MAXIMAL UPPER ASYMPTOTIC DENSITY OF SETS OF INTEGERS WITH MISSING DIFFERENCES FROM A GIVEN SET

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Abstract. Let M be a given nonempty set of positive integers and S any set of nonnegative integers. Let $\overline{\delta}(S)$ denote the upper asymptotic density of S. We consider the problem of finding

$$\mu(M) := \sup_S \overline{\delta}(S),$$

where the supremum is taken over all sets S satisfying that for each $a, b \in S$, $a - b \notin M$. In this paper we discuss the values and bounds of $\mu(M)$ where $M = \{a, b, a + nb\}$ for all even integers and for all sufficiently large odd integers n with $a < b$ and $\gcd(a, b) = 1$.

Keywords: upper asymptotic density; maximal density

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1. Introduction

For any set S of nonnegative integers, we denote by $S(n)$ the number of elements $x \in S$ such that $x \leqslant n$. As usual, we define the upper and lower asymptotic densities of S (denoted by $\overline{\delta}(S)$ and $\underline{\delta}(S)$, respectively) by $\overline{\delta}(S) = \limsup\limits_{n\to\infty} S(n)/n$ and $\underline{\delta}(S) = \liminf\limits_{n\to\infty} S(n)/n$. If $\overline{\delta}(S) = \underline{\delta}(S)$, we denote the common value by $\delta(S)$, and say that S has density $\delta(S)$. Now suppose that M is a given nonempty set of positive integers. Motzkin [7] asks to determine the maximal upper asymptotic density defined by

$$\mu(M) := \sup_S \overline{\delta}(S),$$

where the supremum is taken over all sets S satisfying that for each $a, b \in S$, $a - b \notin M$. Such sets S are called *M-sets* in the literature.

Initial work on this problem is due to Cantor and Gordon [1], in which they show the existence of $\mu(M)$ for each M and also determine $\mu(M)$ when M has one or two elements. They prove that if $|M| = 1$, then $\mu(M) = 1/2$ and if $M = \{a, b\}$ with $\gcd(a, b) = 1$, then $\mu(M) = \lfloor \frac{1}{2}(a+b) \rfloor/(a+b)$. By a result of Cantor and Gordon it is sufficient to consider the problem only for those sets M whose elements are relatively prime. Furthermore, they give the following lower bound for $\mu(M)$.

Lemma 1.1. *Let $M = \{m_1, m_2, m_3, \ldots\}$ and let k, m be positive integers such that* $\gcd(k, m) = 1$. *Then*

$$\mu(M) \geqslant \sup_{(k,m)=1} \frac{1}{m} \min_i |km_i|_m,$$

where $|x|_m$ denotes the absolute value of the absolutely least remainder of x mod m.

The following remark by Haralambis [4] gives three equivalent definitions of the right hand side expression of the inequality in Lemma 1.1. Throughout this paper we use the third definition, i.e., $d_3(M)$.

R e m a r k 1.1. Let $M = \{m_1, m_2, \ldots, m_n\}$, and

$$d_1(M) = \sup_{x \in (0,1)} \min_i \|xm_i\|,$$

$$d_2(M) = \sup_{(k,m)=1} \frac{1}{m} \min_i |km_i|_m,$$

$$d_3(M) = \max_{\substack{m = m_j + m_l \\ 1 \leqslant k \leqslant m/2}} \frac{1}{m} \min |km_i|_m,$$

where for $x \in \mathbb{R}$, $\|x\|$ denotes the distance of x from the nearest integer and m_j, m_l represent distinct elements of M. Then $d_1(M) = d_2(M) = d_3(M)$, and we denote this common value by $d(M)$.

Thus we have $\mu(M) \geqslant d(M)$. At this stage we mention the very first conjecture on this problem by Haralambis [4].

Conjecture. *If $|M| = 3$, then $\mu(M) = d(M)$.*

The above conjecture holds true if $|M| \leqslant 2$ and is false if $|M| = 4$. The proofs and counter examples may be found in [4].
The following lemma in [4] gives an upper bound for $\mu(M)$.

