COEFFICIENT INEQUALITY FOR A FUNCTION WHOSE DERIVATIVE HAS A POSITIVE REAL PART OF ORDER α

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Abstract. The objective of this paper is to obtain sharp upper bound for the function f for the second Hankel determinant $|a_2a_4 - a_3^2|$, when it belongs to the class of functions whose derivative has a positive real part of order α $(0 \leqslant \alpha < 1)$, denoted by $RT(\alpha)$. Further, an upper bound for the inverse function of f for the nonlinear functional (also called the second Hankel functional), denoted by $|t_2t_4 - t_3^2|$, was determined when it belongs to the same class of functions, using Toeplitz determinants.

Keywords: analytic function; upper bound; second Hankel functional; positive real function; Toeplitz determinant

MSC 2010: 30C45, 30C50

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z\colon |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

In 1976, Noonan and Thomas [11] defined the qth Hankel determinant of f for $q \geqslant 1$ and $n \geqslant 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by several authors. For example, Noor [12] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. Ehrenborg [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [7]. One can easily observe that the Fekete-Szegő functional is $H_2(1)$. Fekete-Szegő then further generalized the estimate of $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegő functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order α $(0 < \alpha \leqslant 1)$ denoted by $\widetilde{ST}(\alpha)$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \tag{1.3}$$

Janteng, Halim and Darus [6] have considered the functional $|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by MacGregor [8]. In their work, they have shown that if $f \in RT$ then $|a_2a_4 - a_3^2| \leqslant 4/9$.

The same authors [5] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and showed that $|a_2a_4 - a_3^2| \leqslant 1$ and $|a_2a_4 - a_3^2| \leqslant 1/8$, respectively. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors ([1], [9], [10]).

Motivated by the results obtained by different authors in this direction mentioned above, in the present paper we obtain an upper bound for the nonlinear functional $|a_2a_4 - a_3^2|$ for the function f and its inverse belonging to the class $RT(\alpha)$ $(0 \leqslant \alpha < 1)$, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be a function whose derivative has a positive real part of order α $(0 \leqslant \alpha < 1)$, denoted by $f \in RT(\alpha)$, if and only if

$$\operatorname{Re}\{f'(z)\} > \alpha, \quad \forall\, z \in E.$$

Observe that for $\alpha = 0$, we obtain $RT(0) = RT$.

We first state some preliminary lemmas required for proving our results.

2. Preliminary results

Let $\mathscr{P}$ denote the class of functions p analytic in E for which $\operatorname{Re}\{p(z)\} > 0$,

$$p(z) = (1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots) = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \quad \forall\, z \in E. \tag{2.1}$$

Lemma 2.1 ([13], [14]). *If $p \in \mathscr{P}$, then $|c_k| \leqslant 2$ for each $k \geqslant 1$.*

Lemma 2.2 ([4]). *The power series for p given in (2.1) converges in the unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \ldots & c_n \\ c_{-1} & 2 & c_1 & \ldots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \ldots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum\limits_{k=1}^{m} \varrho_k p_0(\exp(\mathrm{i}t_k)z)$, $\varrho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n \doteq 0$ for $n \geqslant m$.

This necessary and sufficient condition due to Carathéodory and Toeplitz, can be found in [4]. We may assume without restriction that $c_1 > 0$. Using Lemma 2.2 for $n = 2$ and $n = 3$, respectively, we obtain

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2\operatorname{Re}\{c_1^2 c_2\} - 2|c_2|^2 - 4c_1^2] \geqslant 0, \tag{2.2}$$

$$2c_2 \equiv \{c_1^2 + x(4 - c_1^2)\} \quad \text{for some } x,\ |x| \leqslant 1;$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}, \quad D_3 \geqslant 0,$$

$$|(4c_3 - 4c_1 c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leqslant 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 \equiv \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\} \tag{2.4}$$

for some real value of z with $|z| \leqslant 1$.

3. Main results

Theorem 3.1. *If* $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in RT(\alpha)$ *for* $0 \leqslant \alpha < 1$ *then*

$$|a_2 a_4 - a_3^2| \leqslant \frac{4}{9}(1-\alpha)^2$$

and the inequality is sharp.

