Akademie věd České republiky

Disertace
k získání vědeckého titulu "doktor věd"
ve skupině věd fyzikálně-matematických

Mathematical analysis of the motion of viscous fluids:

motion of incompressible fluid around rotating and translating rigid body,

motion of compressible gas, motion of linear viscous fluid in the half space

Komise pro obhajoby doktorských disertací v oboru “ Matematická analýza a příbuzné obory”

Jméno uchazeče RNDr. Šárka Nečasová, CSc.

Pracoviště uchazeče Matematický ústav AV ČR, v. v. i.

Místo a datum Praha 11. 5. 2012

Contents

1 The motion of viscous fluids around a purely rotating body

In the first part of the thesis we shall study the time-periodic Oseen equations past a purely rotating body in the whole space and in an exterior domain.

Let $\widetilde{\Omega}(t) \subset \mathbb{R}^n$ $(n = 2, 3)$ be given, the time-dependent exterior domain past a rotating body $\mathfrak{D}$. We consider that $\partial\widetilde{\Omega}$ is sufficiently smooth. We assume that $\widetilde{\Omega}(t)$ is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations with the velocity v_∞ at infinity. Given the coefficient of viscosity $\nu > 0$ and an external force $\tilde{f} = \tilde{f}(y, t)$, we are looking for the velocity $\tilde{v} := \tilde{v}(y, t)$ and the pressure $\tilde{q} := \tilde{q}(y, t)$ solving the nonlinear system

$$\begin{aligned} \tilde{v}_t - \nu\Delta\tilde{v} + (\tilde{v}\cdot\nabla)\tilde{v} + \nabla\tilde{q} = \tilde{f} \qquad & \text{in } \widetilde{\Omega}(t),\ t > 0, \\ \operatorname{div}\, \tilde{v} = 0 \qquad & \text{in } \widetilde{\Omega}(t),\ t > 0, \\ \tilde{v}(y, t) = \omega \wedge y \quad & \text{on } \partial\widetilde{\Omega}(t),\ t > 0, \\ \tilde{v}(y, t) \to v_\infty \qquad & \text{as } |y| \to \infty\,. \end{aligned} \tag{1.1}$$

Here $\wedge$ denotes the exterior wegde product of $\mathbb{R}^3$, and in the two-dimensional case, $\omega \wedge y \ = (-y_2, y_1)$ for $y = (y_1, y_2)$.

Due to the rotation of the body with the angular velocity ω, we have

$$\widetilde{\Omega}(t) = O_\omega(t)\Omega,$$

where $D \subset \mathbb{R}^n$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|t & -\sin|\omega|t & 0 \\ \sin|\omega|t & \cos|\omega|t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad = \begin{pmatrix} \cos|\omega|t & -\sin|\omega|t \\ \sin|\omega|t & \cos|\omega|t \end{pmatrix} \quad \text{if} \ \ n = 2. \tag{1.2}$$

After the change of variables $x := O_\omega(t)^T y$ and passing to the new functions $u(x, t) := O_\omega^T\tilde{v}(y, t) - v_\infty$ and $p(x, t) := \tilde{q}(y, t)$, as well as to the force term $f(x, t) := O_\omega(t)^T\tilde{f}(y, t)$, we arrive at the modified Navier–Stokes system

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u - ((\omega\wedge x)\cdot\nabla)u + \qquad & \\ +(O_\omega(t)^T v_\infty\cdot\nabla)u + \omega\wedge u + \nabla p = f \qquad & \text{in } \Omega,\ t > 0, \\ \operatorname{div}\, u = 0 \qquad & \text{in } \Omega,\ t > 0, \\ u(x, t) + O_\omega(t)^T v_\infty = \omega\wedge x \quad & \text{on } \partial\Omega,\ t > 0, \\ u(x, t) \to 0 \qquad & \text{as } |x| \to \infty\,. \end{aligned} \tag{1.3}$$

Note that, because of the new coordinate system attached to the rotating body, equation $(1.3)_1$ contains three new terms, the classical Coriolis force term $\omega \wedge u$ (up to a multiplicative constant) and the terms $((\omega \wedge x) \cdot \nabla)u$ and $(O_\omega(t)^T v_\infty \cdot \nabla)u$ which are not subordinate to the Laplacian in unbounded domains.

An important step concerns its linearized and steady versions, i.e.

- either in the whole space $\mathbb{R}^n$ **the modified Stokes systems**,

$$
\begin{aligned}
-\nu\Delta u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u + \nabla p = f \qquad & \text{in } \mathbb{R}^n,\\
\operatorname{div} u = 0 \text{ or } g \qquad & \text{in } \mathbb{R}^n,\\
u \to 0 \qquad & \text{as } |x|\to\infty,
\end{aligned} \tag{1.4}
$$

where $n = 2$ or $n = 3$;

- or in an open set Ω **the modified Oseen systems**,

$$
\begin{aligned}
-\nu\Delta u + k\partial_3 u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u + \nabla p = f \qquad & \text{in } \Omega,\\
\operatorname{div} u = 0 \text{ or } g \qquad & \text{in } \Omega,\\
u(.,t) + u_\infty = \omega\wedge x \qquad & \text{on } \partial\Omega,\\
u \to 0 \qquad & \text{as } |x|\to\infty,
\end{aligned} \tag{1.5}
$$

with an appropriate choice of the constant translational velocity at infinity $u_\infty = ke_3 \neq 0$, therefore parallel to ω.

We follow two different ways to handle this problem. The first approach in an L^2-setting uses variational calculus. This viewpoint has already been applied in [23] by R. Farwig and in [58, 59] by S. Kračmar and P. Penel to solve the scalar model equations

$$-\nu\,\Delta u + k\partial_3 u = f \quad \text{in } \Omega$$

and – with a given non-constant and, in general, non-solenoidal vector function $\mathbf{a}$ –

$$-\nu\,\Delta u + k\partial_3 u - \mathbf{a}\cdot\nabla u = f \quad \text{in } \Omega,$$

respectively, in an exterior domain Ω, together with the boundary conditions $u = 0$ on $\partial\Omega$ and $u \to 0$ as $|\mathbf{x}| \to \infty$.

Second, to consider more general weights in L^q-spaces, we apply weighted multiplier and Littlewood-Paley theory as well as the theory of one-sided Muckenhoupt weights corresponding to one-sided maximal functions. This approach was firstly introduced by Farwig, Hishida, Müller [27] for the case $\mathbf{u}_\infty = 0$ and in [24], [25] when $\mathbf{u}_\infty \neq 0$ without weights and then extended to the weighed case by Krbec, Farwig, Nečasová [31], [30] and Nečasová, Schumacher [68].

1.1 L^q setting

Definition 1. Let A_q, $1 < q < \infty$, the set of Muckenhoupt weights, be given by all strictly positive functions $w \in L^1_{loc}(\mathbb{R}^n)$, for which

$$A_q(w) := \sup_Q \left(|Q|^{-1}w(Q)\right)\left(|Q|^{-1}w'(Q)\right)^{q-1} < \infty. \tag{1.6}$$

where $w' := w^{-\frac{1}{q-1}}$ and the supremum is taken over all cubes Q in $\mathbb{R}^n$. We have excluded the case where w vanishes almost everywhere.

For $q \in (1,\infty)$, $w \in A_q$, $k \geq 1 \in \mathbb{N}$, and an open set Ω, we define

- the Lebesgue space $L^q_w(\Omega) := \{f \in L^1_{loc}(\overline{\Omega}) \ s.t. \ \int_\Omega |f|^q w\,dx \ < \infty\}$, with the norm $\|f\|_{q,w} := \left(\int_\Omega |f|^q w\,dx\right)^{\frac{1}{q}}$,
- the Sobolev space $H^{k,q}_w(\Omega) := \{f \in L^1_{loc}(\overline{\Omega}) \ s.t. \ \nabla^j f \in L^q_w(\Omega), \ j \le k\}$, equipped with the norm $\|u\|_{k,q,w} := \sum_{j=0}^k \|\nabla^j u\|_{q,w}$,
- the homogeneous Sobolev space $\widehat{H}^{k,q}_w(\Omega) := \{f \in L^1_{loc}(\overline{\Omega}) \ s.t. \ \nabla^k f \in L^q_w(\Omega)\}$,
- the space of smooth and compactly supported functions $C_0^\infty(\Omega)$ and its divergence free counterpart $C^\infty_{0,\sigma}(\Omega) := \{\phi \in C_0^\infty(\Omega) \ s.t. \ \text{div}\ \phi = 0\}$,
- and the spaces $\widehat{H}^{k,q}_{w,0}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{\widehat{H}^{k,q}_w}}$, $H^{k,q}_{w,0}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^{k,q}_w}}$.

It is easily seen that

$$(L^q_w(\Omega))' = L^{q'}_{w'}(\Omega) \ \text{ with } \ \frac{1}{q} + \frac{1}{q'} = 1 \ \text{ and } \ w' = w^{-\frac{1}{q-1}}. \tag{1.7}$$

Moreover, by [74], for $1 < q < \infty$ and $w \in A_q$ there exists s such that $1 \le s < q$ and $w \in A_s$. In addition, if Ω is a bounded domain, then it follows from Hölder's inequality that the weighted Lebesgue spaces are embedded into unweighted ones as follows

$$L^q_w(\Omega) \hookrightarrow L^r(\Omega) \ \text{ for every } r < q/s. \tag{1.8}$$

Considering the dual spaces in (1.8) one obtains that for q and w as above there exists $r \in (1,\infty)$ such that $L^r(\Omega) \hookrightarrow L^q_w(\Omega)$.

1.1.1 Strong solution

Oseen system see [30]

The Oseen system (1.5) has been analyzed by Farwig in [24], [25], in L^q-spaces, $1 < q < \infty$, the *a priori* estimates being generalized by Farwig, Krbec, and Nečasová in weighted L^q-spaces

$$\|\nu\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \le c\|f\|_{q,w}, \tag{1.9}$$

$$\|k\partial_3 u\|_{q,w} + \|(\omega \wedge x)\cdot u - \omega \wedge u\|_{q,w} \le c(k,\nu,\omega)\|f\|_{q,w}. \tag{1.10}$$

More precisely,

Theorem 1. *Let the weight function $0 \le w \in L^1_{\text{loc}}(\mathbb{R}^3)$ be independent of the angular variable θ and satisfy the following condition depending on $q \in (1,\infty)$:*

$$\begin{aligned} 2 \le q < \infty: &\quad w^\tau \ \in \widetilde{A}^-_{\tau q/2} \quad \textit{for some} \quad \tau \in [1,\infty) \\ 1 < q < 2: &\quad w^\tau \in \widetilde{A}^-_{\tau q/2} \quad \textit{for some} \quad \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right]. \end{aligned} \tag{1.11}$$

(i) Given $f \in L^q_w(\mathbb{R}^3)^3$ there exists a solution $(u,p) \in L^1_{\rm loc}(\mathbb{R}^3)^3 \times L^1_{\rm loc}(\mathbb{R}^3)$ of (1.5) satisfying the estimate

$$\|\nu\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \le c\|f\|_{q,w} \tag{1.12}$$

with a constant $c = c(q,w) > 0$ independent of ν, k and ω.

(ii) Let $f \in L^{q_1}_{w_1}(\mathbb{R}^3)^3 \cap L^{q_2}_{w_2}(\mathbb{R}^3)^3$ such that both (q_1,w_1) and (q_2,w_2) satisfy the conditions (1.11), and let $u_1,u_2 \in L^1_{\rm loc}(\mathbb{R}^3)^3$ together with corresponding pressure functions $p_1,p_2 \in L^1_{\rm loc}(\mathbb{R}^3)$ be solutions of (1.5) satisfying (1.12) for (q_1,w_1) and (q_2,w_2), respectively. Then there are $\alpha,\beta \in \mathbb{R}$ such that u_1 coincides with u_2 up to an affine linear field $\alpha e_3 + \beta\omega\wedge x$, $\alpha,\beta \in \mathbb{R}$.

Remark 1. Precise definition of $\tilde{A}^-_{\tau q/2}$ is given in [30].

Corollary 1. *Let the weight function $0 \le w \in L^1_{\rm loc}(\mathbb{R}^3)$ be independent of the angular variable θ. Moreover, let w satisfy the following condition depending on $q \in (1,\infty)$:*

$$\begin{aligned} 2 \le q < \infty: &\quad w^\tau \in \widetilde{A}^-_{\tau q/2}(\mathcal{J}) \quad \textit{for some} \quad \tau \in [1,\infty) \\ 1 < q < 2: &\quad w^\tau \in \widetilde{A}^-_{\tau q/2}(\mathcal{J}) \quad \textit{for some} \quad \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right] \end{aligned} \tag{1.13}$$

where the weight class $\widetilde{A}^-_\tau(\mathcal{J})$, $1 \le \tau < \infty$, is defined by

$$\widetilde{A}^-_\tau(\mathcal{J}) = \widetilde{A}^-_\tau(\mathbb{R}^3) \cap A_\tau(\mathcal{J}).$$

Given $f \in L^q_w(\mathbb{R}^3)^3$ there exists a solution $(u,p) \in L^1_{\rm loc}(\mathbb{R}^3)^3 \times L^1_{\rm loc}(\mathbb{R}^3)$ of (1.5) satisfying the estimate

$$\|k\partial_3 u\|_{q,w} + \|(\omega\wedge x)\cdot u - \omega\wedge u\|_{q,w} \le c\left(1 + \frac{k^5}{\nu^{5/2}|\omega|^{5/2}}\right)\|f\|_{q,w} \tag{1.14}$$

with a constant $c = c(q,w) > 0$ independent of ν, k and ω.

We note that the ω-dependent term $1+\frac{k^5}{\nu^{5/2}|\omega|^{5/2}}$ in (1.14) cannot be avoided in general; see [25] for an example in the space $L^2(\mathbb{R}^3)$.

As an example of anisotropic weight functions we consider

$$w(x) = \eta^\alpha_\beta(x) = (1+|x|)^\alpha(1+s(x))^\beta, \quad s(x) = |(x_1,x_2,x_3)| - x_3, \tag{1.15}$$

introduced in [23] to analyze the Oseen equations.

Corollary 2. *The a priori estimates (1.12),(1.14) hold for the anisotropic weights $w = \eta^\alpha_\beta$, see (1.15), provided that*

$$\begin{aligned} 2 \le q < \infty \quad &: -\tfrac{q}{2} < \alpha < \tfrac{q}{2}, \qquad 0 \le \beta < \tfrac{q}{2} \qquad \textit{and} \quad \alpha+\beta > -1, \\ 1 < q < 2 \quad &: -\tfrac{q}{2} < \alpha < q-1, \quad 0 \le \beta < q-1 \quad \textit{and} \quad \alpha+\beta > -\tfrac{q}{2}. \end{aligned}$$

Note that the condition $\beta \ge 0$ will reflect the existence of a wake region in the downstream direction $x_3 > 0$, where the solution of the original nonlinear problem (1.1) will decay slower than in the upstream direction $x_3 < 0$.

1.1.2 Weak solution

Whole space $\mathbb{R}^3$, see [56]

We introduce the following notations. The class $C_0^\infty(\mathbb{R}^3)$ consists of C^∞ functions with compact supports contained in $\mathbb{R}^3$. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue space with norm $\|\cdot\|_q$. We define the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_q} = \{v \in L^q_{\text{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\}/R. \tag{1.16}$$

Definition 2. *Let* $1 < q < \infty$. *Given* $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, *we call* $\{u, p\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ *weak solution to (1.5) if*

$$\begin{aligned} &(1) \qquad \nabla\cdot u = 0 \quad in \quad L^q(\mathbb{R}^3), \\ &(2) \qquad (\omega\wedge x)\cdot\nabla u - \omega\wedge u \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3, \end{aligned} \tag{1.17}$$

$\{u, p\}$ *satisfies* $(1.5)_1$ *in the sense of distributions, that is*

$$\begin{gathered} \nu\langle\nabla u, \nabla\varphi\rangle - \langle(\omega\wedge x)\cdot\nabla u - \omega\wedge u, \varphi\rangle \\ +k\left\langle \frac{\partial u}{\partial x_3}, \varphi\right\rangle - \langle p, \nabla\cdot\varphi\rangle = \langle f, \varphi\rangle, \\ \varphi \in C_0^\infty(\mathbb{R}^3)^3. \end{gathered} \tag{1.18}$$

The main results are the following

Theorem 2. *Let* $1 < q < \infty$ *and suppose* $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$. *Then problem (1.5) possesses a weak solution* $(u, p) \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ *satisfying the estimate*

$$\|\nabla u\|_q + \|p\|_q + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q} + \|\partial_3 u\|_{-1.q} \ \leq\ C\|f\|_{-1,q}, \tag{1.19}$$

with some $C > 0$, *which depends on* q.

Theorem 3. *The solution* $\{u, p\}$ *given by Theorem 2 is unique up to a constant multiple of* ω *for* u.

1.2 L^2 setting

Whole space see [57]
We will introduce notation used in this subsection:
Let $\left(L^2(\Omega;\, w)\right)^3$ be the set of measurable vector functions $\mathbf{f} = (f_1, f_2, f_3)$ in Ω such that

$$\|\mathbf{f}\|^2_{2,\Omega;\,w} = \int_\Omega |\mathbf{f}|^2\, w\, d\mathbf{x} < \infty.$$

We will use the notation $\mathbf{L}^2_{\alpha,\beta}(\Omega)$ instead of $\left(L^2\left(\Omega;\, \eta^\alpha_\beta\right)\right)^3$ and $\|\cdot\|_{2,\alpha,\beta}$ instead of $\|\cdot\|_{\left(L^2\left(\Omega;\,\eta^\alpha_\beta\right)\right)^3}$. Let us define the weighted Sobolev space $\mathbf{H}^1\left(\Omega;\eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ as the set of functions $\mathbf{u} \in$

$\mathbf{L}^2_{\alpha_0,\beta_0}(\Omega)$ with weak derivatives $\partial_i \mathbf{u} \in \mathbf{L}^2_{\alpha_1,\beta_1}(\Omega)$, $i = 1,2,3$. The standard norm of $\mathbf{u} \in \mathbf{H}^1\big(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\big)$ is given by

$$\|\mathbf{u}\|_{\mathbf{H}^1\left(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)} = \left(\int_\Omega |\mathbf{u}|^2\, \eta^{\alpha_0}_{\beta_0}\, d\mathbf{x} + \int_\Omega |\nabla \mathbf{u}|^2\, \eta^{\alpha_1}_{\beta_1}\, d\mathbf{x} \right)^{1/2}.$$

As usual, $\overset{\circ}{\mathbf{H}}{}^1\big(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\big)$ will be the closure of $\mathbf{C}^\infty_0(\Omega)$ in $\mathbf{H}^1\big(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\big)$, where $\mathbf{C}^\infty_0(\Omega)$ is $(C^\infty_0(\Omega))^3$.

For simplicity, we shall use the following abbreviations:

$\mathbf{L}^2_{\alpha,\beta}(\Omega)$ instead of $\big(L^2\big(\Omega;\, \eta^\alpha_\beta\big)\big)^3$ $\|\cdot\|_{2,\alpha,\beta}$ instead of $\|\cdot\|_{\left(L^2\left(\Omega;\, \eta^\alpha_\beta\right)\right)^3}$, $\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}(\Omega)$ instead of $\overset{\circ}{\mathbf{H}}{}^1\big(\Omega;\, \eta^{\alpha-1}_{\beta-1}, \eta^\alpha_\beta\big)$, $\mathbf{V}_{\alpha,\beta}(\Omega)$ instead of $\overset{\circ}{\mathbf{H}}{}^1\big(\Omega;\, \eta^{\alpha-1}_{\beta}, \eta^\alpha_\beta\big)$.

We shall use these last two Hilbert spaces for $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta < 3$.

We will consider the nohomogeneous case div $u = g$.

Theorem 4. (Existence and uniqueness) *Let* $0 < \beta \leq 1$, $0 \leq \alpha < y_1\beta$ *with* y_1 *will be given in see [57]. Moreover, let* $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, $g \in W^{1,2}_0$ *with* $\operatorname{supp} g = K \subset\subset \mathbb{R}^3$, *and* $\int_{\mathbb{R}^3} g\, d\mathbf{x} = 0$. *Then there exists a unique weak solution* $\{\mathbf{u}, p\}$ *of the problem (1.5) such that* $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ *and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \leq C \left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2} \right).$$

An exterior domain see [57]

Theorem 5. *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior domain and* $0 < \beta \leq 1$, $0 \leq \alpha < y_1 \cdot \beta$; y_1 *is given see [57]* , $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$, $g \in W^{1,2}_0(\Omega)$, *with* $\operatorname{supp} g = K \subset\subset \Omega$ *and* $\int_\Omega g\, d\mathbf{x} = 0$. *Then there exists a weak solution* $\{\mathbf{u}, p\}$ *of the problem (1.5) such that* $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}\left(\overline{\Omega}\right)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$, *and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \leq C \left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2} \right).$$

2 Asymptotic behavior of the motion of viscous fluid around a translating and rotating body

For more details see [17].

We consider a stationary linearized variant of (1.3) given by

$$-\Delta u - (U + \omega \times x) \cdot \nabla u + \omega \times u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{2.1}$$

under the assumption that U and ω are parallel. We derive a representation formula for the velocity part u of a solution (u, π) to (2.1). This formula is based on a fundamental solution to (2.1) proposed by Guenther and Thomann in the article [45] where they construct the fundamental solution to a linearized version of the time-dependent problem (1.3). On [45, page 20], they indicate that by integrating this solution with respect to time on $(0, \infty)$, a fundamental solution to (2.1) is obtained. Using our representation formula we prove the asymptotic behavior of the solution.

The result was motivated by references [42, 43], where the linear stationary problem (2.1) as well as the nonlinear stationary variant of (1.3),

$$-\Delta u - (U + \omega \times x) \cdot \nabla u + \omega \times u + (u \cdot \nabla)u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega = \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \tag{2.2}$$

are considered. It is shown in [42] under suitable assumptions on the data, and in the case of (2.2) additionally under some smallness conditions, that solutions to respectively (2.1) and (2.2) exist in certain Sobolev spaces. These solutions are unique in the space of functions (v, ϱ) satisfying relation

$$\sup\{|v(x)| \cdot |x| \, : \, x \in \mathbb{R}^3 \backslash B_S\} < \infty \quad \text{for some } S > 0 \text{ with } \overline{\mathfrak{D}} \subset B_S.$$

Article [43] further shows that under additional assumptions on the data, and after some change of variables, the solutions (u, π) constructed in [42] verify relations

$$\begin{gathered} \sup\{|u(x)| \cdot |x| \cdot \big(1 + Re \cdot (|x| + x_1)\big) \, : \, x \in \mathbb{R}^3 \backslash B_S\} < \infty, \\ \sup\{|\nabla u(x)| \cdot |x|^{3/2} \cdot \big(1 + Re \cdot (|x| + x_1)\big)^{3/2} \, : \, x \in \mathbb{R}^3 \backslash B_S\} < \infty. \end{gathered} \tag{2.3}$$

2.1 Notations, definitions and auxiliary results.

If $x, y \in \mathbb{R}^3$, we write $x \times y$ for the usual vector product of x and y. The open ball centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. The symbol $| \cdot |$ will be used to denote the Euclidean norm of $\mathbb{R}^3$, and it will also stand for the length $\alpha_1 + \alpha_2 + \alpha_3$ of a multiindex $\alpha \in \mathbb{N}_0^3$.

We fix vectors U, $\omega \in \mathbb{R}^3 \backslash \{0\}$ which are parallel: $U = \varrho \cdot \omega$ for some $\varrho \in \mathbb{R} \backslash \{0\}$. By the symbol $\mathfrak{C}$, we denote constants depending only on U and ω. We write $\mathfrak{C}(\gamma_1, ..., \gamma_n)$ for constants which additionally depend on quantities γ_1, ..., $\gamma_n \in \mathbb{R}$, for some $n \in \mathbb{N}$. We further fix an open bounded set $\mathfrak{D}$ in $\mathbb{R}^3$ with Lipschitz boundary $\partial \mathfrak{D}$, the outward unit normal to $\mathfrak{D}$ is denoted by $n^{(\mathfrak{D})}$. For $T \in (0, \infty)$, put $\mathfrak{D}_T := B_T \backslash \overline{\mathfrak{D}}$ ("truncated exterior domain").

Define the matrix Σ by

$$\Sigma := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

such that $\omega \times x = \Sigma \cdot x$ for $x \in \mathbb{R}^3$. For open sets $V \subset \mathbb{R}^3$, sufficiently smooth functions $w : V \mapsto \mathbb{R}^3$, and for $z \in V$, we set

$$L(w)(z) := -\Delta w(z) - (U + \omega \times z) \cdot \nabla w(z) + \omega \times w(z). \tag{2.4}$$

Let K denote the usual fundamental solution to the heat equation, that is,

$$K(z,t) := (4\cdot\pi\cdot t)^{-3/2}\cdot \mathrm{e}^{-|z|^2/(4\cdot t)} \quad \text{for } \ z\in\mathbb{R}^3,\ t\in(0,\infty).$$

In order to introduce the fundamental solution constructed by Guenther, Thomann [45] for the linearized variant of (1.3), we define matrices

$$G^{(1)}(y,z,t) := \big(\,\delta_{jk} - (y-z(t))_j\cdot(y-z(t))_k\cdot|y-z(t)|^{-2}\,\big)_{1\le j,k\le 3}\cdot \mathrm{e}^{-t\cdot\Omega},$$
$$G^{(2)}(y,z,t) := \big(\,\delta_{jk}/3 - (y-z(t))_j\cdot(y-z(t))_k\cdot|y-z(t)|^{-2}\,\big)_{1\le j,k\le 3}\cdot \mathrm{e}^{-t\cdot\Omega}$$

for $y,z\in\mathbb{R}^3,\ t\in(0,\infty)$ with $y\neq z(t)$. Here and in the rest of this paper, we use the abbreviation

$$z(t) := \mathrm{e}^{-t\cdot\Omega}\cdot z - t\cdot U \quad \text{for } \ z\in\mathbb{R}^3,\ t\in[0,\infty). \tag{2.5}$$

The Kummer function ${}_1F_1(1,c,u)$ appearing in the following is defined by

$${}_1F_1(1,c,u) := \sum_{n=0}^{\infty}\big(\,\Gamma(c)/\Gamma(n+c)\,\big)\cdot u^n \quad \text{for } \ u\in\mathbb{R},\ c\in(0,\infty),$$

where the letter Γ denotes the usual Gamma function. As in [45], the same letter Γ is used to denote the fundamental solution introduced in that latter reference for a linearized version of (1.3). This fundamental solution reads

$$\Gamma_{jk}(y,z,t)$$
$$:= K\big(\,y-z(t),\,t\,\big)\cdot\Big(G^{(1)}(y,z,t) - {}_1F_1\big(\,1,5/2,\,|y-z(t)|^2/(4\cdot t)\,\big)\cdot G^{(2)}(y,z,t)\Big)_{jk}$$

for $y,z\in\mathbb{R}^3,\ t\in(0,\infty)$ with $y\neq z(t),\ j,k\in\{1,\,2,\,3\}$.

The following estimates of $|y-z(t)|$ will play a fundamental role in our argument.

Lemma 1. The relation $|\mathrm{e}^{-t\cdot\Omega}\cdot v| = |v|$ holds for $v\in\mathbb{R}^3$.

Let $R\in(0,\infty),\ y,z\in B_R$ with $y\neq z,\ t\in(0,\infty)$ with

$$t \ \le\ \min\big\{\,|y-z|/(2\cdot|U|),\ |y-z|/(24\cdot|\omega|\cdot R),\ \big(\arccos(3/4)\,\big)/|\omega|\,\big\}.$$

Then $|y-z(t)|\ge |y-z|/12$.

2.2 Main theorems

Theorem 6. Let $u\in C^2(\Omega)^3,\ \pi\in C^1(\Omega),\ f\in C^0(\Omega)^3$ with $f=L(u)+\nabla\pi$. Suppose there is $S>0$ with

$\mathfrak{D}\subset B_S$ such that

$$\int_{\mathbb{R}^3\setminus B_S}|z|^{-1/2}\cdot|f(z)|\,dz<\infty,\qquad \int_{\mathbb{R}^3\setminus B_S}|z|^{-2}\cdot|\operatorname{div}u(z)|\,dz<\infty.$$

Further suppose there is a sequence (R_n) in (S,∞) such that

$$R_n^{-1/2}\cdot\int_{\partial B_{R_n}}\big(\,|\nabla u(z)|+|\pi(z)|+|u(z)|\,\big)\,do_z + R_n^{-2}\cdot\int_{\partial B_{R_n}}|\operatorname{div}u(z)|\,do_z \ \longrightarrow\ 0$$

for $n \to \infty$. Let $j \in \{1, 2, 3\}$, $y \in \Omega$. Then

$$
\begin{aligned}
& u_j(y) \\
&= \int_{\mathbb{R}^3 \setminus \overline{\mathfrak{D}}} \Big(\sum_{k=1}^{3} \int_0^\infty \Gamma_{jk}(y,z,t)\, dt \cdot f_k(z) \\
&\qquad\qquad + (4\pi)^{-1} \cdot (y-z)_j \cdot |y-z|^{-3} \cdot \text{ div } u(z) \Big)\, dz \\
&\quad - \int_{\partial \mathfrak{D}} \sum_{k=1}^{3} \Big[\\
&\qquad \sum_{l=1}^{3} \Big(\int_0^\infty \Gamma_{jk}(y,z,t)\, dt \cdot \big(\partial_l u_k(z) - \delta_{kl} \cdot \pi(z) + u_k(z) \cdot (U + \omega \times z)_l \big) \\
&\qquad\qquad - \int_0^\infty \partial_{z_l} \Gamma_{jk}(y,z,t)\, dt \cdot u_k(z) \Big) \cdot n_l^{(\mathfrak{D})}(z) \\
&\qquad -(4 \cdot \pi)^{-1} \cdot (y-z)_j \cdot |y-z|^{-3} \cdot u_k(z) \cdot n_k^{(\mathfrak{D})}(z) \Big]\, do_z.
\end{aligned}
$$

Definition 2.

Let $p \in (1, \infty)$. Define $\mathfrak{M}_p$ as the space of all pairs of functions (u, π) such that $u \in W^{2,p}_{loc}(\overline{\mathfrak{D}}^c)^3$, $\pi \in W^{1,p}_{loc}(\overline{\mathfrak{D}}^c)$,

$$
\begin{gathered}
u|\mathfrak{D}_T \in W^{1,p}(\mathfrak{D}_T)^3, \quad \pi|\mathfrak{D}_T \in L^p(\mathfrak{D}_T), \quad u|\partial\mathfrak{D} \in W^{2-1/p,\,p}(\partial\mathfrak{D})^3, \\
\operatorname{div} u|\mathfrak{D}_T \in W^{1,p}(\mathfrak{D}_T), \quad L(u) + \nabla\pi|\mathfrak{D}_T \in L^p(\mathfrak{D}_T)^3
\end{gathered} \tag{2.6}
$$

for some $T \in (0, \infty)$ with $\overline{\mathfrak{D}} \subset B_T$.

Theorem 7. Let $p \in (1, \infty)$, $(u, \pi) \in \mathfrak{M}_p$. Put $F := L(u) + \nabla\pi$. Suppose there are numbers S_1, S, $\gamma \in (0, \infty)$, $A \in [2, \infty)$, $B \in \mathbb{R}$ such that $S_1 < S$, $\overline{\mathfrak{D}} \subset B_{S_1}$,

$$
\begin{aligned}
& u|B_S^c \in L^6(B_S^c)^3, \ \nabla u|B_S^c \in L^2(B_S^c)^9, \ \pi|B_S^c \in L^2(B_S^c), \quad \text{supp (div } u) \subset B_{S_1}, \\
& A + \min\{1, B\} \geq 3, \ |F(z)| \leq \gamma |z|^{-A} s_\tau(z)^{-B} \text{ for } z \in B_{S_1}^c,
\end{aligned}
$$

where

$$
s_\tau(x) := 1 + \tau(|x| - x_1) \quad \text{for } x \in \mathbb{R}^3.
$$

Put $\delta := \operatorname{dist}(\overline{\mathfrak{D}}, \partial B_S)$. Let $i, j \in \{1, 2, 3\}$, $y \in B_S^c$. Then

$$
\begin{aligned}
|u_j(y)| \leq \mathfrak{C}(S, S_1, A, B, \delta) \big(& \gamma + \|F|B_{S_1}\|_1 + \|\operatorname{div} u\|_1 + \|\nabla u \,|\, \partial\mathfrak{D}\|_1 \\
& + \|\pi|\partial\mathfrak{D}\|_1 + \widetilde{C}(\mathfrak{D}, p) \|u|\partial\mathfrak{D}\|_{2-1/p,p} \big) \big(|y|\, s_\tau(y) \big)^{-1} l_{A,B}(y),
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
& |\partial_i u_j(y)| \\
& \leq \mathfrak{C}(S, S_1, A, B, \delta) \big(\gamma + \|F|B_{S_1}\|_1 + \|\operatorname{div} u\|_1 + \|\nabla u \,|\, \partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 \\
& \qquad + \widetilde{C}(\mathfrak{D}, p) \|u|\partial\mathfrak{D}\|_{2-1/p,p} \big) \big(|y|\, s_\tau(y) \big)^{-3/2} s_\tau(y)^{\max(0,\, 7/2 - A - B)} l_{A,B}(y),
\end{aligned} \tag{2.8}
$$

where $\widetilde{C}(\mathfrak{D}, p)$ was introduced in ([17] Lemma 5.2) and function $l_{A,B}(y)$ see ([17], Theorem 3.3). If the assumption *supp*(div $u) \subset B_{S_1}$ is replaced by the condition

$$
|\text{ div } u(z)| \leq \widetilde{\gamma} |z|^{-C} s_\tau(z)^{-D} \quad \text{for } z \in B_{S_1}^c,
$$

for some $\widetilde{\gamma} \in (0,\infty)$, $C \in (5/2,\infty)$, $D \in \mathbb{R}$ with $C + \min\{1, D\} > 3$, then inequality (2.7) remains valid if the term $\|\operatorname{div} u\|_1$ on the right-hand side of (2.7) is replaced by $\widetilde{\gamma} + \|\operatorname{div} u|B_{S_1}\|_1$. Of course, in that case the constant in (2.7) additionally depends on C and D.

In the next theorem, we present an asymptotic profile of u for the case that $L(u)+\nabla\pi$ and $\operatorname{div} u$ have compact support.

Theorem 8. Let $p \in (1,\infty)$, $(u,\pi) \in \mathfrak{M}_p$, $S, S_1 \in (0,\infty)$ with $S_1 < S$, and put $F := L(u) + \nabla\pi$. Suppose that

$$\overline{\mathfrak{D}} \cup \text{ supp } (F) \cup \text{ supp } (\operatorname{div} u) \subset B_{S_1},$$
$$u|B_S^c \in L^6(B_S^c)^3, \quad \nabla u|B_S^c \in L^2(B_S^c)^9, \quad \pi|B_S^c \in L^2(B_s^c).$$

Then there are coefficients $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and functions $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \in C^0(B_S^c)$ such that for $j \in \{1,\, 2,\, 3\}$, $y \in B_S^c$,

$$u_j(y) = \sum_{k=1}^{3} \beta_k\, \mathfrak{Z}_{jk}(y,0) + \Big(\int_{\partial\mathfrak{D}} u\cdot n^{(\mathfrak{D})}\, do_z + \int_{B_{S_1}} \operatorname{div} u\, dz\Big) E_{4j}(y) + \mathfrak{F}_j(y), \tag{2.9}$$

and

$$\begin{aligned}|\mathfrak{F}_j(y)| \le \mathfrak{C}(S,S_1)\big(\,\|F\|_1 + \|\text{ div } u\|_1 + \|\nabla u\,|\,\partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 \\ +C(\mathfrak{D},p)\,\|u|\partial\mathfrak{D}\|_{2-1/p,p}\,\big)\,\big(\,|y|\,s_\tau(y)\,\big)^{-3/2},\end{aligned} \tag{2.10}$$

where $C(\mathfrak{D},p) > 0$ only depends on $\mathfrak{D}$ and p. (Note that $|E_{4j}(y)| \le \mathfrak{C}|y|^{-2}$ and $|y|^{-2} \le \mathfrak{C}(S)\big(\,|y|\,s_\tau(y)\,\big)^{-1}$ for $y \in B_S^c$; (see Lemma 2.4 [17].)

Finally we obtain a representation formula for weak solutions of the stationary Navier-Stokes system with Oseen and rotational terms.

Theorem 9. Let $u \in W^{1,1}_{loc}(\overline{\mathfrak{D}}^c)^3 \cap L^6(\overline{\mathfrak{D}})^3$ with $\nabla u \in L^2(\overline{\mathfrak{D}})^9$. Let $\pi \in L^2(\overline{\mathfrak{D}})$, $p \in (1,\infty)$, $q \in (1,2)$, $f : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}^3$ a function with $f|\mathfrak{D}_T \in L^p(\mathfrak{D}_T)^3$ for $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$, $f|B_S^c \in L^q(B_S^c)^3$ for some $S \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_S$.