Lemma 1.2. *Let M be a given set of positive integers, α a real number in the interval* $[0, 1]$, *and suppose that for any M-set S with $0 \in S$ there exists a positive integer k (possibly dependent on S) such that $S(k) \leqslant (k+1)\alpha$. Then $\mu(M) \leqslant \alpha$.*

Haralambis [4] gives some general estimates and expressions for $\mu(M)$ for most members of the families $\{1, a, b\}$ and $\{1, 2, a, b\}$. Gupta and Tripathi [3] give the value of $\mu(M)$ when M is finite and the elements of M are in arithmetic progression. Liu and Zhu [5] compute the values of $\mu(M)$ for $M = \{a, 2a, \ldots, (m-1)a, b\}$, $M = \{a, b, a+b\}$, and give bounds of $\mu(M)$ for $M = \{a, b, b-a, b+a\}$ using graph theoretic techniques. They further compute $\mu(M)$ for $M = [1, a] \cup [b, m+1]$, where $a < b$ in [6]. The present author in joint works with Tripathi ([8], [9], [10]) discusses the problem for the family $M = \{a, b, c\}$ with $a < b$, where $c = nb$ or na or $n(a+b)$, and for those families M which are related to finite arithmetic progressions. In the present paper we discuss the problem of finding $\mu(M)$ for $M = \{a, b, a+nb\}$ for all even integers n and for all sufficiently large odd integers n with $a < b$ and $\gcd(a, b) = 1$. In Sections 2, 3 and 4, we give bounds or the exact values of $\mu(M)$.

2. Numbers a and b are of opposite parity and $n \geqslant b - a + 2$ is an odd integer

In this section we study the family $M = \{a, b, a+nb\}$, where $a < b$, $\gcd(a, b) = 1$ and n is a sufficiently large odd integer. Mainly, $d(M)$ is calculated, which is a lower bound of $\mu(M)$ and as we are working in the case where $|M| = 3$, $d(M)$ is conjecturally equal to $\mu(M)$.

Lemma 2.1. *For each $r, s \geqslant 0$, set*

$$A_r = b - a + \{2r(a+b) + 2t \colon\ 1 \leqslant t \leqslant a\},$$
$$B_s = b - a + \{2(s+1)a + 2sb + 2t \colon\ 1 \leqslant t \leqslant b\}.$$

The collection $\{A_0, A_1, \ldots, B_0, B_1, \ldots\}$ partitions $2\mathbb{N} - 1 \setminus \{1, 3, \ldots, b-a\}$.

Proof. Clearly, $|A_r| = a$ and $|B_s| = b$ for each $r, s \geqslant 0$. Also, we have the recurrences $A_{r+1} = A_r + 2(a+b)$ and $B_{s+1} = B_s + 2(a+b)$. Notice that $\{A_0, B_0\}$ partitions the set $[b-a+2, b-a+2(a+b)] \cap (2\mathbb{N} - 1 \setminus \{1, 3, \ldots, b-a\})$. Thus we have the lemma. $\square$

Theorem 2.1. *Let $M = \{a, b, a+nb\}$, where $a < b$, $\gcd(a, b) = 1$, a and b are of opposite parity and $n \geqslant b - a + 2$ is an odd integer. For each $r, s \geqslant 0$, let A_r and B_s be as given in Lemma 2.1. Then*

$$d(M) = \begin{cases} \dfrac{m - ((2r+1)b + 1)}{2m} & \text{if } n \in A_r, \text{ where } m = a + (n+1)b; \\[2ex] \dfrac{m - ((2s+1)b + 2t)}{2m} & \text{if } n \in B_s, \text{ where } m = 2a + nb. \end{cases}$$

P r o o f. *Case I* ($n \in A_r$). To calculate $d(M)$ we use $d_3(M)$. According to the definition of $d_3(M)$, the possible values of m may be $a+(n+1)b$, $2a+nb$, and $a+b$.

$\triangleright$ (1) ($m = a+(n+1)b$). Since $\gcd(b,m)=1$, we can choose an integer x such that

$$bx \equiv \frac{m-((2r+1)b+1)}{2} \pmod m.$$

We have

$$\begin{aligned} ax \equiv -(n+1)bx &\equiv -(n+1)\frac{m-((2r+1)b+1)}{2} \\ &\equiv \frac{(n+1)((2r+1)b+1)}{2} \pmod m. \end{aligned}$$

Since $(n+1)((2r+1)b+1) = (2r+1)(n+1)b+n+1 = (2r+1)m+(2r+1)b+1-2(a-t)$, therefore,

$$ax \equiv \frac{m+(2r+1)b+1-2(a-t)}{2} \equiv -\frac{m-((2r+1)b+1)+2(a-t)}{2} \pmod m.$$

We also have that $(a+nb)x \equiv -bx \pmod m$. Thus

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m-((2r+1)b+1)}{2}.$$

We now show that for all y such that $1 \leqslant y \leqslant m/2$ and $y \neq x$,