Proof. Since $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in RT(\alpha)$, by virtue of Definition 1.1 there exists an analytic function $p \in \mathscr{P}$ in the unit disc E with $p(0) = 1$ and $[\operatorname{Re} p(z)] > 0$ such that

$$\left\{\frac{f'(z) - \alpha}{1-\alpha}\right\} = p(z) \Rightarrow \{f'(z) - \alpha\} = (1-\alpha)p(z). \tag{3.1}$$

Replacing $f'(z)$ and $p(z)$ by their equivalent series expressions in (3.1), we have

$$\left[\left(1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right) - \alpha\right] = (1-\alpha)\left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}.$$

Upon simplification, we obtain

$$[2a_2 + 3a_3 z + 4a_4 z^2 + \ldots] = (1-\alpha)[c_1 + c_2 z + c_3 z^2 + \ldots]. \tag{3.2}$$

Equating the coefficients of the like powers of z^0, z and z^2, respectively, on both sides of (3.2) and simplifying, we get

$$\left\{a_2 = \frac{1-\alpha}{2}c_1;\ a_3 = \frac{1-\alpha}{3}c_2;\ a_4 = \frac{1-\alpha}{4}c_3\right\}. \tag{3.3}$$

Substituting the values of a_2, a_3 and a_4 from (3.3) in the second Hankel functional $|a_2 a_4 - a_3^2|$ for the function $f \in RT(\alpha)$, upon simplification we obtain

$$|a_2 a_4 - a_3^2| = \frac{(1-\alpha)^2}{72} \times |9c_1 c_3 - 8c_2^2|. \tag{3.4}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4), respectively, from Lemma 2.2 in the right hand side of (3.4), we have

$$\begin{aligned} |9c_1c_3 - 8c_2^2| = \Big| 9c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 \\ + 2(4-c_1^2)(1-|x|^2)z\} - 8 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2\Big|. \end{aligned} \tag{3.5}$$

Using the facts $|z| < 1$ and $|xa + yb| \leqslant |x||a| + |y||b|$, where x, y, a and b are real numbers, in the expression (3.5), after simplifying we get

$$
\begin{aligned}
4|9c_1c_3 - 8c_2^2| \leqslant |c_1^4 &+ 18c_1(4-c_1^2) + 2c_1^2(4-c_1^2)|x| \\
&- (c_1+2)(c_1+16)(4-c_1^2)|x|^2|.
\end{aligned} \tag{3.6}
$$

Since $c_1 \in [0,2]$, using the result $(c_1 + a)(c_1 + b) \geqslant (c_1 - a)(c_1 - b)$, where $a, b \geqslant 0$ on the right hand side of (3.6), upon simplification we obtain

$$
\begin{aligned}
4|9c_1c_3 - 8c_2^2| \leqslant |c_1^4 &+ 18c_1(4-c_1^2) + 2c_1^2(4-c_1^2)|x| \\
&- (c_1-2)(c_1-16)(4-c_1^2)|x|^2|.
\end{aligned} \tag{3.7}
$$

Choosing $c_1 = c \in [0,2]$, applying the triangle inequality and replacing x by μ on the right hand side of the above inequality, we have

$$
\begin{aligned}
4|9c_1c_3 - 8c_2^2| &\leqslant [c^4 + \{18c + 2c^2\mu + (c-2)(c-16)\mu^2\} \times (4-c^2)] \\
&= F(c,\mu), \quad \text{for } 0 \leqslant \mu = |x| \leqslant 1.
\end{aligned} \tag{3.8}
$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$
\frac{\partial F}{\partial \mu} = 2[c^2 + (c-2)(c-16)\mu] \times (4-c^2). \tag{3.9}
$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.9) we observe that $\partial F/\partial \mu > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have the maximum value in the interior of the closed region $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2]$ we have

$$
\max_{0 \leqslant \mu \leqslant 1} F(c,\mu) = F(c,1) = G(c). \tag{3.10}
$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification we obtain