Suppose that the pair (u,π) is a weak solution of the Navier-Stokes system with Oseen and rotational terms, and with right-hand side f, that is,

$$\begin{aligned}&\int_{\overline{\mathfrak{D}}^c}\Big(\nabla u\cdot\nabla\varphi + \big(\,\tau\,(u\cdot\nabla)u + \tau\,\partial_1 u - (\omega\times z)\cdot\nabla u + \omega\times u\,\big)\cdot\varphi + \pi\operatorname{div}\varphi\Big)\,dz \\ &= \int_{\overline{\mathfrak{D}}^c} f\cdot\varphi\, dz \quad \text{for } \varphi \in C_0^\infty(\overline{\mathfrak{D}}^c)^3, \quad \operatorname{div} u = 0.\end{aligned}$$

Then

$$u_j(y) = \mathfrak{R}_j\big(\,f - \tau\,(u\cdot\nabla)u\,\big)(y) + \mathfrak{B}_j(u,\pi)(y) \tag{2.11}$$

for $j \in \{1,\, 2,\, 3\}$, a.e. $y \in \overline{\mathfrak{D}}^c$,.

For definition of $E_{4j}(x)$, $\mathfrak{Z}_{jk}(y,z)$, $\mathfrak{R}(f)$, $\mathfrak{B}_j(y)$ see [17].

3 Asymptotic behavior of the viscous fluids in the presence of Coriolis forces

For more details see [69].

3.1 Stokes problem in the whole space $\mathbb{R}^3$

We consider the Stokes problem with the Coriolis force in the whole space $\mathbb{R}^3$. The system reads

$$\begin{aligned} -\mu\Delta u + \omega \times u &= \nabla p + f, \\ \text{div } u &= 0, \end{aligned} \tag{3.1}$$

where ω is given and we set $\omega = \lambda g$, $\lambda > 0$. We assume for the simplicity that $g = e_2$. The motivation of the problem can be found in the work of Weinberger see [80, 81].

Theorem 10. Let $f \in L^q(\mathbb{R}^3)$, $1 < q < \infty$, there exists a pair of functions (u, p) with $u_1, u_3 \in L^q(\mathbb{R}^3)$, $u \in D^{2,q}(\mathbb{R}^3)$,$\nabla p \in L^q(\mathbb{R}^3)$ satisfying the Stokes system (3.1) and moreover

$$|u|_{2,q} + |p|_{1,q} + \|u_1\|_q + \|u_2\|_q \le c\|f\|_q. \tag{3.2}$$

Further, if $1 < q < 3$ then

$$|u_i|_{1,q} + |u_2|_{1,3q/(3-q)} + |u|_{2,q} + |p|_{1,q} + \|u_1\|_q + \|u_2\|_q \le c\|f\|_q, i = 1, 3. \tag{3.3}$$

Finally, if $1 < q < 3/2$ then

$$|u_i|_{1,q} + \|u_i\|_q + |u_2|_{1,3q/(3-q)} + |u_2|_{3q/(3-2q)} + |u|_{2,q} + |p|_{1,q} \le c\|f\|_q, i = 1, 3. \tag{3.4}$$

3.2 Stokes problem in an exterior domain

We consider the Stokes problem in an exterior domain Ω of class C^{m+2}, $m \ge 0$ with data $f \in C_0^\infty(\bar{\Omega}), v_* \in W^{m+2-1/q,q}(\partial\Omega)$. The governing equations are

$$\begin{aligned} -\mu\Delta u + \lambda g \times u &= \nabla p + f, \\ \text{div } u &= 0, \\ u|_{\partial\Omega} &= v_*, \\ \lim_{x\to\infty} u &= 0. \end{aligned} \tag{3.5}$$

Theorem 11. Let Ω be an exterior domain in $\mathbb{R}^3$ of class $C^{m+2}, m \ge 0$. Given $f \in W^{m,p}(\Omega), v_* \in W^{m+2-1/q,q}(\partial\Omega)$, $1 < q < 3/2$ there exists one and only one solution (u, p) to the Stokes problem such that

$$\begin{aligned} &u_i - v_{*i} \in W^{m,q}(\Omega) \cap \Big\{ \textstyle\bigcap_{l=0}^m [D^{l+1,q}(\Omega) \cap D^{l+2,q}] \Big\}, i = 1, 3, \\ &u_2 - v_{*2} \in W^{m,3q/(3-2q)}(\Omega) \cap \Big\{ \textstyle\bigcap_{l=0}^m [D^{l+1,3q/(3-q)}(\Omega) \cap D^{l+2,q}] \Big\}, \\ &p \in \textstyle\bigcap_{l=0}^m D^{l+1,q}(\Omega). \end{aligned} \tag{3.6}$$

Moreover, (u,p) satisfy

$$\|u_i\|_{m,q} + \|u_2\|_{m,3q/(3-2q)} + \sum_{i=0}^{m}\{|u_i\|_{l+1,q} + \|u_2\|_{l+1,3q/(3-q)} + |u|_{l+2,q} + |p|_{l+1,q}\}$$
$$\leq c(\|f\|_{m,q} + \|v_*\|_{m+2-1/q,q,\partial\Omega}),\ i=1,3, \tag{3.7}$$

where c depends on m,n,q,Ω. Moreover, let $f \in L^1(\Omega)$ then for $x \in B_1$ (by B_1 we denote the ball with radius 1 and B_1^c its complement)

$$\begin{array}{l} |u(x)| \leq c_{m,n}|x|^{-1} \\ |D^\beta u(x)| \leq c_{m,n}|x|^{-1-\beta}, 0 < |\beta| \geq 2 \end{array} \tag{3.8}$$

and for $x \in B_1^c, 0 \leq \beta \leq 2$

$$|D^\beta u(x)| \leq c_{m,n}|x|^{-2-\beta}. \tag{3.9}$$

3.3 Oseen problem in the whole space $\mathbb{R}^3$

We investigate the Oseen problem with the Coriolis force in the whole space $\mathbb{R}^3$. The system reads

$$\begin{array}{rcl} \frac{\partial u}{\partial x_2} - \mu\Delta u + \lambda g \times u &=& \nabla p + f, \\ \text{div } u &=& 0. \end{array} \tag{3.10}$$

We assume for the simplicity $g = e_2$.

Theorem 12. Let $f \in L^q(\mathbb{R}^3)$, $1 < q < \infty$, there exists a pair of functions (u,p) with $u_1, u_3, \partial u/\partial x_2 \in L^q(\mathbb{R}^3)$, $u \in D^{2,q}(\mathbb{R}^3)$,$\nabla p \in L^q(\mathbb{R}^3)$ satisfying the Oseen system (3.10) and moreover

$$\left\|\frac{\partial u}{\partial x_2}\right\|_q + \left\|\frac{\partial u_i}{\partial x_l}\right\|_q + |p|_{1,q} + \|u_1\|_q + \|u_2\|_q \leq c\|f\|_q, i = 1,3. \tag{3.11}$$

Further, if $1 < q < 4$ then

$$|u_i|_{1,q} + |u_2|_{1,4q/(4-q)} + |u|_{2,q} + |p|_{1,q} + \left\|\frac{\partial u}{\partial x_2}\right\|_q + \left\|\frac{\partial u_i}{\partial x_l}\right\|_q \leq c\|f\|_q, i,l = 1,3. \tag{3.12}$$

Finally, if $1 < q < 2$ then

$$|u_i|_{1,q} + |u_2|_{1,4q/(4-q)} + |u_2|_{2q/(2-q)} + |u|_{2,q} + |p|_{1,q} + \left\|\frac{\partial u}{\partial x_2}\right\|_q + \left\|\frac{\partial u_i}{\partial x_l}\right\|_q \leq c\|f\|_q, i,l = 1,3. \tag{3.13}$$

3.4 Oseen problem in an exterior domain

We consider the Oseen problem in an exterior domain Ω of class C^{m+2}, $m \geq 0$ with data $f \in C_0^\infty(\overline{\Omega}), v_* \in W^{m+2-1/q,q}(\partial\Omega)$, $v_\infty \neq 0$. The governing equations are

$$\begin{array}{rcl} \frac{\partial u}{\partial x_2} - \mu\Delta u + \lambda g \times u &=& \nabla p + f, \\ \text{div } u &=& 0, \\ u|_{\partial\Omega} &=& v_*, \\ \lim_{|x|\to\infty} u &=& v_\infty. \end{array} \tag{3.14}$$

Theorem 13. Let Ω be an exterior domain in $\mathbb{R}^3$ of class $C^{m+2}, m \geq 0$. Given $f \in W^{m,p}(\Omega), v_* \in W^{m+2-1/q,q}(\partial\Omega)$, $1 < q < 2$, $v_\infty \in R^3$ there exists one and only one solution (u, p) to the Oseen problem such that

$$\begin{aligned}
&u_i - v_{*i} \in W^{m,q}(\Omega) \cap \Big\{ \bigcap_{l=0}^m [D^{l+1,q}(\Omega) \cap D^{l+2,q}] \Big\}, i = 1, 3, \\
&u_2 - v_{*2} \in W^{m,2q/(2-q)}(\Omega) \cap \Big\{ \bigcap_{l=0}^m [D^{l+2,q}] \Big\}, \\
&\tfrac{\partial u}{\partial x_2} \in W^{m,q}(\Omega), \\
&\tfrac{\partial u_2}{\partial x_l} \in W^{m,4q/(4-q)}(\Omega), \\
&p \in \textstyle\bigcap_{l=0}^m D^{l+1,q}(\Omega).
\end{aligned} \tag{3.15}$$

Moreover, (u, p) satisfy

$$\begin{aligned}
&\|u_i - v_{\infty i}\|_{m,q} + \|u_2 - v_{\infty 2}\|_{m,2q/(2-q)} + \textstyle\sum_{i=0}^m \{|u_i\|_{l+1,q} + \|\frac{\partial u_2}{\partial x_l}\|_{l,4q/(4-q)} + |u|_{l+2,q} + |p|_{l+1,q}\} \\
&\leq c(\|f\|_{m,q} + \|v_* - v_\infty\|_{m+2-1/q,q,\partial\Omega}),\ i, l = 1, 3,
\end{aligned} \tag{3.16}$$

where c depends on m, n, q, Ω.

4 Compressible motion

There are several results concerning one dimensional situation, let us mention work of Kazhikov and Shelukhin in 1977 [53], who firstly proved the global existence in one dimension for smooth initial data and for discontinuous data we can refer to work of Serre and Hoff see [73, 47]. The significant progress was made during last twenty years on the compressible Navier-Stokes system or Navier-Stokes-Fourier system in dimension 2 and 3. We mention the work of Matsumura, Nishida [63, 64, 65] and fundamental work of P. L. Lions [61] which was extended by Feireisl [33, 34, 35]. We would like to mention that for large initial data the global existence and large-time behavior of solutions to the Navier-Stokes-Fourier system have also been obtained in the spherically symmetric case (see [48, 47, 37]). For other references see [50, 51, 62, 70, 75, 76, 77]. In case when viscosity coefficients dependent on the density and viscosity coefficients vanish on vacuum and new entropy inequality was proved to provide the regularity for the density see Bresch and Desjardins [11, 12].

Recently Mellet and Vasseur [66] proved the existence of a solution for the barotropic Navier-Stokes system, when the viscosity coefficients are density dependent functions related by the Bresch-Desjardins relation [11], [12], for any "physical" adiabatic exponent $\gamma > 1$.

4.1 Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent viscosity isentropic case

For more details see [70].

In this part we consider the following system of equations

$$\begin{cases} \frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial\xi} = 0, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial\xi}(\rho u^2 + p) = \frac{\partial}{\partial\xi}\big(\mu\frac{\partial u}{\partial\xi}\big) - \rho g, \end{cases} \tag{4.1}$$

where $t > 0$, $0 < \xi < y(t)$. The unknown functions ρ, u represent the density and the velocity, respectively, $p = a\rho^\gamma$ and $\mu = b\rho^\beta$ are the pressure and the viscosity coefficient, a, b are positive constants, $\gamma > 1$, and $0 < \beta < \gamma - 1$. The constant g is the gravitation constant, $\xi = 0$ is the fixed boundary

$$u(t, 0) = 0,$$

and $\xi = y(t)$ is the free boundary, i.e. the interface of the gas and the vacuum;

$$\frac{dy}{dt} = u(t, y(t)),\ \Big(p - \mu\frac{\partial u}{\partial\xi}\Big)(t, y(t)) = 0.$$

We rewrite the equations in the Lagrangean mass coordinate:

$$x = \int_0^\xi \rho(t, \varsigma)d\varsigma.$$

Assuming that

$$\int_0^{y(t)} \rho(t, \xi)d\xi = 1,$$

the above problem is transformed to the following fixed boundary problem

$$\begin{cases} \frac{\partial\rho}{\partial t} + \rho^2\frac{\partial u}{\partial x} = 0, \\ \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\Big(\mu\rho\frac{\partial u}{\partial x}\Big) - g, \end{cases} \tag{4.2}$$

in $t > 0$ and $0 < x < 1$, where $p = a\rho^\gamma$, $\mu = b\rho^b$ with the boundary conditions

$$u(t, 0) = 0,\ \Big(p - \mu\rho\frac{\partial u}{\partial x}\Big)(t, 1) = 0 \tag{4.3}$$

and the initial condition

$$(\rho, u)(0, x) = (\rho_0, u_0)(x),\ 0 \le x \le 1. \tag{4.4}$$

We consider the following assumptions (A.1), (A.2) and (A.3) for the initial data and β:

- (A.1) $\rho_0 \in Lip[0, 1]$ and $\rho_0(x) \ge \underline{\rho}$ ($\underline{\rho}$ is a positive constant),
- (A.2) $u_0 \in C^1[0, 1]$ and $\frac{du_0}{dx} \in Lip[0, 1]$,
- (A.3) $0 < \beta < \frac{1}{3}$.

Definition 3.
A couple (ρ, u) is called a global weak solution for (4.2) if

$$\rho,\, u \in L^\infty([0,T]\times[0,1]) \cap C^1([0,T]; L^2(0,1)), \tag{4.5}$$

$$\rho^{\beta+1} u_x \in L^\infty([0,T]\times[0,1]) \cap C^{1/2}([0,T]; L^2(0,1)), \tag{4.6}$$

for any T,

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial x} = 0, \tag{4.7}$$

for a.e. $x \in (0,1)$ and for any $t \geq 0$, and

$$\int_0^1 [\phi u_t - \phi_x(p - \mu\rho u_x) + \phi g] dx = 0 \tag{4.8}$$

with $\phi \in C_0^\infty((0,1])$ and for a.e. $t \in [0,T]$.

Theorem 14. If the assumptions (A.1)–(A.3) hold, then the initial - boundary value problem (4.2), (4.4), (4.3) admits a global weak solution in the sense (4.5) - (4.8).

Theorem 15. Let us assume (A.1), (A.2), (A.3) and let there exists a constant $C(T)$ such that

$$\frac{1}{C(T)} \leq \rho(t,x) \leq C(T), |u_x(t,x)| \leq C(T). \tag{4.9}$$

4.2 Free boundary problem for the equation of spherically symmetric motion of viscous gas

See [71, 72]. We consider the following model of compressible symmetrical motion, which are described by the following system of equations

$$\begin{array}{rcl} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{2}{r}\rho u &=& 0, \\ \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}) + \frac{\partial p}{\partial r} &=& \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2}{r^2}u\right) - \frac{\rho M}{r^2}, \\ p &=& a\rho^\gamma, \end{array} \tag{4.10}$$

where ν, a, γ are positive constants and $1 < \gamma \leq 2$, ρ is the density and u the velocity field. We consider the boundary condition

$$u|_{r=1} = 0 \tag{4.11}$$

and the initial conditions

$$\rho|_{t=0} = \rho^0(r), u|_{t=0} = u^0(r). \tag{4.12}$$

We are interested in the class of initial data which includes the stationary solutions

$$\rho = \left\{ \begin{array}{rl} [\frac{(\gamma-1)M}{a\gamma}\left(\frac{1}{r} - \frac{1}{R}\right)]^{1/(\gamma-1)} & (r \leq R), \\ 0 & (R < r), \end{array} \right. \quad u = 0. \tag{4.13}$$

We rewrite the equations in the Lagrange mass coordinates:

$$x = 4\pi \int_0^r \rho(t,s) s^2 ds.$$

The above problem is transformed to the following fixed boundary problem

$$\begin{cases} \frac{\partial \rho}{\partial t} + 4\pi\rho^2 \frac{\partial}{\partial x}(r^2 u) = 0, \\ \frac{\partial u}{\partial t} + 4\pi r^2 \frac{\partial p}{\partial x} = 16\pi^2 \nu \frac{\partial}{\partial x}\left(r^4 \rho \frac{\partial u}{\partial x}\right) - 2\nu \frac{u}{r^2 \rho} - \frac{M}{r^2}, \\ p = a\rho^\gamma, \end{cases} \tag{4.14}$$

where

$$r = [1 + \frac{3}{4\pi}\int_0^x \frac{d\xi}{\rho(t,\xi)}]^{1/3}.$$

By normalizing the total mass, we consider the equations (4.14) in $0 \leq x \leq 1$ with the boundary conditions

$$u|_{x=0} = 0, \rho|_{x=1} = 0 \tag{4.15}$$

and the initial conditions

$$\rho|_{t=0} = \rho_0(x), u|_{t=0} = u_0(x). \tag{4.16}$$

We consider the following assumptions

- (A.1) $\rho_0 \in C[0,1]$ and $\rho_0(x) \geq 0$ for $x \in [0,1), \rho_0(1) = 0$, total variation $[\rho] < \infty$ and there exists a monotone decreasing function $\lambda(x)$ such that $0 \leq \lambda(x) \leq \rho(x)$ and $\int_0^1 \frac{dx}{\lambda(x)} < \infty$,
- (A.2) $u_0 \in C[0,1]$,
- (A.3) assume $\rho_0 = a\rho_0^\gamma \in C^1[0,1]$ and $u_0 = 0$.

Definition 4.
A couple (ρ, u) is called a global weak solution for (4.14) if

$$\rho,\, u \in L^\infty([0,T] \times [0,1]) \cap C^1([0,T]; L^2(0,1)), \tag{4.17}$$

$$\rho u_x \in L^\infty([0,T] \times [0,1]) \cap C^{1/2}([0,T]; L^2(0,1)), \tag{4.18}$$

$$\text{there exists a constant } C(T) \text{ with } \frac{1}{C(T)}\rho_0(x) \leq \rho(t,x) \leq C(T)\rho_0(x), \tag{4.19}$$

for a.e. $x \in (0,1)$ and for any $t \geq 0$, and satisfying

$$\frac{\partial \rho}{\partial t} + 4\pi r^2 \rho^2 u_x + \frac{2u\rho}{r} = 0 \text{ for a.e. } x \in (0,1) \text{ and for any } t \leq 0, \tag{4.20}$$

$$\int_0^1 [\phi u_t - (4\pi r^2 \phi_x + \tfrac{2\phi}{r\rho})p + 16\pi^2 \nu \phi_x r^4 \rho u_x + 2\nu\phi \tfrac{u}{r^2\rho} + \phi \tfrac{M}{r^2}]dx = 0 \tag{4.21}$$

with $\phi \in C_0^\infty(0,1)$ and for any $t \geq 0$,

$$\rho(0,x) = \rho_0(x) \text{ and } u(0,x) = u_0(x) \text{ for any } x \in [0,1], \tag{4.22}$$

and

$$u(t,0) = 0 \text{ for any } t \geq 0. \tag{4.23}$$

Theorem 16. Assume (A1), (A2), (A3). Let (ρ_1, u_1) and (ρ_2, u_2) be solutions of (4.20),(4.21), (4.22), (4.23) satisfying (4.17, (4.18), (4.19) for any T. Then we have $\rho_1 = \rho_2$, $u_1 = u_2$.

Let us consider the following additional assumption

- (A4) $\int_0^1 \rho_0 \Big(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}\Big)^2 \frac{dx}{(1-x)^\mu} < +\infty$ for some $\mu > \frac{3}{4}$.

Theorem 17. Let (A1)–(A4) be satisfied, suppose that a is sufficiently small and let

$$La < M, \text{ where } L = L(\gamma, \nu, E^*, M^*, \bar{R}) \tag{4.24}$$

provided $E_0 \leq E^*$ and $M \leq M^*$. Let us assume that the initial pressure p_0 satisfies

$$La(1-x) \leq p_0 \leq M(1-x), \tag{4.25}$$

where

$$E_0 = \int_0^1 \Big(\frac{1}{2}u_0^2 + \frac{1}{\gamma - 1}\frac{p_0}{\rho_0} - \frac{M}{r_0}\Big)dx,$$

where L is a suitable constant depending on $\gamma, \nu, E^*, M^*, \bar{R}$ for definition of $E^*, \bar{R}$ see [71, 72]. Then the global solution (ρ, u) satisfies

$$\begin{array}{l} \int_0^1 u(x,t)^2 dx \to 0, \\ \int_0^1 \rho_0(x)\Big(\frac{1}{\rho(x,t)} - \frac{1}{\bar{\rho}(x)}\Big)^2 \frac{dx}{(1-x)^{3/4}} \to 0 \text{ as } t \to \infty. \end{array} \tag{4.26}$$

4.3 Global existence of solutions for the one-dimensional motions of a compressible viscous gas with radiation: an "infrarelativistic model"

The aim of radiation hydrodynamics is to include the effects of radiation into the hydrodynamical framework. When the equilibrium holds between the matter and the radiation, a simple way to do that is to include local radiative terms into the state functions and the transport coefficients. One knows from quantum mechanics that radiation is described by its quanta, the photons, which are massless particles traveling at the speed c of light, characterized by their frequency ν, their energy $E = h\nu$ (where h is the Planck's constant), their momentum $\vec{p} = \frac{\hbar\nu}{c}\,\vec{\Omega}$, where $\vec{\Omega}$ is a unit vector. Statistical mechanics allows us to describe macroscopically an assembly of massless photons of energy E and momentum $\vec{p}$ by using a distribution function: the radiative intensity $I(r, t, \vec{\Omega}, \nu)$. Using this fundamental quantity, one can derive global quantities by integrating with respect to the angular and frequency variables: the spectral radiative energy density $E_R(r,t)$ per unit volume is then $E_R(r,t) := \frac{1}{c}\int\int I(r,t,\vec{\Omega},\nu)\; d\Omega\; d\nu$, and the spectral radiative flux $\vec{F}_R(r,t) = \int\int \vec{\Omega}\; I(r,t,\vec{\Omega},\nu)\; d\Omega\; d\nu$. If matter is in thermodynamic equilibrium at constant temperature T and if radiation is also in thermodynamic equilibrium with matter, its temperature is also T and statistical mechanics tells us that the distribution function for photons is given by the Bose-Einstein statistics with zero chemical potential.

In the absence of radiation, one knows that the complete hyrodynamical system is derived from the standard conservation laws of mass, momentum and energy by using the Boltzmann's equation satisfied by the $f_m(r, \vec{v}, t)$ and Chapman-Enskog's expansion [38]. One gets then formally the compressible Navier-Stokes system

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = -\nabla \cdot \overset{\Rightarrow}{\Pi} + \vec{f}, \\ \partial_t(\rho \varepsilon) + \nabla \cdot (\rho \varepsilon \vec{u}) = -\nabla \vec{q} - \overset{\Rightarrow}{D} : \overset{\Rightarrow}{\Pi} + g, \end{cases} \tag{4.27}$$

where $\overset{\Rightarrow}{\Pi} = -p(\rho, T) \ \overset{\Rightarrow}{I} + \overset{\Rightarrow}{\pi}$ is the material stress tensor for a Newtonian fluid with the viscous contribution $\overset{\Rightarrow}{\pi} = 2\mu \ \overset{\Rightarrow}{D} + \lambda \nabla \cdot \vec{u} \ \ \overset{\Rightarrow}{I}$ with $3\lambda + 2\mu \geq 0$ and $\mu > 0$, and the strain tensor $\overset{\Rightarrow}{D}$ such that $\overset{\Rightarrow}{D}_{ij} (\vec{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. $\vec{q}$ is the thermal heat flux and $\vec{F}$ and g are external force and energy source terms.

When radiation is present, the terms $\vec{f}$ and g include the coupling terms between the matter and the radiation (neglecting any other external field), depending on I, and I is driven by a transport equation: the so called radiative transfert integro-differential equation discussed by Chandrasekhar in [13].

Supposing that the matter is at LTE, the coupled system reads

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \\ \partial_t(\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = -\nabla \cdot \overset{\Rightarrow}{\Pi} - \vec{S_F}, \\ \partial_t(\rho \varepsilon) + \nabla \cdot (\rho \varepsilon \vec{u}) = -\nabla \vec{q} - \overset{\Rightarrow}{D} : \overset{\Rightarrow}{\Pi} - S_E, \\ \dfrac{1}{c}\dfrac{\partial}{\partial t} I\left(r, t, \vec{\Omega}, \nu\right) + \vec{\Omega} \cdot \nabla I\left(r, t, \vec{\Omega}, \nu\right) = S_t\left(r, t, \vec{\Omega}, \nu\right), \end{cases} \tag{4.28}$$

where the coupling terms are

$$S_t(r, t, \vec{\Omega}, \nu) = \sigma_a(\nu, \vec{\Omega}, \rho, T, \frac{\vec{\Omega} \cdot \vec{u}}{c}) \left[B(\nu, T) - I\left(r, t, \vec{\Omega}, \nu\right)\right]$$

$$+ \int\int \sigma_s\left(r, t, \rho, \vec{\Omega}' \cdot \vec{\Omega}, \nu' \to \nu\right) \left\{\frac{\nu}{\nu'} \ I\left(r, t, \vec{\Omega}', \nu'\right) \ I\left(r, t, \vec{\Omega}, \nu\right)\right.$$

$$\left. - \sigma_s\left(r, t, \rho, \vec{\Omega} \cdot \vec{\Omega}', \nu \to \nu'\right) I\left(r, t, \vec{\Omega}, \nu\right) I\left(r, t, \vec{\Omega}', \nu'\right)\right\} \ d\Omega' \ d\nu',$$

the radiative energy source

$$S_E(r, t) := \int\int S_t(r, t, \vec{\Omega}, \nu) \ d\Omega \ d\nu,$$

the radiative flux

$$\vec{S_F}(r, t) := \frac{1}{c}\int\int \vec{\Omega} \ S_t(r, t, \vec{\Omega}, \nu) \ d\Omega \ d\nu.$$

In the radiative transfer equation (the last equation (4.28)) the functions σ_a and σ_s describe in a phenomenological way the absorption-emission and scattering properties of the interaction photon-matter and the Planck's function $B(\nu, \theta)$ describe the frequency-temperature black body distribution.

In 1D the previous system reads

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + (\rho v^2)_y + p_y = \mu v_{yy} - (S_F)_R, \\ \left[\rho\left(e + \frac{1}{2}\,v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}\,v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \\ \frac{1}{c} I_t + \omega I_y = S, \end{cases} \tag{4.29}$$

In this study we only consider an "infra-relativistic" model of compressible Navier - Stokes system for a 1D flow coupled to a the radiative transfer equation. As in the model studied by Amosov [1], we suppose that the fluid motion is so small with respect to the velocity of light c that one can drop all the $\frac{1}{c}$ factors in the previous formulation. We get then

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + (\rho v^2)_y + p_y = \mu v_{yy}, \\ \left[\rho\left(e + \frac{1}{2}\,v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}\,v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \\ \omega I_y = S, \end{cases} \tag{4.30}$$

in the domain $\mathcal{O} \times \mathbb{R}_+$ with $\mathcal{O} := (0, L)$, where the density ρ, the velocity v, the temperature θ depend on the coordinates (y, τ). The radiative intensity $I = I(y, \tau, \nu, \omega)$, depends also on two extra variables: the radiation frequency $\nu \in \mathbb{R}_+$ and the angular variable $\omega \in S^1 := [-1, 1]$. The state functions are the pressure p, the internal energy e, the heat conductivity κ and the viscosity coefficient μ .

In the standard radiative transfer equation, the source term is

$$S(y, \tau, \nu, \omega) := S_{a,e}(y, \tau, \nu, \omega) + S_s(y, \tau, \nu, \omega), \tag{4.31}$$

where the absorption-emission term is

$$S_{a,e}(y, \tau, \nu, \omega) = \sigma_a(\nu, \omega; \rho, \theta)\,[B(\nu; \theta) - I(y, \tau, \nu, \omega)]\,, \tag{4.32}$$

and the scattering term is

$$S_s(y, \tau, \nu, \omega) = \sigma_s(\nu; \rho, \theta)\ \left[\tilde{I}(y, \tau, \nu, \theta) - I(y, \tau, \nu, \omega)\right], \tag{4.33}$$

where $\tilde{I}(y, \tau, \nu) := \frac{1}{2}\int_{-1}^{1} I((y, \tau, \nu, \omega)\ d\omega$ and B is a function of temperature and frequency describing the equilibrium state. We suppose that $\sigma_a(\nu, \omega; \rho, \theta)$ and $\sigma_s(\nu; \rho, \theta)$

are positive functions. We also define the radiative energy

$$E_R := \int_{-1}^{1} \int_{0}^{\infty} I(y, \tau, \nu, \omega) \; d\nu \; d\omega, \tag{4.34}$$

the radiative flux

$$F_R := \int_{-1}^{1} \int_{0}^{\infty} \omega I(y, \tau, \nu, \omega) \; d\nu \; d\omega, \tag{4.35}$$

and the radiative energy source

$$(S_E)_R := \int_{-1}^{1} \int_{0}^{\infty} S(y, \tau, \nu, \omega) \; d\nu \; d\omega. \tag{4.36}$$

It is convenient to switch now to Lagrange (mass) coordinates relative to matter flow: $(y, \tau) \to (x, t)$. With the transformation rules [8]: $\partial_y \to \rho \partial_x$ and $\partial_\tau + v \partial_y \to \partial_x$, the previous system reads now

$$\begin{cases} \eta_t = v_x, \\ v_t = \sigma_x, \\ \left(e + \dfrac{1}{2} \; v^2\right)_t = (\sigma v - q)_x - \eta (S_E)_R, \\ \omega I_x = \eta S, \end{cases} \tag{4.37}$$

in the transformed domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0, M)$ (M is the total mass of matter), where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v, the temperature θ and the radiative intensity I depend now on the Lagrangian mass coordinates (x, t). We also denote by $\sigma := -p + \mu \; \frac{v_x}{\eta}$ the stress and by $q := -\kappa \; \frac{\theta_x}{\eta}$ the heat flux, and the source term in the last equation is

$$\begin{aligned} S(x, t, \nu, \omega) = \sigma_a(\nu, \omega; \eta, \theta) \left[B(\nu; \theta) - I(x, t; \nu, \omega)\right] \\ + \sigma_s(\nu; \eta, \theta) \; \left[\tilde{I}(x, t, \nu) - I(x, t, \nu, \omega)\right], \end{aligned} \tag{4.38}$$

We consider Dirichlet-Neumann boundary conditions for the fluid unknowns

$$\begin{cases} v|_{x=0} = v|_{x=M} = 0, \\ q|_{x=0} = q|_{x=M} = 0, \end{cases} \tag{4.39}$$

and transparent boundary conditions for the radiative intensity

$$\begin{cases} I|_{x=0} = 0 \quad \text{for } \omega \in (0, 1) \\ I|_{x=M} = 0 \quad \text{for } \omega \in (-1, 0), \end{cases} \tag{4.40}$$

for $t > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad \theta|_{t=0} = \theta^0(x), \quad \text{on } \; \Omega. \tag{4.41}$$

and

$$I|_{t=0} = I(x, \nu, \omega) \quad \text{on} \quad \Omega \times \mathbb{R}_+ \times S^1. \tag{4.42}$$

Pressure and energy are related by the thermodynamical relation

$$e_\eta(\eta, \theta) = -p(\eta, \theta) + \theta p_\theta(\eta, \theta). \tag{4.43}$$

Finally we assume that state functions e, p and κ (resp. $\sigma_{a,e}$ and σ_s) are C^2 (resp C^0) functions of their arguments for $0 < \eta < \infty$ and $0 \leq \theta < \infty$, and we suppose the following growth conditions

$$\begin{cases} e(\eta, 0) \geq 0, \quad c_1(1+\theta^r) \leq e_\theta(\eta, \theta) \leq C_1(1+\theta^r), \\ -c_2\eta^{-2}(1+\theta^{1+r}) \leq p_\eta(\eta, \theta) \leq -C_2\eta^{-2}(1+\theta^{1+r}), \\ |p_\theta(\eta, \theta)| \leq C_3\eta^{-1}(1+\theta^r), \\ c_4(1+\theta^{1+r}) \leq \eta p(\eta, \theta) \leq C_4(1+\theta^{1+r}), \quad p_\eta(\eta, \theta_0) \leq 0, \\ 0 \leq p(\eta, \theta) \leq C_5(1+\theta^{1+r}), \\ c_6(1+\theta^q) \leq \kappa(\eta, \theta) \leq C_6(1+\theta^q), \\ |\kappa_\eta(\eta, \theta)| + |\kappa_{\eta\eta}(\eta, \theta)| \leq C_7(1+\theta^q), \\ \eta\sigma_a(\nu, \omega; \eta, \theta) B^m(\nu; \theta) \leq C_8|\omega|\theta^{\alpha+1} f(\nu, \omega) \quad \text{for } m = 1, 2, \\ \sigma_a(\nu, \omega; \eta, \theta) \leq C_9 g(\nu, \omega), \\ \left[\left|(\sigma_a)_\eta\right| + |(\sigma_a)_\theta|\right](\nu, \omega; \eta, \theta)\left[1 + B(\nu; \theta) + |B_\theta(\nu; \theta|\right] \leq C_{10} h(\nu, \omega), \\ \sigma_s(\nu; \eta, \theta) \leq C_{11} k(\nu, \omega), \end{cases} \tag{4.44}$$

where $r \in [0, 1]$, $q \geq 2r + 1$, $0 \leq \alpha < r$, the numbers c_j, C_j, $j = 1, ..., 10$ are positive constants and the functions f, g, h, k are such that

$$f, g, h \ \in L^1(\mathbb{R}_+ \times S^1) \cap L^\infty(\mathbb{R}_+ \times S^1),$$

and

$$k \ \in L^{1+\gamma}(\mathbb{R}_+ \times S^1) \cap L^\infty(\mathbb{R}_+ \times S^1),$$

for an arbitrary small $\gamma > 0$.

We suppose also that the viscosity coefficient is a positive constant.

We study weak solutions for the above problem with properties

$$\begin{cases} \eta \in L^\infty(Q_T), \quad \eta_t \in L^\infty([0,T], L^2(\Omega)), \\ v \in L^\infty([0,T], L^4(\Omega)), \quad v_t \in L^\infty([0,T], L^2(\Omega)), \quad v_x \in L^\infty([0,T], L^2(\Omega)), \\ \sigma_x \in L^\infty([0,T], L^2(\Omega)), \\ \theta \in L^\infty([0,T], L^2(\Omega)), \quad \theta_x \in L^\infty([0,T], L^2(\Omega)), \\ I \in L^1(\Omega \times \mathbb{R}_+ \times S^1) \end{cases} \tag{4.45}$$

where $Q_T := \Omega \times (0,T)$.

We also assume the following conditions on the data:

$$\begin{cases} \eta^0 > 0 \text{ on } \Omega, \ \eta^0 \in L^1(\Omega), \\ v_0 \in L^2(\Omega), \quad v_x^0 \in L^2(\Omega), \\ \theta^0 \in L^2(\Omega), \quad \inf_\Omega \theta^0 \geq 0. \end{cases} \tag{4.46}$$

Then our definition of a weak solution for the previous problem is the following

Definition 5. *We call (η, v, θ, I) a weak solution of (4.37) if it satisfies*

$$\eta(x,t) = \eta^0(x) + \int_0^t v_x \, ds, \tag{4.47}$$

for a.e. $x \in \Omega$ and any $t > 0$, and if, for any test function $\phi \in L^2([0,T], H^1(\Omega))$ with $\phi_t \in L^1([0,T], L^2(\Omega))$ such that $\phi(\cdot, T) = 0$, one has

$$\int_Q \left[\phi_t v + \phi_x p - \frac{\mu \phi_x}{\eta} v_x\right] \, dx \, dt$$
$$= \int_\Omega \phi(0,x) \ v^0(x) \ dx, \tag{4.48}$$

$$\int_Q \left[\phi_t \left(e + \frac{1}{2} \ v^2\right) + \phi_x \left(\sigma v - q\right) + \phi \eta (S_E)_R\right] \, dx \, dt$$
$$= \int_\Omega \phi(0,x) \left(e^0(x) + \frac{1}{2} \ v^0(x)^2\right) \, dx, \tag{4.49}$$

and if, for any test function $\psi \in H^1(\Omega) \times L^1(\mathbb{R}_+ \times S^1))$, one has

$$\int_{\mathbb{R}_+ \times S^1} \left[\psi_x \omega I + \psi \eta S\right] \, d\nu \, d\omega \, dx = 0. \tag{4.50}$$

In the following we use the following notation for the integrated radiative intensity

$$\mathcal{I}(x,t) := \int_0^\infty \int_{S^1} I(x,t;\omega,\nu) \, d\omega \, d\nu.$$

Then our main result is the following

Theorem 18. Suppose that the initial data satisfy (4.46) and that T is an arbitrary positive number. Then the problem (4.37), (4.39)–(4.42) possesses a global weak solution satisfying (4.45) together with properties (4.47), (4.48) and (4.49).

Moreover one has the uniqueness result

Theorem 19. Suppose that the initial data satisfy (4.46) and that T is an arbitrary positive number. Then the problem (4.37),(4.39)–(4.42) possesses a global unique weak solution satisfying (4.45) together with properties (4.47), (4.48) and (4.49).

5 Laplace equation and Stokes problem in the half space

For more details see [2, 6, 7].