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-((2r+1)b+1)}{2}.$$

Let $l := (2r+1)b+1$, and $1 \leqslant y \leqslant m/2$. Suppose for some integer i,

$$by \equiv \frac{m}{2} - \frac{l}{2} + i \pmod m.$$

This gives

$$ay \equiv \frac{m}{2} + \frac{l}{2} - (a-t) - (n+1)i \pmod m.$$

If $m/2 - l/2 + i$ modulo m is in $[m/2-l/2, m/2+l/2]$, then $0 \leqslant i \leqslant l$. Since we have that $(a+nb)y \equiv -by \pmod m$, the inequality will be valid if we show that $m/2+l/2-(a-t)-(n+1)i$ modulo m is in $[-(m/2-l/2), m/2-l/2]$ for each $1 \leqslant i \leqslant l$. First, let $i = l$. In this case, the congruences become

$$by \equiv \frac{m}{2} - \frac{l}{2} + l \equiv -\Big(\frac{m}{2} - \frac{l}{2}\Big) \pmod m,$$

$$(a+nb)y \equiv -by \equiv \frac{m}{2} - \frac{l}{2} \pmod m,$$

and

$$ay \equiv \frac{m}{2} + \frac{l}{2} - (a-t) - (n+1)l \pmod{m}.$$

Since $(n+1)l = (2r+1)m + l - 2(a-t)$,

$$ay \equiv \frac{m}{2} - \frac{l}{2} + (a-t) \pmod{m}.$$

Therefore, we have the inequality in this case. Next, let $1 \leqslant i \leqslant l-1$. Observe that

$$\{1, 2, \ldots, l-1\} \subseteq \bigcup_{p=0}^{2r} I_p,$$

where $I_p = [pb + ((p-1)a + t + l)/(n+1), (p+1)b + (pa+t)/(n+1)]$. Indeed, since the largest integer in I_p is $(p+1)b$, we only need to verify that $(p+1)b+1$ is in I_{p+1}. Notice that $(pa+t+l)/(n+1) \leqslant 1$ if and only if $pa \leqslant n+1-t-l = (2r-1)a + t \leqslant 2ra$, i.e., $p \leqslant 2r$, which is true. Hence $(pa+t+l)/(n+1) \leqslant 1$. This implies $(p+1)b + (pa+t+l)/(n+1) \leqslant (p+1)b+1$, and hence $(p+1)b+1$ is in I_{p+1} and it is the smallest integer of the interval.

As $1 \leqslant i \leqslant l-1$, therefore, for some $0 \leqslant p \leqslant 2r$, $i \in I_p$, i.e.,

$$pb + \frac{(p-1)a+t+l}{n+1} \leqslant i \leqslant (p+1)b + \frac{pa+t}{n+1},$$

therefore

$$\frac{pm + l - (a-t)}{n+1} \leqslant i \leqslant \frac{(p+1)m - (a-t)}{n+1}.$$

This gives

$$\begin{aligned} \frac{m}{2} + \frac{l}{2} - (a-t) - (n+1)\frac{(p+1)m-(a-t)}{n+1} &\leqslant \frac{m}{2} + \frac{l}{2} - (a-t) - (n+1)i \\ &\leqslant \frac{m}{2} + \frac{l}{2} - (a-t) - (n+1)\frac{pm+l-(a-t)}{n+1}, \end{aligned}$$

so

$$-(p+1)m + \frac{m}{2} + \frac{l}{2} \leqslant \frac{m}{2} + \frac{l}{2} - (a-t) - (n+1)i \leqslant -pm + \frac{m}{2} - \frac{l}{2},$$

thus

$$-pm - \left(\frac{m}{2} - \frac{l}{2}\right) \leqslant \frac{m}{2} + \frac{l}{2} - (a-t) - (n+1)i \leqslant -pm + \frac{m}{2} - \frac{l}{2}.$$

Therefore, $m/2 + l/2 - (a-t) - (n+1)i$ modulo m is in $[-(m/2 - l/2), m/2 - l/2]$ for each $1 \leqslant i \leqslant l-1$. Hence, we have the desired inequality. Thus we see that

$$\max_{1 \leqslant y \leqslant m/2} (\min\{|ay|_m, |by|_m, |(a+nb)y|_m\}) = \frac{m - ((2r+1)b+1)}{2}.$$

