

Summarizing, we have proved that every single component of z_n goes to zero in its norm, contradicting (6.3). □

Remark 6.6. Following the argument devised in [14], one can see that the conclusion of Theorem 6.4 still holds if (6.2) is replaced by the weaker condition (4.5).

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