5.1 Notations

For any real number $p > 1$, we always take p' to be the Hölder conjugate of p, *i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let Ω be an open set of $\mathbb{R}^N$, $N \geq 2$. Writing a typical point $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, we will especially look on the upper half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N;\ x_N > 0\}$. We let $\overline{\mathbb{R}^N_+}$ denote the closure of $\mathbb{R}^N_+$ in $\mathbb{R}^N$ and let $\Gamma = \{x \in \mathbb{R}^N;\ x_N = 0\} \equiv \mathbb{R}^{N-1}$ denote its boundary. Let $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$ denote the Euclidean norm of x, we will use two basic weights

$$\varrho = (1 + |x|^2)^{1/2} \quad \text{and} \quad \lg\varrho = \ln(2 + |x|^2).$$

Weighted Sobolev spaces

For any nonnegative integer m, real numbers $p > 1$, α and β, we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \Big\{ & u \in \mathcal{D}'(\Omega);\ 0 \leq |\lambda| \leq k,\ \varrho^{\alpha-m+|\lambda|}\,(\lg\varrho)^{\beta-1}\,\partial^\lambda u \in L^p(\Omega); \\ & k+1 \leq |\lambda| \leq m,\ \varrho^{\alpha-m+|\lambda|}\,(\lg\varrho)^{\beta}\,\partial^\lambda u \in L^p(\Omega) \Big\}, \end{aligned} \tag{5.1}$$

where

$$k = \begin{cases} -1 & \text{if } \frac{N}{p} + \alpha \notin \{1, \ldots, m\}, \\ m - \dfrac{N}{p} - \alpha & \text{if } \frac{N}{p} + \alpha \in \{1, \ldots, m\}. \end{cases}$$

Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{0 \leq |\lambda| \leq k} \|\varrho^{\alpha-m+|\lambda|}\,(\lg\varrho)^{\beta-1}\,\partial^\lambda u\|^p_{L^p(\Omega)} + \sum_{k+1 \leq |\lambda| \leq m} \|\varrho^{\alpha-m+|\lambda|}\,(\lg\varrho)^{\beta}\,\partial^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p}.$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda|=m} \|\varrho^{\alpha}\,(\lg\varrho)^{\beta}\,\partial^{\lambda}u\|^{p}_{L^{p}(\Omega)} \Big)^{1/p}.$$

The weights in the definition (5.1) are chosen so that the corresponding space satisfies two fundamental properties. On the one hand, $\mathcal{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. On the other hand, the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ (see [2], Theorem 1.1): if

$$\frac{N}{p} + \alpha \notin \{1, \dots, m\} \quad \text{or} \quad (\beta - 1)p \neq -1, \tag{5.2}$$

then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \quad \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}} \leq C\,|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}, \tag{5.3}$$

with $q^* = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. Now, we define the space

$$\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}};$$

which will be characterized [see Lemma 2.2 [6]] as the subspace of functions with null traces in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. From that, we can introduce the space $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$ as the dual space of $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. In addition, under the assumption (5.2), $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ which is equivalent to the full norm $\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$. We will now recall some properties of the weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$.

Remark 2. In the case $\beta = 0$, we simply denote the space $W^{m,p}_{\alpha,0}(\Omega)$ by $W^{m,p}_{\alpha}(\Omega)$.

The spaces of traces

We define the traces of functions of $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$. For any real number $\alpha \in \mathbb{R}$, we define the space:

$$W^{\sigma,p}_{\alpha}(\mathbb{R}^N) = \Big\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w^{\alpha-\sigma}u \in L^p(\mathbb{R}^N), \\ \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\varrho^{\alpha}(x)\,u(x) - \varrho^{\alpha}(y)\,u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\},$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho\,(\lg\varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$W^{s,p}_{\alpha}(\mathbb{R}^N) = \Big\{ u \in \mathcal{D}'(\mathbb{R}^N);\ 0 \leq |\lambda| \leq k,\ \varrho^{\alpha-s+|\lambda|}\,(\lg\varrho)^{-1}\,\partial^{\lambda}u \in L^p(\mathbb{R}^N); \\ k+1 \leq |\lambda| \leq [s]-1,\ \varrho^{\alpha-s+|\lambda|}\,\partial^{\lambda}u \in L^p(\mathbb{R}^N);\ |\lambda| = [s],\ \partial^{\lambda}u \in W^{\sigma,p}_{\alpha}(\mathbb{R}^N) \Big\},$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \ldots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W^{s,\,p}_{\alpha}(\mathbb{R}^N)} = \Big(\sum_{0 \le |\lambda| \le k} \|\varrho^{\alpha - s + |\lambda|} \, (\lg \varrho)^{-1} \, \partial^{\lambda} u\|^p_{L^p(\mathbb{R}^N)}$$
$$+ \sum_{k+1 \le |\lambda| \le [s]-1} \|\varrho^{\alpha - s + |\lambda|} \, \partial^{\lambda} u\|^p_{L^p(\mathbb{R}^N)} \Big)^{1/p} + \sum_{|\lambda| = [s]} \|\partial^{\lambda} u\|_{W^{\sigma,\,p}_{\alpha}(\mathbb{R}^N)}.$$

We can similarly define, for any real number β, the space:

$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^N) = \Big\{ v \in \mathcal{D}'(\mathbb{R}^N); \; (\lg \varrho)^{\beta} \, v \in W^{s,\,p}_{\alpha}(\mathbb{R}^N) \Big\}.$$

5.2 Laplace equation

The aim of this section is to study the problem

$$(P) \left\{ \begin{aligned} -\Delta \mathbf{u} \;&= f \quad \text{in } \mathbb{R}^N_+, \\ \mathbf{u} \;&= \mathbf{g} \quad \text{on } \Gamma = \mathbb{R}^{N-1}. \end{aligned} \right.$$

Theorem 20. Let $l \ge 1$ be an integer and assume that

$$\frac{N}{p} \notin \{1, ..., -l\}. \tag{5.4}$$

Then for any $f \in W^{-1,p}_{-l}(\mathbb{R}^N_+)$ and $g \in W^{\frac{1}{p'},p}_{-l}(\mathbb{R}^{N-1})$, problem (P) has a unique solution $u \in W^{1,p}_{-l}(\mathbb{R}^N_+)/A^{\Delta}_{[l+1-N/p]}$ and there exists a constant C independent of u, f and g such that

$$\inf_{q \in A^{\Delta}_{[l+1-\frac{N}{p}]}} \|u + q\|_{W^{1,p}_{-l}(\mathbb{R}^N_+)} \le C(\|f\|_{W^{-1,p}_{-l}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'},p}_{-l}(\mathbb{R}^{N-1})}). \tag{5.5}$$

For definition of $A^{\Delta}_{[l+1-N/p]}$ see [2].

Theorem 21. Let m be a nonnegative integer, let g belong to $W^{\frac{1}{m'}+m,p}(\mathbb{R}^{N-1})$ and assume that

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} \ne 1 \text{ or } m = 0, \tag{5.6}$$

or

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \cap W^{-1,p}_{0}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} = 1 \text{ and } m \ne 0. \tag{5.7}$$

Then problem (P) has a unique solution $u \in W^{1+m,p}_{m}(\mathbb{R}^N_+)$ and u satisfies

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \le C(\|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\mathbb{R}^{N-1})}) \text{ if } \frac{N}{p'} \ne 1 \text{ or } m = 0 \tag{5.8}$$

and

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \le C(\|f\|_{W^{1,p}_{p}(\mathbb{R}^N_+)} + \|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\mathbb{R}^{N-1})}) \tag{5.9}$$

if $\frac{N}{p'} = 1$ and $m \ne 0$.

5.3 Stokes system

The purpose of this part is the study of the Stokes system

$$(S^+)\quad \begin{cases} -\Delta \mathbf{u} + \nabla\pi &= \mathbf{f} \quad \text{in } \mathbb{R}^N_+,\\ \operatorname{div}\mathbf{u} &= h \quad \text{in } \mathbb{R}^N_+,\\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma = \mathbb{R}^{N-1}, \end{cases}$$

with data and solutions which live in weighted Sobolev spaces, expressing at the same time their regularity and their behavior at infinity. We will naturally base on the previously established results on the harmonic and biharmonic operators (see [2], [3], [4], [5]). We will also concentrate on the basic weights because they are the most usual and they avoid the question of the kernel for this operator and symmetricaly the compatibility condition for the data.

Among the first works on the Stokes problem in the half-space, we can cite Cattabriga. In [15], he applies the potential theory to get explicit solution of the velocity fields and pressure. For the homogeneous problem ($\mathbf{f} = 0$ and $h = 0$), for instance, he shows that if $\mathbf{g} \in \mathbf{L^p(\Gamma)}$ and the semi-norm $|\mathbf{g}|_{\mathbf{W_0^{1-1/p,\,p}(\Gamma)}} < \infty$, then $\nabla\mathbf{u} \in \mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)$ and $\pi \in L^p(\mathbb{R}^N_+)$.

Similar results are given by Farwig-Sohr (see [28]) and Galdi (see [39]), who also have chosen the setting of homogeneous Sobolev spaces. On the other hand, Maz'ya-Plamenevskiĭ-Stupyalis (see [67]), work within the suitable setting of weighted Sobolev spaces and consider different types of boundary conditions. However, their results are limited to the dimension 3 and to the Hilbertian framework in which they give generalized and strong solutions. This is also the case of Boulmezaoud (see [9]), who only gives strong solutions. Otherwise, always in dimension 3, by Fourier analysis techniques, Tanaka considers the case of very regular data, corresponding to velocities which belong to $\mathbf{W_2^{m+3,\,2}}(\mathbb{R}^\mathbf{3}_+)$, with $m \geq 0$ (see [78]).

Let us also quote, for the evolution Stokes or Navier-Stokes problems, Fujigaki-Miyakawa (see [29]), who are interested in the behaviour in $t \to +\infty$; Bochers-Miyakawa (see [10]) and Kozono (see [54]), for the L^N-decay property; Ukai (see [79]), for the L^p-L^q estimates and Giga (see [44]), for the estimates in Hardy spaces.

5.3.1 Generalized solutions to the Stokes system in $\mathbb{R}^N_+$

Theorem 22. *For any* $\mathbf{f} \in \mathbf{W_0^{-1,\,p}}(\mathbb{R}^\mathbf{N}_+)$, $\mathbf{h} \in \mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)$ *and* $\mathbf{g} \in \mathbf{W_0^{1-1/p,\,p}(\Gamma)}$, *problem* (S^+) *admits a unique solution* $(\mathbf{u},\, \pi) \in \mathbf{W_0^{1,\,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)$, *and there exists a constant* C *such that*

$$\|\mathbf{u}\|_{\mathbf{W_0^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi\|_{\mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)} \leq C\left(\|\mathbf{f}\|_{\mathbf{W_0^{-1,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|h\|_{L^p(\mathbb{R}^N_+)} + \|\mathbf{g}\|_{\mathbf{W_0^{1-1/p,\,p}(\Gamma)}}\right). \quad (5.10)$$

5.3.2 Strong solutions and regularity for the Stokes system in $\mathbb{R}^N_+$

In this section, we are interested in the existence of strong solutions (and then to regular solutions, see Corollaries 3 and 4), *i.e.* of solutions $(\mathbf{u},\, \pi) \in \mathbf{W_{\ell+1}^{2,\,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{W_{\ell+1}^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)$. Here, we limit ourselves to the two cases $\ell = 0$ or $\ell = -1$. Note that in the case

$\ell = 0$, we have $W_1^{2,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{1,p}(\mathbb{R}^N_+)$ and $W_1^{1,p}(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$. The proposition and theorem which follow show that the generalized solution of Theorem 20, with a stronger hypothesis on the data, is in fact a strong solution. First, we introduce the homogeneous case:

$$-\Delta \mathbf{u} + \nabla \pi = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{5.11}$$
$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{5.12}$$
$$\mathbf{u} = \mathbf{g} \qquad \text{on } \Gamma. \tag{5.13}$$

Proposition 1. *Assume that* $\dfrac{N}{p'} \neq 1$. *For any* $\mathbf{g} \in \mathbf{W_1^{2-1/p,\,p}(\Gamma)}$, *the Stokes problem (5.11)–(5.13) has a unique solution* $(\mathbf{u}, \pi) \in \mathbf{W_1^{2,\,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{W_1^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)$, *with the estimate*

$$\|\mathbf{u}\|_{\mathbf{W_1^{2,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi\|_{\mathbf{W_1^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)} \;\leq\; \mathbf{C}\; \|\mathbf{g}\|_{\mathbf{W_1^{2-1/p,\,p}(\Gamma)}}.$$

Now, we can study the strong solutions for the complete problem (S^+). As for the generalized solutions, we will show that it is equivalent to an homogeneous problem, solved by Proposition 1. The following theorem was established in the case $N = 3$, $p = 2$, by Maz'ya-Plamenevskiĭ-Stupyalis (see [67]).

Theorem 23. *Assume that* $\dfrac{N}{p'} \neq 1$. *For any* $\mathbf{f} \in \mathbf{W_1^{0,\,p}}(\mathbb{R}^\mathbf{N}_+)$, $\mathbf{h} \in \mathbf{W_1^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)$ *and* $\mathbf{g} \in \mathbf{W_1^{2-1/p,\,p}(\Gamma)}$, *problem* (S^+) *admits a unique solution* $(\mathbf{u}, \pi)$ *which belongs to* $\mathbf{W_1^{2,\,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{W_1^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)$, *with the estimate*

$$\|\mathbf{u}\|_{\mathbf{W_1^{2,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi\|_{\mathbf{W_1^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)} \;\leq\; C\left(\|\mathbf{f}\|_{\mathbf{W_1^{0,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|h\|_{W_1^{1,\,p}(\mathbb{R}^N_+)} + \|\mathbf{g}\|_{\mathbf{W_1^{2-1/p,\,p}(\Gamma)}}\right).$$

Corollary 3. *Let* $m \in \mathbb{N}$ *and assume that* $\dfrac{N}{p'} \neq 1$ *if* $m \geq 1$. *For any* $\mathbf{f} \in \mathbf{W_m^{m-1,\,p}}(\mathbb{R}^\mathbf{N}_+)$, $\mathbf{h} \in \mathbf{W_m^{m,\,p}}(\mathbb{R}^\mathbf{N}_+)$ *and* $\mathbf{g} \in \mathbf{W_m^{m+1-1/p,\,p}(\Gamma)}$, *problem* (S^+) *admits a unique solution* $(\mathbf{u}, \pi) \in \mathbf{W_m^{m+1,\,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{W_m^{m,\,p}}(\mathbb{R}^\mathbf{N}_+)$, *with the estimate*

$$\|\mathbf{u}\|_{\mathbf{W_m^{m+1,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi\|_{\mathbf{W_m^{m,\,p}}(\mathbb{R}^\mathbf{N}_+)} \;\leq\; C\left(\|\mathbf{f}\|_{\mathbf{W_m^{m-1,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|h\|_{W_m^{m,\,p}(\mathbb{R}^N_+)} + \|\mathbf{g}\|_{\mathbf{W_m^{m+1-1/p,\,p}(\Gamma)}}\right).$$

Now, we examine the basic case $\ell = -1$, corresponding to $f \in \mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)$. More precisely, we have the following result, corresponding to Theorem 23:

Theorem 24. *For any* $\mathbf{f} \in \mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)$, $\mathbf{h} \in \mathbf{W_0^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)$ *and* $\mathbf{g} \in \mathbf{W_0^{2-1/p,\,p}(\Gamma)}$, *problem* (S^+) *admits a solution* $(\mathbf{u}, \pi) \in \mathbf{W_0^{2,\,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{W_0^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)$, *unique if* $N > p$, *unique up to an element of* $(\mathbb{R}\, x_N)^{N-1} \times \{0\} \times \mathbb{R}$ *if* $N \leq p$, *with the following estimate if* $N \leq p$ *(eliminate* (λ, μ) *if* $N > p$*):*

$$\inf_{(\lambda,\,\mu)\in(\mathbb{R}\,\mathbf{x_N})^{\mathbf{N-1}}\times\{\mathbf{0}\}\times\mathbb{R}} \left(\|\mathbf{u} + \lambda\|_{\mathbf{W_0^{2,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi + \mu\|_{\mathbf{W_0^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)}\right) \;\leq\; C\left(\|\mathbf{f}\|_{\mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)} + \|h\|_{W_0^{1,\,p}(\mathbb{R}^N_+)} + \|\mathbf{g}\|_{\mathbf{W_0^{2-1/p,\,p}(\Gamma)}}\right).$$

Corollary 4. *Let $m \in \mathbb{N}$. For any $\mathbf{f} \in \mathbf{W^{m,\,p}_m}(\mathbb{R}^N_+)$, $\mathbf{h} \in \mathbf{W^{m+1,\,p}_m}(\mathbb{R}^N_+)$ and $\mathbf{g} \in \mathbf{W^{m+2-1/p,\,p}_m}(\boldsymbol{\Gamma})$, problem (S^+) admits a solution $(\mathbf{u},\,\pi) \in \mathbf{W^{m+2,\,p}_m}(\mathbb{R}^N_+)\times \mathbf{W^{m+1,\,p}_m}(\mathbb{R}^N_+)$, unique if $N > p$, unique up to an element of $(\mathbb{R}\,x_N)^{N-1}\times\{0\}\times\mathbb{R}$ if $N \le p$, with the following estimate if $N \le p$ (eliminate $(\lambda,\,\mu)$ if $N > p$):*

$$\inf_{(\lambda,\,\mu)\in(\mathbb{R}\,\mathbf{x_N})^{\mathbf{N-1}}\times\{\mathbf{0}\}\times\mathbb{R}} \left(\|\mathbf{u}+\lambda\|_{\mathbf{W^{m+2,\,p}_m}(\mathbb{R}^N_+)} + \|\pi+\mu\|_{\mathbf{W^{m+1,\,p}_m}(\mathbb{R}^N_+)}\right) \le$$
$$C\left(\|\mathbf{f}\|_{\mathbf{W^{m,\,p}_m}(\mathbb{R}^N_+)} + \|h\|_{W^{m+1,\,p}_m(\mathbb{R}^N_+)} + \|\mathbf{g}\|_{\mathbf{W^{m+2-1/p,\,p}_m}(\boldsymbol{\Gamma})}\right).$$

5.3.3 Very weak solutions for the Stokes system

Proposition 2. *Assume that $\dfrac{N}{p} \neq 1$. For any $\mathbf{g} \in \mathbf{W^{-1/p,\,p}_{-1}}(\boldsymbol{\Gamma})$ such that $g_N = 0$, the Stokes problem (5.11)–(5.13) has a unique solution $(\mathbf{u},\,\pi) \in \mathbf{W^{0,\,p}_{-1}}(\mathbb{R}^N_+)\times \mathbf{W^{-1,\,p}_{-1}}(\mathbb{R}^N_+)$, with the estimate*

$$\|\mathbf{u}\|_{\mathbf{W^{0,\,p}_{-1}}(\mathbb{R}^N_+)} + \|\pi\|_{\mathbf{W^{-1,\,p}_{-1}}(\mathbb{R}^N_+)} \le \mathbf{C}\,\|\mathbf{g}\|_{\mathbf{W^{-1/p,\,p}_{-1}}(\boldsymbol{\Gamma})}.$$

Theorem 25. *Assume that $\dfrac{N}{p} \neq 1$. For any $\mathbf{g} \in \mathbf{W^{-1/p,\,p}_{-1}}(\boldsymbol{\Gamma})$, the Stokes problem (5.11)–(5.13) has a unique solution $(\mathbf{u},\,\pi) \in \mathbf{W^{0,\,p}_{-1}}(\mathbb{R}^N_+)\times \mathbf{W^{-1,\,p}_{-1}}(\mathbb{R}^N_+)$, with the estimate*

$$\|\mathbf{u}\|_{\mathbf{W^{0,\,p}_{-1}}(\mathbb{R}^N_+)} + \|\pi\|_{\mathbf{W^{-1,\,p}_{-1}}(\mathbb{R}^N_+)} \le \mathbf{C}\,\|\mathbf{g}\|_{\mathbf{W^{-1/p,\,p}_{-1}}(\boldsymbol{\Gamma})}.$$

Proposition 3. *For any $\mathbf{g} \in \mathbf{W_0^{-1/p,\,p}}(\boldsymbol{\Gamma})$ such that $g_N = 0$, and $\mathbf{g}' \perp \mathbb{R}^{N-1}$ if $N \le p'$, the Stokes problem (5.11)–(5.13) has a unique solution $(\mathbf{u},\,\pi) \in \mathbf{L^p}(\mathbb{R}^N_+)\times \mathbf{W_0^{-1,\,p}}(\mathbb{R}^N_+)$, with the estimate*

$$\|\mathbf{u}\|_{\mathbf{L^p}(\mathbb{R}^N_+)} + \|\pi\|_{\mathbf{W_0^{-1,\,p}}(\mathbb{R}^N_+)} \le \mathbf{C}\,\|\mathbf{g}\|_{\mathbf{W_0^{-1/p,\,p}}(\boldsymbol{\Gamma})}.$$

Theorem 26. *For any $\mathbf{g} \in \mathbf{W_0^{-1/p,\,p}}(\boldsymbol{\Gamma})$ such that $\mathbf{g} \perp \mathbb{R}^N$ if $N \le p'$, the Stokes problem (5.11)–(5.13) has a unique solution $(\mathbf{u},\,\pi) \in \mathbf{L^p}(\mathbb{R}^N_+)\times \mathbf{W_0^{-1,\,p}}(\mathbb{R}^N_+)$, with the estimate*

$$\|\mathbf{u}\|_{\mathbf{L^p}(\mathbb{R}^N_+)} + \|\pi\|_{\mathbf{W_0^{-1,\,p}}(\mathbb{R}^N_+)} \le \mathbf{C}\,\|\mathbf{g}\|_{\mathbf{W_0^{-1/p,\,p}}(\boldsymbol{\Gamma})}.$$

5.4 Stokes problem with Navier condition

For the stationary Stokes problem

$$-\Delta\mathbf{u} + \nabla\pi = \mathbf{f} \quad\text{and}\quad \operatorname{div}\mathbf{u} = \mathbf{h} \quad\text{in } \boldsymbol{\Omega},$$

where Ω is a domain of $\mathbb{R}^N$, there are several possible boundary conditions. Under the hypothesis of impermeability of the boundary, the velocity field $\mathbf{u}$ satisfies

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{0} \quad\text{on } \partial\boldsymbol{\Omega}, \tag{5.14}$$

where $\boldsymbol{n}$ stands for the outer normal vector. According to the idea that the fluid cannot slip on the wall due to its viscosity, we get the no-slip condition

$$\mathbf{u}_\tau = \mathbf{0} \quad\text{on } \partial\Omega, \tag{5.15}$$

where $\mathbf{u}_\tau = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}$ denotes, as usual, the tangential component of $\mathbf{u}$. The Dirichlet boundary value problem, which was suggested by Stokes, is the combination of (5.14) and (5.15). Concerning this problem, the literature is well known and extensive. Especially in the case of the half-space, we would like to mention the works of Cattabriga [15], Tanaka [78], Farwig and Sohr [28], and Galdi [39], where the solution of the problem is investigated in homogeneous Sobolev spaces, whereas in the works of Maz'ya, Plamenevskiĭ, and Stupyalis [67] and Boulmezaoud [9], we can find results in weighted Sobolev spaces. This is also the functional framework of our previous work (see [6]) and also see Section 5.3.

The correctness of the no-slip hypothesis has been a subject of discussion for over two centuries among many distinguished scientists. Instead of (5.15), Navier had already proposed the following condition saying that the velocity on the boundary is proportional to the tangential component of the stress:

$$(\mathbb{T}\cdot\boldsymbol{n})_\tau + \beta\,\mathbf{u}_\tau = \mathbf{0} \quad \text{on } \partial\Omega, \tag{5.16}$$

where $\mathbb{T}$ denotes the viscous stress tensor and β is a friction coefficient. For the incompressible isotropic fluids, the viscous stress tensor is given by

$$\mathbb{T} = -\pi\,\mathbb{I} + \nu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathbf{T}}).$$

The case $\beta = 0$ is termed *complete slip*, while (5.16) reduces to (5.15) in the asymptotic limit $\beta \to \infty$. The aim of this paper is to investigate the Stokes problem in the half-space with the following type of slip boundary conditions:

$$(\mathrm{S}^\sharp)\quad \begin{cases} -\Delta\mathbf{u} + \nabla\pi = \mathbf{f} & \text{and} \quad \operatorname{div}\mathbf{u} = \mathbf{h} \quad \text{in } \mathbb{R}^n_+, \\ u_n = g_n & \text{and} \quad \partial_n\mathbf{u}' = \mathbf{g}' \quad \text{on } \Gamma. \end{cases}$$

Similarly as in Section 5.3 the weak, strong and very weak solution were investigated.

5.4.1 Weak solutions

Proposition 5.1. *For any $g_n \in W_0^{1-1/p,p}(\Gamma)$ and $\mathbf{g}' \in \mathbf{W_0^{-1/p,p}(\Gamma)}$ such that $\mathbf{g}' \perp \mathbb{R}^{N-1}$ if $N \le p'$, the Stokes problem*

$$-\Delta\mathbf{u} + \nabla\pi = \mathbf{0} \qquad \text{in } \mathbb{R}^N_+, \tag{5.17a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \mathbb{R}^N_+, \tag{5.17b}$$

$$u_n = g_n \qquad \text{on } \Gamma, \tag{5.17c}$$

$$\partial_n\mathbf{u}' = \mathbf{g}' \qquad \text{on } \Gamma \tag{5.17d}$$

has a solution $(\mathbf{u},\,\pi) \in \mathbf{W_0^{1,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1}\times\{0\}^2$ if $N \le p$, with the estimate

$$\inf_{\mathbf{h}\in\mathbb{R}^{N-1}\times\{0\}} \|\mathbf{u} + \|_{\mathbf{W_0^{1,p}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi\|_{\mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)} \le C\,\left(\|g_n\|_{W_0^{1-1/p,p}(\Gamma)} + \|\mathbf{g}'\|_{\mathbf{W_0^{-1/p,p}(\Gamma)}}\right)$$

if $N \le p$, and the corresponding estimate without inf $(\mathbf{h} = 0)$ *if $N > p$.*

Theorem 27. Assume that $\frac{N}{p'} \neq 1$. For any $\mathbf{f} \in \mathbf{W_1^{0,\,p}}(\mathbb{R}^\mathbf{N}_+)$, $h \in W_1^{1,\,p}(\mathbb{R}^N_+)$, $g_n \in W_0^{1-1/p,\,p}(\Gamma)$, and $\mathbf{g}' \in \mathbf{W_0^{-1/p,\,p}(\Gamma)}$, satisfying the following compatibility condition if $N < p'$:

$$\forall i \in \{1, \ldots, N-1\}, \quad \int_{\mathbb{R}^N_+} f_i \,\mathrm{d}x = \langle g_i,\, 1\rangle_{W_0^{-1/p,\,p}(\Gamma)\times W_0^{1/p,\,p'}(\Gamma)}\,, \tag{5.18}$$

problem ($\mathrm{S}^\sharp$) admits a solution $(\mathbf{u},\, \pi) \in \mathbf{W_0^{1,\,p}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)$, unique if $N > p$, unique up to an element of $\mathbb{R}^{N-1} \times \{0\}^2$ if $N \le p$, with the estimate

$$\begin{aligned}\inf_{\mathbf{h}\in\mathbb{R}^{N-1}\times\{0\}} \|\mathbf{u}+\mathbf{h}\|_{\mathbf{W_0^{1,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi\|_{\mathbf{L^p}(\mathbb{R}^\mathbf{N}_+)}\\ \le\; C\;\Big(\|\mathbf{f}\|_{\mathbf{W_1^{0,\,p}}(\mathbb{R}^\mathbf{N}_+)} + \|h\|_{W_1^{1,\,p}(\mathbb{R}^N_+)} + \|g_n\|_{W_0^{1-1/p,\,p}(\Gamma)} + \|\mathbf{g}'\|_{\mathbf{W_0^{-1/p,\,p}(\Gamma)}}\Big)\end{aligned}$$

if $N \le p$, and the corresponding estimate without inf ($\mathbf{h} = 0$) if $N > p$.

5.4.2 Strong solutions

Theorem 28. Let $\ell \in \mathbb{Z}$ with hypothesis

$$N/p' \notin \{1, \ldots, \ell+1\} \quad\text{ and }\quad N/p \notin \{1, \ldots, -\ell-1\}. \tag{5.19}$$

For any $\mathbf{f} \in \mathbf{W^{0,\,p}_{\ell+1}}(\mathbb{R}^\mathbf{N}_+)$, $h \in W^{1,\,p}_{\ell+1}(\mathbb{R}^N_+)$, $g_n \in W^{2-1/p,\,p}_{\ell+1}(\Gamma)$, $\mathbf{g}' \in \mathbf{W^{1-1/p,\,p}_{\ell+1}(\Gamma)}$, satisfying condition

$$\begin{aligned}&\forall\varphi \in \mathcal{N}^{\mathbf{\Delta}}_{[\mathbf{1}+\ell-\mathbf{N/p'}]} \times \mathcal{A}^{\mathbf{\Delta}}_{[\mathbf{1}+\ell-\mathbf{N/p'}]},\\ &\int_{\mathbb{R}^N_+} (\mathbf{f}-\nabla h)\cdot\varphi\,\mathrm{d}\mathbf{x} \;+\; \langle \mathrm{div}\,\mathbf{f},\, \mathbf{\Pi_N}\mathrm{div}\,\varphi\rangle_{\mathbf{W^{-1,\,p}_{\ell+1}}(\mathbb{R}^\mathbf{N}_+)\times\mathring{\mathbf{W}}^{\mathbf{1},\,\mathbf{p'}}_{-\ell-\mathbf{1}}(\mathbb{R}^\mathbf{N}_+)}\\ &\quad+\int_\Gamma g_n\,\partial_n\varphi_n\,\mathrm{d}x' \;-\; \langle \mathbf{g}',\, \varphi'\rangle_{\mathbf{W_\ell^{-1/p,\,p}(\Gamma)}\times\mathbf{W^{1-1/p',\,p'}_{-\ell}(\Gamma)}} \;=\; 0,\end{aligned} \tag{5.20}$$

problem $(S^\sharp)$ admits a solution $(\mathbf{u},\, \pi) \in \mathbf{W^{2,\,p}_{\ell+1}}(\mathbb{R}^\mathbf{N}_+) \times \mathbf{W^{1,\,p}_{\ell+1}}(\mathbb{R}^\mathbf{N}_+)$, unique up to an element of $\mathcal{S}^\sharp_{[1-\ell-N/p]}$, with the estimate

$$\begin{aligned}\inf_{(\lambda,\,\mu)\in\mathcal{S}^\sharp_{[\mathbf{1}-\ell-\mathbf{N/p}]}} \Big(\|\mathbf{u}+\lambda\|_{\mathbf{W^{2,\,p}_{\ell+1}}(\mathbb{R}^\mathbf{N}_+)} + \|\pi+\mu\|_{\mathbf{W^{1,\,p}_{\ell+1}}(\mathbb{R}^\mathbf{N}_+)}\Big)\\ \le C\,\Big(\|\mathbf{f}\|_{\mathbf{W^{0,\,p}_{\ell+1}}(\mathbb{R}^\mathbf{N}_+)} + \|h\|_{W^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)} + \|g_n\|_{W^{2-1/p,\,p}_{\ell+1}(\Gamma)} + \|\mathbf{g}'\|_{\mathbf{W^{1-1/p,\,p}_{\ell+1}(\Gamma)}}\Big)\,.\end{aligned}$$

5.4.3 Very weak solutions

Theorem 29. Let $\ell \in \mathbb{Z}$ and assume that

$$N/p' \notin \{1, \ldots, \ell+1\} \quad\text{ and }\quad N/p \notin \{1, \ldots, -\ell+1\}.$$

For any $\mathbf{f} \in \mathbf{W}^{\mathbf{0},\,\mathbf{p}}_{\ell+\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+)$, $h \in W^{1,\,p}_{\ell+1}(\mathbb{R}^N_+)$, $g_n \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$, $\mathbf{g}' \in \mathbf{W}^{-\mathbf{1}-\mathbf{1}/\mathbf{p},\,\mathbf{p}}_{\ell-\mathbf{1}}(\mathbf{\Gamma})$, satisfying the compatibility condition

$$\forall \varphi \in \mathcal{N}^{\mathbf{\Delta}}_{[\mathbf{1}+\ell-\mathbf{N}/\mathbf{p}']} \times \mathcal{A}^{\mathbf{\Delta}}_{[\mathbf{1}+\ell-\mathbf{N}/\mathbf{p}']},$$

$$\int_{\mathbb{R}^N_+} (-\nabla h)\cdot \varphi \,\mathrm{d}\mathbf{x} \;+\; \langle \operatorname{div}\mathbf{f},\, \mathbf{\Pi_N}\operatorname{div}\varphi\rangle_{\mathbf{W}^{-\mathbf{1},\,\mathbf{p}}_{\ell+\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+)\times \mathring{\mathbf{W}}^{\mathbf{1},\,\mathbf{p}'}_{-\ell-\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+)}$$
$$+\; \langle g_n,\, \partial_n\varphi_n\rangle_{\mathbf{W}^{-\mathbf{1}/\mathbf{p},\,\mathbf{p}}_{\ell-\mathbf{1}}(\mathbf{\Gamma})\times \mathbf{W}^{\mathbf{1}-\mathbf{1}/\mathbf{p}',\,\mathbf{p}'}_{-\ell+\mathbf{1}}(\mathbf{\Gamma})}$$
$$-\; \langle \mathbf{g}',\, \varphi'\rangle_{\mathbf{W}^{-\mathbf{1}-\mathbf{1}/\mathbf{p},\,\mathbf{p}}_{\ell-\mathbf{1}}(\mathbf{\Gamma})\times \mathbf{W}^{\mathbf{2}-\mathbf{1}/\mathbf{p}',\,\mathbf{p}'}_{-\ell+\mathbf{1}}(\mathbf{\Gamma})} \;=\; 0,$$

problem $(S^\sharp)$ admits a solution $(\mathbf{u},\, \pi) \in \mathbf{W}^{\mathbf{0},\,\mathbf{p}}_{\ell+\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+) \times \mathbf{W}^{-\mathbf{1},\,\mathbf{p}}_{\ell-\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+)$, unique up to an element of $\mathcal{S}^\sharp_{[1-\ell-N/p]}$, with the estimate

$$\inf_{(\lambda,\,\mu)\in \mathcal{S}^\sharp_{[\mathbf{1}-\ell-\mathbf{N}/\mathbf{p}]}} \Big(\|\mathbf{u}+\lambda\|_{\mathbf{W}^{\mathbf{0},\,\mathbf{p}}_{\ell-\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+)} + \|\pi+\mu\|_{\mathbf{W}^{-\mathbf{1},\,\mathbf{p}}_{\ell-\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+)} \Big)$$
$$\le C \Big(\|\mathbf{f}\|_{\mathbf{W}^{\mathbf{0},\,\mathbf{p}}_{\ell+\mathbf{1}}(\mathbb{R}^{\mathbf{N}}_+)} + \|h\|_{W^{0,\,p}_{\ell+1}(\mathbb{R}^N_+)} + \|g_n\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)} + \|\mathbf{g}'\|_{\mathbf{W}^{-\mathbf{1}-\mathbf{1}/\mathbf{p},\,\mathbf{p}}_{\ell-\mathbf{1}}(\mathbf{\Gamma})} \Big).$$

6 List of publications of the dissertation

1. **R. Farwig, M. Krbec, Š. Nečasová.** *A weighted L^q - approach to Oseen flow around a rotating body.* Math. Methods Appl. Sci., **31**, 5, 551–574, 2008.
2. **S. Kračmar, Š. Nečasová, P. Penel.** *Anisotropic L^2-estimates of weak solutions to the stationary Oseen-type equations in 3D-exterior domain for a rotating body.* J. of J. Math. Soc. of Japan, **62**, 1, 239–268, 2010.
3. **S. Kračmar, Š. Nečasová, P. Penel.** *L^q-approach to weak solutions of the Oseen flow around a rotating body.* Parabolic and Navier-Stokes equations. Part 1, 259-276, Banach Center Publ., **81**, Part 1, Polish Acad. Sci. Inst. Math., Warsaw, 2008.
4. **P. Deuring, S. Kračmar, Š. Nečasová.** *On pointwise decay of linearized stationary incompressible viscous flow around rotating and translating bodies.* SIAM J. Math. Anal., **43,** 2, 705-738, 2011.
5. **Š. Nečasová.** *Asymptotic properties of the steady fall of a body in viscous fluids.* Math. Methods Appl. Sci., **27**, 17, 1969–1995, 2004.
6. **M. Okada, Š. Matušů-Nečasová, T. Makino.** *Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent temperature.* Ann. Univ. Ferrara - Sez. VII - Sc. Mat., **48**, 1–20, 2002.
7. **M. Okada, Š. Matušů-Nečasová, T. Makino.** *Free boundary problem for the equation of spherically symmetric motion of viscous gas. II.* Japan J. Indust. Appl. Math., **12**, 2, 195–203, 1995.
8. **M. Okada, Š. Matušů-Nečasová, T. Makino.** *Free boundary problem for the equation of spherically symmetric motion of viscous gas III.* Japan J. Indust. Appl. Math., **14**, 2, 199–213, 1997.

9. **B. Ducomet, Š. Nečasová.** *Global existence of solutions for the one-dimensional motions of a compressible viscous gas with radiation: and "infrarelativistic model".* Nonlinear Analysis, **72,** 7-8, 3258–3274, 2010.
10. **C. Amrouche, Š. Nečasová.** *Laplace equation in the half-space with nonhomogeneous Dirichlet boundary condition.* Mathematica Bohemica, **126**, 2, 265–274, 2001.
11. **C. Amrouche, Š. Nečasová, Y. Raudin.** *Very weak generalized and strong solutions to the Stokes system in the half space.* J. of Diff. Eq., **244**, 887–915, 2008.
12. **C. Amrouche, Š. Nečasová, Y. Raudin.** *From strong to very weak solutions to the Stokes system with Navier boundary conditions in the half-space.* SIAM J. Math. Anal., **41**, 5, 1792-1815, 2009.