$\triangleright$ (2) ($m = 2a + nb$). Choose an integer x such that

$$bx \equiv \frac{m - ((2r+1)b + 2)}{2} \pmod{m}.$$

Such an x exists. For, let $d = \gcd(b, m)$, and $d \neq 1$. Then $d \mid 2a$. If b is odd, then as $d \mid b$, $d \geqslant 3$ hence $d \mid a$, which shows that $\gcd(a, b) \neq 1$, which is false. Hence, $d = 1$ and hence the congruence in this case is true. Now, let b be even. Since $d \mid 2a$ and a is odd with $\gcd(a, b) = 1$, we have $d = 2$. Notice that $2 \mid (m - ((2r+1)b+2))/2$, and hence the congruence is again true. We have

$$2ax \equiv -nbx \equiv -n\frac{m - ((2r+1)b+2)}{2} \equiv -\frac{m - (2r+1)nb - 2n}{2} \pmod{m},$$

which implies

$$2ax \equiv -\frac{m - (2r+1)m + 2(2r+1)a - 2n}{2} \equiv n - (2r+1)a \pmod{m}.$$

Now $n - (2r+1)a = b - a + 2r(a+b) + 2t - (2r+1)a = (2r+1)b - 2(a-t) = (2r+1)b + 2 - 2(a - t + 1)$. This gives

$$2ax \equiv (2r+1)b + 2 - 2(a-t+1) \equiv -(m - ((2r+1)b+2) + 2(a-t+1)) \pmod{m},$$

therefore,

$$ax \equiv -\frac{m - ((2r+1)b+2) + 2(a-t+1)}{2} \pmod{m}.$$

Since $(a + nb)x \equiv -ax \pmod{m}$, we have

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m - ((2r+1)b+2)}{2}.$$

Also, as in (1), it can be shown that for all y such that $1 \leqslant y \leqslant m/2$ and $y \neq x$,

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m - ((2r+1)b+2)}{2}.$$

Thus we see that

$$\max_{1 \leqslant y \leqslant m/2}(\min\{|ay|_m, |by|_m, |(a+nb)y|_m\}) = \frac{m - ((2r+1)b+2)}{2}.$$

$\triangleright$ (3) ($m = a + b$). Choose an integer x such that

$$ax \equiv -bx \equiv \frac{a+b-1}{2} \pmod{m}.$$

We have

$$(a+nb)x \equiv (n-1)bx \equiv \frac{n-1}{2} \pmod{m}.$$

Thus we see that if $n=(2r+1)(a+b)$ (which is obtained by taking $t=a$ in A_r) then

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{a+b-1}{2}.$$

Moreover, it can be shown that if $n=(2r+1)(a+b)$ then

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-1}{2}$$

for all y; $1 \leqslant y \leqslant m/2$. Thus we see that

$$\max_{1\leqslant y\leqslant m/2}(\min\{|ay|_m, |by|_m, |(a+nb)y|_m\}) = \frac{a+b-1}{2}.$$

On the other hand, if $n \neq (2r+1)(a+b)$ then it is obvious that

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-3}{2}$$

for each y. Thus we see that

$$\max_{1\leqslant y\leqslant m/2}(\min\{|ay|_m, |by|_m, |(a+nb)y|_m\}) = \frac{a+b-3}{2}.$$

To calculate $d(M)$ we apply the definition $d_3(M)$. Let us denote m values in (1), (2), and (3) by m_1, m_2, and m_3, respectively, i.e., $m_1 = a+(n+1)b$, $m_2 = 2a+nb$, and $m_3 = a+b$. Then

$$\begin{aligned} d(M) &= \max\Big(\frac{m_1-((2r+1)b+1)}{2m_1}, \frac{m_2-((2r+1)b+2)}{2m_2}, \frac{a+b-\varepsilon}{2m_3}\Big) \\ &= \frac{m_1-((2r+1)b+1)}{2m_1}. \end{aligned}$$

Here $\varepsilon=1$ if $n=(2r+1)(a+b)$ and $\varepsilon=3$ if $n\neq(2r+1)(a+b)$.

Case II ($n\in B_s$). To calculate $d(M)$ we use $d_3(M)$ and hence as in the previous case we consider the following values of m.