$$
G(c) = (-2c^4 - 20c^2 + 128), \tag{3.11}
$$

$$
G'(c) = (-8c^3 - 40c). \tag{3.12}
$$

From (3.12), we observe that $G'(c) \leqslant 0$ for every $c \in [0,2]$. Therefore, $G(c)$ is a decreasing function of c in the interval $c \in [0,2]$, whose maximum value occurs at $c = 0$. From (3.11), at $c = 0$ we obtain the G-maximum as

$$
G_{\max} = G(0) = 128. \tag{3.13}
$$

From the relations (3.8) and (3.13), after simplifying, we get

$$|9c_1c_3 - 8c_2^2| \leqslant 32. \tag{3.14}$$

From the expressions (3.4) and (3.14), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \leqslant \frac{4}{9}(1-\alpha)^2. \tag{3.15}$$

By setting $c_1 = c = 0$ and selecting $x = -1$ in the expressions (2.2) and (2.4), we find that $c_2 = -2$ and $c_3 = 0$, respectively. Using these values in (3.14), we observe that equality is attained, which shows that our result is sharp. This completes the proof of our Theorem 3.1. □

R e m a r k 3.2. For the choice of $\alpha = 0$, we get $RT(0) = RT$, for which, from (3.15), we obtain $|a_2a_4 - a_3^2| \leqslant 4/9$. This inequality is sharp and the result coincides with that of Janteng, Halim and Darus [6].

Theorem 3.2. *If* $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in RT(\alpha)$ $(0 \leqslant \alpha < 1/4)$ *and* $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} t_n w^n$ *near* $w = 0$ *is the inverse function of* f, *then*

$$|t_2t_4 - t_3^2| \leqslant \Big[\frac{(1-\alpha)^2(432\alpha^2 - 312\alpha - 137)}{144(9\alpha^2 - 6\alpha - 2)}\Big].$$

P r o o f. Since $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in RT(\alpha)$, from the definition of the inverse function of f we have

$$\begin{aligned} w = f\{f^{-1}(w)\} \Leftrightarrow \{&(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 \\ &+ (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 + \ldots\} = 0. \end{aligned} \tag{3.16}$$

Equating the coefficients of the like powers of w^2, w^3 and w^4 on both sides of (3.16), respectively, after simplifying we get

$$\{t_2 = -a_2;\ t_3 = \{-a_3 + 2a_2^2\};\ t_4 = \{-a_4 + 5a_2a_3 - 5a_2^3\}\}. \tag{3.17}$$

Using the values of a_2, a_3 and a_4 in (3.3) along with (3.17), upon simplification we obtain

$$\begin{aligned} \Big\{t_2 &= -\frac{(1-\alpha)}{2}c_1;\ t_3 = -\frac{(1-\alpha)}{6}\{3(1-\alpha)c_1^2 - 2c_2\}; \\ t_4 &= -\frac{(1-\alpha)}{24}\{-6c_3 + 20(1-\alpha)c_1c_2 - 15(1-\alpha)^2c_1^3\}\Big\}. \end{aligned} \tag{3.18}$$

Substituting the values of t_2, t_3 and t_4 from (3.18) in the second Hankel functional $|t_2t_4 - t_3^2|$ for the inverse function of $f \in RT(\alpha)$, after simplifying we get

$$|t_2t_4 - t_3^2| = \frac{(1-\alpha)^2}{144} \times |18c_1c_3 - 12(1-\alpha)c_1^2c_2 - 16c_2^2 + 9(1-\alpha)^2c_1^4|.$$

The above expression is equivalent to

$$|t_2t_4 - t_3^2| = \frac{(1-\alpha)^2}{144} \times |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \tag{3.19}$$

where

$$\{d_1 = 18;\ d_2 = -12(1-\alpha);\ d_3 = -16;\ d_4 = 9(1-\alpha)^2\}. \tag{3.20}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4), respectively, from Lemma 2.2 in the right hand side of (3.19), applying the same procedure as described in Theorem 3.1, we obtain

$$\begin{aligned} |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leqslant |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + [2d_1c_1 \\ &+ 2(d_1 + d_2 + d_3)c_1^2|x| - \{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\}|x|^2] \times (4 - c_1^2)|. \end{aligned} \tag{3.21}$$