References

[1] A. A. Amosov. Well-posedness "in the large" initial and boundary-value problems for the system of dynamical equations of a viscous radiating gas, *Sov. Physics Dokl.*, **30**, 129–131, 1985.

[2] C. Amrouche, Š. Nečasová. Laplace equation in the half-space with a nonhomogeneous Dirichlet boundary condition, *Mathematica Bohemica*, **126**, 2, 265–274, 2001.

[3] C. Amrouche. The Neumann problem in the half-space, *C. R. Acad. Sci. Paris, Série I*, **335**, 151–156, 2002.

[4] C. Amrouche, Y. Raudin. Nonhomogeneous biharmonic problem in the half-space, L^p theory and generalized solutions, *Journal of Differential Equations,* **236**, 1, 57–81, 2007.

[5] C. Amrouche, Y. Raudin. Singular boundary conditions and regularity for the biharmonic problem in the half-space, *Com. Pure Appl. Anal.,* **6**, 4, 957–982, 2007.

[6] C. Amrouche, Š. Nečasová, Y. Raudin. Very weak generalized and strong solutions to the Stokes system in the half space, *J. of Diff. Eq.*, **244**, 887–915, 2008.

[7] C. Amrouche, Š. Nečasová, Y. Raudin. From strong to very weak solution to the Stokes system with Navier boundary conditions in R^n_+, *SIAM J. Math. Anal.*, **41**, Issue 5 1792–1815, 2010.

[8] S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov. *Boundary value problems in mechanics of nonhomogeneous fluids*, North-Holland, Amsterdam, New-York, Oxford, Tokyo, 1990.

[9] T. Z. Boulmezaoud. On the Stokes system and the biharmonic equation in the half-space: an approach via weighted Sobolev spaces, *Mathematical Methods in the Applied Sciences*, **25**, 373–398, 2002.

[10] W. Bochers, T. Miyakawa. L^2 Decay for the Navier-Stokes flow in halfspace, *Math. Ann.*, **282**, 139–155, 1988.

[11] D. Bresch, B. Desjardins. Some diffusive capillary models of Korteweg type, *C. R. Acad. Sci. Paris, Section Mécanique,* **332**, 881–886, 2004.

[12] D. Bresch, B. Desjardins, D. Gérard-Varet. On compressible Navier-Stokes equations with density dependent viscosities in bounded domains, *J. Math. Pures Appl.,* **87**, 227–235, 2007.

[13] S. Chandrasekhar. *Radiative transfer.* Dover Publications, Inc., New York, 1960.

[14] S. Chapman, T. G. Cowling. *The mathematical theory of non-uniform gases, Dynamics of viscous compressible fluids,* Cambridge University Press, 1995.

[15] L. Cattabriga. Su un problema al contorno relativo al sistema di equazioni di Stokes, *Rend. Sem. Mat. Padova*, **31**, 308–340, 1961.

[16] P. Deuring, S. Kračmar, Š. Nečasová. A representation formula for linearized stationary incompressible viscous flows around rotating and translating bodies, *Discrete Contin. Dyn. Syst. Ser. S*, **3**, 2, 237-253, 2010.

[17] P. Deuring, S. Kračmar, Š. Nečasová. On pointwise decay of linearized stationary incompressible viscous flow around rotating and translating bodies, *SIAM J. Math. Anal.,* **43**, 2, 705-738, 2011.

[18] B. Ducomet, Š. Nečasová. Thermalization in a model of neutron star *Discrete and Continuous Dynamical Systems B*, **16,** 3, 2011.

[19] B. Ducomet, Š. Nečasová, A. Vasseur. On global motions of a compressible barotropic and self-gravitating gas with density-dependent viscosities, *Z. Angew. Math. Phys.*, **61**, 3, 479-491, 2010.

[20] B. Ducomet, Š. Nečasová, A. Vasseur. On spherically symmetric motions of a viscous comperssible barotropic and selfgraviting gas, *J. of Mathematical Fluid Mech.*, **13**, 191–211, 2011.

[21] B. Ducomet, E. Feireisl, A. Petzeltová, I. Straškraba. Global in time weak solutions for compressible barotropic self-gravitating fluid, *Discrete and Continuous Dynamical Systems,* **11**, 113–130, 2004.

[22] B. Ducomet, Š. Nečasová. On a fluid model of neutron star, *Annali di Ferrara,* **55**, 153–193, 2009.

[23] R. Farwig. The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces, *Math. Z.,* **211**, 409–447, 1992.

[24] R. Farwig. An L^q-analysis of viscous fluid flow past a rotating obstacle, *Tôhoku Math. J.*, **58**, 129–147, 2006.

[25] R. Farwig. Estimates of lower order derivatives of viscous fluid flow past a rotating obstacle, *Banach Center Publications*, **70**, Warsaw, 73–84, 2005.

[26] R. Farwig, T. Hishida. Stationary Navier-Stokes flow around a rotating obstacle, Funkcialaj Ekvacioj., **50**, 3, 371–403, 2007.

[27] R. Farwig, T. Hishida, and D. Müller. L^q-Theory of a singular "winding" integral operator arising from fluid dynamics, *Pacific J. Math.*, **215**, 297–312, 2004.

[28] R. Farwig, H. Sohr. On the Stokes and Navier-Stokes System for Domains with Noncompact Boudary in L^q-spaces, *Math. Nachr.*, **170**, 53–77, 1994.

[29] Y. Fujigaki, T. Miyakawa. Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the half-space, *Methods and Applications of Analysis*, **8-1**, 121–158, 2001.

[30] R. Farwig, M. Krbec, Š. Nečasová. A weighted L^q - approach to Oseen flow around a rotating body, *Math. Methods Appl. Sci.,* **31**, 5, 551–574, 2008.

[31] R. Farwig, M. Krbec, Š. Nečasová. A weighted L^q - approach to Stokes flow around a rotating body, *Ann. Univ. Ferrara - Sez. VII. Mat.,* **54**, 1, 61–84, 2008.

[32] E. Feireisl. Compressible Navier-Stokes Equations with a Non-Monotone Pressure Law, *Journal of Diff. Equ.,* **184**, 97–108, 2002.

[33] E. Feireisl. On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable, *Comment. Math. Univ. Carolina,* **42**, 83–98, 2001.

[34] E. Feireisl. *Dynamics of Viscous Compressible Fluids,* Oxford Lecture Series in Mathematics and its Applications, 2004.

[35] E. Feireisl. On the motion of a viscous, compressible and heat-conducting fluid, *Indiana Univ. Math J.,* **53**, 1705–1738, 2004.

[36] E. Feireisl, H. Petzeltová. On the zero-velocity limit solutions to the Navier-Stokes equations of compressible flow, *Manuscripta Mathematica,* **97**, 109-116, 1998.

[37] H. Fujita-Yashima, R. Benabidallah. Equation à symétrie sphérique d'un gaz visqueux et calorifère avec la surface libre, *Annali di Matematica pura ed applicata,* **168**, 75–117, 1995.

[38] G. Gallavotti. *Foundations of fluid mechanics.* Springer-Verlag Berlin, Heidelberg, New-York, 2002.

[39] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations: Linearised steady problems.* Springer Tracts in Natural Philosophy, Vol. 38, 2nd edition, Springer 1998.

[40] G. P. Galdi. On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications. *Handbook of Mathematical Fluid Dynamics*, Vol. 1, Ed. by S. Friedlander, D. Serre, Elsevier 2002.

[41] G. P. Galdi. Steady flow of a Navier-Stokes fluid around a rotating obstacle, *J. Elasticity,* **71**, 1–31, 2003.

[42] G. P. Galdi, A. L. Silvestre. On the steady motion of a Navier- Stokes liquid around a rigid body, *Archive for Rational Mechanics and Analysis*, **184**, 3, June, 371–400, 2006.

[43] Galdi, G. P., Silvestre, S. A., Further results on steady-state flow of a Navier-Stokes liquid around a rigid body. Existence of the wake, *RIMS Kôkyûroku Bessatsu,* B1, 108-127, 2008.

[44] Y. Giga, S. Matsui, Y. Shimizu. On Estimates in Hardy Spaces for the Stokes Flow in a Half Space, *Math. Z.*, **231-2**, 383–396, 1999.

[45] R. B. Guenther and E. A. Thomann The fundamental solution of the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions – time dependent case, *J. Math. Fluid Mech.,* **7,** 1-22, 2005.

[46] Z. Guo, Q. Jiu, Z. Xin. Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *SIAM J. Math. Anal.,* **39**, 1402–1427, 2008.

[47] D. Hoff. Global solutions of the Navier - Stokes equations for multidimensional compressible flow with discontiuous initial data, *J. Diff. Eqs.,* **120**, 215–254, 1995.

[48] S. Jiang. Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain, *Commun. Math. Phys.,* **178**, 339–374, 1996.

[49] S. Jiang. Large-time behavior of solutions to the equations of a viscous polytropic ideal gas, *Ann. Mat. Pura Appl.,* **175,** 253–275, 1998.

[50] S. Jiang. Global smooth solutions to the equations of a viscous heat-conducting one-dimensional gas with density-dependent viscosity, *Math. Nachr.,* **190**, 163–183, 1998.

[51] S. Jiang, Z. Xin, P. Zhang. Global weak solution to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity, *Methods and Applications of Analysis,* **12**, 239–252, 2005.

[52] S. Jiang, P. Zhang. On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.,* **215**, 559–581, 2001.

[53] A.V. Kazhikov, V.V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *Prikl. Mat. Meh.,* **41**, 2, 282–291, 1977.

[54] H. Kozono. Global L^n-Solution and Its Decay Property for Navier-Stokes Equations in Half-Space $\mathbb{R}^n_+$, *Journal of Differential Equations*, **79**, 79–88, 1989.

[55] S. Kračmar, Š. Nečasová, P. Penel. L^q - approach of weak solutions of Oseen flow around a rotating body, *Quartely of Applied Mathematics,* **LXVIII,** 3 , 421–437, 2010.

[56] S. Kračmar, Š. Nečasová, P. Penel. L^q-approach of weak solutions of Oseen Flow Around a Rotating Body, *Lecture Notes of Banach Centrum,* **81**, 259–276, 2008.

[57] S. Kračmar, Š. Nečasová, P. Penel. Anisotropic L^2-estimates of weak solutions to the stationary Oseen-type equations in 3D-exterior domain for a rotating body, *J. of J. Math. Soc. of Japan,* **62**, 1, 239–268, 2010.

[58] S. Kračmar, P. Penel. Variational properties of a generic model equation in exterior 3D domains, *Funkcial. Ekv.,* **47**, 499–523, 2004.

[59] S. Kračmar, P. Penel. New regularity results for a generic model equation in exterior 3D domains, *Banach Center Publications Warsaw,* **70**, 139–155, 2005.

[60] J. M. Lattimer, K. A. Van Riper, M. Prakash, M. Prakash. Rapid cooling and the structure of neutron stars, *The Astrophysical Journal,* **425**, 802–813, 1994.

[61] P. L. Lions. *Mathematical Topics in Fluid Mechanics,* Vol. 2, Oxford University Press, New York, 1998.

[62] T. P. Liu, Z. P. Xin, T. Yang. Vacuum states of compressible flow, *Discrete Continuous Dynam. Systems,* **4,** 1–31, 1990.

[63] A. Matsumura, T. Nishida. The initial value problem for the equtions of motion of compressible viscous and heat - conductive fluids, *Proc. Japan Acad. Ser. A,* **55,** 337–342, 1979.

[64] A. Matsumura, T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.,* **20,** 67–104, 1980.

[65] A. Matsumura, T. Nishida. Initial value problems for the equations of motion of compressible viscous and heat-conductive fluids, *Commun. Math. Phys.,* **89**, 445–46, 1983.

[66] A. Mellet, A. Vasseur. On the isentropic compressible Navier-Stokes equation, *Comm. Partial Differential Equations,* **32**, 431–452, 2007.

[67] V. G. Maz'ya, B. A. Plamenevskiĭ, L. I. Stupyalis. The three-dimentional problem of steady-state motion of a fluid with a free surface, *Amer. Math. Soc. Transl.,* **123**, 171–268, 1984.

[68] Š. Nečasová, K. Schumacher. Strong solution of the Stokes Flow Around a Rotating Body in weighted L^q-spaces, *Mathemath. Nachr.,* **284**, 13, 1701–1714, 2011.

[69] Š. Nečasová Asymptotic properties of the steady fall of a body in a viscous fluids, *Math. Methods in the Appl. Sciences*, **27**, 1969–1995, 2004.

[70] M. Okada, Š. Matušů-Nečasová, T. Makino. Free boundary problem for the equation of one-dimensional motion of compressible gas with density-dependent temperature, *Ann. Univ. Ferrara - Sez. VII - Sc. Mat.,* **48**, 1–20, 2002.

[71] M. Okada, Š. Matušů-Nečasová, T. Makino. Free boundary problem for the equation of spherically symmetric motion of viscous gas III, *Japan J. Indust. Appl. Math.,* **14**, 2, 199–213, 1997.

[72] M. Okada, Š. Matušů-Nečasová, T. Makino. Free boundary problem for the equation of spherically symmetric motion of viscous gas II, Japan J. Indust. Appl. Math., **12**, 2, 195–203, 1995.

[73] D. Serre. Solutions faibles globales des equations de Navier-Stokes pour un fluide compressible, *C. R. Acad. Sci. Paris Sér. I Math.,* **303,** 13, 639–642, 1986.

[74] E. M. Stein. Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, N.J., (1993).

[75] S. W. Vong, T. Yang, C. J. Zhu. Compressible Navier- Stokes equations with degenerate viscosity coefficient and vacuum II, *J. Differential Equations,* **192**, 475–501, 2003.

[76] T. Yang, H. J. Zhao. A vacuum problem for the one - dimensional compressible Noavier- Stokes equations with density- dependent viscosity, *J. Differential Equations,* **184**, 163–184, 2002.

[77] T. Yang, C. J. Zhu. Compressible Navier- Stokes equations with degenerate viscosity coefficient and vacuum, *Comm. Math. Phys.,* **230**, 329–363, 2002.

[78] N. Tanaka. On the boundary value problem for the stationary Stokes system in the half-space, *Journal of Differential Equations*, **115**, 70–74, 1995.

[79] S. Ukai. A Solution Formula for the Stokes Equation in $\mathbb{R}^n_+$, *Communications on Pure and Applied Mathematics,* **XL**, 611-621, 1987.

[80] Weinberger, H. F. Variational properties of steady fall in Stokes flow, *J. Fluid Mech.,* **52**, 2, 321-344, 1972.

[81] Weinberger, H. F. On the steady fall of a body in a Navier-Stokes fluid, *Partial differential equations* (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), 421-439. Amer. Math. Soc., Providence, R. I., 1973.

[82] S. Yanagi. Asymptotic stability of the spherically symmetric solutions for a viscous polytropic gas in a field of external forces, *Transport Theory and Statistical Physics,* **29**, 333–353, 2000.

[83] T. Zhang, D. Fang. A note on spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *Nonlinear Analysis: Real World Applications* (in press).

7 Articles included in the dissertation

MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2008; **31**:551–574
Published online 9 July 2007 in Wiley InterScience
(www.interscience.wiley.com) DOI: 10.1002/mma.925
MOS subject classification: primary 76 D 05; secondary 35 Q 30

A weighted L^q-approach to Oseen flow around a rotating body

R. Farwig[1], M. Krbec[2] and Š. Nečasová[2,*,†]

[1]*Department of Mathematics, Darmstadt University of Technology, 64289 Darmstadt, Germany*
[2]*Mathematical Institute of Academy of Sciences, Žitná 25, 11567 Prague 1, Czech Republic*

Communicated by W. Wendland

SUMMARY

We study time-periodic Oseen flows past a rotating body in $\mathbb{R}^3$ proving weighted *a priori* estimates in L^q-spaces using Muckenhoupt weights. After a time-dependent change of coordinates the problem is reduced to a stationary Oseen equation with the additional terms $(\omega \wedge x)\cdot\nabla u$ and $-\omega\wedge u$ in the equation of momentum where ω denotes the angular velocity. Due to the asymmetry of Oseen flow and to describe its wake we use anisotropic Muckenhoupt weights, a weighted theory of Littlewood–Paley decomposition and of maximal operators as well as one-sided univariate weights, one-sided maximal operators and a new version of Jones' factorization theorem. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: Littlewood–Paley theory; maximal operators; rotating obstacles; stationary Oseen flow; anisotropic Muckenhoupt weights; one-sided weights; one-sided maximal operators

1. INTRODUCTION

We consider a three-dimensional rigid body $K\subset\subset\mathbb{R}^3$ rotating with angular velocity $\omega=\tilde{\omega}(0,0,1)^T$, $\tilde{\omega}\neq 0$, and assume that the complement $\mathbb{R}^3\backslash K$ is filled with a viscous incompressible fluid modelled by the Navier–Stokes equations. Then we will analyse the viscous flow either past the rotating body K with velocity $u_\infty = ke_3\neq 0$ at infinity or around a rotating body K which is moving in the direction of its axis of rotation. Given the coefficient of viscosity $v>0$ and an external force $\tilde{f}=\tilde{f}(y,t)$, we are looking for the velocity $v=v(y,t)$ and the pressure $q=q(y,t)$ solving the

*Correspondence to: Š. Nečasová, Mathematical Institute of Academy of Sciences, Žitná 25, 11567 Prague 1, Czech Republic.
†E-mail: matus@math.cas.cz

Contract/grant sponsor: Academy of Sciences of the Czech Republic; contract/grant numbers: AV0Z10190503, D-CZ 3/05-06
Contract/grant sponsor: Grant Agency of the Academy of Sciences; contract/grant number: IAA100190505
Contract/grant sponsor: DAAD; contract/grant number: D/04/25763

Received 8 February 2007

nonlinear system

$$\begin{aligned} v_t - \nu\Delta v + v\cdot\nabla v + \nabla q &= \widetilde{f} && \text{in } \Omega(t),\ t>0\\ \operatorname{div} v &= 0 && \text{in } \Omega(t),\ t>0\\ v(y,t) &= \omega\wedge y && \text{on } \partial\Omega(t),\ t>0\\ v(y,t) &\to u_\infty \neq 0 && \text{as } |y|\to\infty \end{aligned} \tag{1}$$

Here the time-dependent exterior domain $\Omega(t)$ is given—due to the rotation with angular velocity ω—by

$$\Omega(t) = O_\omega(t)\Omega$$

where $\Omega\subset\mathbb{R}^3$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix:

$$O_\omega(t) = \begin{pmatrix} \cos\widetilde{\omega}t & -\sin\widetilde{\omega}t & 0\\ \sin\widetilde{\omega}t & \cos\widetilde{\omega}t & 0\\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Introducing the change of variables and the new functions

$$x = O_\omega(t)^{\mathrm{T}} y \quad \text{and} \quad u(x,t) = O^{\mathrm{T}}_{\omega(t)}(v(y,t)-u_\infty), \quad p(x,t) = q(y,t) \tag{3}$$

respectively, as well as the force term $f(x,t)=O(t)^{\mathrm{T}}\widetilde{f}(y,t)$ we arrive at the modified Navier–Stokes system

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f && \text{in } \Omega\times(0,\infty)\\ \operatorname{div} u &= 0 && \text{in } \Omega\times(0,\infty)\\ u(x,t) &\to 0 && \text{as } |x|\to\infty \end{aligned} \tag{4}$$

with boundary condition $u(x,t)=\omega\wedge x - u_\infty$ on $\partial\Omega$ in the exterior time-independent domain Ω.

Due to the new coordinate system attached to the rotating body the nonlinear system (4) contains two new linear terms, the classical Coriolis force term $\omega\wedge u$ (up to a multiplicative constant) and the term $(\omega\wedge x)\cdot\nabla u$ which is *not* subordinate to the Laplacian in unbounded domains. Linearizing (4) in u at $u\equiv 0$ and considering only the stationary problem we arrive at the modified Oseen system

$$\begin{aligned} -\nu\Delta u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f && \text{in } \Omega\\ \operatorname{div} u &= 0 && \text{in } \Omega\\ u &\to 0 && \text{at } \infty \end{aligned} \tag{5}$$

together with the boundary condition $u(x,t)=\omega\wedge x - u_\infty$ on $\partial\Omega$. Note that there is no boundary condition in the case $\Omega=\mathbb{R}^3$.

DOI: 10.1002/mma

The linear system (5) has been analysed in classical L^q-spaces, $1<q<\infty$, for the whole space case in [1, 2] proving the *a priori* estimate

$$\begin{aligned} \|\nu\nabla^2 u\|_q + \|\nabla p\|_q &\leqslant c\|f\|_q \\ \|k\partial_3 u\|_q + \|(\omega\wedge x)\cdot\nabla u + \omega\wedge u\|_q &\leqslant c\left(1+\frac{k^4}{\nu^2|\omega|^2}\right)\|f\|_q \end{aligned} \tag{6}$$

with a constant $c>0$ independent of ν, k and ω. For a discussion of weak solutions, we refer to [3, 4]; the spectrum of the linear operator defined by (5) is considered in [5]. The corresponding case when $u_\infty=0$ has recently been analysed in [6–13]. For a more comprehensive introduction including physical considerations and nonNewtonian fluids we refer to [14].

The aim of this paper is to generalize the *a priori* estimate (6) to weighted L^q-spaces for the whole space $\mathbb{R}^3$. For this reason, we introduce the weighted Lebesgue space

$$L^q_w(\mathbb{R}^3) = L^q_w = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^3) :\ \|u\|_{q,w} = \left(\int_{\mathbb{R}^n} |u(x)|^q w(x)\,\mathrm{d}x\right)^{1/q} < \infty \right\}$$

where $w \in L^1_{\text{loc}}$ is a nonnegative weight function and should reflect the anisotropy of the flow and the existence of a wake region in the downstream direction $x_3>0$. Our tools will include Littlewood–Paley theory, singular integral operators, multiplier operators and maximal operators in weighted spaces so that we need weight functions satisfying Muckenhoupt-type conditions. For a totally different approach using variational methods see [15].

Definition 1.1
Let $\mathscr{R}$ be a collection of bounded sets R in $\mathbb{R}^n$, each of positive Lebesgue measure $|R|$. A weight function $0\leqslant w\in L^1_{\text{loc}}$ belongs to the *Muckenhoupt class* $A_q(\mathscr{R}) = A_q(\mathbb{R}^n,\mathscr{R})$, $1\leqslant q<\infty$, if there exists a constant $C>0$ such that

$$\sup_R \left(\frac{1}{|R|}\int_R w(x)\,\mathrm{d}x\right)\left(\frac{1}{|R|}\int_R w^{-1/(q-1)}\,\mathrm{d}x\right)^{q-1} \leqslant C \text{ for any } R\in\mathscr{R}$$

if $1<q<\infty$, and

$$\sup_{R\in\mathscr{R},\, R\ni x_0} \frac{1}{|R|}\int_R w(x)\,\mathrm{d}x \leqslant C w(x_0) \quad \text{for a.a. } x_0\in\mathbb{R}^n$$

if $q=1$, respectively.

Due to the anisotropic nature of our problem we shall need a variant of the classical Muckenhoupt class $A_q(\mathscr{C}) = A_q(\mathbb{R}^3,\mathscr{C})$, where $\mathscr{C}$ is the set of all cubes $Q\subset\mathbb{R}^3$ with edges parallel to the coordinate axes. Namely, $\mathscr{C}$ is replaced by $\mathscr{J}$, the set of all bounded intervals (rectangles) in $\mathbb{R}^3$, leading to the class $A_q(\mathscr{J}) = A_q(\mathbb{R}^3,\mathscr{J})$. Obviously, $A_q(\mathbb{R}^3,\mathscr{J}) \subsetneq A_q(\mathbb{R}^3,\mathscr{C})$.

Moreover, to describe the anisotropy of the wake region more precisely by weights we have to introduce in addition to the weights on $\mathbb{R}^n$ *one-sided Muckenhoupt weights* and *one-sided maximal operators* on the real line, see Definition 1.2, Theorem 2.3 and Lemma 2.4.

DOI: 10.1002/mma

Definition 1.2

(i) For every locally integrable function u on the real line let, M^+u be defined by

$$M^+u(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |u(t)|\,\mathrm{d}t$$

Analogously,

$$M^-u(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |u(t)|\,\mathrm{d}t$$

(ii) A weight function $0 \leqslant w \in L^1_{loc}(\mathbb{R})$ lies in the *weight class* A_1^- if there exists a constant $c>0$ such that $M^+w(x) \leqslant cw(x)$ for almost all $x \in \mathbb{R}$. Analogously, $w \in A_1^+$ if and only if $M^-w(x) \leqslant cw(x)$ for almost all $x \in \mathbb{R}$. The smallest constant $c \geqslant 0$ satisfying $M^{\pm}w(x) \leqslant cw(x)$ for almost all $x \in \mathbb{R}$ is called the $A_1^{\mp}$-*constant of* w.

(iii) A weight function $0 \leqslant w \in L^1_{\text{loc}}$ belongs to the *one-sided Muckenhoupt class* A_q^+, $1<q<\infty$, if there exists a constant $C>0$ such that for all $x \in \mathbb{R}$

$$\sup_{h>0} \left(\frac{1}{h} \int_{x-h}^{x} w(t)\,\mathrm{d}t \right) \left(\frac{1}{h} \int_x^{x+h} w(t)^{-1/(q-1)}\,\mathrm{d}t \right)^{q-1} \leqslant C$$

The smallest constant $C \geqslant 0$ satisfying this estimate is called the A_q^+-*constant* of w. By analogy, we define the set of weights A_q^- and the A_q^--constant of a weight in A_q^-.

Now we are in a position to describe the most general weights considered in this paper. Note that these weights are independent of the angular variable θ in the cylindrical coordinate system $(r, \theta, x_3) \in [0, \infty) \times [0, 2\pi] \times \mathbb{R}$ attached to the axis of revolution $e_3 = (0, 0, 1)^{\mathrm{T}}$. Hence, we will write $w(x) = w(x_1, x_2, x_3) = w_r(x_3)$ for $r = |(x_1, x_2)|$, $x = (x_1, x_2, x_3)$.

Definition 1.3
For $1 \leqslant q < \infty$, let

$$\begin{aligned} \widetilde{A}_q^- = \widetilde{A}_q^-(\mathbb{R}^3) = \{ & w \in A_q(\mathbb{R}^3) : w \text{ is } \theta\text{-independent for a.a. } r>0 \\ & w(x_1, x_2, \cdot) = w_r(\cdot) \in A_q^-(\mathbb{R}) \\ & \text{with } A_q^-(\mathbb{R})\text{-constant essentially bounded in } r \} \end{aligned} \tag{7}$$

Theorem 1.4
Let the weight function $0 \leqslant w \in L^1_{\text{loc}}(\mathbb{R}^3)$ be independent of the angular variable θ and satisfy the following condition depending on $q \in (1, \infty)$:

$$\begin{aligned} 2 \leqslant q < \infty: \quad & w^\tau \in \widetilde{A}^-_{\tau q/2} \quad \text{for some } \tau \in [1, \infty) \\ 1 < q < 2: \quad & w^\tau \in \widetilde{A}^-_{\tau q/2} \quad \text{for some } \tau \in \left(\frac{2}{q}, \frac{2}{2-q} \right] \end{aligned} \tag{8}$$

DOI: 10.1002/mma

(i) Given $f \in L^q_w(\mathbb{R}^3)^3$ there exists a solution $(u, p) \in L^1_{\text{loc}}(\mathbb{R}^3)^3 \times L^1_{\text{loc}}(\mathbb{R}^3)$ of (5) satisfying the estimate

$$\|v\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \leqslant c\|f\|_{q,w} \tag{9}$$

with a constant $c = c(q, w) > 0$ independent of v, k and ω.

(ii) Let $f \in L^{q_1}_{w_1}(\mathbb{R}^3)^3 \cap L^{q_2}_{w_2}(\mathbb{R}^3)^3$ such that both (q_1, w_1) and (q_2, w_2) satisfy conditions (8), and let $u_1, u_2 \in L^1_{\text{loc}}(\mathbb{R}^3)^3$ together with corresponding pressure functions $p_1, p_2 \in L^1_{\text{loc}}(\mathbb{R}^3)$ be solutions of (5) satisfying (9) for (q_1, w_1) and (q_2, w_2), respectively. Then there are $\alpha, \beta \in \mathbb{R}$ such that u_1 coincides with u_2 up to an affine linear field $\alpha e_3 + \beta\omega \wedge x$, $\alpha, \beta \in \mathbb{R}$.

Corollary 1.5
Let the weight function $0 \leqslant w \in L^1_{\text{loc}}(\mathbb{R}^3)$ be independent of the angular variable θ. Moreover, let w satisfy the following condition depending on $q \in (1, \infty)$:

$$\begin{aligned} 2 \leqslant q < \infty: \;\; & w^\tau \in \widetilde{A}^-_{\tau q/2}(\mathscr{J}) \quad \text{for some } \tau \in [1, \infty) \\ 1 < q < 2: \;\; & w^\tau \in \widetilde{A}^-_{\tau q/2}(\mathscr{J}) \quad \text{for some } \tau \in \left(\frac{2}{q}, \frac{2}{2-q}\right] \end{aligned} \tag{10}$$

where the weight class $\widetilde{A}^-_\tau(\mathscr{J})$, $1 \leqslant \tau < \infty$, is defined by

$$\widetilde{A}^-_\tau(\mathscr{J}) = \widetilde{A}^-_\tau(\mathbb{R}^3) \cap A_\tau(\mathscr{J})$$

Given $f \in L^q_w(\mathbb{R}^3)^3$ there exists a solution $(u, p) \in L^1_{\text{loc}}(\mathbb{R}^3)^3 \times L^1_{\text{loc}}(\mathbb{R}^3)$ of (5) satisfying the estimate

$$\|k\partial_3 u\|_{q,w} + \|(\omega \wedge x) \cdot u - \omega \wedge u\|_{q,w} \leqslant c\left(1 + \frac{k^5}{v^{5/2}|\omega|^{5/2}}\right)\|f\|_{q,w} \tag{11}$$

with a constant $c = c(q, w) > 0$ independent of v, k and ω.

We remark that the ω-dependent term $1 + k^5/v^{5/2}|\omega|^{5/2}$ in (11) cannot be avoided in general; see [2] for an example in the space $L^2(\mathbb{R}^3)$.

As an example of anisotropic weight functions we consider

$$w(x) = \eta^\alpha_\beta(x) = (1 + |x|)^\alpha (1 + s(x))^\beta, \quad s(x) = |(x_1, x_2, x_3)| - x_3 \tag{12}$$

introduced in [16] to analyse the Oseen equations; see also [3, 15].

Corollary 1.6
The *a priori* estimate (9) holds for the anisotropic weights $w = \eta^\alpha_\beta$, see (12), provided that

$$\begin{aligned} 2 \leqslant q < \infty: & -\frac{q}{2} < \alpha < \frac{q}{2}, \quad 0 \leqslant \beta < \frac{q}{2} \text{ and } \alpha + \beta > -1 \\ 1 < q < 2: & -\frac{q}{2} < \alpha < q - 1, \quad 0 \leqslant \beta < q - 1 \text{ and } \alpha + \beta > -\frac{q}{2} \end{aligned}$$

Note that the condition $\beta \geqslant 0$ will reflect the existence of a wake region in the downstream direction $x_3 > 0$ where the solution of the original nonlinear problem (1) will decay slower than in the upstream direction $x_3 < 0$.

DOI: 10.1002/mma

2. PRELIMINARIES

To prove Theorem 1.4 we need several properties of Muckenhoupt weights and of maximal operators. Recall that $\mathscr{J}$ stands for the set of all nondegenerate rectangles in $\mathbb{R}^n$ with edges parallel to the coordinate axes.

Proposition 2.1

(1) Let μ be a nonnegative regular Borel measure such that the strong centred Hardy–Littlewood maximal operator

$$\mathscr{M}_{\mathscr{J}}\mu(x) = \sup_{R\in\mathscr{J}, R\ni x} \frac{1}{|R|}\int_R \mathrm{d}\mu$$

is finite for almost all $x \in \mathbb{R}^n$; here R runs through the collection $\mathscr{J}$ of rectangles containing additionally the point x, and $|R|$ denotes the Lebesgue measure of R. Then $(\mathscr{M}_{\mathscr{J}}\mu)^\gamma \in A_1(\mathscr{J})$ for all $\gamma \in [0, 1)$.

(2) For all $1<q<\tau$, we have $A_1(\mathscr{J}) \subset A_q(\mathscr{J}) \subset A_\tau(\mathscr{J})$.

(3) Let $1<q<\infty$ and $w \in A_q(\mathscr{J})$. Then there are $w_1, w_2 \in A_1(\mathscr{J})$ such that

$$w = \frac{w_1}{w_2^{q-1}}$$

Conversely, given $w_1, w_2 \in A_1(\mathscr{J})$, the weight $w = w_1 w_2^{1-q}$ belongs to $A_q(\mathscr{J})$.

For the proofs see [17, Chapter IV, Section 6]. Claim (3) is a variant of Jones' factorization theorem, see [17, Chapter IV, Theorem 6.8].

For a rapidly decreasing function $u \in \mathscr{S}(\mathbb{R}^n)$, let

$$\mathscr{F}u(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi}u(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^n$$

be the Fourier transform of u. Its inverse will be denoted by $\mathscr{F}^{-1}$. Moreover, we define the centred Hardy–Littlewood maximal operator

$$\mathscr{M}u(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |u(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n$$

for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where Q runs through the set of all closed cubes centred at x.

Theorem 2.2
Let $1<q<\infty$ and $w \in A_q$.

(i) The operator $\mathscr{M}$, defined e.g. on $\mathscr{S}(\mathbb{R}^n)$, is a bounded operator from L^q_w to L^q_w.

(ii) Let $m \in C^n(\mathbb{R}^n\backslash\{0\})$ satisfy the pointwise Hörmander–Mikhlin multiplier condition

$$|\xi|^{|\alpha|}\,|D^\alpha m(\xi)| \leqslant c_\alpha \quad \text{for all } \xi \in \mathbb{R}^n\backslash\{0\}$$

and all multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant n_1 \in \mathbb{N}$, where $n_1 \geqslant n/2$. Then the multiplier operator $u \mapsto \mathscr{F}^{-1}(m\hat{u}),\ u \in \mathscr{S}(\mathbb{R}^n)$, can be extended to a bounded linear operator from L^q_w to L^q_w.

DOI: 10.1002/mma

(iii) Let m be of class C^n in each 'quadrant' of $\mathbb{R}^n$ and let a constant $B\geqslant 0$ exist such that $\|m\|_\infty \leqslant B$,

$$\sup_{x_{k+1},\dots,x_n} \int_{\mathscr{I}} \left| \frac{\partial^k m(x)}{\partial x_1 \cdots \partial x_k} \right| \mathrm{d}x_1 \dots \mathrm{d}x_k \leqslant B$$

for any dyadic interval $\mathscr{I}$ in $\mathbb{R}^k$, $1\leqslant k\leqslant n$, and also for any permutation of the variables $x_1,\dots,x_k$ within $x_1,\dots,x_n$. If $1<p<\infty$ and $w\in A_p(\mathbb{R}^n,\mathscr{J})$, then m defines a bounded multiplier operator from $L^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$.

Proof
(i) See [17, Theorem IV 2.8], [18, Theorem 9] and (ii) see [17, Theorem IV 3.9] or [19, Theorem 4]. Note that the pointwise condition on m implies the integral condition in [17, 19]. For the proof of (iii) see [19]. □

Concerning *one-sided weights* and *one-sided maximal operators* on the real line, see Definition 1.2, we first recall the following duality property: $w\in A_q^+$ if and only if $w^{-q'/q}=w^{-1/(q-1)}\in A_{q'}^-$. Moreover, we will need the following results:

Theorem 2.3 (*Theorem 1 of* [*20*])
Let $1<p<\infty$ and $p'=p/(p-1)$.

(i) Let $w_1\in A_1^+$, $w_2\in A_1^-$. Then $w_1/w_2^{p-1}\in A_p^+$. Conversely, given $w\in A_p^+$ there exist $w_1\in A_1^+$, $w_2\in A_1^-$ such that $w=w_1/w_2^{p-1}$.
(ii) The operator M^+ is continuous from $L^p_w(\mathbb{R})$ to itself if and only if $w\in A_p^+$. Analogously, $M^-: L^p_w(\mathbb{R})\to L^p_w(\mathbb{R})$ if and only if $w\in A_p^-$.