▷ (1) ($m=a+(n+1)b$). Choose x such that

$$bx \equiv \frac{m-((2s+1)b+1)}{2} \pmod{m}.$$

We have

$$ax \equiv -(n+1)bx \equiv \; -(n+1)\frac{m-((2s+1)b+1)}{2}$$
$$\equiv \frac{(n+1)((2s+1)b+1)}{2} \pmod{m}.$$

Since $(n+1)((2s+1)b+1) = (2s+1)m-(2s+1)a+n+1 = (2s+1)m+(2s+1)b+1+2t$,

$$ax \equiv \frac{m+(2s+1)b+1+2t}{2} \equiv -\frac{m-((2s+1)b+1+2t)}{2} \pmod{m}.$$

We also have that $(a+nb)x \equiv -bx \pmod{m}$. Thus

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m-((2s+1)b+1+2t)}{2}.$$

Moreover, it can also be shown as in the Case I that

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-((2s+1)b+1+2t)}{2}$$

for each y; $1 \leqslant y \leqslant m/2$. Thus we see that

$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{m-((2s+1)b+1+2t)}{2}.$$

$\triangleright$ (2) $(m = 2a+nb)$. Choose an integer x such that

$$bx \equiv \frac{m-((2s+1)b+2)}{2} \pmod{m}.$$

Such an x exists. For, arguments are similar to (2) of Case I. We have

$$2ax \equiv -nbx \equiv -n\frac{m-((2s+1)b+2)}{2} \equiv -\frac{m-(2s+1)nb-2n}{2} \pmod{m}.$$

This implies

$$2ax \equiv -\frac{m-(2s+1)m+2(2s+1)a-2n}{2} \equiv n-(2s+1)a \pmod{m}.$$

Since $n-(2s+1)a = b-a+2(s+1)a+2sb+2t-(2s+1)a = (2s+1)b+2t$,

$$2ax \equiv (2s+1)b+2t \equiv -(m-((2s+1)b+2t)) \pmod{m}.$$

Therefore,

$$ax \equiv -\frac{m-((2s+1)b+2t)}{2} \pmod{m}.$$

Since $(a+nb)x \equiv -ax \pmod{m}$, we have

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m-((2s+1)b+2t)}{2}.$$

Also, it can be shown that for all y such that $1 \leqslant y \leqslant m/2$ and $y \neq x$,

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-((2s+1)b+2t)}{2}.$$

Thus we see that

$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{m-((2s+1)b+2t)}{2}.$$

▷ (3) $(m = a+b)$. Choose an integer x such that

$$ax \equiv -bx \equiv \frac{a+b-1}{2} \pmod{m}.$$

We have

$$(a+nb)x \equiv (n-1)bx \equiv \frac{n-1}{2} \pmod{m}.$$

Thus we see that if $n = (2s+1)(a+b)+2$ (which is obtained by taking $t=1$ in B_s) then

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{a+b-1}{2}.$$

Moreover, it can be shown that if $n = (2s+1)(a+b)+2$ then

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-1}{2}$$

for all y; $1 \leqslant y \leqslant m/2$. Thus we see that

$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{a+b-1}{2}.$$

On the other hand, if $n \neq (2s+1)(a+b)+2$ then it is obvious that

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-3}{2}$$

for each y. Thus we see that

$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{a+b-3}{2}.$$

To calculate $d(M)$ we again apply the definition $d_3(M)$. Let us denote m values in (1), (2), and (3) by m_1, m_2, and m_3, respectively, i.e., $m_1 = a+(n+1)b$, $m_2 = 2a+nb$, and $m_3 = a+b$. Then

$$\begin{aligned} d(M) &= \max\Big(\frac{m_1 - ((2s+1)b+1+2t)}{2m_1}, \frac{m_2 - ((2s+1)b+2t)}{2m_2}, \frac{a+b-\varepsilon}{2m_3}\Big) \\ &= \frac{m_2 - ((2s+1)b+2t)}{2m_2}. \end{aligned}$$

Here $\varepsilon = 1$ if $n = (2s+1)(a+b)+2$ and $\varepsilon = 3$ if $n \neq (2s+1)(a+b)+2$. This completes the proof of the theorem. □

Corollary 2.1. *Let $M = \{a, b, a+nb\}$, where $a < b$, $\gcd(a,b) = 1$, a and b are of opposite parity and $n \in \{(2r+1)(a+b), (2s+1)(a+b)+2\}$. Then $\mu(M) = \frac{1}{2}(a+b-1)/(a+b)$.*

Proof. If $n \in \{(2r+1)(a+b), (2s+1)(a+b)+2\}$ then it follows from the theorem that $\mu(M) \geqslant d(M) = \frac{1}{2}(a+b-1)/(a+b)$. On the other hand, we always have $\mu(M) \leqslant \mu(\{a,b\}) = \lfloor \frac{1}{2}(a+b) \rfloor/(a+b)$. Thus we have the corollary. □