Using the values of d_1, d_2, d_3 and d_4 from the relation (3.20), upon simplification we obtain

$$\begin{gathered} \{(d_1 + 2d_2 + d_3 + 4d_4) = (18\alpha^2 - 24\alpha + 7);\quad d_1 = 18; \\ (d_1 + d_2 + d_3) = (12\alpha - 10)\}, \end{gathered} \tag{3.22}$$

$$\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\} = \{(c_1 - 2)(c_1 - 16)\}. \tag{3.23}$$

Substituting the calculated values from (3.22) and (3.23) in the right hand side of (3.21), we have

$$\begin{aligned} 2|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leqslant |(18\alpha^2 - 24\alpha + 7)c_1^4 + \{18c_1 + (12\alpha - 10)c_1^2|x| \\ - (c_1 - 2)(c_1 - 16)|x|^2\} \times (4 - c_1^2)|. \end{aligned}$$

Choosing $c_1 = c \in [0, 2]$, applying the triangle inequality and replacing $|x|$ by μ on the right hand side of the above inequality, we get

$$\begin{aligned} &2|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\ &\leqslant [(18\alpha^2 - 24\alpha + 7)c^4 + \{18c + (10 - 12\alpha)c^2\mu + (c-2)(c-16)\mu^2\}(4 - c^2)] \\ &= F(c, \mu), \quad \text{for } 0 \leqslant \mu = |x| \leqslant 1 \end{aligned} \tag{3.24}$$

where

$$F(c,\mu) = [(18\alpha^2 - 24\alpha + 7)c^4 + \{18c + (10 - 12\alpha)c^2\mu + (c-2)(c-16)\mu^2\} \times (4 - c^2)]. \tag{3.25}$$

Applying the same procedure as described in Theorem 3.1, we get

$$\frac{\partial F}{\partial \mu} = [(10 - 12\alpha)c^2 + 2\{(c-2)(c-16)\}\mu] \times (4 - c^2). \tag{3.26}$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and $0 \leqslant \alpha < 1/4$, from (3.26) we observe that $\partial F/\partial\mu > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have the maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. Further, for a fixed $c \in [0,2]$, we have

$$\max_{0 \leqslant \mu \leqslant 1} F(c,\mu) = F(c,1) = G(c). \tag{3.27}$$

Therefore, from (3.25) and (3.27), upon simplification, we obtain

$$G(c) = \{2(9\alpha^2 - 6\alpha - 2)c^4 + 12(1 - 4\alpha)c^2 + 128\}, \tag{3.28}$$

$$G'(c) = \{8(9\alpha^2 - 6\alpha - 2)c^3 + 24(1 - 4\alpha)c\}, \tag{3.29}$$

$$G''(c) = \{24(9\alpha^2 - 6\alpha - 2)c^2 + 24(1 - 4\alpha)\}. \tag{3.30}$$

For the extreme values of $G(c)$, consider $G'(c) = 0$. From (3.29), we get

$$8c\{(9\alpha^2 - 6\alpha - 2)c^2 + 3(1 - 4\alpha)\} = 0. \tag{3.31}$$

We now discuss the following cases.

Case 1. If $c = 0$, then, from (3.30), we obtain

$$G''(c) = 24(1 - 4\alpha) > 0 \quad \text{for } 0 \leqslant \alpha < \frac{1}{4}.$$

From the second derivative test, $G(c)$ has the minimum value at $c = 0$.