Obviously, $A_p\subset A_p^\pm$ where A_p denotes the usual Muckenhoupt class on the real line. Hence $|x|^\alpha$, $(1+|x|)^\alpha\in A_p^\pm$ if $-1<\alpha<p-1$, $1<p<\infty$. However, in view of the anisotropic weight $w=\eta^\alpha_\beta$ on $\mathbb{R}^3$, see (12), we have to consider also one-dimensional anisotropic weight functions such as

$$\tilde{w}_{\alpha,\beta}(x)=\tilde{w}_{\alpha,\beta}(x;r)=(r^2+x^2)^{\alpha/2}(\sqrt{r^2+x^2}-x)^\beta, \quad x\in\mathbb{R},\ r>0 \tag{13}$$

Lemma 2.4

(i) For every $r>0$, the univariate weight $\tilde{w}_{\alpha,\beta}(x;r)$ lies in A_1^- if and only if $\beta\geqslant 0$, $\alpha\leqslant\beta$ and $\alpha+\beta>-1$. Moreover, the A_1^--constant of $\tilde{w}_{\alpha,\beta}$ is uniformly bounded in r.
(ii) For every $r>0$, the univariate weight

$$w_{\alpha,\beta}(x)=w_{\alpha,\beta}(x;r)=(1+r^2+x^2)^{\alpha/2}(1+\sqrt{r^2+x^2}-x)^\beta$$

lies in A_1^- with an A_1^--constant independent of $r>0$ if and only if

$$\alpha\leqslant 0\leqslant\beta \quad \text{and} \quad \alpha+\beta>-1 \tag{14}$$

DOI: 10.1002/mma

(iii) Let $1<p<\infty$. Then for every $r>0$

$$\begin{aligned} w_{\alpha,\beta}(\cdot\,;r)\in A_p^+ &\quad \text{for } \alpha>-1,\ \beta\leqslant 0,\ \alpha+\beta<p-1 \\ w_{\alpha,\beta}(\cdot\,;r)\in A_p^- &\quad \text{for } \alpha<p-1,\ \beta\geqslant 0,\ \alpha+\beta>-1 \end{aligned} \tag{15}$$

Moreover, the $A_p^{\pm}$-constant is uniformly bounded in $r>0$.

Proof
(i) A simple scaling argument shows that it suffices to look at the weight $\widetilde{w}=\widetilde{w}_{\alpha,\beta}$ in (13) for $r=1$ only and that the A_1^--constant is independent of $r>0$. We will consider three cases.

Case 1: $x>0$. Then $\widetilde{w}(x)\sim(1+|x|)^{\alpha-\beta}$, i.e. there exists a constant $c>0$ independent of $x>0$ such that $(1/c)(1+|x|)^{\alpha-\beta}\leqslant\widetilde{w}(x)\leqslant c(1+|x|)^{\alpha-\beta}$ for all $x>0$. Hence, for all $h>0$

$$\frac{1}{h}\int_x^{x+h}\widetilde{w}(t)\,\mathrm{d}t\sim\frac{1}{h}\int_x^{x+h}(1+t)^{\alpha-\beta}\,\mathrm{d}t$$

If $\alpha-\beta>0$, then the term on the right-hand side is strictly increasing to $+\infty$ as $h\to\infty$. Thus, we are led to the condition $\alpha\leqslant\beta$.

Now let $\alpha\leqslant\beta$. Then for all $h>0$

$$\frac{1}{h}\int_x^{x+h}(1+t)^{\alpha-\beta}\,\mathrm{d}t\leqslant\frac{1}{h}\int_x^{x+h}(1+x)^{\alpha-\beta}\,\mathrm{d}t=(1+|x|)^{\alpha-\beta}\sim\widetilde{w}(x)$$

Case 2: $x<0$ and $0<h<|x|$. Then $\widetilde{w}(t)\sim(1+|t|)^{\alpha+\beta}$ for all $t\in(x,x+h)$. Assume that $\alpha+\beta=-1$ and let $h=|x|$. Then

$$\frac{1}{|x|}\int_x^0(1+|t|)^{-1}\,\mathrm{d}t=\frac{\log(1+|x|)}{|x|}$$

is not bounded by $c\widetilde{w}(x)=c/|x|$ uniformly in $x<0$ for any constant $c>0$. Analogously, if $\alpha+\beta<-1$, then for $h=|x|$ we see that $(1/|x|)\int_x^0(1+|t|)^{\alpha+\beta}\,\mathrm{d}t\sim 1/|x|$ is not bounded by $c\widetilde{w}(x)=c(1+|x|)^{\alpha+\beta}$ uniformly in $x<0$. Hence, in the following we have to assume that $\alpha+\beta>-1$. We shall consider two subcases: $h>0$ small with respect to $|x|$ and h comparable with $|x|$. If $0<h<|x|/2$, then

$$\frac{1}{h}\int_x^{x+h}(1+|t|)^{\alpha+\beta}\,\mathrm{d}t\sim\frac{1}{h}\int_x^{x+h}(1+|x|)^{\alpha+\beta}\,\mathrm{d}t=(1+|x|)^{\alpha+\beta}\sim\widetilde{w}(x)$$

For the second subcase, assume that $|x|/2<h<|x|$. Then we are led to the integral

$$\begin{aligned} &\frac{1}{|x|}\int_x^{x+h}(1+|t|)^{\alpha+\beta}\,\mathrm{d}t \\ &\quad\leqslant\frac{1}{|x|}\int_x^0(1+|t|)^{\alpha+\beta}\,\mathrm{d}t\sim\begin{cases}\dfrac{(1+|x|)^{\alpha+\beta+1}}{|x|}, & |x|>1\\ 1, & |x|<1\end{cases}\sim\widetilde{w}(x)\end{aligned}$$

DOI: 10.1002/mma

Case 3: $x<0$ *and* $h>|x|$. In this case, we have to consider the sum

$$\frac{1}{h}\int_x^0 \widetilde{w}\,\mathrm{d}t+\frac{1}{h}\int_0^{x+h}\widetilde{w}\,\mathrm{d}t\leqslant\frac{1}{|x|}\int_x^0\widetilde{w}\,\mathrm{d}t+\frac{c}{h}\int_0^{x+h}(1+t)^{\alpha-\beta}\,\mathrm{d}t=:I_1+I_2$$

where the first integral I_1 is bounded by $c\widetilde{w}(x)$ uniformly in $x<0$, see *Case 2*, and where for $|x|<1$ the second integral I_2 is bounded by $c\sim\widetilde{w}(x)$. Therefore, let $|x|>1$ in the following. If $\alpha-\beta\leqslant-1$, then the condition $\alpha+\beta>-1$ implies that $\beta>0$; moreover, I_2 is easily shown to be bounded by $c\widetilde{w}(x)\sim(1+|x|)^{\alpha+\beta}$ uniformly in $x<0$ and $h>|x|$.

Now consider the case $\alpha-\beta>-1$. We shall investigate three possibilities of the position of h with respect to $|x|$. If $h=2|x|$, then

$$\frac{1}{|x|}\int_0^{|x|}(1+t)^{\alpha-\beta}\,\mathrm{d}t=\frac{c}{|x|}((1+|x|)^{\alpha-\beta+1}-1)$$

Since $1/|x|=o(|x|^{\alpha+\beta})=o(\widetilde{w}(x))$ by the condition that $\alpha+\beta>-1$, the assertion $I_2\leqslant c\widetilde{w}(x)\sim|x|^{\alpha+\beta}$ necessarily implies that $|x|^{\alpha-\beta}\leqslant c|x|^{\alpha+\beta}$ for $|x|>1$. Thus, β must be nonnegative.

Next, if $|x|<h<2|x|$, then, since $\alpha-\beta\leqslant\alpha+\beta$ and $\alpha+\beta>-1$,

$$I_2\leqslant\frac{c}{|x|}\int_0^{|x|}(1+t)^{\alpha-\beta}\,\mathrm{d}t\leqslant c|x|^{\alpha+\beta}\sim\widetilde{w}(x)$$

Finally, if $h>2|x|>2$, then

$$I_2\leqslant\frac{c}{h}(1+x+h)^{\alpha-\beta+1}\leqslant ch^{\alpha-\beta}\leqslant c|x|^{\alpha+\beta}\sim\widetilde{w}(x)$$

since $\alpha\leqslant\beta$ (see *Case 1*). Summarizing the previous cases and estimates we have proved that there exists $c>0$ such that $M^+\widetilde{w}(x)\leqslant c\widetilde{w}(x)$ for a.a. $x\in\mathbb{R}$, and that this results holds if and only if $\beta\geqslant0$, $\alpha\leqslant\beta$ and $\alpha+\beta>-1$.

(ii) To verify the necessity of (14) let $r=1$ and $w=w_{\alpha,\beta}$. For $x>0$ when $(1+\sqrt{r^2+x^2}-x)^\beta\sim1$, we have to estimate

$$\frac{1}{h}\int_x^{x+h}w(t)\,\mathrm{d}t\sim\frac{1}{h}\int_x^{x+h}(1+t)^\alpha\,\mathrm{d}t$$

by $cw(x)\sim(1+x)^\alpha$. As in *Case 1* of Part (i) (with $\beta=0$) we get the necessary condition $\alpha\leqslant0$.

Let $x<0$. Again we shall distinguish according to the size of h with respect to $|x|$. If $0<h<|x|$, then $w(t)\sim(1+|t|)^{\alpha+\beta}$ for all $t\in(x,x+h)$, and

$$\frac{1}{h}\int_x^{x+h}w(t)\,\mathrm{d}t\sim\frac{1}{h}\int_x^{x+h}(1+|t|)^{\alpha+\beta}\,\mathrm{d}t$$

is bounded by $cw(x)\sim(1+|x|)^{\alpha+\beta}$ only when $\alpha+\beta>-1$; cf. *Case 2* of Part (i). Finally, when $x<0$ and $h>|x|$, say $h=2|x|>2$, and when $\alpha>-1$, then

$$\frac{1}{h}\int_x^{x+h}w(t)\,\mathrm{d}t\sim\frac{1}{h}\int_x^0(1+|t|)^{\alpha+\beta}\,\mathrm{d}t+\frac{1}{h}\int_0^{x+h}(1+t)^\alpha\,\mathrm{d}t\leqslant cw(x)+c|x|^\alpha$$

which is bounded by $cw(x)\sim(1+|x|)^{\alpha+\beta}$ only if $\beta\geqslant0$. However, if $\alpha\leqslant-1$, then the condition $\alpha+\beta>-1$ implies that even $\beta>0$. Hence, the conditions (14) are necessary to prove that $w\in A_1^-$.

DOI: 10.1002/mma

We shall prove that conditions (14) are sufficient for $w_{\alpha,\beta}(x;r)\in A_1^-$ with an A_1^--constant independent of $r>0$. Let us assume that (14) holds and let first $0<r<1$. Then

$$w(t)\sim(1+|t|)^{\alpha}\cdot\begin{cases}1, & t>0\\ (1+|t|)^{\beta}, & t<0\end{cases}$$
$$\sim(1+|t|)^{\alpha+\beta/2}\cdot\begin{cases}(1+|t|)^{-\beta/2}, & t>0\\ (1+|t|)^{\beta/2}, & t<0\end{cases}\sim\widetilde{w}_{\alpha',\beta'}(t;r)$$

where $\alpha'=\alpha+\beta/2$, $\beta'=\beta/2$. Since assumptions (14) on α, β imply that α', β' satisfy the assumptions in (i), $w\in A_1^-$ with an A_1^--constant independent of $0<r<1$.

Next, let $r\geqslant 1$. An elementary calculation shows that

$$w(t)\sim\begin{cases}\widetilde{w}_{\alpha,\beta}(t;r), & t<r^2\\ \widetilde{w}_{\alpha,0}(t;r), & t>r^2\end{cases}$$

Then we will consider three cases.

Case 1: $x<r^2$ *and* $x+h<r^2$. In this case, by Part (i),

$$\frac{1}{h}\int_x^{x+h}w(t)\,\mathrm{d}t\sim\frac{1}{h}\int_x^{x+h}\widetilde{w}_{\alpha,\beta}(t;r)\,\mathrm{d}t\leqslant c\widetilde{w}_{\alpha,\beta}(x;r)\sim cw(x)$$

with $c>0$ independent of $r>1$.

Case 2: $x>r^2$ *and* $x+h>r^2$. Now

$$\frac{1}{h}\int_x^{x+h}w(t)\,\mathrm{d}t\sim\frac{1}{h}\int_x^{x+h}\widetilde{w}_{\alpha,0}(t;r)\,\mathrm{d}t\leqslant c\widetilde{w}_{\alpha,0}(x;r)\sim cw(x)$$

due to *Case 1* in Part (i).

Case 3: $x<r^2$ *but* $x+h>r^2$. Then

$$\frac{1}{h}\int_x^{x+h}w(t)\,\mathrm{d}t\sim\frac{1}{h}\int_x^{r^2}\widetilde{w}_{\alpha,\beta}(t;r)\,\mathrm{d}t+\frac{1}{h}\int_{r^2}^{x+h}\widetilde{w}_{\alpha,0}(t;r)\,\mathrm{d}t$$

By Part (i), the first integral on the right-hand side is bounded by $((r^2-x)/h)\widetilde{w}_{\alpha,\beta}(x;r)\leqslant\widetilde{w}_{\alpha,\beta}(x;r)\leqslant cw(x)$. Hence, it suffices to prove that

$$\frac{1}{h}\int_{r^2}^{x+h}\widetilde{w}_{\alpha,0}(t;r)\,\mathrm{d}t\leqslant cw(x)$$

If $|x|\leqslant r^2$, then Part (i) implies that

$$\frac{1}{h}\int_{r^2}^{x+h}\widetilde{w}_{\alpha,0}(t;r)\,\mathrm{d}t\leqslant\frac{x+h-r^2}{h}\widetilde{w}_{\alpha,0}(r^2;r)\leqslant\widetilde{w}_{\alpha,0}(r^2;r)\leqslant cr^{2\alpha}$$

where $r^{2\alpha}\leqslant(r+|x|)^{\alpha}\leqslant cw(x)$ since $\alpha\leqslant 0\leqslant\beta$.

DOI: 10.1002/mma

If $x < -r^2$, then $w(x) \sim |x|^{\alpha+\beta}$, and a simple scaling argument and the condition $\beta \geqslant 0$ allow to reduce the problem to the case $r = 1$. Actually, it suffices to show the existence of $c > 0$ such that

$$J := \int_1^{x+h} t^\alpha \, \mathrm{d}t \leqslant ch|x|^{\alpha+\beta} \quad \text{when } x \leqslant -1, \quad x + h \geqslant 1$$

If $\alpha < -1$, then J is bounded by $(1/|\alpha+1|) \leqslant c|x|^{\alpha+\beta+1} \leqslant ch|x|^{\alpha+\beta}$, since $\alpha+\beta > -1$ and $h > |x| > 1$. In the case $\alpha = -1$ the integral J equals

$$\log(x+h) \sim \log h + \frac{x}{h} \leqslant c\big(1 + h^{\min(\beta,1)}\big) \leqslant ch|x|^{\beta-1}$$

since $\beta > -1 - \alpha = 0$. Finally, for $\alpha > -1$, we may bound J by $c(x+h)^{\alpha+1}$. If $1 < |x| < h < 2|x|$, this term is bounded by $c|x| \leqslant ch|x|^\alpha \leqslant ch|x|^{\alpha+\beta}$. In the remaining case when $h > 2|x|$, we get that $(x+h)^{\alpha+1} \leqslant ch^{\alpha+1} \leqslant ch|x|^{\alpha+\beta}$, since $\alpha \leqslant 0 \leqslant \beta$.

Now (ii) is completely proved.

(iii) By Theorem 2.3 (i) and Part (ii) of this Lemma

$$w(x) = \frac{(1 + r^2 + x^2)^{\alpha_1/2}}{(1 + r^2 + x^2)^{\alpha_2(p-1)/2}(1 + \sqrt{r^2 + x^2} - x)^{\beta_2(p-1)}} \in A_p^+$$

for all $\alpha_1, \alpha_2, \beta_2$ satisfying $-1 < \alpha_1 \leqslant 0$, $\alpha_2 \leqslant 0 \leqslant \beta_2$ and $\alpha_2 + \beta_2 > -1$. Hence, with $\alpha = \alpha_1 - \alpha_2(p-1)$, $\beta = -\beta_2(p-1)$, we get that $w = w_{\alpha,\beta}(\cdot\,; r) \in A_p^+$ for all α, β satisfying $\alpha > -1$, $\beta \leqslant 0$ and $\alpha + \beta < p - 1$. By analogy,

$$w(x) = \frac{(1 + r^2 + x^2)^{\alpha_1/2}(1 + \sqrt{r^2 + x^2} - x)^{\beta_1}}{(1 + r^2 + x^2)^{\alpha_2(p-1)/2}} \in A_p^-$$

for all $\alpha_1, \alpha_2, \beta_1$ satisfying $\alpha_1 \leqslant 0 \leqslant \beta_1$, $\alpha_1 + \beta_1 > -1$, $-1 < \alpha_2 \leqslant 0$. Hence, $w = w_{\alpha,\beta}(\cdot\,; r) \in A_p^-$ for all α, β such that $\beta \geqslant 0$, $\alpha < p - 1$ and $\alpha + \beta > -1$. Moreover, in both cases the $A_p^\pm$-constant of the weight is uniformly bounded in $r > 0$. □

Note that the univariate weights $\widetilde{w}_{\alpha,\beta}$ and $w_{\alpha,\beta}$ mainly differ for large $x > 0$. While $\widetilde{w}_{0,\beta}$ decays as $(1/x)^\beta$ as $x \to \infty$ for every fixed $r > 0$, the weight $w_{0,\beta}$ is bounded below by 1 as $x \to \infty$. The reason to consider the weights $w_{\alpha,\beta}$ rather than $\widetilde{w}_{\alpha,\beta}$ is based on the use of the anisotropic weights η_β^α on $\mathbb{R}^3$, see Corollary 1.5, when fixing $r = |(x_1, x_2)|$, $x_1, x_2 \in \mathbb{R}$, so that $\eta_\beta^\alpha(x_1, x_2, x_3) = w_{\alpha,\beta}(x_3; r)$.

Due to the geometry of the problem we introduce cylindrical coordinates $(r, x_3, \theta) \in (0, \infty) \times \mathbb{R} \times [0, 2\pi)$ and write $u(x_1, x_2, x_3) = u(r, x_3, \theta)$. Then the term $(e_3 \wedge x) \cdot \nabla u = -x_2 \partial_1 u + x_1 \partial_2 u$ may be rewritten in the form $(e_3 \wedge x) \cdot \nabla u = \partial_\theta u$ using the angular derivative ∂_θ applied to $u(r, x_3, \theta)$. Working first of all formally or in the space $\mathscr{S}'(\mathbb{R}^3)$ of tempered distributions we apply the Fourier transform $\mathscr{F} = \widehat{\ }$ to (5). With the Fourier variable $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $s = |\xi|$ we get from (5)

$$(vs^2 + \mathrm{i}k\xi_3)\widehat{u} - \widetilde{\omega}(\partial_\varphi \widehat{u} - e_3 \wedge \widehat{u}) + \mathrm{i}\xi\widehat{p} = \widehat{f}, \quad \mathrm{i}\xi \cdot \widehat{u} = 0 \tag{16}$$

Here $(e_3 \wedge \xi) \cdot \nabla_\xi = -\xi_2 \partial/\partial\xi_1 + \xi_1 \partial/\partial\xi_2 = \partial_\varphi$ is the angular derivative in Fourier space when using cylindrical coordinates $(s, \xi_3, \phi,) \in \mathbb{R}_+ \times \mathbb{R} \times [0, 2\pi)$. Since $\mathrm{i}\xi \cdot \widehat{u} = 0$ implies $\mathrm{i}\xi \cdot (\partial_\varphi \widehat{u} - \omega \wedge \widehat{u}) = 0$,

DOI: 10.1002/mma

the unknown pressure p is given by $-|\xi|^2\widehat{p} = \mathrm{i}\xi\cdot\widehat{f}$, i.e.

$$\widehat{\nabla p}(\xi) = \mathrm{i}\xi\cdot\widehat{p} = \frac{(\xi\cdot\widehat{f})\widehat{f}}{|\xi|^2}$$

Then the Hörmander–Mikhlin multiplier theorem on weighted L^q-spaces (Theorem 2.2 (ii)) yields for every weight $w\in A_q(\mathbb{R}^3,\mathscr{C})$ the estimate

$$\|\nabla p\|_{q,w}\leqslant c\|f\|_{q,w} \tag{17}$$

where $c=c(q,w)>0$; in particular $\nabla p\in L^q_w$.

Hence, u may be considered as a (solenoidal) solution of the reduced problem

$$-\nu\Delta u + k\partial_3 u - \widetilde{\omega}(\partial_\theta u - e_3\wedge u) = F := f - \nabla p \quad \text{in } \mathbb{R}^3 \tag{18}$$

or—in Fourier space—

$$(\nu s^2 + \mathrm{i}k\xi_3)\widehat{u} - \widetilde{\omega}(\partial_\varphi\widehat{u} - e_3\wedge\widehat{u}) = \widehat{F}$$

As shown in [1] this inhomogeneous linear differential equation of first order with respect to φ has the unique 2π-periodic solution

$$\begin{aligned}\widehat{u}(\xi) &= \frac{1}{1-\mathrm{e}^{-2\pi(\nu|\xi|^2+\mathrm{i}k\xi_3)/\widetilde{\omega}}}\int_0^{2\pi/\widetilde{\omega}} \mathrm{e}^{-(\nu|\xi|^2+\mathrm{i}k\xi_3)t}\,O_\omega^{\mathrm{T}}(t)\mathscr{F}F(O_\omega(t)\xi)\,\mathrm{d}t\\ &= \int_0^\infty \mathrm{e}^{-\nu|\xi|^2t}\,O_\omega^{\mathrm{T}}(t)(\mathscr{F}F(O_\omega(t)\cdot -kte_3))(\xi)\,\mathrm{d}t\end{aligned} \tag{19}$$

Finally, note that $\mathrm{e}^{-\nu|\xi|^2t}$ is the Fourier transform of the heat kernel $E_t(x)=(4\pi\nu t)^{-3/2}\mathrm{e}^{-|x|^2/4\nu t}$ yielding

$$u(x) = \int_0^\infty E_t * O_\omega^{\mathrm{T}}(t)F(O_\omega(t)\cdot -kte_3)(x)\,\mathrm{d}t \tag{20}$$

Since $F=f-\nabla p$ is solenoidal, the identity $\mathrm{i}\xi\cdot\widehat{F}=0$ easily implies that also u is solenoidal.

The main ingredients of the proof of Theorem 1.4 are a weighted version of Littlewood–Paley theory and a decomposition of the integral operator

$$\begin{aligned}Tf(x) &= \int_0^\infty \widehat{\psi}_{\nu t}(\xi)O_\omega^{\mathrm{T}}(t)\mathscr{F}f(O_\omega(t)\cdot -kte_3)(\xi)\,\frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \widehat{\psi}_t(\xi)O_{\omega/\nu}^{\mathrm{T}}(t)\mathscr{F}f\left(O_{\omega/\nu}(t)\cdot -\frac{k}{\nu}te_3\right)(\xi)\,\frac{\mathrm{d}t}{t}\end{aligned} \tag{21}$$

where

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}}|\xi|^2\mathrm{e}^{-|\xi|^2} \quad\text{and}\quad \widehat{\psi}_t(\xi)=\widehat{\psi}(\sqrt{t}\xi),\quad t>0 \tag{22}$$

DOI: 10.1002/mma

are the Fourier transforms of the function $\psi = -\Delta E_1 \in \mathscr{S}(\mathbb{R}^3)$ and of $\psi_t(x) = t^{-3/2}\psi(x/\sqrt{t})$, $t>0$, respectively. Note that due to Theorem 1.4 it suffices to find an estimate of $\|\Delta u\|_{q,w}$ in order to estimate all second-order derivatives $\partial_j\partial_k u$ of u.

To decompose $\widehat{\psi}_t$ choose $\widetilde{\chi} \in C_0^\infty(\frac{1}{2}, 2)$ satisfying $0 \leqslant \widetilde{\chi} \leqslant 1$ and $\sum_{j=-\infty}^{\infty} \widetilde{\chi}(2^{-j}s) = 1$ for all $s>0$. Then define χ_j, $j \in \mathbb{Z}$, by its Fourier transform

$$\widehat{\chi}_j(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \xi \in \mathbb{R}^n$$

yielding $\sum_{j=-\infty}^{\infty} \widehat{\chi}_j = 1$ on $\mathbb{R}^n \backslash \{0\}$ and

$$\operatorname{supp} \widehat{\chi}_j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^3 : 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\} \tag{23}$$

Using χ_j, we define for $j \in \mathbb{Z}$

$$\psi^j = \frac{1}{(2\pi)^{3/2}} \chi_j * \psi \quad (\widehat{\psi} = \widehat{\chi}_j \cdot \widehat{\psi}) \tag{24}$$

Obviously, $\sum_{j=-\infty}^{\infty} \psi^j = \psi$ on $\mathbb{R}^3$. Finally, in view of (21), (24), we define the linear operators

$$\begin{aligned} T_j f(x) &= \int_0^\infty \widehat{\psi}_{vt}^j(\xi) O_\omega^{\mathrm{T}}(t) \mathscr{F} f(O_\omega(t) \cdot -kte_3)(\xi) \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \widehat{\psi}_t^j(\xi) O_{\omega/v}^{\mathrm{T}}(t) \mathscr{F} f\left(O_{\omega/v}(t) \cdot -\frac{k}{v} te_3\right)(\xi) \frac{\mathrm{d}t}{t} \end{aligned} \tag{25}$$

Since formally $T = \sum_{j=-\infty}^{\infty} T_j$, we have to prove that this infinite series converges even in the operator norm on L_w^q.

For later use we cite the following lemma, see [7].

Lemma 2.5
The functions ψ^j, ψ_t^j, $j \in \mathbb{Z}$, $t>0$, have the following properties:

(i) $\operatorname{supp} \widehat{\psi}_t^j \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}}\right)$

(ii) For $m>\frac{3}{2}$ let $h(x) = (1+|x|^2)^{-m}$ and $h_t(x) = t^{-3/2} h\left(\frac{x}{\sqrt{t}}\right)$, $t>0$. Then there exists a constant $c>0$ independent of $j \in \mathbb{Z}$ such that

$$\begin{aligned} |\psi^j(x)| &\leqslant c 2^{-2|j|} h_{2^{-2j}}(x), \quad x \in \mathbb{R}^3 \\ \|\psi^j\|_1 &\leqslant c 2^{-2|j|} \end{aligned} \tag{26}$$

To introduce a weighted Littlewood–Paley decomposition of L_w^q choose $\widetilde{\varphi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \leqslant \widetilde{\varphi} \leqslant 1$ and $\int_0^\infty \widetilde{\varphi}(s)^2 \, \mathrm{d}s/s = \frac{1}{2}$. Then define $\varphi \in \mathscr{S}(\mathbb{R}^3)$ by its Fourier transform $\widehat{\varphi}(\xi) = \widetilde{\varphi}(|\xi|)$ yielding for every $s>0$

$$\widehat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|), \quad \operatorname{supp} \widehat{\varphi}_s \subset A\left(\frac{1}{2\sqrt{2}}, \frac{2}{\sqrt{2}}\right) \tag{27}$$

and the normalization $\int_0^\infty \widehat{\varphi}_s(\xi)^2 \mathrm{d}s/s = 1$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$.

DOI: 10.1002/mma

Theorem 2.6
Let $1<q<\infty$ and $w\in A_q(\mathbb{R}^3)$. Then there are constants $c_1, c_2>0$ depending on q, w and φ such that for all $f\in L^q_w$

$$c_1\|f\|_{q,w}\leqslant\left\|\left(\int_0^\infty|\varphi_s*f(\cdot)|^2\,\frac{\mathrm{d}s}{s}\right)^{1/2}\right\|_{q,w}\leqslant c_2\|f\|_{q,w} \tag{28}$$

where $\varphi_s\in\mathscr{S}(\mathbb{R}^n)$ is defined by (27).

Proof
See [21, Proposition 1.9, Theorem 1.10], and also [19, 22]. □

3. PROOFS

As a preliminary version of Theorem 1.4 we prove the following proposition. The extension to more general weights based on complex interpolation of L^q_w-spaces will be postponed to the end of Section 3.

Proposition 3.1
Let the weight $w\in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be independent of the angle θ and define $w_r(x_3):=w(x_1,x_2,x_3)$ for fixed $r=|(x_1,x_2)|>0$. Assume that

$$\begin{aligned} &w\in\widetilde{A}^-_{q/2} && \text{if } q>2\\ &w\in\widetilde{A}^-_1 \quad\text{or}\quad \frac{1}{w}\in\widetilde{A}^+_1 && \text{if } q=2\\ &w^{2/(2-q)}\in\widetilde{A}^-_{q/(2-q)} && \text{if } 1<q<2\end{aligned} \tag{29}$$

Then the linear operator T defined by (21) satisfies the estimate

$$\|Tf\|_{q,w}\leqslant c\|f\|_{q,w}\quad\text{for all } f\in L^q_w \tag{30}$$

with a constant $c=c(q,w)>0$ independent of f.

Proof
Step 1: First we consider the case $q>2$, $w\in\widetilde{A}^-_{q/2}\subset A_q$, and define the sublinear operator $\mathscr{M}^j$, a modified maximal operator, by

$$\mathscr{M}^j g(x)=\sup_{s>0}\int_{A_s}(|\psi^j_t|*|g|)\left(O^{\mathrm{T}}_{\omega/v}(t)x+\frac{k}{v}te_3\right)\frac{\mathrm{d}t}{t} \tag{31}$$

where $A_s=[s/16, 16s]$. Then we will prove the preliminary estimate

$$\|T_jf\|_{q,w}\leqslant c\|\psi^j\|_1^{1/2}\|\mathscr{M}^j\|^{1/2}_{L_v^{(q/2)'}}\|f\|_{q,w},\quad j\in\mathbb{Z} \tag{32}$$

where v denotes the θ-independent weight

$$v=w^{-(q/2)'/(q/2)}=w^{-2/(q-2)}\in\widetilde{A}^+_{(q/2)'}=\widetilde{A}^+_{q/(q-2)} \tag{33}$$

DOI: 10.1002/mma

To prove (32) we use the Littlewood–Paley decomposition of L^q_w, see (28), applied to $T_j f$. By a duality argument we find some function $0\leqslant g\in L_v^{(q/2)'}=(L_w^{(q/2)})^*$ with $\|g\|_{(q/2)',v}=1$ such that

$$\left\|\int_0^\infty |\varphi_s * T_j f(\cdot)|^2\frac{\mathrm{d}s}{s}\right\|_{q/2,w}=\int_0^\infty\int_{\mathbb{R}^3}|\varphi_s * T_j f(x)|^2 g(x)\,\mathrm{d}x\frac{\mathrm{d}s}{s} \tag{34}$$

To estimate the right-hand side of (34) note that

$$\varphi_s * T_j f(x)=\int_0^\infty O^{\mathrm{T}}_{\omega/v}(t)(\varphi_s*\psi_t^j*f)\left(O_{\omega/v}(t)x-\frac{k}{v}te_3\right)\frac{\mathrm{d}t}{t}$$

where $\varphi_s*\psi_t^j=0$ unless $t\in A(s,j):=[2^{2j-4}s,2^{2j+4}s]$. Since $\int_{t\in A(s,j)}\mathrm{d}t/t=\log 2^8$ for every $j\in\mathbb{Z}$, $s>0$, we get by the inequality of Cauchy–Schwarz and the associativity of convolutions that

$$\begin{aligned}|\varphi_s*T_jf(x)|^2&\leqslant c\int_{A(s,j)}\left|(\psi_t^j*(\varphi_s*f))\left(O_{\omega/v}(t)x-\frac{k}{v}te_3\right)\right|^2\frac{\mathrm{d}t}{t}\\&\leqslant c\|\psi^j\|_1\int_{A(s,j)}(|\psi_t^j|*|\varphi_s*f|^2)\left(O_{\omega/v}(t)x-\frac{k}{v}te_3\right)\frac{\mathrm{d}t}{t}\end{aligned}$$

here we used the estimate $|(\psi_t^j*(\varphi_s*f))(y)|^2\leqslant\|\psi_t^j\|_1(|\psi_t^j|*|\varphi_s*f|^2)(y)$ and the identity $\|\psi_t^j\|_1=\|\psi^j\|_1$, see (26). Thus,

$$\begin{aligned}&|T_jf\|^2_{q,w}\\&\leqslant c\|\psi^j\|_1\int_0^\infty\int_{A(s,j)}\int_{\mathbb{R}^3}(|\psi_t^j|*|\varphi_s*f|^2)\left(O_{\omega/v}(t)x-\frac{k}{v}te_3\right)g(x)\,\mathrm{d}x\frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s}\\&\leqslant c\|\psi^j\|_1\int_0^\infty\int_{A(s,j)}\int_{\mathbb{R}^3}(|\psi_t^j|*|\varphi_s*f|^2)(x)g\left(O^{\mathrm{T}}_{\omega/v}(t)x+\frac{k}{v}te_3\right)\mathrm{d}x\frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s}\\&\leqslant c\|\psi^j\|_1\int_{\mathbb{R}^3}\int_0^\infty|\varphi_s*f|^2(x)\int_{A(s,j)}(|\psi_t^j|*g)\left(O^{\mathrm{T}}_{\omega/v}(t)x+\frac{k}{v}te_3\right)\frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s}\,\mathrm{d}x\end{aligned} \tag{35}$$

since ψ_t^j is radially symmetric. By definition of $\mathscr{M}^j$ the innermost integral is bounded by $\mathscr{M}^j g(x)$ uniformly in $s>0$. Hence, we may proceed in (35) using Hölder's inequality as follows:

$$\begin{aligned}\|T_jf\|^2_{q,w}&\leqslant c\|\psi^j\|_1\int_{\mathbb{R}^3}\left(\int_0^\infty|\varphi_s*f|^2(x)\frac{\mathrm{d}s}{s}\right)\mathscr{M}^jg(x)\,\mathrm{d}x\\&\leqslant c\|\psi^j\|_1\left\|\int_0^\infty|\varphi_s*f|^2(x)\frac{\mathrm{d}s}{s}\right\|_{q/2,w}\|\mathscr{M}^jg\|_{(q/2)',v}\end{aligned} \tag{36}$$

Now (28) and the normalization $\|g\|_{(q/2)',v}=1$ complete the proof of (32).

DOI: 10.1002/mma

Step 2: We estimate $\|\mathscr{M}^j g\|_{(q/2)',v}$. For functions γ depending on θ, x_3 only let $\mathscr{M}_{\text{hel}}$ denote the 'helical' maximal operator

$$\mathscr{M}_{\text{hel}}\gamma(\theta, x_3) = \sup_{s>0} \frac{1}{s} \int_{A_s} |\gamma| \left(\theta - \frac{\omega}{v}t, x_3 + \frac{k}{v}t\right) \mathrm{d}t$$

where $A_s = [s/16, 16s]$. Then, writing $p := (q/2)'$, we claim that

$$\mathscr{M}^j g(x) \leqslant c2^{-2|j|}\mathscr{M}(\mathscr{M}_{\text{hel}}g)(x) \quad \text{for a.a. } x \in \mathbb{R}^n \tag{37}$$

$$\|\mathscr{M}^j g\|_{p,v} \leqslant c2^{-2|j|}\|g\|_{p,v} \tag{38}$$

where in (37) $\mathscr{M}_{\text{hel}}g$ is considered as $\mathscr{M}_{\text{hel}}g(r, \cdot, \cdot)$ for almost all $r>0$.

To prove (37) we use the pointwise estimate $|\psi_t^j(x)| \leqslant c2^{-2|j|}h_{t2^{-2j}}(x)$, see Lemma 2.5(ii). Hence,

$$\mathscr{M}^j g(x) \leqslant c2^{-2|j|} \sup_{s>0} \int_{A_s} (h_{t2^{-2j}} * |g|) \left(O^{\mathrm{T}}_{\omega/v}(t)x + \frac{k}{v}te_3\right) \frac{\mathrm{d}t}{t}$$

Moreover, there exists a constant $c>0$ independent of $s>0$, $j \in \mathbb{Z}$, such that $h_{t2^{-2j}} \leqslant ch_{s2^{-2j}}$ for all $t \in A_s$. Consequently,

$$\begin{aligned}\mathscr{M}^j g(x) &\leqslant c2^{-2|j|} \sup_{s>0} h_{s2^{-2j}} * \int_{A_s} |g| \left(O^{\mathrm{T}}_{\omega/v}(t)x + \frac{k}{v}te_3\right) \frac{\mathrm{d}t}{t} \\ &\leqslant c2^{-2|j|} \sup_{t>0} h_t * \mathscr{M}_{\text{hel}}g(x)\end{aligned}$$

Since h is nonnegative, radially decreasing, and $\|h_t\|_1 = \|h\|_1 = c_0 > 0$ for all $t>0$, a well-known convolution estimate, see [23, II Section 2.1], yields the pointwise estimate (37).