3. Numbers a and b are of opposite parity and n is an even integer

Theorem 3.1. *Let $M = \{a, b, a+nb\}$, where $a < b$, $\gcd(a,b) = 1$, a and b are of opposite parity and n is even. For each $r, s \geqslant 0$, set*

$$A'_r = \{2(ra+rb+t)\colon\ 1 \leqslant t \leqslant b\}, \quad \text{and} \quad B'_s = \{2(sa+(s+1)b+t)\colon\ 1 \leqslant t \leqslant a\}.$$

Then

$$d(M) = \begin{cases} \dfrac{m - 2(rb+t)}{2m} & \text{if } n \in A'_r, \text{ where } m = 2a+nb; \\ \dfrac{m - (2(s+1)b+1)}{2m} & \text{if } n \in B'_s, \text{ where } m = a+(n+1)b. \end{cases}$$

Proof. As in Lemma 2.1 it can be shown that the collection $\{A'_0, A'_1, \ldots, B'_0, B'_1, \ldots\}$ partitions the set $2\mathbb{N}$.

The method of proof of this theorem is similar to that of the previous theorem. Therefore, we omit the similar calculations here.

Case I ($n \in A'_r$). To calculate $d(M)$ we consider the following three values of m.

▷ (1) ($m = a+(n+1)b$). Since $\gcd(b,m) = 1$, we can choose an x such that

$$bx \equiv \frac{m - (2rb+1)}{2} \pmod{m}.$$

We have

$$ax \equiv -(n+1)bx \equiv -(n+1)\frac{m-(2rb+1)}{2}$$
$$\equiv -\frac{m-(n+1)(2rb+1)}{2} \pmod{m}.$$

Since $(n+1)(2rb+1) = 2rm + 2rb + 1 + 2t$,

$$ax \equiv -\frac{m-(2rb+1+2t)}{2} \pmod{m}.$$

We also have that $(a+nb)x \equiv -bx \pmod{m}$. Thus

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m-(2rb+1+2t)}{2}.$$

Moreover, for all y such that $1 \leqslant y \leqslant m/2$ and $y \neq x$,

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-(2rb+1+2t)}{2}.$$

Thus we see that

$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{m-(2rb+1+2t)}{2}.$$

▷ (2) $(m = 2a+nb)$. Choose an integer x such that

$$bx \equiv \frac{m-2(rb+1)}{2} \pmod{m}.$$

We have

$$2ax \equiv -nbx \equiv -n\frac{m-2(rb+1)}{2} \equiv n(rb+1) \pmod{m}.$$

Since $n(rb+1) = rm + 2rb + 2t$,

$$2ax \equiv 2rb + 2t \equiv -(m-2(rb+t)) \pmod{m},$$

therefore,

$$ax \equiv -\frac{m-2(rb+t)}{2} \pmod{m}.$$

We also have $(a+nb)x \equiv -ax \pmod{m}$. Thus

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m-2(rb+t)}{2}.$$

Also, it can be shown that for all y such that $1 \leqslant y \leqslant m/2$ and $y \neq x$,

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-2(rb+t)}{2}.$$

Thus we see that

$$\max_{1 \leqslant y \leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{m-2(rb+t)}{2}.$$

▷ (3) $(m = a+b)$. Choose an integer x such that

$$ax \equiv -bx \equiv \frac{a+b-1}{2} \pmod{m}.$$

We have

$$(a+nb)x \equiv (n-1)bx \equiv \frac{n+a+b-1}{2} \pmod{m}.$$

Thus we see that if $n = 2r(a+b)+2$ (which is obtained by taking $t = 1$ in A'_r) then

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{a+b-1}{2}.$$

Moreover, it can be shown that if $n = 2r(a+b)+2$ then

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-1}{2}$$

for all y; $1 \leqslant y \leqslant m/2$. Thus we see that

$$\max_{1 \leqslant y \leqslant m/2} (\min\{|ay|_m, |by|_m, |(a+nb)y|_m\}) = \frac{a+b-1}{2}.$$

On the other hand, if $n \neq 2r(a+b)+2$ then it is obvious that

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-3}{2}$$

for each y. Thus we see that

$$\max_{1 \leqslant y \leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{a+b-3}{2}.$$

To calculate $d(M)$ we apply the definition $d_3(M)$. Let us denote m values in (1), (2), and (3) by m_1, m_2, and m_3, respectively. Then

$$d(M) = \max\left(\frac{m_1-(2rb+1+2t)}{2m_1}, \frac{m_2-2(rb+t)}{2m_2}, \frac{a+b-\varepsilon}{2m_3}\right) = \frac{m_2-2(rb+t)}{2m_2}.$$

Here $\varepsilon = 1$ if $n = 2r(a+b)+2$ and $\varepsilon = 3$ if $n \neq 2r(a+b)+2$.