Case 2. If $c \neq 0$, then, from (3.31), we get

$$c^2 = \left\{-\frac{3(1 - 4\alpha)}{(9\alpha^2 - 6\alpha - 2)}\right\} \in [0,2] \quad \text{for } 0 \leqslant \alpha < \frac{1}{4}. \tag{3.32}$$

Using the value of c^2 given in (3.32) in (3.31), upon simplification we obtain

$$G''(c) = -48(1 - 4\alpha) < 0 \quad \text{for } 0 \leqslant \alpha < \frac{1}{4}.$$

By the second derivative test, $G(c)$ has the maximum value at c, where c^2 given in (3.32). Using the value of c^2 in (3.28), after simplifying we get

$$\max_{0\leqslant c\leqslant 2} G(c)=\Big[\frac{2(432\alpha^2-312\alpha-137)}{(9\alpha^2-6\alpha-2)}\Big]. \tag{3.33}$$

Considering the maximum value of $G(c)$ only at c^2, from (3.24) and (3.33), upon simplification we obtain

$$|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\leqslant\Big[\frac{(432\alpha^2-312\alpha-137)}{(9\alpha^2-6\alpha-2)}\Big]. \tag{3.34}$$

From (3.19) and (3.34) we get

$$|t_2t_4-t_3^2|\leqslant\Big[\frac{(1-\alpha)^2(432\alpha^2-312\alpha-137)}{144(9\alpha^2-6\alpha-2)}\Big]. \tag{3.35}$$

This completes the proof of our theorem. □

Remark 3.4. Choosing $\alpha=0$, we have $RT(0)=RT$, for which, from (3.35), we get $|t_2t_4-t_3^2|\leqslant 137/288$.

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References

[1] *A. Abubaker, M. Darus*: Hankel determinant for a class of analytic functions involving a generalized linear differential operator. Int. J. Pure Appl. Math. *69* (2011), 429–435. zbl MR

[2] *R. M. Ali*: Coefficients of the inverse of strongly starlike functions. Bull. Malays. Math. Sci. Soc. (2) *26* (2003), 63–71. zbl MR

[3] *R. Ehrenborg*: The Hankel determinant of exponential polynomials. Am. Math. Mon. *107* (2000), 557–560. zbl MR

[4] *U. Grenander, G. Szegő*: Toeplitz Forms and Their Applications. Chelsea Publishing Co., New York, 1984. zbl MR

[5] *A. Janteng, S. A. Halim, M. Darus*: Hankel determinant for starlike and convex functions. Int. J. Math. Anal., Ruse *1* (2007), 619–625. zbl MR

[6] *A. Janteng, S. A. Halim, M. Darus*: Coefficient inequality for a function whose derivative has a positive real part. J. Inequal. Pure Appl. Math. (electronic only) *7* (2006), Article 50, 5 pages. zbl MR

[7] *J. W. Layman*: The Hankel transform and some of its properties. J. Integer Seq. (electronic only) *4* (2001), Article 01.1.5, 11 pages. zbl MR

[8] *T. H. MacGregor*: Functions whose derivative has a positive real part. Trans. Am. Math. Soc. *104* (1962), 532–537. zbl MR

[9] *A. K. Mishra, P. Gochhayat*: Second Hankel determinant for a class of analytic functions defined by fractional derivative. Int. J. Math. Math. Sci. *2008* (2008), Article ID 153280, 10 pages. zbl MR

[10] *G. Murugusundaramoorthy, N. Magesh*: Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant. Bull. Math. Anal. Appl. *1* (2009), 85–89. MR

[11] *J. W. Noonan, D. K. Thomas*: On the second Hankel determinant of areally mean p-valent functions. Trans. Am. Math. Soc. *223* (1976), 337–346. zbl MR

[12] *K. I. Noor*: Hankel determinant problem for the class of functions with bounded boundary rotation. Rev. Roum. Math. Pures Appl. *28* (1983), 731–739. zbl MR

[13] *C. Pommerenke*: Univalent Functions. With a Chapter on Quadratic Differentials by Gerd Jensen. Studia Mathematica/Mathematische Lehrbücher. Band 25, Vandenhoeck & Ruprecht, Göttingen, 1975. zbl MR

[14] *B. Simon*: Orthogonal Polynomials on the Unit Circle. Part 1: Classical Theory. American Mathematical Society Colloquium Publications 54, AMS, Providence, 2005. zbl MR

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