Step 3: Note that up to now we have not yet used any specific properties of the weight $v \in A_p$. To estimate $\mathscr{M}_{\text{hel}}g$, we shall work with a suitable one-sided maximal operator since our weight belongs to a Muckenhoupt class in $\mathbb{R}^3$ but a problem occurs when the weight is considered with respect to x_3 only. This naturally corresponds to the physical circumstances of the problem, where in the Oseen case the wake should appear. To estimate $\mathscr{M}_{\text{hel}}g$, we write $g_r(\theta, x_3) = g(r, \theta, x_3) = g(x)$ and $v_r(x_3) = v(x)$, $r = |(x_1, x_2)| > 0$, for the θ-independent weight v. Then by the 2π-periodicity of g_r and v_r with respect to θ we get for almost all $r>0$

$$\begin{aligned}&\int_{\mathbb{R}} \int_0^{2\pi} \mathscr{M}_{\text{hel}}g_r(\theta, x_3)^p v_r(x_3)\, \mathrm{d}\theta\, \mathrm{d}x_3 \\ &\leqslant \int_{\mathbb{R}} \int_0^{2\pi} \left| \sup_{s>0} \frac{1}{s} \int_0^{16s} |g_r| \left(\theta - \frac{\omega}{k}\left(x_3 + \frac{k}{v}t\right), x_3 + \frac{k}{v}t\right) \mathrm{d}t \right|^p v_r(x_3)\, \mathrm{d}\theta\, \mathrm{d}x_3 \\ &= \int_{\mathbb{R}} \int_0^{2\pi} \left| \sup_{s>0} \frac{1}{s} \int_0^{16s} \gamma_{r,\theta}\left(x_3 + \frac{k}{v}t\right) \mathrm{d}t \right|^p \mathrm{d}\theta\, v_r(x_3)\, \mathrm{d}x_3 \\ &= 16 \int_0^{2\pi} \int_{\mathbb{R}} |M^+\gamma_{r,\theta}(x_3)|^p v_r(x_3)\, \mathrm{d}x_3\, \mathrm{d}\theta\end{aligned}$$

DOI: 10.1002/mma

where $\gamma_{r,\theta}(y_3)=|g_r|(\theta-(\omega/k)y_3, y_3)$ and M^+ denotes the one-sided maximal operator, see Definition 1.2. Since $w_r\in A^-_{q/2}$, by (33) and Theorem 2.3(i) $v_r=w_r^{-(q/2)'/(q/2)}\in A^+_{(q/2)'}=A^+_p$ with an A^+_p-constant uniformly bounded in $r>0$. Then Theorem 2.3(ii) yields the estimate

$$\int_{\mathbb{R}}\int_0^{2\pi}\mathscr{M}_{\text{hel}}g_r(\theta,x_3)^p v_r(x_3)\,\mathrm{d}\theta\,\mathrm{d}x_3$$
$$\leqslant c\int_0^{2\pi}\int_{\mathbb{R}}|\gamma_{r,\theta}(x_3)|^p v_r(x_3)\,\mathrm{d}x_3\,\mathrm{d}\theta=c\|g_r\|^p_{L^p(\mathbb{R}\times(0,2\pi),v_r(x_3))}$$

where $c>0$ is independent of k, v. Integrating with respect to $r\,\mathrm{d}r$, $r\in(0,\infty)$, Fubini's theorem allows to consider an extension of $\mathscr{M}_{\text{hel}}$ to a bounded operator from $L^p_v(\mathbb{R}^3)$ to itself with an operator norm bounded uniformly in k, v. Moreover, $\mathscr{M}:L^p_v(\mathbb{R}^3)\to L^p_v(\mathbb{R}^3)$ is bounded by Theorem 2.3(ii). Hence, (37) implies (38), and by (32) as well as Lemma 2.5(ii) we get the estimate

$$\|T_j f\|_{q,w}\leqslant c2^{-2|j|}\|f\|_{q,w}$$

for all $f\in L^q_w(\mathbb{R}^3)$ with a constant $c>0$ independent of $j\in\mathbb{Z}$. Summarizing the previous inequalities we proved (30) for $q>2$.

Step 4: Now let $q=2$, $w\in\widetilde{A}^-_1$. In this case, the Littlewood–Paley decomposition of $T_j f$ in L^2_w implies that

$$\|T_j f\|^2_{2,w}\leqslant c\int_0^\infty\int_{\mathbb{R}^n}|\varphi_s * T_j f|^2(x)g(x)\,\mathrm{d}x\,\frac{\mathrm{d}s}{s}$$

where

$$g\in L^\infty_v,\quad v=\frac{1}{w}\quad\text{and}\quad \|g\|_{\infty,v}=\operatorname{ess\,sup}_{\mathbb{R}^3}|gv|=1$$

By the same reasoning as before we arrive at (32), i.e.

$$\|T_j f\|_{2,w}\leqslant c2^{-|j|}\|\mathscr{M}^j g\|^{1/2}_{\infty,v}\|f\|_{2,w}\tag{39}$$

and at (37). Concerning $\mathscr{M}_{\text{hel}}$ we use the pointwise estimate $g_r(\theta,x_3)\leqslant w_r(x_3)$ for a.a. $\theta\in(0,2\pi)$, $x_3\in\mathbb{R}$, and get that

$$\mathscr{M}_{\text{hel}}g_r(\theta,x_3)\leqslant\sup_{s>0}\frac{1}{s}\int_0^{16s}w_r\left(x_3+\frac{k}{v}t\right)\mathrm{d}t\leqslant 16M^+w_r(x_3)\leqslant cw_r(x_3)$$

with a constant $c>0$ independent of $r>0$. Since w is an $A_1(\mathbb{R}^3)$-weight, (37) implies that

$$\mathscr{M}^j g(x)\leqslant c2^{-2|j|}\mathscr{M}w(x)\leqslant c2^{-2|j|}w(x)$$

and consequently that $\|\mathscr{M}^j g\|_{\infty,v}\leqslant c2^{-2|j|}$ with a constant $c>0$ independent of $j\in\mathbb{Z}$. Hence, $\|T_j f\|_{2,w}\leqslant c2^{-2|j|}$ proving (30) when $q=2$.

Step 5: The remaining estimates are proved by duality arguments. Obviously, the dual operator to T is defined by

$$T^* f(x)=\int_0^\infty(-\Delta)O_\omega(t)E_t * f(O^{\mathrm{T}}_\omega(t)x+kte_3)\,\mathrm{d}t$$

DOI: 10.1002/mma

which has the same structure as K, but with an 'opposite orientation'. Hence, T^* is bounded on L^q_w for $q\geqslant 2$ and all weights $w\in\widetilde{A}^+_{q/2}$. Now let $1<q<2$ and $w^{2/(2-q)}\in\widetilde{A}^-_{q/(2-q)}=\widetilde{A}^-_{(q'/2)'}$. Then by simple duality arguments $w'=w^{-q'/q}\in\widetilde{A}^+_{(q'/2)}$ and

$$|\langle Tf, g\rangle| = |\langle f, T^*g\rangle| \leqslant \|f\|_{q,w}\|T^*g\|_{q',w'} \leqslant c\|f\|_{q,w}\|g\|_{q',w'}$$

Finally, let $q=2$ and $1/w\in\widetilde{A}^+_1$. As before,

$$|\langle Tf, g\rangle| \leqslant \|f\|_{2,w}\|T^*g\|_{2,1/w} \leqslant c\|f\|_{2,w}\|g\|_{2,1/w}$$

Now Proposition 3.1 is completely proved. □

Lemma 3.2 (*Bergh and Löfström* [*24*])
Let $1\leqslant p_1, p_2<\infty$, let $0<w_1, w_2$ be weight functions, $\delta\in(0,1)$, and

$$\frac{1}{p}=\frac{1-\delta}{p_1}+\frac{\delta}{p_2},\quad w^{1/p}=w_1^{(1-\delta)/p_1}\cdot w_2^{\delta/p_2}$$

Then

$$[L^{p_1}_{w_1}, L^{p_2}_{w_2}]_\delta = L^p_w$$

in the sense of complex interpolation.

In the following, we shall derive an anisotropic variant of Jones's factorization theorem tailored to our situation, when we need to work with one-sided Muckenhoupt weights with respect to x_3, satisfying the usual Muckenhoupt condition in three dimensions.

Lemma 3.3 (*Anisotropic version of Jones' factorization theorem*)
Suppose that $w\in\widetilde{A}^-_q$. Then there exist weights $w_1\in\widetilde{A}^-_1$ and $w_2\in\widetilde{A}^+_1$ such that

$$w=\frac{w_1}{w_2^{q-1}}$$

Here $\widetilde{A}^+_1$ is defined by analogy with $\widetilde{A}^-_1$, cf. Definition 1.2, by assuming for $w_2\in\widetilde{A}^+_1$ that $(w_2)_r\in A^+_1$ with A^+_1-constant uniformly bounded in $r>0$. An analogous result holds for $w\in\widetilde{A}^+_q$.

Proof
Let $q\geqslant 2$. Given $w\in\widetilde{A}^-_q$ we consider the operator T defined by

$$\begin{aligned} Tf &= (w^{-1/q}\mathscr{M}(f^{q/q'}w^{1/q}))^{q'/q} + w^{1/q}\mathscr{M}(fw^{-1/q}) \\ &\quad + (w^{-1/q}M_1^+(f_r^{q/q'}w_r^{1/q}))^{q'/q} + w^{1/q}M_1^-(f_r w_r^{-1/q}) \end{aligned}$$

where $r=|(x_1,x_2)|$. Then for all $f\in L^q(\mathbb{R}^3)$

$$\begin{aligned} \|Tf\|_q^q &\leqslant c\left\{\int_{\mathbb{R}^3} w^{-q'/q}(\mathscr{M}(f^{q/q'}w^{1/q}))^{q'}\,\mathrm{d}x + \int_{\mathbb{R}^3} w(\mathscr{M}(fw^{-1/q}))^q\,\mathrm{d}x \right. \\ &\quad + \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} w_r^{-q'/q}(M_1^+(f_r^{q/q'}w_r^{1/q}))^{q'}\,\mathrm{d}x_3\right)\mathrm{d}(x_1,x_2) \end{aligned}$$

DOI: 10.1002/mma

$$+\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} w_r(M_1^+(f_r w_r^{-1/q}))^q\,\mathrm{d}x_3\right)\mathrm{d}(x_1,x_2)\Bigg\}$$
$$\leqslant A^q\|f\|_q^q$$

with a constant $A=A(q,w)>0$.

Let us fix a nonnegative θ-independent function $f\in L^q(\mathbb{R}^3)$ with $\|f\|_q=1$ and define

$$\eta=\sum_{k=1}^{\infty}(2A)^{-k}T^k(f)$$

where $T^k(f)=T(T^{k-1}(f))$. Obviously, Tf and therefore also η are θ-independent. Moreover, $\eta\in L^q(\mathbb{R}^3)$ and $\|\eta\|_q\leqslant\sum_{k=1}^{\infty}2^{-k}=1$. In particular, $\eta(x)<\infty$ for a.a. $x\in\mathbb{R}^3$, $\eta_r(\cdot)\in L^q(\mathbb{R})$ for a.a. $(x_1,x_2)\in\mathbb{R}^2$ and $\eta_r(x_3)<\infty$ for a.a. $x_3\in\mathbb{R}$. Since T is subadditive and positivity-preserving, we get the pointwise inequality

$$T\eta\leqslant\sum_{k=1}^{\infty}(2A)^{-k}T^{k+1}(f)=\sum_{k=2}^{\infty}(2A)^{1-k}T^k(f)\leqslant(2A)\eta$$

Now let $w_1:=w^{1/q}\eta^{q/q'}$ and $w_2:=w^{-1/q}\eta$ such that $w=w_1/w_2^{q-1}$. Then

$$\mathscr{M}(w_1)\leqslant w^{1/q}(T\eta)^{q/q'}\leqslant w^{1/q}\eta^{q/q'}(2A)^{q/q'}=(2A)^{q/q'}w_1$$
$$M_1^+((w_1)_r)\leqslant w^{1/q}(T\eta)^{q/q'}\leqslant w^{1/q}\eta^{q/q'}(2A)^{q/q'}=(2A)^{q/q'}(w_1)_r$$
$$\mathscr{M}(w_2)\leqslant w^{-1/q}T(\eta)\leqslant w^{-1/q}\eta 2A=2Aw_2$$
$$M_1^-((w_2)_r)\leqslant w^{-1/q}T(\eta)\leqslant w^{-1/q}\eta 2A=2A(w_2)_r$$

proving that $w_1\in\widetilde{A}_1^-$, $w_2\in\widetilde{A}_1^+$.

The case $1\leqslant q<2$ follows by a simple duality argument, since $w\in\widetilde{A}_q^-$ is equivalent to $w^{-q'/q}\in\widetilde{A}_{q'}^+$. □

Proof of Theorem 1.4

(i) Let $q\in(1,\infty)$ and $w\in A_q$ such that the L_w^q-estimate of ∇p holds, see (17). Hence, it suffices to consider u defined by (19)–(20). We consider arbitrary $q_1,q_2\in(1,\infty)$ and $\delta\in(0,1)$ with

$$1<q_1<q<q_2<\infty,\quad q_1\leqslant 2\leqslant q_2\quad\text{and}\quad\frac{1}{q}=\frac{1-\delta}{q_1}+\frac{\delta}{q_2}\tag{40}$$

and assume that $w^\tau\in\widetilde{A}_{\tau q/2}^-$ with $\tau=2/(2-q(1-\delta))\in[1,\infty)$. By Lemma 3.3 there exist weights $u\in\widetilde{A}_1^-$, $v\in\widetilde{A}_1^+$ such that

$$w^\tau=\frac{u}{v^{\tau q/2-1}}=\frac{u}{v^{q/(2-q(1-\delta))-1}}$$

Then we define the weights w_1,w_2 by

$$w_1^{2/(2-q_1)}=\frac{u}{v^{2(q_1-1)/(2-q_1)}}\quad\text{and}\quad w_2=\frac{u}{v^{(q_2-2)/2}}$$

yielding

$$w_1^{2/(2-q_1)} \in \widetilde{A}^-_{q_1/(2-q_1)}, \quad w_2 \in \widetilde{A}^-_{q_2/2}$$

Since, due to an elementary calculation, with $w = w_1^{q(1-\delta)/q_1} \cdot w_2^{q\delta/q_2}$, Lemma 3.3 and Proposition 3.1 we can prove that T is bounded on $L^q_w(\mathbb{R}^3)$. Since $u_1 \in \widetilde{A}^-_1$, $v_1 \in \widetilde{A}^+_1$ are arbitrary, we proved the boundedness of T on L^q_w for arbitrary w if

$$w^\tau \in \widetilde{A}^-_{\tau q/2}, \quad \tau = \frac{2}{2 - q(1-\delta)} \in [1, \infty)$$

Now we have to find all admissible τ subject to the restrictions given by (40). For this reason, consider the easier term

$$s = 2\left(1 - \frac{1}{\tau}\right) = q(1-\delta) = q\,\frac{1/q - 1/q_2}{1/q_1 - 1/q_2}$$

First Case $1<q<2$, in which $1<q_1<q$ and $q_2 \geqslant 2$. Due to monotonicity properties of s as a function of $1/q_1$ and of $1/q_2$ it suffices to check s at the corners of the rectangle $(1/q, 1) \times (0, \frac{1}{2}]$. The corresponding function values are q, 1 and $2-q$. Hence, the range of s equals the interval $(2-q, q)$ yielding for $\tau = 2/(2-s)$ the condition

$$\frac{2}{q} < \tau < \frac{2}{2-q}$$

Note that the limiting value $\tau = 2/(2-q)$ is allowed due to Proposition 3.1. Finally, the condition $w^\tau \in \widetilde{A}^-_{\tau q/2}$, $2/q < \tau \leqslant 2/(2-q)$, easily implies that $w \in A_q$: By Lemma 3.3, there exist $u_1 \in \widetilde{A}^-_1$, $v_1 \in \widetilde{A}^+_1$ such that

$$w = \frac{u_1^{1/\tau}}{v_1^{q/2 - 1/\tau}} \tag{41}$$

where $u_1^{1/\tau} \in \widetilde{A}^-_1$ and $q/2 - 1/\tau \leqslant q-1$ yielding $v_1^{(q/2-1/\tau)/(q-1)} \in \widetilde{A}^+_1$.

Second Case $q>2$, in which $1<q_1 \leqslant 2$ and $q_2>q$. In this case, the values of s at the corners of the rectangle $[\frac{1}{2}, 1) \times (0, 1/q)$ in the $(1/q_1, 1/q_2)$-plane are 0, 1 and 2. Hence,

$$1 < \tau < \infty$$

and we observe that $\tau = 1$ is admissible due to Proposition 3.1. Finally, note that the condition $w^\tau \in A_{\tau q/2}$ implies also $w \in \widetilde{A}^-_q$: there exist $u_1 \in \widetilde{A}^-_1$, $v_1 \in \widetilde{A}^+_1$ such that w satisfies (41), where again $q/2 - 1/\tau + 1 \leqslant q$ for all $\tau \in (1, \infty)$.

Third Case $q=2$. In this case it suffices to interpolate between $L^2_{w_1}$ and $L^2_{w_2}$, where $w_1 \in \widetilde{A}^-_1$ and $1/w_2 \in \widetilde{A}^+_1$, see Proposition 3.1. Then T is bounded on L^2_w for all

$$w = \frac{w_1^{1-\delta}}{w_2^\delta}, \quad 0<\delta<1$$

DOI: 10.1002/mma

Then $w^{1/(1-\delta)} = w_1/w_2^{\delta/(1-\delta)}$, or with $\tau = \frac{1}{1-\delta} \in (1, \infty)$,

$$w^\tau = \frac{w_1}{w_2^{\tau-1}} \in \widetilde{A}_\tau^- = \widetilde{A}_{\tau q/2}^-$$

(ii) Note that $L^{q_i}_{w_i}(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$, $i = 1, 2$; indeed, $w_i \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\int_{|x|\geqslant 1} w_i(x)|x|^{-nq_i}\,\mathrm{d}x < \infty$, see [17, IV.3 (30)]. Since Equation (5) is linear, it suffices to consider $f = 0$ and a solution $u \in S'(\mathbb{R}^n)^n$ of (8). In the proof of [7], Theorem 1.1 (2), (3), it was shown that under these assumptions u is a polynomial and that $u(x) = \alpha\omega + \beta\omega \wedge x + \gamma(x_1, x_2, -2x_3)^{\mathrm{T}}$, $\alpha, \beta, \gamma \in \mathbb{R}$ ($u(x) = \beta(-x_2, x_1)$ if $n = 2$). □

Proof of Corollary 1.5

Considering *a priori* estimates for $\partial u/\partial x_3$ we use representation (19) of u. In order to analyse the dependence of the following estimates on the parameters k, v and $\widetilde{\omega}$ let

$$k' = k/\widetilde{\omega}, \quad v' = v/\widetilde{\omega} \quad \text{and} \quad D(\xi) = 1 - \mathrm{e}^{-2\pi(v'|\xi|^2 + \mathrm{i}k'\xi_3)}$$

Then for $f \in \mathscr{S}(\mathbb{R}^3)^3$ we get the identity

$$\widehat{k\partial_3 u(\xi)} = \frac{\mathrm{i}k'\xi_3}{D(\xi)} \int_0^{2\pi} \mathrm{e}^{-(v'|\xi|^2 + \mathrm{i}k'\xi_3)t}\, O_{e_3}^{\mathrm{T}}(t)\, \widehat{F}(O_{e_3}(t)\xi)\,\mathrm{d}t \tag{42}$$

where $F = f - \nabla p$, see (18). Choose a cut-off function $\eta \in C_0^\infty(B_1(0))$ with $\eta(\xi) = 1$ for $\xi \in B_{1/2}(0)$ and define the multiplier functions

$$m_0(\xi) = \frac{\mathrm{i}k'\xi_3\eta_{v'}(\xi)}{D(\xi)}, \quad m_1(\xi) = \frac{k'}{\sqrt{v'}} \frac{1 - \eta_{v'}(\xi)}{D(\xi)}$$

where $\eta_{v'}(\xi) = \eta(\sqrt{v'}\xi)$, as well as

$$\mu_{0,t}(\xi) = \mathrm{e}^{-(v'|\xi|^2 + \mathrm{i}k'\xi_3)t}, \quad \mu_{1,t}(\xi) = \mathrm{i}\xi_3\sqrt{v'}\mathrm{e}^{-(v'|\xi|^2 + \mathrm{i}k'\xi_3)t}, \quad t \in (0, 2\pi)$$

Then we get

$$\widehat{k\partial_3 u(\xi)} = m_0(\xi)\widehat{I_0}(\xi) + m_1(\xi)\widehat{I_1}(\xi)$$

where $I_0(x)$, $I_1(x)$ are defined by their Fourier transforms

$$\widehat{I_0}(\xi) = \int_0^{2\pi} \mu_{0,t}(\xi) O_{e_3}^{\mathrm{T}}(t) \widehat{F}(O_{e_3}(t)\cdot)(\xi)\,\mathrm{d}t$$

$$\widehat{I_1}(\xi) = \int_0^{2\pi} \mu_{1,t}(\xi) O_{e_3}^{\mathrm{T}}(t) \widehat{F}(O_{e_3}(t)\cdot)(\xi)\,\mathrm{d}t$$

Concerning the multiplier function $\mu_{0,t}$ we note that e.g.

$$\left|\xi_3 \frac{\partial \mu_{0,t}}{\partial \xi_3}\right| = |(-2v't\xi_3^2 - \mathrm{i}k't\xi_3)\mathrm{e}^{-(v'|\xi|^2 + \mathrm{i}k'\xi_3)t}|$$

DOI: 10.1002/mma

$$\leqslant C\left(\nu' t|\xi|^2 + \frac{k'}{\sqrt{\nu'}}\sqrt{\nu' t}|\xi_3|\right)\mathrm{e}^{-\nu'|\xi|^2 t}$$
$$\leqslant C\left(1+\frac{k'}{\sqrt{\nu'}}\right)$$

with a constant $C>0$ independent of $\xi\neq 0$, $t\in(0,2\pi)$, $k'>0$ and $\nu'>0$. Then it is easily seen that $\mu_{0,t}$, $\mu_{1,t}$ satisfy the pointwise multiplier estimates

$$\sup_{t\in(0,2\pi)}\max_{\alpha}\sup_{\xi\neq 0}(|\xi^\alpha D_\xi^\alpha \mu_{0,t}(\xi)| + \sqrt{t}|\xi^\alpha D_\xi^\alpha \mu_{1,t}(\xi)|)\leqslant C\left(1+\frac{k}{\sqrt{\nu|\omega|}}\right)$$

uniformly in $k'>0$ and $\nu'>0$, where $\alpha\in\mathbb{N}_0^3$ runs through the set of all multi-indices $\alpha\in\{0,1\}^3$. Hence, Theorems 2.2(iii) and (17) show that

$$\|I_0\|_{q,w}\leqslant c\left(1+\frac{k}{\sqrt{\nu|\omega|}}\right)\int_0^{2\pi}\|F(O_{e_3}(t)\cdot)\|_{q,w}\,\mathrm{d}t\leqslant c\left(1+\frac{k}{\sqrt{\nu|\omega|}}\right)\|f\|_{q,w}$$
$$\|I_1\|_{q,w}\leqslant c\left(1+\frac{k}{\sqrt{\nu|\omega|}}\right)\int_0^{2\pi}\frac{1}{\sqrt{t}}\|F(O_{e_3}(t)\cdot)\|_{q,w}\,\mathrm{d}t\leqslant c\left(1+\frac{k}{\sqrt{\nu|\omega|}}\right)\|f\|_{q,w}$$

where $c>0$ is independent of k,ω and ν. Moreover, a lengthy, but elementary calculation proves that m_0, m_1 satisfy the pointwise estimates

$$\max_{j=0,1}\max_{\alpha}\sup_{\xi\neq 0}|\xi^\alpha D_\xi^\alpha m_j(\xi)|\leqslant C\left(1+\frac{k^4}{\nu^2|\omega|^2}\right)$$

with $c>0$ independent of ν,ω,k; for details see [1]. Now another application of Theorem 2.2(iii) yields the estimate

$$\|k\partial_3 u\|_{q,w}\leqslant c\left(1+\frac{k^5}{\nu^{5/2}|\omega|^{5/2}}\right)\|f\|_{q,w}$$

for $f\in S(\mathbb{R}^3)^3$, with a constant $c>0$ independent of f,k,ν and ω. Since $S(\mathbb{R}^3)$ is dense in $L_w^q(\mathbb{R}^3)$, this result extends to all $f\in L_w^q$; for its proof we refer to [1]. However, note that we did not estimate $\widehat{F}(O_\omega(t)\cdot - kte_3)\xi$ in $L^q(\Omega)$ as in [1]; instead we have to deal with $\widehat{F}(O_{e_3}(t)\cdot)$, and the shift operator is estimated with the help of multipliers.

Now Corollary 1.5 is completely proved. □

Proof of Corollary 1.6

We have to check for which α,β the weight $w(x)=\eta_\beta^\alpha(x)=(1+|x|)^\alpha(1+s(x))^\beta$ satisfies the conditions needed in Theorem 1.4. From [16] and [25, Theorem 5.2] we know that $w=\eta_\beta^\alpha\in A_p$, $1<p<\infty$, if and only if $-1<\beta<p-1$ and $-3<\alpha+\beta<3(p-1)$; moreover, by Lemma 2.4(iii) we have to satisfy the conditions $\alpha<p-1$, $\beta\geqslant 0$, $\alpha+\beta>-1$ to get that $w_r(\cdot)\in A_p^-$.

DOI: 10.1002/mma

Let $q>2$. Then in view of (8) and (15) we have to analyse the convex set

$$\mathscr{C}=\left\{(\alpha,\beta);\ \alpha<\frac{q}{2}-\frac{1}{\tau},\ \beta\geqslant 0,\ \alpha+\beta>-\frac{1}{\tau},\ -\frac{1}{\tau}<\beta<\frac{q}{2}-\frac{1}{\tau},\right.$$
$$\left.-\frac{3}{\tau}<\alpha+\beta<\frac{3q}{2}-\frac{3}{\tau}\text{ for some }\tau\in[1,\infty)\right\}$$

Obviously, the conditions $\beta>-1/\tau$ and $-3/\tau<\alpha+\beta<3q/2-3/\tau$ are redundant since $q/2-1/\tau$ is positive; moreover, the conditions $\alpha+\beta>-1/\tau$ and $\beta<q/2-1/\tau$ yield $\alpha>-q/2$. We will see that

$$\mathscr{C}=\left\{(\alpha,\beta);\ -\frac{q}{2}<\alpha<\frac{q}{2},\ 0\leqslant\beta<\frac{q}{2},\ \alpha+\beta>-1\right\}$$

Indeed, it suffices to consider pairs (α,β) with $\alpha<0$. If moreover $\alpha+\beta<0$, we find $\tau_0>1$ such that $\alpha+\beta=-1/\tau_0$. Then $\beta=-1/\tau_0-\alpha<-1/\tau_0+q/2$ and $\alpha<0<q/2-1/\tau_0$; consequently $(\alpha,\beta)\in\mathscr{C}$. If $\alpha+\beta\geqslant 0$, we may choose τ sufficiently large to show that $(\alpha,\beta)\in\mathscr{C}$.

Now consider the case $1<q<2$. As in the previous case we have to analyse the set $\mathscr{C}$ where now τ runs in the interval $(2/q, 2/(2-q))]$. Since $\tau>2/q$, the same conditions as before are redundant; moreover, $\alpha>-q/2$. Then we will show that

$$\mathscr{C}=\left\{(\alpha,\beta);\ -\frac{q}{2}<\alpha<q-1,\ 0\leqslant\beta<q-1,\ \alpha+\beta>-\frac{q}{2}\right\}$$

Indeed, if e.g. $\alpha<0$ and $\alpha+\beta\leqslant q/2-1<0$, then there exists $\tau_0\in(2/q,(2-q)/2]$ such that $\alpha+\beta=-1/\tau_0$, $\beta=-1/\tau_0-\alpha<-1/\tau_0+q/2$ and $\alpha<0<q/2-1/\tau_0$; however, when $\alpha+\beta>q/2-1$, we may choose $\tau=2/(2-q)$ to see that $(\alpha,\beta)\in\mathscr{C}$. □

ACKNOWLEDGEMENTS

The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan no. AV0Z10190503, by the Grant Agency of the Academy of Sciences No. IAA100190505, and by the joint research project of DAAD (D/04/25763) and the Academy of Sciences of the Czech Republic (D-CZ 3/05-06).

REFERENCES

1. Farwig R. An L^q-analysis of viscous fluid flow past a rotating obstacle. *Tôhoku Mathematics Journal* 2005; **58**:129–147.
2. Farwig R. *Estimates of Lower Order Derivatives of Viscous Fluid Flow Past a Rotating Obstacle*, vol. 70. Banach Center Publications: Warsaw, 2005; 73–82.
3. Kračmar S, Nečasová Š, Penel P. Anisotropic L^2 estimates of weak solutions to the stationary Oseen-type equations in $\mathbb{R}^3$ for a rotating body. *Preprint 165*, Academy of Sciences of the Czech Republic, Mathematical Institute, 2006; *Proceedings of the RIMS*, Kyoto University, 2007; **B1**:219–235.
4. Kračmar S, Nečasová Š, Penel P. L^q approach of weak solutions of Oseen flow around a rotating body. *Preprint*, 2006, submitted.
5. Farwig R, Neustupa J. On the spectrum of an Oseen-type operator arising from flow past a rotating body. *Preprint no. 2484*, FB Mathematik, TU Darmstadt, 2006, submitted
6. Farwig R, Hishida T. Stationary Navier–Stokes flow around a rotating obstacle. *Preprint no. 2445*, FB Mathematik, TU Darmstadt, 2006; *Funkcialaj Ekvacioj*, in press.

7. Farwig R, Hishida T, Müller D. L^q-theory of a singular 'winding' integral operator arising from fluid dynamics. *Pacific Journal of Mathematics* 2004; **215**:297–312.
8. Farwig R, Krbec M, Nečasová Š. A weighted L^q approach to Stokes flow around a rotating body. *Preprint no. 2422*, FB Mathematik, TU Darmstadt, 2005, submitted.
9. Hishida T. L^q estimates of weak solutions to the stationary Stokes equations around a rotating body. *Journal of the Mathematical Society of Japan* 2006; **58**:743–767.
10. Hishida T, Shibata Y. L_p–L_q estimate of the Stokes operator and Navier–Stokes flows in the exterior of a rotating obstacle. *Preprint*, 2006.
11. Nečasova Š. Some remarks on the steady fall of a body in Stokes and Oseen flow. *Preprint 143*, Academy of Sciences of the Czech Republic, Mathematical Institute, 2001; *Proceedings of AIMS Conference,* 2006.
12. Nečasova Š. On the problem of the Stokes flow and Oseen flow in $\mathbb{R}^3$ with Coriolis force arising from fluid dynamics. *IASME Transactions* 2005; **2**:1262–1270.
13. Nečasova Š. Asymptotic properties of the steady fall of a body in viscous fluids. *Mathematical Methods in the Applied Sciences* 2004; **27**:1969–1995.
14. Galdi GP. On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications. In *Handbook of Mathematical Fluid Dynamics*, Friedlander S, Serre D (eds), vol. 1. Elsevier: Amsterdam, 2002.
15. Kračmar S, Nečasova Š, Penel P. Estimates of weak solutions in anisotropically weighted Sobolev spaces to the stationary rotating Oseen equations. *IASME Transactions* 2005; **2**:854–861.
16. Farwig R. The stationary exterior 3D-problem of Oseen and Navier–Stokes equations in anisotropically weighted Sobolev spaces. *Mathematische Zeitschrift* 1992; **211**:409–447.
17. Garcia-Cuerva J, Rubio de Francia JL. *Weighted Norm Inequalities and Related Topics*. North-Holland: Amsterdam, 1985.
18. Muckenhoupt B. Weighted norm inequalities for the Hardy maximal function. *Transactions of the American Mathematical Society* 1972; **165**:207–226.
19. Kurtz DS. Littlewood–Paley and multiplier theorems on weighted L^p spaces. *Transactions of the American Mathematical Society* 1980; **259**:235–254.
20. Sawyer E. Weighted inequalities for the one-sided Hardy–Littlewood maximal function. *Transactions of the American Mathematical Society* 1986; **297**:53–61.
21. Rychkov V. Littlewood–Paley theory and function spaces with A_p^{loc} weights. *Mathematische Nachrichten* 2001; **224**:145–180.
22. Strömberg J-O, Torchinsky A. *Weighted Hardy Spaces*. Lecture Notes in Mathematics, vol. 1381. Springer: Berlin, 1989.
23. Stein EM. *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality*, *and Oscillatory Integrals*. Princeton University Press: Princeton, NJ, 1993.
24. Bergh J, Löfström J. *Interpolation Spaces*. Springer: New York, 1976.
25. Kračmar S, Novotný A, Pokorný M. Estimates of Oseen kernels in weighted L^p spaces. *Journal of the Mathematical Society of Japan* 2001; **53**:59–111.

DOI: 10.1002/mma

PARABOLIC AND NAVIER–STOKES EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 81
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2008

L^q-APPROACH TO WEAK SOLUTIONS OF THE OSEEN FLOW AROUND A ROTATING BODY

STANISLAV KRAČMAR

Department of Technical Mathematics
Karlovo nám. 13, 121 35 Prague 2, Czech Republic
E-mail: kracmar@marian.fsik.cvut.cz

ŠÁRKA NEČASOVÁ

Mathematical Institute of Academy of Sciences
Žitná 25, 11567 Prague 1, Czech Republic
E-mail: matus@math.cas.cz

PATRICK PENEL

Department of Mathematics, Université de Sud Toulon-Var
B.P. 20132, 83957 La Garde Cedex, France
E-mail: penel@univ-tln.fr

Abstract. We consider the time-periodic Oseen flow around a rotating body in $\mathbb{R}^3$. We prove *a priori* estimates in L^q-spaces of weak solutions for the whole space problem under the assumption that the right-hand side has the divergence form. After a time-dependent change of coordinates the problem is reduced to a stationary Oseen equation with the additional term $-(\omega \wedge x) \cdot \nabla u + \omega \wedge u$ in the equation of momentum where ω denotes the angular velocity. We prove the existence of generalized weak solutions in L^q-space using Littlewood-Paley decomposition and maximal operators.

2000 *Mathematics Subject Classification*: Primary 76D05; Secondary 35Q30.

Key words and phrases: Littlewood-Paley theory, maximal operators, rotating obstacles, stationary Oseen flow, weak solution, Bogovskii operator.

Š. N. and S. K. were supported by the Grant Agency of the Academy of Sciences No. IAA100190505 and IAA100190804. The research of S. K. was supported by the research plans of the Ministry of Education of the Czech Republic N. 6840770010 and the research of Š. N. was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan N. AV0Z10190503.

The paper is in final form and no version of it will be published elsewhere.

1. Introduction. Many physical phenomena involve moving or deformable structures interacting with fluids and are of great concern for aerospace, mechanical, biomedical applications, sedimentation. From the mathematical point of view, they have been studied extensively over the last few years. When the domain depends on time, we refer to [21], [14], [22], [4]. In this paper we consider the case when we fix a rotation of a body and we investigate the flow around the body. In recent years the analysis of the Navier-Stokes equations describing the flow around or past a rotating body has attracted much attention, see [18], [12], [10], [11], [29]–[31], [19], [26], [28], [7], [8], [9], [16], [17], [32], [34], [13]. Further references on moving bodies in fluids are given in [16].

We study the stationary Oseen system in the whole three-dimensional space:

$$\begin{aligned} -\nu\Delta u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p &= f \quad \text{in } \mathbb{R}^3 \\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^3 \\ u &\to 0 \quad \text{as } |x|\to\infty. \end{aligned} \tag{1.1}$$

Here, $\wedge$ denotes the usual exterior product of three-dimensional vectors. Note that the second and the third terms are linearized convective terms, and that in unbounded domains they are not subordinated to the Laplacian. Let us also note that

$$\nabla\cdot[-(\omega\wedge x)\cdot\nabla u + \omega\wedge u] = 0. \tag{1.2}$$

As a consequence $\Delta p = \nabla\cdot f$ and we can write the reduced equation

$$-\nu\Delta u + k\partial_3 u - (\omega\wedge x)\cdot\nabla u + \omega\wedge u = g \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

where $g = f - \nabla p$.

The linear system (1.1) has been analyzed in L^q-spaces, $1 < q < \infty$, in [10] proving the *a priori* estimates

$$\begin{aligned} &\|\nu\nabla^2 u\|_q + \|\nabla p\|_q \le c\|f\|_q, \\ &\|k\partial_3 u\|_q + \|-(\omega\wedge x)\cdot\nabla u + \omega\wedge u\|_q \le c\Big(1 + \frac{k^4}{\nu^2|\omega|^2}\Big)\|f\|_q \end{aligned} \tag{1.4}$$

with the constant $c > 0$ independent of ω, ν, k. Further the results were improved in [7] in weighted spaces and the authors have obtained the following *a priori* estimates

$$\begin{aligned} &\|\nu\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \le c\|f\|_{q,w}, \\ &\|k\partial_3 u\|_{q,w} + \|-(\omega\wedge x)\cdot\nabla u + \omega\wedge u\|_{q,w} \le c\Big(1 + \frac{k^5}{\nu^2|\omega|^{5/2}}\Big)\|f\|_{q,w}, \end{aligned}$$

where the weights (denoted by w) belong to the more general Muckenhoupt class $\tilde{A}_q^-$, see [7], with the constant $c > 0$ independent of ν, ω, k.

Let us recall in two steps the natural introduction of the previous Oseen system starting with a viscous flow either past a threedimensional rigid body, rotating with an angular velocity $\omega = |\omega|(0,0,1)^T, |\omega| \neq 0$, or around a rotating body which is moving in the direction of its axis of rotation. We assume this viscous flow modelled by the incompressible Navier-Stokes equations with the velocity $u_\infty = ke_3 \neq 0$ at infinity. Then, given the coefficient of viscosity $\nu > 0$ and an external force $\tilde{f} = \tilde{f}(y,t)$, the velocity

$v = v(y,t)$ and the pressure $q = q(y,t)$ solve the well known nonlinear system:

$$\begin{aligned} \partial_t v - \nu\Delta v + (v\cdot\nabla)\, v + \nabla q &= \tilde f && \text{in } (0,+\infty)\times\Omega(t),\\ \operatorname{div} v &= 0 && \text{in } (0,+\infty)\times\Omega(t),\\ v(y,t) &= \omega\wedge y && \text{on } (0,+\infty)\times\partial\Omega(t),\\ v(y,t) &\to u_\infty \neq 0 && \text{as } |y|\to\infty. \end{aligned} \tag{1.5}$$

Due to the rotation with angular velocity ω, the time-dependent exterior domain $\Omega(t)$ is given by

$$\Omega(t) = O_\omega(t)\Omega,$$

where $\Omega \subset \mathbb{R}^3$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|\,t & -\sin|\omega|\,t & 0\\ \sin|\omega|\,t & \cos|\omega|\,t & 0\\ 0 & 0 & 1 \end{pmatrix}. \tag{1.6}$$

Introducing the change of variables

$$x = O_\omega(t)^T y \tag{1.7}$$

and the new functions

$$u(x,t) = O_\omega^T(t)(v(y,t) - u_\infty),\;\; p(x,t) = q(y,t),\;\; f(x,t) = O_\omega(t)^T \tilde f(y,t) \tag{1.8}$$

we arrive at the modified Navier-Stokes system, this is the first step:

$$\begin{aligned} \partial_t u - \nu\Delta u \;+\; &(u\cdot\nabla)\, u + (O_\omega^T(t)u_\infty)\cdot\nabla u\\ -\; &(\omega\wedge x)\cdot\nabla u + \omega\wedge u + \nabla p = f && \text{in } (0,+\infty)\times\Omega,\\ \operatorname{div} u &= 0 && \text{in } (0,+\infty)\times\Omega,\\ u(x,t) &= \omega\wedge x - O_\omega^T(t)u_\infty && \text{on } (0,+\infty)\times\partial\Omega,\\ u(x,t) &\to 0 && \text{as } |x|\to\infty. \end{aligned} \tag{1.9}$$

Due to the new coordinate system attached to the rotating body, equation in (1.9) contains two new linear terms, the classical Coriolis force term $\omega\wedge u$ (up to a multiplicative constant) and the additional term $(\omega\wedge x)\cdot\nabla u$.