Case II ($n \in B'_s$). To calculate $d(M)$ we use $d_3(M)$.

▷ (1) ($m = a + (n+1)b$). Choose x such that

$$bx \equiv \frac{m - (2(s+1)b+1)}{2} \pmod{m}.$$

We have

$$\begin{aligned} ax \equiv -(n+1)bx \equiv & -(n+1)\frac{m-(2(s+1)b+1)}{2} \\ \equiv & -\frac{m-(2(s+1)b+1)(n+1)}{2} \pmod{m}. \end{aligned}$$

Since $(n+1)(2(s+1)b+1) = 2(s+1)(m-a)+n+1 = 2(s+1)m+2(s+1)b+1-2(a-t)$,

$$ax \equiv -\frac{m-(2(s+1)b+1)+2(a-t)}{2} \pmod{m}.$$

We also have that $(a+nb)x \equiv -bx \pmod{m}$. Thus

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m-(2(s+1)b+1)}{2}.$$

Moreover, it can also be shown that

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-(2(s+1)b+1)}{2}$$

for each y; $1 \leqslant y \leqslant m/2$. Thus we see that

$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{m-(2(s+1)b+1)}{2}.$$

▷ (2) ($m = 2a + nb$). Choose an integer x such that

$$bx \equiv \frac{m - 2((s+1)b+1)}{2} \pmod{m}.$$

We have

$$2ax \equiv -nbx \equiv -n\frac{m-2((s+1)b+1)}{2} \equiv (s+1)nb + n \pmod{m}.$$

Since $(s+1)nb+n = (s+1)(m-2a)+2sa+2(s+1)b+2t = (s+1)m+2(s+1)b-2(a-t)$,

$$2ax \equiv 2(s+1)b - 2(a-t) \equiv -(m-2((s+1)b+1)+2(a-t+1)) \pmod{m},$$

therefore,
$$ax \equiv -\frac{m - 2((s+1)b+1) + 2(a-t+1)}{2} \pmod{m}.$$
We also have $(a+nb)x \equiv -ax \pmod{m}$. Thus
$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m - 2((s+1)b+1)}{2}.$$
Also, it can be shown that for all y such that $1 \leqslant y \leqslant m/2$ and $y \neq x$,
$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m - 2((s+1)b+1)}{2}.$$
Thus we see that
$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{m - 2((s+1)b+1)}{2}.$$

▷ (3) $(m = a+b)$. Choose an integer x such that
$$ax \equiv -bx \equiv \frac{a+b-1}{2} \pmod{m}.$$
We have
$$(a+nb)x \equiv (n-1)bx \equiv \frac{n+a+b-1}{2} \pmod{m}.$$
Thus we see that if $n = 2(s+1)(a+b)$ (which is obtained by taking $t = a$ in B_s') then
$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{a+b-1}{2}.$$
Moreover, it can be shown that if $n = 2(s+1)(a+b)$ then
$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-1}{2}$$
for all y; $1 \leqslant y \leqslant m/2$. Thus we see that
$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{a+b-1}{2}.$$

On the other hand, if $n \neq 2(s+1)(a+b)$ then it is obvious that
$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{a+b-3}{2}$$

for each y. Thus we see that

$$\max_{1\leqslant y\leqslant m/2} \min\{|ay|_m, |by|_m, |(a+nb)y|_m\} = \frac{a+b-3}{2}.$$

To calculate $d(M)$ we apply the definition $d_3(M)$. Let us denote m values in (1), (2), and (3) by m_1, m_2, and m_3, respectively. Then

$$\begin{aligned} d(M) &= \max\Big(\frac{m_1-(2(s+1)b+1)}{2m_1}, \frac{m_2-2((s+1)b+1)}{2m_2}, \frac{a+b-\varepsilon}{2m_3}\Big) \\ &= \frac{m_1-(2(s+1)b+1)}{2m_1}. \end{aligned}$$

Here $\varepsilon = 1$ if $n = 2(s+1)(a+b)$ and $\varepsilon = 3$ if $n \neq 2(s+1)(a+b)$. This completes the proof. $\square$

Corollary 3.1. *Let $M = \{a, b, a+nb\}$, where $a < b$, $\gcd(a,b) = 1$, a and b are of opposite parity and $n \in \{k(a+b), k(a+b)+2\colon\ k \in 2\mathbb{N}\}$. Then $\mu(M) = \frac{1}{2}(a+b-1)/(a+b)$.*