The second step consists of the linearization of equation (1.9) at $u = 0$, assuming the case $u_\infty || \omega$ and then $O_\omega^T(t)u_\infty = ke_3$, for all $t > 0$, and assuming $\Omega = \mathbb{R}^3$. Thus we get the modified Oseen system (1.1).

REMARK 1. The study of the whole space problem is of interest because we need the results about existence, uniqueness and boundedness of a solution in order to get respective results also in the case of exterior domains. This complete study will be the object of a forthcoming paper [27], we will use the so called localization procedure, see [25].

REMARK 2. We would like to mention that there exists another type of transformation (a local transformation) which was introduced by Inou and Wakimoto [24]. The transformation is applied by several authors, see e.g. [38].

We introduce notation. The class $C_0^\infty(\mathbb{R}^3)$ consists of C^∞ functions with compact supports contained in $\mathbb{R}^3$. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue space with norm $\|\cdot\|_q$.

We define the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_q} = \{v \in L^q_{\mathrm{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\}/\mathbb{R}. \tag{1.10}$$

REMARK 3. Another possibility of the definition of the homogeneous Sobolev spaces can be found in the work of Galdi [15]. He defines the homogeneous Sobolev spaces in the following way

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_q}$$

and from Theorem II.6.3, and Remark II.6.2 [15] he gives the following characterisation

$$\begin{aligned}\widehat{W}^{1,q}(\mathbb{R}^3) &= \{v \in L^1_{\mathrm{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\},\ q \ge 3,\\ &= \{v \in L^1_{\mathrm{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3, v \in L^{\frac{3q}{3-q}}(\mathbb{R}^3)\},\ q < 3.\end{aligned} \tag{1.11}$$

We mention [25], Proposition 2.4 for characterisation of the spaces $\widehat{W}^{1,q}(\mathbb{R}^3)$.

LEMMA 1.1.

- *For $1 < r < n$ we have $\widehat{W}^{1,r}(\mathbb{R}^3) = \{u \in L^s(\mathbb{R}^3) : \nabla u \in L^r(\mathbb{R}^3)\}$ where $s = \frac{3r}{3-r}$.*
- *Let $r \ge n$. Suppose $u_k \in C_0^\infty(\mathbb{R}^3)$, $k = 1, 2, \dots$ is a Cauchy sequence in $\widehat{W}^{1,r}(\mathbb{R}^3)$. Then there is a Cauchy sequence $w_k \in C_0^\infty$ with $\nabla u \in L^r(\mathbb{R}^3)$ satisfying*

$$\begin{aligned}&\|\nabla u_k - \nabla w_k\|_{L^r(\mathbb{R}^3)} \to 0,\\ &w_k \to u \ \text{in}\ L^r_{loc}(\mathbb{R}^3),\\ &\nabla w_k \to \nabla u \ \text{in}\ L^r(\mathbb{R}^3) \ \text{as}\ k \to \infty.\end{aligned} \tag{1.12}$$

Such a u is unique up to additive constants. In this case, we have the inclusion $\widehat{W}_0^{1,r}(\mathbb{R}^3) \subset \{[u] \in L^r_{loc}(\mathbb{R}^3)/\mathbb{R} : \nabla u \in L^r(\mathbb{R}^3)\}$ where $[u] = \{w \in L^r_{loc}(\mathbb{R}^3) : w - u \in \mathbb{R}\}$.

REMARK 4. We would like to mention that definitions (1.10) and (1.11) are equivalent in the following sense. In definition (1.10) the elements of space are classes of functions since we factorized the homogeneous spaces $\widehat{W}^{1,r}$ by constants. In definition (1.11) we divide into two cases:

- the case $1 < r < n$ where Sobolev imbedding is valid
- the case of $r \ge n$ where limits of Cauchy sequences are unique up to constant, see previous Lemma.

REMARK 5. We would like to mention that a different approach was given by Girault and collaborators. They introduce Sobolev spaces with weights where the density of weighted Sobolev spaces in C_0^∞ is satisfied automatically from the definition, see [1].

Their dual space is defined in the following way

$$\widehat{W}^{-1,q}(\mathbb{R}^3) = (\widehat{W}^{1,q/(q-1)}(\mathbb{R}^3))^*, \text{ with norm } \|\cdot\|_{-1,q}. \tag{1.13}$$

A characterisation of the normed dual spaces can be found in [15] page 72–74.

REMARK 6. The definition of dual spaces is important for extension of Bogovskii operator to negative homogeneous spaces; for more details see [5, 20].

We will use systematic notations ∂_j for partial derivatives in Cartesian coordinates and ∂_r or ∂_θ in cylindrical coordinates. We are now interested in the weak solution to (1.1).

DEFINITION 1.1. Let $1 < q < \infty$. Given $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we call $\{u,p\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ a *weak solution* to (1.1) if

$$\begin{aligned} &(1) \qquad \nabla \cdot u = 0 \quad \text{in} \quad L^q(\mathbb{R}^3), \\ &(2) \qquad (\omega \wedge x)\cdot \nabla u - \omega \wedge u \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3, \end{aligned} \tag{1.14}$$

$\{u,p\}$ satisfies $(1.1)_2$ in the sense of distributions, that is,

$$\begin{aligned} &\nu\langle \nabla u, \nabla\varphi\rangle - \langle (\omega\wedge x)\cdot\nabla u - \omega\wedge u, \varphi\rangle \\ &+k\left\langle \frac{\partial u}{\partial x_3}, \varphi\right\rangle - \langle p, \nabla\cdot\varphi\rangle = \langle f,\varphi\rangle, \\ &\varphi \in C_0^\infty(\mathbb{R}^3)^3. \end{aligned} \tag{1.15}$$

In fact, as usual, equation (1.15) holds by density for all $\varphi \in \widehat{W}^{1,q/(q-1)}(\mathbb{R}^3)^3$.

In Definition 1.1 we use that functions from $\varphi \in \widehat{W}^{1,q/(q-1)}(\mathbb{R}^3)^3$ can be approximated by functions from C_0^∞, for more details see [31].

In the work of Galdi [15], the author defines q-generalized solutions (see page 189) which are similar to our Definition 1.1.

The main results are the following

THEOREM 1.1. *Let $1 < q < \infty$ and suppose $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, then the problem* (1.1) *possesses a weak solution $(u,p) \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ satisfying the estimate*

$$\|\nabla u\|_q + \|p\|_q + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q} \;\le\; C\|f\|_{-1,q}, \tag{1.16}$$

with some $C > 0$, which depends on q.

THEOREM 1.2. *The solution $\{u,p\}$ given by Theorem 1.1 is unique up to a constant multiple of ω for u.*

COROLLARY 1.1. *Let $1 < q < 4$, $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ and let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be the unique weak solution to problem* (1.1). *Then there exists $\alpha \in \mathbb{R}$ such that*

$$u - \alpha e_3 \in L^s(\mathbb{R}^3)^3 \textit{ for all } s > 1, 1/s \in 1/q - [1/4, 1/3].$$

Moreover

$$\|u - \alpha e_3\|_s \le C\|f\|_{-1,q}$$

with a constant $C = C(\nu,k,\omega,s) > 0$.

COROLLARY 1.2. *Let $1 < q < 3$, $\nu > 0$, $k > 0$, $f \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, and let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be the unique weak solution to problem* (1.1). *Then*

$$\|u/|x|\|_q \le \frac{c}{\nu}\|f\|_{-1,q}$$

with $c = c(q,\omega) > 0$.

In Theorem 1.1 and Theorem 1.2, due to our choice of the right-hand side, we improve Farwig's a priori estimates (1.4) . We extend to the Oseen problem the analysis done by Hishida [31] for the Stokes problem.

2. Mathematical preliminaries

2.1. *Definition of Littlewood-Paley decomposition*

DEFINITION 2.1. Let $\chi \in \mathcal{D}(\mathbb{R}^d)$, $d \in \mathbb{N}$, be such that $|\xi| \le 1/2$ implies $\chi(\xi) = 1$ and $|\xi| \ge 1$ implies $\chi(\xi) = 0$. Let ψ be defined as $\psi(\xi) = \chi(\xi/2) - \chi(\xi)$. Let S_j and Δ_j be defined as the Fourier multipliers $\mathcal{F}(S_j f) = \chi(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j f) = \psi(\xi/2^j)\mathcal{F}f$. Then for all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^d)$ we have $f = S_N f + \sum_{j \ge N} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^d)$, this equality is called the Littlewood-Paley decomposition of the distribution f.

THEOREM 2.1 (Littlewood-Paley decomposition of $L^p(\mathbb{R}^d)$). *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $1 < p < \infty$. Then the following assertions are equivalent:*

(i) $f \in L^p(\mathbb{R}^d)$,
(ii) $S_0 \in L^p(\mathbb{R}^d)$ *and* $(\sum_{j\in\mathbb{N}} |\Delta_j f|^2)^{1/2} \in L^p(\mathbb{R}^d)$,
(iii) $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ *and* $(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2)^{1/2} \in L^p(\mathbb{R}^d)$.

Moreover, the following norms are equivalent on L^p :

$$\|f\|_p, \qquad \|S_0 f\|_p + \Big\| \Big(\sum_{j\in\mathbb{N}} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p \quad \textit{and} \quad \Big\| \Big(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p .$$

Proof. See [33].

2.2. *Bogovskii operator*

DEFINITION 2.2. Let $\mathcal{D}(\Delta_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ denote the usual domain of definition of the Laplace operator $\Delta = \Delta_q$ in L^q space with zero Dirichlet boundary condition. We set

$$L_0^q(\Omega) = \left\{ u \in L_q(\Omega) : \int_\Omega u \, dx = 0 \right\}.$$

We introduce the Bogovskii operator and we recall its properties. For a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary in $C^{0,1}$ Bogovskii [2], [3] constructed a bounded linear operator $\mathcal{R} : L_0^q(\Omega) \to W_0^{1,q}(\Omega)^n$ such that $u = \mathcal{R}g$ is a solution of

$$\begin{aligned} \operatorname{div} u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

satisfying $\|\mathcal{R}g\|_{W^{1,q}(\Omega)^n} \le c\|g\|_q$. Additionally $\mathcal{R}$ maps $W_0^{1,q}(\Omega) \cap L_0^q(\Omega)$ into $W_0^{2,q}(\Omega)$, see [2].

The Bogovskii operator was studied in a more general class of domains, see e.g. [5].

ASSUMPTIONS I. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain with boundary $\partial\Omega \in C^{1,1}$, and suppose one of the following cases

(i) Ω is bounded,
(ii) Ω is an exterior domain, i.e., a domain having a compact nonempty complement.

(iii) Ω is a perturbed half space, i.e., there exists some open ball B such that $\Omega \setminus B = \mathbb{R}^n_+ \setminus B$.

LEMMA 2.1 (Farwig, Sohr). *Let $\Omega = \mathbb{R}^n$ or let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain satisfying Assumption I, further let $1 < q < \infty$. Then there exists a linear bounded operator $\mathcal{R} : W^{1,q}(\Omega) \cap \widehat{W}^{-1,q}(\Omega) \to \mathcal{D}(\Delta_q)^n$ if Ω is unbounded or $\mathcal{R} : W^{1,q}(\Omega) \cap L^q_0(\Omega) \to \mathcal{D}(\Delta_q)^n$ if Ω is bounded such that $u = \mathcal{R}g$ is a solution of (2.1) for all $g \in W^{1,q}(\Omega) \cap \widehat{W}^{-1,q}(\Omega)$ or $g \in W^{1,q}(\Omega) \cap L^q_0(\Omega)$ respectively; $u = \mathcal{R}g$ satisfies the estimates*

$$\|u\|_q \leq c\|g\|_{-1,q} \quad \text{and} \quad \|u\|_{W^{2,q}(\Omega)} \leq c(\|\nabla g\|_q + \|g\|_{-1,q}),$$

where $c = c(\Omega, q) > 0$ is a constant.

Proof. See [5].

2.3. *Maximal operator.* For a rapidly decreasing function $u \in \mathcal{S}(\mathbb{R}^n)$ let

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transform of u. Its inverse is denoted by $\mathcal{F}^{-1}$. Moreover, we define the centered Hardy-Littlewood maximal operator

$$\mathcal{M}u(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |u(y)|\, dy, \quad x \in \mathbb{R}^n,$$

for $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ where again Q runs through the set of all cubes centered at x.

3. Computation of ∇u. Using the fact that the space $\{g \mid g = \nabla \cdot G,\ G \in C_0^\infty(\mathbb{R}^3)^{3\times 3}\}$ is dense in $\widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we can write either f in the divergence form in the Oseen system (1.1) or $g = \nabla \cdot G$ in the reduced Oseen system 1.3, assuming firstly $G \in C_0^\infty\left(\mathbb{R}^3\right)^{3\times 3}$. We will work in the space of tempered distributions because we have in mind to apply the Fourier transform. We will derive the following formal expressions of $\widehat{u}$, u, and ∇u:

$$\widehat{u}(\xi) = \int_0^\infty e^{-(\nu|\xi|^2 + ik\xi_3)t} O_\omega^T(t)\, \widehat{g}(O_\omega(t)\xi)\, dt,$$

yielding $u(\cdot)$ in the form

$$u(x) = \int_0^\infty E_t * O_\omega^T(t)\, g(O_\omega(t). - kte_3)(x)\, dt,$$

where

$$E_t(x) = \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{|x|^2}{4\nu t}}.$$

Observing the previous integral the solution can be rewritten as

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x, y)\ \nabla \cdot G(y)\ dy,$$

where

$$\Gamma(x, y) = \int_0^\infty E_t\left(O_\omega(t)x - y - kte_3\right)\, O_\omega^T(t)\, dt.$$

Therefore we can compute the gradient of u,

$$\nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma(x, y) : G(y)\ dy,$$

and come back to its Fourier transform.

Let us compute explicitly $\widehat{u}$. First of all, due to the geometry of the problem it is reasonable to introduce cylindrical coordinates $(r, \theta, x_3) \in (0, \infty) \times [0, 2\pi) \times \mathbb{R}$. Then $(\omega \wedge x) \cdot \nabla u = |\omega| \, (-x_2 \partial_1 u + x_1 \partial_2 u)$ may be rewritten in the form

$$(\omega \wedge x) \cdot \nabla u = |\omega| \, \partial_\theta u$$

using the angular derivative ∂_θ applied to $u(r, x_3, \theta)$.

With the Fourier variable, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ we get from (1.1)

$$(\nu|\xi|^2 + ik\xi_3)\widehat{u} - |\omega|\partial_\varphi \widehat{u} + |\omega| e_3 \wedge \widehat{u} + i\xi\widehat{p} = \widehat{f}, \quad i\xi \cdot \widehat{u} = 0. \tag{3.1}$$

It is clear that $(e_3 \wedge \xi) \cdot \nabla_\xi = -\xi_2 \partial/\partial\xi_1 + \xi_1 \partial/\partial\xi_2 = \partial_\varphi$ is the angular derivative in Fourier space when using cylindrical coordinates. Since (1.2) we have $i\xi \cdot (\partial_\varphi \widehat{u} - e_3 \wedge \widehat{u}) = 0$, the unknown pressure p is explicitly given by $-|\xi|^2 \widehat{p} = i\xi \cdot \widehat{f}$. Denoting $g = f - \nabla p$ then we get

$$-\partial_\varphi \widehat{u} + \frac{1}{|\omega|}(\nu|\xi|^2 + ik\xi_3)\widehat{u} + e_3 \wedge \hat{u} = \frac{1}{|\omega|}\widehat{g}, \tag{3.1$'$}$$

a first order differential equation with respect to φ for $\widehat{u} := \widehat{u}(\sqrt{\xi_1^2 + \xi_2^2}, \varphi, \xi_3)$.

To deal with the term $\omega \wedge u$ note that $\partial_\phi O(\varphi) = \omega \wedge O(\varphi)$ in the sense of linear maps. Applying the $O(\varphi)$ to (3.1)$'$ the unknown $\widehat{v}(\varphi) = O(\varphi)^T \widehat{u}(\varphi)$ solves the problem

$$-\partial_\varphi \widehat{v} + \frac{1}{|\omega|}(\nu|\xi|^2 + ik\xi_3)\widehat{v} = \frac{1}{|\omega|}\widehat{g}.$$

This inhomogeneous, linear ordinary differential equation of first order with respect to φ has a unique 2π-periodic solution

$$\widehat{v}(\varphi) = \frac{1/|\omega|}{1 - e^{-2\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}} \int_0^{2\pi} e^{(-\nu|\xi|^2 + ik\xi_3)t} O_{e_3}^T(\varphi + t)\widehat{g}(O_{e_3}(t)\xi)dt.$$

Then

$$\widehat{u} = \frac{1}{1 - e^{-2\pi(\nu|\xi|^2 + ik\xi_3)/|\omega|}} \int_0^{2\pi/|\omega|} e^{-(\nu|\xi|^2 + ik\xi_3)t} O_{|\omega|}^T(t)\widehat{g}(O_{|\omega|})(t)\xi)dt.$$

Applying the geometric series and the $2\pi/|\omega|$-periodicity of the map $t \mapsto O_{|\omega}^T(t)\widehat{g}(O_{|\omega|}(t)\xi)$ we get the unique $2\pi/|\omega|$-periodic solution

$$\widehat{u}(\xi) = \int_0^\infty e^{-(\nu|\xi|^2 + ik\xi_3)t} O_\omega^T(t) \, \widehat{g}(O_\omega(t)\xi) \, dt. \tag{3.2}$$

$\widehat{u}$ solves the reduced equation (3.1)$'$ in the Fourier space, we have

$$u(x) = \int_0^\infty \frac{1}{(4\pi\nu t)^{3/2}} \exp\left(-\frac{|\,.\,|^2}{4\nu t}\right) * O_\omega^T(t) \, g(O_\omega(t). - kte_3)(x) \, dt, \tag{3.3}$$

$$\nabla u(x) = \int_0^\infty \nabla \frac{1}{(4\pi\nu t)^{3/2}} \exp\left(-\frac{|\,.\,|^2}{4\nu t}\right) * O_\omega^T(t) \, g(O_\omega(t). - kte_3)(x) \, dt, \tag{3.4}$$

yielding

$$\nabla u(x) = \int_{\mathbb{R}^3} \nabla_x \Gamma(x, y) \, g(y) \, dy, \tag{3.5}$$

with $g = \nabla \cdot G$ and

$$\Gamma(x, y) = \int_0^\infty \frac{1}{(4\pi\nu t)^{3/2}} \exp\left(-\frac{|O_\omega(t)\, x - y - k\, t\, e_3|^2}{4\nu t}\right) O_\omega^T(t) \, dt.$$

REMARK 7. Taking into account the expression $g = \nabla \cdot G$, $G \in C_0^\infty(\mathbb{R}^3)^9$, we can integrate by parts:

$$\nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x \nabla_y \Gamma(x,y) : G(y)\, dy$$

Observing that $|y - O_\omega(t)x + kte_3| = |x - O_\omega^T(t)(y - kte_3)| = |x - O_\omega^T(t)y - kte_3|$ we get $\nabla_x \nabla_y \Gamma(x,y) = \nabla_x^2 \Gamma(x,y)$. So we have

$$\nabla u(x) = -\int_{\mathbb{R}^3} \nabla_x^2 \Gamma(x,y) : G(y)\, dy = -\nabla^2 \int_{\mathbb{R}^3} \Gamma(x,y) : G(y)\, dy.$$

4. Proof of the main theorem. In this section we estimate the L^q-norm of each component of $TG(x) := \Delta \int_{\mathbb{R}^3} \Gamma(x,y) : G(y)\, dy$, say $TG_{i,k}(\cdot)$ by the L^q-norm of $G_{i,k}(\cdot)$, and then we apply these results for L^q-estimate of ∇u.

To this end, we follow the way used by Farwig, Hishida, Müller [12] and by Hishida [31] because till now we do not have a more direct analysis. By means of the Fourier transform we have

$$\widehat{TG}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty |\xi|^2 \exp(-\nu |\xi|^2 t) O_\omega^T(t)\, \widehat{G}\left((O_\omega(t) \cdot -k\, t\, e_3)\, \xi\right) dt.$$

Which we can rewrite as

$$\widehat{TG}(\xi) = \frac{1}{\nu(2\pi)^{3/2}} \int_0^\infty |\xi|^2 \exp(-|\xi|^2 t) O_{\omega/\nu}^T(t)\, \widehat{G}\left(\left(O_{\omega/\nu}(t) \cdot -k\, \frac{t}{\nu}\, e_3\right) \xi\right) dt.$$

Let us temporarily denote $TG_{i,k}(x)$ by $F(x)$. A deep tool from harmonic analysis requires us to define an appropriate function $\varphi(\cdot) \in C_0^\infty((0,\infty); \mathcal{S}(\mathbb{R}^3))$ such that with the so called square operator

$$S(F)(x) = \int_0^\infty |\varphi(t,\cdot) * F(x)|^2\, \frac{dt}{t}$$

we obtain the equivalence of L^q-norms given by the theorem of E. M. Stein, Chapter I, Section 8.23 [37],

$$c_1 \|F\|_q \le \|S(F)^{1/2}\|_q \le c_2 \|F\|_q.$$

The necessary properties of $\varphi(t,\cdot)$ for $t > 0$ are

$$\operatorname{supp} \widehat{\varphi}(t,\cdot) \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{2\sqrt{t}} < |\xi| < \frac{2}{2\sqrt{t}} \right\},$$

$$\int_0^\infty \widehat{\varphi}(t,\xi)^2 \frac{dt}{t} = 1, \qquad \int_{\mathbb{R}^3} \varphi(t,x)\, dx = 0.$$

We start with Littlewood-Paley decomposition. We define $\psi \in \mathcal{S}(\mathbb{R}^3)$ by its Fourier transform

$$\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}} |\xi|^2 e^{-|\xi|^2} \quad \text{and} \quad \widehat{\psi_t}(\xi) = \widehat{\psi}(\sqrt{t}\xi) \text{ for } t > 0, \tag{4.1}$$

and so for all $t > 0$

$$\psi_t(x) = t^{-3/2} \psi\left(\frac{x}{\sqrt{t}}\right), \quad \widehat{\psi_t}(\xi) = \frac{1}{(2\pi)^{3/2}} t |\xi|^2 e^{-\nu t |\xi|^2}. \tag{4.2}$$

So we get

$$\widehat{TG}(\xi) = \frac{1}{\nu}\int_0^\infty \widehat{\psi}_t(\xi)\, O^T_{\omega/\nu}(t)\, \widehat{G}\left(\left(O_{\omega/\nu}(t). - k\frac{t}{\nu}e_3\right)\xi\right)\frac{dt}{t}. \tag{4.3}$$

We define the multiplier operator Δ_j such that

$$\widehat{\Delta_j f(\xi)} := \widehat{\chi}^j(\xi)\widehat{f}(\xi) \tag{4.4}$$

where

$$\widehat{\chi}^j(\xi) = \widehat{\chi}_0\left(\frac{\xi}{2^{j+1}}\right) - \widehat{\chi}_0\left(\frac{|\xi|}{2^j}\right) \tag{4.5}$$

with

$$\widehat{\chi}_0(.) : |\xi| \to \mathbb{R}, \chi \in C^\infty, \widehat{\chi}_0|_{\{|\xi|\le\frac{1}{2}\}} = 1, \widehat{\chi}_0|_{\{|\xi|\ge 1\}} = 0. \tag{4.6}$$

Note that

$$\sum_{j\in\mathbb{Z}} \widehat{\chi}^j(\xi) = 1. \tag{4.7}$$

and

$$f(.) = \sum_{j\in\mathbb{Z}} \Delta_j f(.). \tag{4.8}$$

We define χ^j for $\xi \in \mathbb{R}^3$ and $j \in \mathbb{Z}$ by its Fourier transform

$$\widehat{\chi}^j(\xi) = \widetilde{\chi}(2^{-j}|\xi|), \quad \xi \in \mathbb{R}^3,$$

yielding $\sum_{j=-\infty}^{\infty} \widehat{\chi}_j = 1$ on $\mathbb{R}^3 \setminus \{0\}$ and

$$\operatorname{supp} \widehat{\chi}^j \subset A(2^{j-1}, 2^{j+1}) := \{\xi \in \mathbb{R}^3 : 2^{j-1} < |\xi| < 2^{j+1}\}. \tag{4.9}$$

Using χ^j we define for $j \in \mathbb{Z}$

$$\psi^j = \frac{1}{(2\pi)^{n/2}}\chi_j * \psi_t, \quad \widehat{\psi^j} = \widehat{\Delta_j\psi(.)} = \widehat{\chi}^j \cdot \widehat{\psi}_t. \tag{4.10}$$

Obviously, $\sum_{j=-\infty}^{\infty} \psi^j = \psi$ on $\mathbb{R}^3$. We start with the procedure of the Littlewood-Paley decomposition of $F = TG_{i,k}$. For $G_{i,k} \in \mathcal{S}'(\mathbb{R}^3)$ the property $TG_{i,k} \in L^p(\mathbb{R}^3)$ is equivalent to the property

$$TG_{i,k} = \sum_{j=-\infty}^{+\infty} \Delta_j TG_{i,k} \quad \text{and} \quad \left(\sum_{j=-\infty}^{+\infty} |\Delta_j TG_{i,k}|^2\right)^{1/2} \in L^p(\mathbb{R}^3).$$

where

$$\Delta^j = \mathcal{F}^{-1}\widehat{\psi^j}\left(\frac{\xi}{2^j}\right)\mathcal{F}, \quad \Delta^j_t = \mathcal{F}^{-1}\widehat{\psi}^j_t\left(\frac{\xi}{2^j}\right)\mathcal{F}.$$

We define

$$\Delta\Gamma = \sum_{j\in Z} \Delta_j \Delta\Gamma \tag{4.11}$$

leading to

$$\Delta\int_{\mathbb{R}^3} \Gamma(x,y) : G(y)dy, \quad G \in C_0^\infty(\mathbb{R}^3)^9, (G_{ik})_{1\le i\le 3, 1\le k\le 3}. \tag{4.12}$$

We define the linear operator

$$TG_{ik}(x) = \int_{\mathbb{R}^3} \Delta\Gamma(x,y)_{ki} G_{ik}(y)\, dy. \tag{4.13}$$

Since formally $T = \sum_{j=-\infty}^{\infty} T_j$, we have to prove that this infinite series converges even in the operator norm on L^q.

For later use we cite the following lemma, see [12].

LEMMA 4.1. *The functions Δ^j, Δ^j_t, $j \in \mathbb{Z}$, $t > 0$, have the following properties:*

(i) $\operatorname{supp} \widehat{\Delta}^j_t \subset A\left(\frac{2^{j-1}}{\sqrt{t}}, \frac{2^{j+1}}{\sqrt{t}}\right)$.

(ii) *For $m > \frac{n}{2}$ let $h(x) = (1+|x|^2)^{-m}$ and $h_t(x) = t^{-n/2} h(\frac{x}{\sqrt{t}})$, $t > 0$. Then there exists a constant $c > 0$ independent of $j \in \mathbb{Z}$ such that*

$$|\Delta^j(x)| \le c 2^{-2|j|}\, h_{2^{-2j}}(x), \quad x \in \mathbb{R}^n,$$
$$\|\Delta^j\|_1 \le c 2^{-2|j|}.$$

Proof. See [12].

From the general definition of a Littlewood-Paley decomposition of L^q choose $\widetilde{\varphi} \in C_0^\infty(\frac{1}{2}, 2)$ such that $0 \le \widetilde{\varphi} \le 1$ and

$$\int_0^\infty \widetilde{\varphi}(s)^2 \frac{ds}{s} = \frac{1}{2}.$$

Then define $\varphi \in \mathcal{S}(\mathbb{R}^n)$ by its Fourier transform $\widehat{\varphi}(\xi) = \widetilde{\varphi}(|\xi|)$ yielding for every $s > 0$

$$\widehat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|), \quad \operatorname{supp} \widehat{\varphi}_s \subset A\left(\frac{1}{2\sqrt{s}}, \frac{2}{\sqrt{s}}\right), \tag{4.14}$$

and the normalization $\int_0^\infty \widehat{\varphi}_s(\xi)^2 \frac{ds}{s} = 1$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$.

THEOREM 4.1. *Let $1 < q < \infty$. Then there are constants $c_1, c_2 > 0$ depending on q and φ such that for all $f \in L^q$*

$$c_1 \|f\|_q \le \left\| \left(\int_0^\infty \int_0^\infty |\varphi_s * f(.)|^2 \frac{ds}{s} \right) \right\|_{q/2} \le c_2 \|f\|_q$$

where $\varphi_s \in \mathcal{S}(\mathbb{R}^n)$ is defined by (4.14).

Proof. See [37], Chapter I, Section 8.23.

We apply Theorem 4.1 to the operator $T_j G_{ik}$:

$$c_1 \|T_j G_{ik}\|_q \le \left\| \int_0^\infty |(\varphi(t,.) * T_j G_{ik})(x)|^2 \frac{dt}{t} \right\|_{q/2} \le c_2 \|T_j G_{ik}\|_q. \tag{4.15}$$

5. Proofs. As a preliminary version of Theorem 1. 1 we prove the following proposition.

PROPOSITION 5.1. *Let $j \in \mathcal{Z}$. The linear operator T defined by* (4.3) *satisfies the estimate*

$$\|T_j G_{ik}\|_q \le c \|G_{ik}\|_q \quad \textit{for all} \quad G \in L^q,\ q \in (2,\infty)$$

with a constant $c = c(q,w) > 0$ independent of f.

Proof. We define the sublinear operator $\mathcal{M}^j$, a modified maximal operator, by

$$\mathcal{M}^j\varphi(x) = \sup_{s>0} \int_{A_s} (|\Delta_t^j| * |\varphi|) \left(O_{\omega/\nu}(t)^T x + \frac{k}{\nu} te_3 \right) \frac{dt}{t}\,, \tag{5.1}$$

where $A_s = [\frac{s}{16}, 16s]$.

First step. We will prove the preliminary estimate

$$\|T_j G_{ik}\|_q \le c\|\Delta^j\|_1^{1/2} \|\mathcal{M}^j\|_{L^{(q/2)'}}^{1/2} \|G_{ik}\|_q\,, \quad j \in \mathbb{Z}. \tag{5.2}$$

To prove (5.2) we use the Littlewood-Paley decomposition of L^q,

$$c_1^2\|f\|_q^2 \le \left\| \int_0^\infty |\varphi_s * f(\cdot)|^2 \frac{ds}{s} \right\|_{q/2} \le c_2^2 \|f\|_q^2. \tag{5.3}$$

By a duality argument we find some function $0 \le g \in L^{(q/2)'}$ with $\|g\|_{(q/2)'} = 1$ such that

$$\left\| \int_0^\infty |\varphi_s * T_j G_{ik}(\cdot)|^2 \frac{ds}{s} \right\|_{q/2} = \int_0^\infty \int_{\mathbb{R}^n} |\varphi_s * T_j G_{ik}(x)|^2 g(x)\, dx\, \frac{ds}{s}. \tag{5.4}$$

To estimate the right-hand side of (5.4) note that

$$\varphi_s * T_j G(x) = \int_0^\infty O(t)_{\omega/\nu}^T(t)(\varphi_s * \Delta_t^j * G_{ik}) \left(O_{\omega/\nu}(t)x - \frac{k}{\nu} te_3 \right) \frac{dt}{t}\,,$$

where $\varphi_s * \Delta_t^j = 0$ unless $t \in A(s,j) := [2^{2j-4}s, 2^{2j+4}s]$. Since $\int_{t\in A(s,j)} \frac{dt}{t} = \log 2^8$ for every $j \in \mathbb{Z}$, $s > 0$, we get by the inequality of Cauchy-Schwarz and the associativity of convolutions that

$$\begin{aligned} |\varphi_s * T_j G_{ik}(x)|^2 &\le c \int_{A(s,j)} \left| (\Delta_t^j * (\varphi_s * G_{ik})) \left(O_{\omega/\nu}(t)x - \frac{k}{\nu} te_3 \right) \right|^2 \frac{dt}{t} \\ &\le c\|\Delta^j\|_1 \int_{A(s,j)} (|\Delta_t^j| * |\varphi_s * G_{ik}|^2) \left(O(t)_{\omega/\nu} x + \frac{k}{\nu} te_3 \right) \frac{dt}{t}; \end{aligned}$$

here we used the estimate $|(\Delta_t^j * (\varphi_s * G_{ik}))(y)|^2 \le \|\Delta_t^j\|_1 (|\Delta_t^j| * |\varphi_s * G_{ik}|^2)(y)$ and the identity $\|\Delta_t^j\|_1 = \|\Delta^j\|_1$, see Lemma 4.1. Thus

$$\begin{aligned} &\|T_j G_{ik}\|_q^2 \\ &\le c \int_0^\infty \int_{A(s,j)} \int_{\mathbb{R}^n} (|\Delta_t^j| * |\varphi_s * G_{ik}|^2)(x) \left(O(t)_{\omega/\nu}^T x - \frac{k}{\nu} te_3 \right) g(x)\, dx \frac{dt}{t} \frac{ds}{s} \end{aligned} \tag{5.5}$$

since Δ_t^j is radially symmetric. By definition of $\mathcal{M}^j$ the innermost integral is bounded by $\mathcal{M}^j g(x)$ uniformly in $s > 0$. Hence we may proceed in (5.5) using Hölder's inequality as follows:

$$\|T_j G\|_q^2 \le c\|\Delta^j\|_1 \int_{\mathbb{R}^n} \left(\int_0^\infty |\varphi_s * G|^2(x) \frac{ds}{s} \right) \mathcal{M}^j g(x)\, dx. \tag{5.6}$$

Now (5.3) and the normalization $\|g\|_{(q/2)'} = 1$ complete the proof of (5.2).

Second step. We investigate the estimate $\|\mathcal{M}^j g\|_{(q/2)'}$. Since $\frac{q}{2} \in (1, \infty)$ is arbitrary, we have to consider $\|\mathcal{M}^j\|_{L^p}$ for arbitrary $p \in (1, \infty)$. For this reason we define the classical Hardy-Littlewood maximal operator $\mathcal{M}$ on $L^p(\mathbb{R}^3)$ by

$$\mathcal{M}g(x) := \sup_{s>0} \frac{1}{|B_s(x)|} \int_{B_s(x)} |g(x)|\, dy$$

and a "helical" maximal operator

$$\mathcal{M}_{\text{hel}}g(\theta,x_3) := \sup_{s>0}\frac{1}{s}\int_{A_s}|g|\left|\theta-\frac{\omega}{\nu}t, x_3+\frac{k}{\nu}t\right|,$$

for functions g depending on (θ, x_3), which are 2π-periodic in θ. Since $0 \le h \in L^1(\mathbb{R}^3)$ is radially symmetric and strictly decreasing,

$$\sup_{r>0} h_r * u(x) \le c\mathcal{M}u(x).$$

Then

$$\mathcal{M}_j g(x) \le c2^{-2|j|}\mathcal{M}(\mathcal{M}_{hel}g_r(.,.))(x),$$

where $g_r(\theta,x_3) = g(r,\theta,x_3) = g(x)$ is considered as a function of θ, x_3 and

$$\|\mathcal{M}_j g\|_p \le C2^{-2j}\|\mathcal{M}_{\text{hel}}g_r(\cdot,\cdot)\|_{L^p(\mathbb{R}^3)}.$$

due to L^p continuity of $\mathcal{M}$.

To estimate $\mathcal{M}_{\text{hel}}g_r(\cdot,\cdot)$ in $L^p(\mathbb{R}^3)$, fix $r>0$ and use the 2π-periodicity of g_r with respect to θ to get that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^{2\pi}|\mathcal{M}_{\text{hel}}g_r(\theta,x_3)|^p d\theta dx_3\\
&\le \int_{\mathbb{R}}\int_0^{2\pi}\left|\sup_{s>0}\frac{1}{s}\int_{-16s}^{16s}|g_r|\left(\theta-\frac{\omega}{k}\left(x_3+\frac{k}{\nu}t\right), x_3+\frac{k}{\nu}t\right)dt\right|^p d\theta dx_3\\
&= \int_{\mathbb{R}}\int_0^{2\pi}\left|\sup_{s>0}\frac{1}{s}\int_{-16s}^{16s}\gamma_{r,\theta}\left(x_3+\frac{k}{\gamma}t\right)dt\right|^p d\theta dx_3,
\end{aligned}$$

where

$$\gamma_{r,\theta}(y_3) = |g_r|\left(\theta-\frac{\omega}{k}y_3, y_3\right).$$

Thus we get applying Hardy-Littlewood maximal operator on $\mathbb{R}^1$ that

$$\|\mathcal{M}_{\text{hel}}g_r(\cdot,\cdot)\|_{L^p(\mathbb{R}^3)} \le c\|g\|_{L^p(\mathbb{R}^3)}. \tag{5.7}$$

From Lemma 4.1 and Proposition 5.1 the operator T_j satisfies the estimates

$$\|T_j G_{ik}\|_q \le C2^{-2|j|}\|G_{ik}\|_q,\ j\in\mathcal{Z},\ q\in(2,\infty), c=c(q)>0.$$

Then $T=\sum_{j=-\infty}^{\infty}T_j$ converges in the operator norm on $L^q(\mathbb{R}^3)^3$ and $\|TG\|_q \le c\|G\|_q$, for every $G\in S(\mathbb{R}^3)^3$.