P r o o f. If $n \in \{k(a+b), k(a+b)+2\colon\ k \in 2\mathbb{N}\}$ then it follows from the theorem that $\mu(M) \geqslant d(M) = \frac{1}{2}(a+b-1)/(a+b)$. On the other hand, we always have $\mu(M) \leqslant \mu(\{a,b\}) = \lfloor\frac{1}{2}(a+b)\rfloor/(a+b)$. Thus we have the corollary. $\square$

4. Both a and b are odd integers

Theorem 4.1. *Let $M = \{a, b, a+nb\}$, where $a < b$, $\gcd(a,b) = 1$, and a, b are odd integers. Then*

$$d(M) = \begin{cases} \dfrac{1}{2} = \mu(M) & \text{if } n \text{ is even;} \\[2ex] \dfrac{a+nb}{2\{a+(n+1)b\}} & \text{if } n \geqslant \dfrac{(b-2)(a+b)}{2b} \text{ and odd.} \end{cases}$$

P r o o f. Suppose that n is even. Observe that all three elements of M are odd. Therefore, any set S of nonnegative integers which contains elements of the same parity is an M-set and hence $\overline{\delta}(S) \leqslant 1/2$. On the other hand, if we take $S = \{1, 3, 5, \ldots\}$ then $\overline{\delta}(S) = 1/2$. Hence $\mu(M) = 1/2$. Now taking $x = 1/2$ in the definition of $d_1(M)$ we get $1/2 \leqslant d_1(M) = d(M)$. But we always have $d(M) \leqslant \mu(M) = 1/2$. Consequently, $d(M) = 1/2$. Next, suppose that $n \geqslant \frac{1}{2}(b-2)(a+b)/b$ and odd. To calculate $d(M)$ we consider the following possible values of m.

▷ (1) ($m = 2a + nb$). Choose x such that $x \equiv (m-1)/2 \pmod{m}$. This gives $bx \equiv (m-b)/2 \pmod{m}$, and $ax \equiv (m-a)/2 \pmod{m}$. Since $(a+nb)x \equiv -ax \pmod{m}$, therefore

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} = \frac{m-b}{2}.$$

Also it can be seen that

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-b}{2}$$

for each y; $1 \leqslant y \leqslant m/2$.

▷ (2) ($m = a + (n+1)b$). The proof is identical to the one in (1), and therefore omitted. We have

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-b}{2}$$

for each y; $1 \leqslant y \leqslant m/2$.

▷ (3) ($m = a + b$). Observe that m is even. Now we claim that

$$\min\{|ax|_m, |bx|_m, |(a+nb)x|_m\} \neq \frac{m}{2}$$

for any x.

Suppose that for some x, $ax \equiv -bx \equiv m/2 \pmod{m}$. This gives $(a+nb)x \equiv m/2 - nm/2 \equiv 0 \pmod{m}$. Hence the claim is true in this case. The other possibility we can have is that for some x, $(a+nb)x \equiv m/2 \pmod{m}$. The claim will be false only if $ax \equiv -bx \equiv m/2 \pmod{m}$. But this is not possible. Therefore, we have the claim and hence,

$$\min\{|ay|_m, |by|_m, |(a+nb)y|_m\} \leqslant \frac{m-2}{2} = \frac{a+b-2}{2}$$

for each y; $1 \leqslant y \leqslant m/2$.

To calculate $d(M)$ we apply the definition $d_3(M)$. Let us denote m values in (1), (2), and (3) by m_1, m_2, and m_3, respectively. Then

$$d(M) = \max\left(\frac{m_1-b}{2m_1}, \frac{m_2-b}{2m_2}, \frac{m_3-2}{2m_3}\right) = \frac{m_2-b}{2m_2} = \frac{a+nb}{2\{a+(n+1)b\}}.$$

For, we always have $\frac{1}{2}(m_2-b)/m_2 \geqslant \frac{1}{2}(m_1-b)/m_1$, and $\frac{1}{2}(m_2-b)/m_2 \geqslant \frac{1}{2}(m_3-2)/m_3$ if and only if $2m_2 \geqslant b(a+b)$ if and only if $n \geqslant \frac{1}{2}(b-2)(a+b)/b$. Thus we have the theorem. □

5. Concluding remark

Using $\mu(M)$ for $M=\{a,b,a+nb\}$ is a generalization of $\mu(M)$ for $M=\{a,b,a+b\}$ which was discussed earlier by Rabinowitz and Proulx [11], Gupta [2], and Liu and Zhu [5]. We are unable to calculate the values or bounds of $\mu(M)$ for some finite number of odd integers n.

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