Third step. For $1<q<2$ we use the adjoint operator T^* given by

$$T^*G(x) = \int_0^\infty (\Delta_t * O_{\omega/\nu}(t)G)\left(O^T_{\omega/\nu}(t)x+\frac{k}{\nu}te_3\right)\frac{dt}{t}, \tag{5.8}$$

with $G\in\mathcal{S}(\mathbb{R}^3)^3$ and then by same argument we get that T^* is bounded in $L^{\frac{q}{q-1}}(\mathbb{R}^3)^9$, so T is L^q bounded for $1<q<2$. This implies the following estimate

$$\|\nabla u\|_q \le \|G\|_q. \tag{5.9}$$

Fourth step. Now, using Farwig-Sohr Lemma 2.1 we know that there is $G\in L^q(\mathbb{R}^3)^9$ such that

$$\nabla\cdot G = f,\ \|G\|_{q,\mathbb{R}^3} \le C\|f\|_{-1,q,\mathbb{R}^3}.$$

Let $G_k \in C_0^\infty(\mathbb{R}^3)^9$ such that $\|G_k - G\|_{q,\mathbb{R}^3} \to 0$ as $k \to \infty$. Let u_k be solution of our fundamental solution (3.3) with $f = \nabla \cdot G_k$. For each k and $m \in \mathbb{N}$, we choose a constant vector $b_k^m \in \mathbb{R}^3$ satisfying

$$\int_{B_m} (u_k(x) + b_k^m)dx = 0$$

so that

$$\|u_k + B_k^m\|_{q,B_m} \le C_m \|\nabla u_k\|_{q,B_m} \le C_m \|\nabla u_k\|_{q,\mathbb{R}^3} \le c_m \|G_k\|_{q,\mathbb{R}^3}$$

by Poincaré inequality and by (5.9).

Therefore, there exist $u^{(m)} \in W^{1,q}(B_m)^3$ and $V \in L^q(\mathbb{R}^3)^9$ such that

$$\|u_k + b_k^{(m)} - u^{(m)}\|_{q,B_m} \to 0, \quad \|\nabla u_k - V\|_{q,\mathbb{R}^3} \to 0, \quad k \to \infty,$$

with

$$\nabla u^{(m)}(x) = V(x) \text{ (a.a. } x \in B_m).$$

We first set

$$\tilde{u} = u^{(1)} \text{ on } B_1; \ b_k = b_k^{(1)}.$$

Consider next the case $m = 2$; since $\nabla u^{(2)}(x) = V(x) = \nabla u^{(1)}(x) = \nabla \tilde{u}(x)$ for a.a. $x \in B_1 \subset B_2$, the difference $u^{(2)}(x) - \tilde{u}(x) =: a$ is a constant vector and

$$\begin{aligned} &|B_1|^{1/q} |b_k^{(2)} - b_k - a| = \|b_k^{(2)} - b_k - a\|_{q,B_1} \le \\ &\|u_k + b_k - \tilde{u}\|_{q,B_1} + \|u_k + b_k^{(2)} - u^{(2)}\|_{q,B_2} \to 0, \ k \to \infty \end{aligned} \tag{5.10}$$

One extends $\tilde{u}$ by

$$\tilde{u} = u^{(2)} - a \text{ on } B_2.$$

Then (5.10) implies

$$\|u_k + b_k - \tilde{u}\|_{q,B_2} \le \|u_k + b_k^{(2)} - u^{(2)}\|_{q,B_2} + |B_2|^{1/q} |b_k^{(2)} - b_k - a| \to 0 \tag{5.11}$$

as $k \to \infty$. By induction there exists a function $\tilde{u} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ so that

$$\|u_k + b_k - \tilde{u}\|_{q,B_m} + |\nabla u_k - \nabla \tilde{u}\|_{q,\mathbb{R}^3} \to 0, \quad k \to \infty, \tag{5.12}$$

for all $m \in \mathbb{N}$. We define

$$L = -\Delta - \frac{\partial}{\partial x_3} - (\omega \wedge x) \cdot \nabla + \omega \wedge .$$

From definition of L together with $Lu_k = \nabla \cdot G_k$ we have

$$Lb_k = \omega \wedge \dot{b}_k = L(u_k + b_k) - \nabla \cdot G_k \to L\tilde{u} - \nabla \cdot G \text{ in } \mathcal{D}'(\mathbb{R}^3)^3 \text{ as } k \to \infty.$$

Since there is a constant vector $b \in \mathbb{R}^3$ such that

$$\omega \wedge b_k \to \omega \wedge b = Lb$$

as $k \to \infty$. Consequently, we get

$$L(\tilde{u} - b) = \nabla \cdot G \text{ in } \mathcal{D}'(\mathbb{R}^3)^3$$

and $u = \tilde{u} - b$ is the desired solution. By (5.12) we have $\|u_k - \nabla u\|_{q,\mathbb{R}^3} \to 0$ and, therefore, the estimate (1.14) holds.

Fifth step. It remains to prove the uniqueness. We use the duality method. We consider the adjoint equation

$$L^*v \equiv -\Delta v + (\omega \wedge x)\cdot \nabla v - \omega \wedge v + \frac{\partial u}{\partial x_3} = \nabla \cdot G \tag{5.13}$$

with $G \in C_0^\infty(\mathbb{R}^3)^9$. It has the solution

$$\widehat{v}(\xi) = \int_0^\infty e^{-\nu|\xi|^2 t} O_\omega(t)(\mathcal{G}f(O_\omega^T(t). - kte_3))(\xi)dt. \tag{5.14}$$

Applying the same argument we get

$$\|\nabla v\|_{r,\mathbb{R}^3} \le C\|G\|_{r,\mathbb{R}^3}, \text{for all } v \in \widehat{W}^{1,r}(\mathbb{R}^3),\ r \in (1,\infty). \tag{5.15}$$

Let $u \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be a weak solution of $Lu = 0$ in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$. One can take as a test function to get

$$\langle Lu, v\rangle = 0.$$

Similarly, one takes u as a test function for (5.13) in $\widehat{W}^{-1,q/(q-1)}(\mathbb{R}^3)^3$ to obtain

$$\langle u, L^*v\rangle = \langle u, \nabla\cdot G\rangle.$$

Therefore,

$$\langle u, \nabla \cdot G\rangle = 0.$$

Since $G \in (C_0^\infty)^9$ is arbitrary, we obtain $u = 0$ in $\widehat{W}^{1,q}(\mathbb{R}^3)^3$ by Theorem 1.2. u is a constant vector, but it is a constant multiple of ω because $\omega \wedge u = 0$.

To complete the proof of Theorem 1.1, we have to show the following lemma

LEMMA 5.1. *Let $v \in \mathcal{S}(\mathbb{R}^3)$ be the solution of*

$$-\Delta v + \frac{\partial v}{\partial x_3} - (\omega \wedge x)\cdot\nabla v = 0 \quad in\ \mathbb{R}^3.$$

Then $\operatorname{supp}\widehat{v} \subset \{0\}$.

Proof. This was proved in [10].

Proof of Theorem 1.1. As we explained before, the term $-(\omega\wedge x)\cdot\nabla u + \omega\wedge u$ is divergence free. The pressure is formally obtained from the problem

$$p = -\nabla\cdot(-\Delta)^{-1}f.$$

Since $(-\Delta)^{-1}$ can be justified as a bounded operator from $\widehat{W}^{-1,q}(\mathbb{R}^3)$ to $\widehat{W}^{1,q}(\mathbb{R}^3)$ we get

$$\|\nabla p\|_q \le c\|f\|_{1,q},$$

which implies that

$$\|f - \nabla p\|_{-1,q} \le c\|f\|_{-1,q}.$$

This completes the proof of Theorem 1.1.

Proof of Corollary 1.1. From [6] there exists $\alpha \in \mathbb{R}^3$ such that

$$v = u - \alpha \in L^s(\mathbb{R}^3), \text{ for all } s > 1,\ 1/s \in 1/q - [1/4, 1/3].$$

Let

$$\tilde{L}u' := -\partial_\theta u' + u^\perp.$$

Since $\tilde{L}v' = -\partial_\theta v' + v^\perp = \tilde{L}u' - a^\perp$ and applying integration with respect to θ we get

$$2\pi a' = \int_0^{2\pi} \tilde{L}u' d\theta - \int_0^{2\pi} v' d\theta \in L^q + L^s,$$

which implies $a' = 0$.

Proof of Corollary 1.2. From Theorem II5.1 of [15] yields the estimate

$$\left(\int_{\mathbb{R}^3} \frac{|u(x) - u_\infty|}{|x|} dx \right)^{1/q} \leq \frac{q}{3-q} \left(\int_{\mathbb{R}^3} |\nabla u(x)|^q dx \right)^{1/q}.$$

Moreover, by lemma 5.2 of [15]

$$\int_{y=1} |u(Ry)|^q do(y) = o(R^{q-3})$$

as $R \to \infty$. Since $u \in L^s(\mathbb{R}^3)$ that u_∞ vanishes.

References

[1] C. Amrouche, V. Girault and G. Giroire, *Weighted Sobolev spaces for Laplace's equation in* $\mathbb{R}^N$, J. Math. Pures Appl. 73 (1994), 579–606.

[2] M. E. Bogovskii, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Soviet Math. Dokl. 20 (1979), 1094–1098.

[3] M. E. Bogovskii, *Solution of some vector analysis problems connected with operators div and grad*, Trudy Seminara S. L. Soboleva 1 (1980), 5–40 (in Russian).

[4] C. Conca, H. San Martin and M. Tucsnak, *Existence of solutions for the equations modelling the motion of a rigid fluid* Comm. Partial Differential Equations 25 (2000), 1019–1042.

[5] R. Farwig and H. Sohr, *Generalized resolvent estimates for the Stokes system in bounded and unbounded domains*, J. Math. Soc. Japan 48 (1994), 607–643.

[6] R. Farwig, *The stationary Navier-Stokes equations in a 3D-exterior domain*, in: Recent Topics on Mathematical Theory of Viscous Incompressible Fluid, Lecture Notes in Num. Appl. Anal. 16 (1998), 53–115.

[7] R. Farwig, M. Krbec and Š. Nečasová, *A weighted* L^q *approach to Oseen flow around a rotating body*, Math. Meth. Appl. Sci. 31 (2008), 551–574.

[8] R. Farwig, M. Krbec and Š. Nečasová, *A weighted* L^q *approach to Stokes flow around a rotating body*, Annali di Ferrara 54 (2008), 61–84.

[9] R. Farwig and J. Neustupa, *On the spectrum of a Stokes-type operator arising from flow around a rotating body*, Preprint 2005, Univ. of Darmstadt.

[10] R. Farwig, *An* L^q*-analysis of viscous fluid flow past a rotating obstacle*, Tôhoku Math. J. 58 (2006), 129–147.

[11] R. Farwig, *Estimates of lower order derivatives of viscous fluid flow past a rotating obstacle*, in: Regularity and Other Aspects of the Navier-Stokes Equations, Banach Center Publ. 70, Polish Acad. Sci., Warsaw, 2005, 73–84.

[12] R. Farwig, T. Hishida and D. Müller, L^q*-Theory of a singular "winding" integral operator arising from fluid dynamics*, Pacific J. Math. 215 (2004), 297–312.

[13] R. Farwig and T. Hishida, *Stationary Navier-Stokes flow around a rotating obstacles*, FB Mathematik, TU Darmstadt, Preprint no. 2445 (2006).

[14] E. Feireisl, *On the motion of rigid bodies in viscous compressible fluid*, Arch. Rational Mech. Anal. 167 (2003), 281–308.

[15] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Linearised Steady Problems*, Springer Tracts in Natural Philosophy, 38, 2nd edition, Springer, 1998.

[16] G. P. Galdi, *On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications,* Handbook of Mathematical Fluid Dynamics, Volume 1, S. Friedlander and D. Serre (eds.), Elsevier, 2002.

[17] G. P. Galdi, *Steady flow of a Navier-Stokes fluid around a rotating obstacle*, in: Essays and Papers Dedicated to the Memory of Clifford Ambrose Truesdell III, Vol. II, J. Elasticity 71 (2003), 1–31.

[18] G. P. Galdi and A. L. Silvestre, *On the steady motion of a Navier-Stokes liquid around a rigid body*, Archive for Rational Mechanics and Analysis 184 (2007), 371–400.

[19] M. Geissert, H. Heck and M. Hieber, *L^p theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle*, J. Reine A. Math. 596 (2006), 45–62.

[20] M. Geissert, H. Heck and M. Hieber, *On the equation div = 0 and Bogovskii's operator in Sobolev spaces of negative order*, in: Partial Differential Equations and Functional Analysis, Oper. Theory Adv. Appl. 168, Birkhäuser, Basel, 2006, 113–121.

[21] C. Grandmont, *Existence for a three-dimensional steady state fluid-structure interaction problem*, J. Math. Fluid Mech. 4 (2002), 76 –94.

[22] M. D. Gunzburger, H. C. Lee and G. A. Seregin, *Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions*, J. Math. Fluid Mech. 2 (2000), 291–266.

[23] K. H. Hoffmann and V. N. Starovoitov, *On a motion of a solid body in a viscous fluid. Two-dimensional case*, Adv. Math. Sci. Appl. 9 (1999), 633–648.

[24] A. Inoue and M. Wakimoto, *On existence of solutions of the Navier-Stokes equation in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1977), 303–319.

[25] H. Kozono and H. Sohr, *New apriori estimates for the Stokes equations in exterior domains*, Indiana Univ. Math. J. 41 (1991), 1–27.

[26] S. Kračmar, Š. Nečasová and P. Penel, *Estimates of weak solutions in anisotropically weighted Sobolev spaces to the stationary rotating Oseen equations.* IASME Transactions 2 (2005), 854–861.

[27] S. Kračmar, Š. Nečasová and P. Penel, *L^q approach of weak solutions of Oseen flow around a rotating body in an exterior domain*, Preprint 2007.

[28] S. Kračmar, Š. Nečasová and P. Penel, *Anisotropic L^2 estimates of weak solutions to the stationary Oseen type equations in $\mathbb{R}^3$ for a rotating body*, RIMS Kokyuroku Bessatsu, B1, (2007), 219–235.

[29] T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Rational Mech. Anal. 150 (1999), 307–348.

[30] T. Hishida, *The Stokes operator with rotating effect in exterior domains*, Analysis 19 (1999), 51–67.

[31] T. Hishida, *L^q estimates of weak solutions to the stationary Stokes equations around a rotating body*, Hokkaido Univ. Preprint series in Math., No. 691 (2004).

[32] Š. Nečasová, *On the problem of the Stokes flow and Oseen flow in $\mathbb{R}^3$ with Coriolis force arising from fluid dynamics*, IASME Trans. 2 (2005), 1262–1270.

[33] P. G. Lemarié-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman &Hall/CRC, Research Notes in Mathematics Series, 2002.

[34] Š. Nečasová, *Asymptotic properties of the steady fall of a body in viscous fluids*, Math. Meth. Appl. Sci. 27 (2004), 1969–1995.

[35] H. Sohr, Private communication.

[36] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[37] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, 1993.

[38] T. Takahashi, M. Tucsnak, *Global strong solutions for the two dimensional motion of an infinite cylinder in a viscous fluid*, J. Math. Fluid Mech. 6 (2004), 53–77.

[39] W. von Wahl, *Vorlesungen über das Aussenraumproblem für die instationären Gleichungen von Navier-Stokes*, SFB 256 Nichtlineare partielle Differentialgleichungen, Vorlesungsreihe Nr. 11, Universität Bonn, 1989.

J. Math. Soc. Japan
Vol. 62, No. 1 (2010) pp. 239–268
doi: 10.2969/jmsj/06210239

Anisotropic L^2-estimates of weak solutions to the stationary Oseen-type equations in 3D-exterior domain for a rotating body

By Stanislav KRAČMAR, Šárka NEČASOVÁ and Patrick PENEL

(Received Jan. 4, 2008)
(Revised Dec. 14, 2008)

Abstract. We study the Oseen problem with rotational effect in exterior three-dimensional domains. Using a variational approach we prove existence and uniqueness theorems in anisotropically weighted Sobolev spaces in the whole three-dimensional space. As the main tool we derive and apply an inequality of the Friedrichs-Poincaré type and the theory of Calderon-Zygmund kernels in weighted spaces. For the extension of results to the case of exterior domains we use a localization procedure.

1. Introduction.

1.1. Formulation of the problem.

In a three-dimensional exterior domain $\Omega \subset \boldsymbol{R}^3$, the classical Oseen problem [**30**] describes the velocity vector $\boldsymbol{v}$ and the associated pressure π by a linearized version of the incompressible Navier-Stokes equations as a perturbation of $\boldsymbol{v}_\infty$ the velocity at infinity; $\boldsymbol{v}_\infty$ is generally assumed to be constant in a fixed direction, say the first axis, $\boldsymbol{v}_\infty = |\boldsymbol{v}_\infty| \boldsymbol{e}_1$. In the next we denote $|\boldsymbol{v}_\infty|$ by k, and we will write the Oseen operator $k\,\partial_1 \boldsymbol{v}$. On the other hand it is known that for various flows past a rotating obstacle, the Oseen operator appears with some concrete non-constant coefficient functions, e.g. $\boldsymbol{a}(\boldsymbol{x}) = \omega \times \boldsymbol{x}$, where ω is a given vector, see [**17**], [**29**]; in view of industrial applications $\boldsymbol{a}(\boldsymbol{x})$ can also play the role of an "experimental" known velocity field, see [**20**].

This paper is devoted to the study of the following problem in Ω for (non-solenoidal) vector function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ and scalar function $p = p(\boldsymbol{x})$:

$$-\nu\,\Delta \boldsymbol{u} + k\partial_1 \boldsymbol{u} - (\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} + \omega \times \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = g \quad \text{in} \quad \Omega \tag{1.2}$$

$$\boldsymbol{u} \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty \tag{1.3}$$

$$\boldsymbol{u} = (\omega \times \boldsymbol{x}) - k\boldsymbol{e}_1 \quad \text{on} \quad \partial\Omega, \tag{1.4}$$

2000 *Mathematics Subject Classification.* Primary 35Q35; Secondary 35B45, 76D99.
Key Words and Phrases. Oseen problem, rotating body, anisotropically weighted L^2 spaces.

where $\omega = (\widetilde{\omega}, 0, 0)$ is a constant vector, ν, k and $\widetilde{\omega}$ are some positive constants, and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$ a given vector function, $g = g(\boldsymbol{x})$ a given scalar function.

We restrict ourselves to the assumption of compact support of g when Ω is an exterior domain. The system arises from the Navier-Stokes system modelling viscous fluid around a rotating body which is moving with a given non-zero velocity in the direction of its axis of rotation. An appropriate coordinate transform and a linearization yield in the stationary case equations (1.1) and (1.2), for details see [**3**], [**17**]. The third term together with the fourth one (the Coriolis force $\omega \times \boldsymbol{u}$) in (1.1) arise from the influence of rotation of the body.

Let us begin with some comments and relevant process of analysis of the problem (1.1)–(1.4).

- The governing equations of fluid motion are stationary and linear, but in unbounded domains the convective operators, $k\,\partial_1$ and $(\omega \times \boldsymbol{x}) \cdot \nabla$, *cannot be treated as perturbations* of lower order of the Laplacian.
- The fundamental tensor (similarly as the fundamental tensor to the Oseen problem) exhibits *the anisotropic behavior* in the three-dimensional space. To reflect the decay properties near the infinity we introduce the following weight functions:

$$\eta_\beta^\alpha(\boldsymbol{x}) = \eta_{\beta,\varepsilon}^{\alpha,\delta}(\boldsymbol{x}) = (1 + \delta r)^\alpha (1 + \varepsilon s)^\beta,$$

  with $r = r(\boldsymbol{x}) = |\boldsymbol{x}| = (\sum_{i=1}^3 x_i^2)^{1/2}$, $s = s(\boldsymbol{x}) = r - x_1$, $\boldsymbol{x} \in \boldsymbol{R}^3$, $\varepsilon, \delta > 0$, $\alpha, \beta \in \boldsymbol{R}$. Discussing the range of the exponents α and β, the corresponding weighted spaces $L^q(\boldsymbol{R}^3; \eta_\beta^\alpha)$ give the appropriate framework to test the solutions to (1.1)–(1.3). This paper is concerned with $q = 2$. Let us mention also that η_β^α belongs to the Muckenhoupt class A_2 of weights in $\boldsymbol{R}^3$ if $-1 < \beta < 1$ and $-3 < \alpha + \beta < 3$.
- In this paper we will prefer *the variational approach.* To avoid the difficulties with the pressure part of the solution p we solve firstly the problem in $\boldsymbol{R}^3$. Using the theory of Calderon-Zygmund integrals in corresponding weighted spaces, we determine the pressure p of the problem in $\boldsymbol{R}^3$ to be from the same space as the right-hand side of (1.1). This first step cannot be done directly in an exterior domain. Then we apply the variational approach for the velocity part of the solution.
- For the extension of the results to the case of exterior domains we use *the localization procedure*, see [**22**].

1.2. Short bibliographical remarks.

The weighted estimates of the solution to the stationary classical Oseen problem were firstly obtained by Finn in 1959, see [**9**]. *The variational approach* to the model equation $-\nu\,\Delta u + k\partial_1 u = f$ in an exterior domain in anisotropically weighted L^2-spaces was applied by Farwig, see [**1**]. The same variational viewpoint has been also applied in [**27**], [**28**] by Kračmar and Penel to solve the generic scalar model equation $-\nu\,\Delta u + k\partial_1 u - \boldsymbol{a}\cdot\nabla u = f$ with a given non-constant and, in general, non-solenoidal vector function $\boldsymbol{a}$ in an exterior domain. Both model equations are assumed with boundary conditions $u = 0$ on $\partial\Omega$ and $u \to 0$ as $|\boldsymbol{x}| \to \infty$.

Another common approach to study the asymptotic properties of the solutions to the Dirichlet problem of the classical steady Oseen flow is the use of *the potential theory*, i.e. convolutions with Oseen fundamental tensor and its first and second gradients for the velocity (or with the fundamental solution of Laplace equation for the pressure): the L^2-estimates in anisotropically weighted Sobolev spaces in $\boldsymbol{R}^3$ were derived by Farwig [**2**], the L^q-estimates in these spaces were proved in $\boldsymbol{R}^3$ and $\boldsymbol{R}^n$ by Kračmar, Novotný and Pokorný in [**25**] and [**26**], respectively. Different approach was used by Kobayashi and Shibata [**21**].

The fundamental solution to rotating Oseen problem in the time dependent case is known due to Guenther and Thomann, see [**32**], but, unfortunately, the respective stationary kernel does not seem to be of Calderon-Zygmund type. *The Littlewood-Paley decomposition technique* offers another approach for an L^q-analysis: Thus, L^q-estimates in non-weighted spaces were derived for the rotating Stokes problem by Hishida [**17**], by Farwig, Hishida, and Müller [**5**], and for the rotating Oseen problem in $\boldsymbol{R}^3$ by Farwig [**3**], [**4**]. L^q-estimates of the pressure and the gradient of the velocity for the exterior Stokes flow around a rotating body without translation were derived in [**19**]. L^q-setting with non-integrable right-hand side in non-homogeneous case was investigated by Kračmar, Nečasová and Penel in [**24**]. The Littlewood-Paley decomposition technique for L^q-weighted estimates with anisotropic weight functions was used by Farwig, Krbec and Nečasová [**7**], [**8**].

Another approach based on the use of the *non-stationary equations* in both the linear and also non-linear cases is proposed by Galdi and Silvestre in [**11**], [**12**], [**13**], [**14**]. The last paper showed the existence of the wake region for the Navier-Stokes flow for small data.

We would like also to mention that the problem was solved by the *semigroup theory* in L^2-setting in particular by Hishida [**18**], and then the respective results were extended to L^q case by Geissert, Heck and Hieber [**15**].

1.3. Basic notations and elementary properties.

Let us outline our notations. Let $\mathscr{S}'$ be the space of the moderate distributions in $\boldsymbol{R}^3$. Let Ω be an exterior domain with a boundary of the class $\mathscr{C}^2$, and

$$\widehat{W}^{m,q}(\Omega) = \{u \in L^1_{loc}(\Omega) : \ D^l u \in L^q(\Omega), \quad |l| = m\}$$

with the seminorm $|u|_{m,q} = (\sum_{|l|=m} \int_\Omega |u|^q)^{1/q}$. It is known that $\widehat{W}^{m,q}(\Omega)$ is a Banach space (and if $q = 2$ the space $\widehat{H}^m(\Omega) = \widehat{W}^{m,2}(\Omega)$ a Hilbert space), provided we identify two functions u_1, u_2 whenever $|u_1 - u_2|_{m,q} = 0$, i.e. u_1, u_2 differ (at most) on a polynomial of the degree $m-1$. As usual, we denote by $\widehat{W}^{m,q}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $\widehat{W}^{m,q}(\Omega)$.

Let $(L^2(\Omega;\, w))^3$ be the set of measurable vector functions $\boldsymbol{f} = (f_1,\, f_2,\, f_3)$ in Ω such that

$$\|\boldsymbol{f}\|^2_{2,\Omega;\, w} = \int_\Omega |\boldsymbol{f}|^2 w \, d\boldsymbol{x} < \infty.$$

We will use the notation $\boldsymbol{L}^2_{\alpha,\beta}(\Omega)$ instead of $(L^2(\Omega;\, \eta^\alpha_\beta))^3$ and $\|\cdot\|_{2,\alpha,\beta}$ instead of $\|\cdot\|_{(L^2(\Omega;\eta^\alpha_\beta))^3}$. Let us define the weighted Sobolev space $\boldsymbol{H}^1(\Omega; \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$ as the set of functions $\boldsymbol{u} \in \boldsymbol{L}^2_{\alpha_0,\beta_0}(\Omega)$ with the weak derivatives $\partial_i \boldsymbol{u} \in \boldsymbol{L}^2_{\alpha_1,\beta_1}(\Omega)$. The norm of $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$ is given by

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1\left(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)} = \left(\int_\Omega |\boldsymbol{u}|^2 \eta^{\alpha_0}_{\beta_0} \, d\boldsymbol{x} + \int_\Omega |\nabla \boldsymbol{u}|^2 \eta^{\alpha_1}_{\beta_1} \, d\boldsymbol{x}\right)^{1/2}.$$

As usual, $\overset{\circ}{\boldsymbol{H}}{}^1(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$, will be the closure of $\boldsymbol{C}^\infty_0(\Omega)$ in $\boldsymbol{H}^1(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$, where $\boldsymbol{C}^\infty_0(\Omega)$ is $(C^\infty_0(\Omega))^3$, and $\overset{\circ}{\boldsymbol{H}}{}^1(\overline{\Omega};\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$ will be the closure of $\boldsymbol{C}^\infty_0(\overline{\Omega})$ in $\boldsymbol{H}^1(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$.

For simplicity, we shall use the following abbreviations:

$\boldsymbol{L}^2_{\alpha,\beta}(\Omega)$	instead of	$\left(L^2\left(\Omega;\, \eta^\alpha_\beta\right)\right)^3$
$\|\cdot\|_{2,\alpha,\beta;\,\Omega}$	instead of	$\|\cdot\|_{(L^2(\Omega;\,\eta^\alpha_\beta))^3}$
$\overset{\circ}{\boldsymbol{H}}{}^1_{\alpha,\,\beta}(\Omega)$	instead of	$\overset{\circ}{\boldsymbol{H}}{}^1(\Omega;\, \eta^{\alpha-1}_{\beta-1}, \eta^\alpha_\beta)$
$\boldsymbol{V}_{\alpha,\beta}(\Omega)$	instead of	$\overset{\circ}{\boldsymbol{H}}{}^1(\Omega;\, \eta^{\alpha-1}_\beta, \eta^\alpha_\beta)$
$\boldsymbol{V}_{\alpha,\beta}(\overline{\Omega})$	instead of	$\overset{\circ}{\boldsymbol{H}}{}^1(\overline{\Omega};\, \eta^{\alpha-1}_\beta, \eta^\alpha_\beta)$.

We shall use these last two Hilbert spaces for $\alpha \geq 0,\ \beta > 0,\ \alpha+\beta < 3$. If no confusion can occur, we omit the domain in the notation of the norm $\|\cdot\|_{2,\alpha,\beta;\,\Omega}$. The notation $\boldsymbol{H}^1(\Omega)$ and $\overset{\circ}{\boldsymbol{H}}{}^1(\Omega)$ mean, as usual, the non-weighted spaces $(H^1(\Omega;\,1,\,1))^3$ and $(\overset{\circ}{H}{}^1(\Omega;\,1,\,1))^3$, respectively. As usual, omitting the domain Ω in the notation of spaces will indicate that $\Omega = \boldsymbol{R}^3$, so e.g. $\boldsymbol{H}^1 = \boldsymbol{H}^1(\boldsymbol{R}^3)$.

Concerning the weight functions η^α_β, we will use two notations $\eta^\alpha_\beta(x)$ and $\eta^{\alpha,\delta}_{\beta,\varepsilon}(x)$ taking the advantages of the following remark:

Remark 1.1. Let us note that for $\eta^{\alpha,\delta}_{\beta,\varepsilon}$ and for any $\delta_1,\delta_2,\varepsilon_1,\varepsilon_2 > 0$ one has

$$c_{\min}\cdot\eta^{\alpha,\delta_2}_{\beta,\varepsilon_2} \leq \eta^{\alpha,\delta_1}_{\beta,\varepsilon_1} \leq c_{\max}\cdot\eta^{\alpha,\delta_2}_{\beta,\varepsilon_2},$$

$c_{\min} = \min(1,(\delta_1/\delta_2)^\alpha)\cdot\min(1,(\varepsilon_1/\varepsilon_2)^\beta)$, $\quad c_{\max} = \max(1,(\delta_1/\delta_2)^\alpha)\cdot\max(1,(\varepsilon_1/\varepsilon_2)^\beta)$. The parameters δ and ε are useful to re-scale separately the isotropic and anisotropic parts of the weight function η^α_β.

We also use the notation of sets $B_R = \{\boldsymbol{x}\in\boldsymbol{R}^3; |\boldsymbol{x}| \leq R\}$, $B^R = \{\boldsymbol{x}\in\boldsymbol{R}^3; |\boldsymbol{x}| \geq R\}$, $\Omega_R = B_R\cap\Omega$, $\Omega^R = B^R\cap\Omega$, $B^{R_1}_{R_2} = B^{R_1}\cap B_{R_2}$, $\Omega^{R_1}_{R_2} = B^{R_1}_{R_2}\cap\Omega$, for positive numbers R, R_1, R_2.

1.4. Main results.

In the first part of the paper (chapters 2–4) we study the problem in $\boldsymbol{R}^3$. Let us assume for a moment that pressure p is known. In solving the problem (1.1)–(1.3) with respect to $\boldsymbol{u}$ and p by means of a pure variational approach, we shall deal with the following equation:

$$\begin{aligned}\nu\int_{\boldsymbol{R}^3}|\nabla\boldsymbol{u}|^2 w\,d\boldsymbol{x} + \nu\int_{\boldsymbol{R}^3}(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\cdot\nabla w\,d\boldsymbol{x} - \frac{k}{2}\int_{\boldsymbol{R}^3}|\boldsymbol{u}|^2\partial_1 w\,d\boldsymbol{x}\\ - \frac{1}{2}\int_{\boldsymbol{R}^3}|\boldsymbol{u}|^2\mathrm{div}(w[\omega\times\boldsymbol{x}])d\boldsymbol{x} = \int_{\boldsymbol{R}^3}\boldsymbol{f}\cdot\boldsymbol{u}\,w\,d\boldsymbol{x} - \int_{\boldsymbol{R}^3}\nabla p\cdot\boldsymbol{u}\,w\,d\boldsymbol{x}\end{aligned} \tag{1.5}$$

as we get integrating formally the product of (1.1) by $w\,\boldsymbol{u}$ with w an appropriate weight function. First, let us note that $\mathrm{div}(\eta^\alpha_\beta\,[\omega\times\boldsymbol{x}])$ equals zero for $w = \eta^\alpha_\beta$. The left hand side can be estimated from below by:

$$\frac{\nu}{2}\int_{\boldsymbol{R}^3}|\nabla\boldsymbol{u}|^2 w\,d\boldsymbol{x} + \frac{1}{2}\int_{\boldsymbol{R}^3}|\boldsymbol{u}|^2\Big(-\nu|\nabla w|^2/w - k\partial_1 w\Big)d\boldsymbol{x}. \tag{1.6}$$

Because the term $-\nu|\nabla w|^2/w - k\partial_1 w$ is known explicitly, we have the possibility to evaluate it from below by a small negative quantity in the form $-C\,\eta^{\alpha-1}_{\beta-1}$ without any constraint in $s(\cdot)$ (see Lemma 2.5).

An improved weighted Friedrichs-Poincaré type inequality in $\overset{\circ}{\boldsymbol{H}}{}^1_{\alpha,\beta}$ is necessary. The obtained inequality allows us to compensate by the viscous Dirichlet integral the "small" negative contribution in the second integral of (1.6). We finally prove the existence of a weak solution (1.1)–(1.3) in $\boldsymbol{V}_{\alpha,\beta}$ by the Lax-Milgram theorem.

The main results of the first part of the paper can be summarized in the following theorems (parameters α, β, δ, ε are specified in Section 1.3):

THEOREM 1.2. *Let $\beta > 0$. There are positive constants R_0, c_0, c_1 depending on α, β, δ, ε (explicit expressions of these constants are given by Lemma 2.3, essentially $c_0 = O(\varepsilon^{-2} + \delta^{-2})$ and $c_1 = O(\varepsilon^{-1}\delta^{-1})$ for δ and ε tending to zero) such that for all $\boldsymbol{v} \in \overset{\circ}{\boldsymbol{H}}{}^1_{\alpha,\beta}$*

$$\|\boldsymbol{v}\|^2_{2,\alpha-1,\beta-1} \le c_0 \int_{B_{R_0}} |\nabla \boldsymbol{v}|^2 \eta^\alpha_\beta \, d\boldsymbol{x} + c_1 \int_{B^{R_0}} |\nabla \boldsymbol{v}|^2 \eta^\alpha_\beta \, d\boldsymbol{x}. \tag{1.7}$$

THEOREM 1.3 (Existence and uniqueness). *Let $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$, $\boldsymbol{f} \in \boldsymbol{L}^2_{\alpha+1,\beta}$, $g \in H^1_{loc}$ such that $\nu\nabla g - kg\,\boldsymbol{e}_1 + g(\omega \times \boldsymbol{x}) \in L^2_{\alpha+1,\beta}$; y_1 will be given in Lemma 4.3. Then there exists a unique weak solution $\{\boldsymbol{u}, p\}$ of the problem (1.1)–(1.3) such that $\boldsymbol{u} \in \boldsymbol{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}$ and*

$$\begin{aligned} &\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla \boldsymbol{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \\ &\quad \le C\Big(\|\boldsymbol{f}\|_{2,\alpha+1,\beta} + \|\nu\nabla g - kg\,\boldsymbol{e}_1 + g(\omega\times\boldsymbol{x})\|_{2,\alpha+1,\beta}\Big). \end{aligned}$$

In the second part of the paper (chapters 5, 6) we extend the results of the first part onto exterior domains.

THEOREM 1.4. *Let $\Omega \subset \boldsymbol{R}^3$ be an exterior domain and $0 < \beta \le 1$, $0 \le \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\boldsymbol{f} \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$, $g \in \overset{\circ}{H}{}^1(\Omega)$ with $\operatorname{supp} g = K \subset\subset \Omega$ and $\int_\Omega g\,d\boldsymbol{x} = 0$. Then there exists a weak solution $\{\boldsymbol{u}, p\}$ of the problem (1.1)–(1.4) such that $\boldsymbol{u} \in \boldsymbol{V}_{\alpha,\beta}(\overline{\Omega})$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$\begin{aligned} &\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla \boldsymbol{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \\ &\quad \le C\Big(\|\boldsymbol{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2} + \omega^2 + \omega + k^2 + k\Big). \end{aligned}$$

REMARK 1.5. Concerning $\partial_1 \boldsymbol{u}$ and $\nabla \boldsymbol{u}$, our analysis did not catch any difference in the dependence of the parameters α and β. The reason appears inside the proofs of the Theorems 1.3 and 1.4 when we ask for the coercivity of the bilinear form $\widetilde{Q}(\cdot,\cdot)$, testing equation (4.20) by $\boldsymbol{u}\eta^{\alpha}_{\beta}$. On the other hand, we have no heuristic argument for not expecting better decay behavior of $\partial_1 \boldsymbol{u}$ like ∇p as in Farwig's result, see [**2**].

Remark 1.6. The important feature of the Friedrichs-Poincaré type inequality is that we are able to evaluate its coefficients, precisely expressed in Lemma 2.3 separately near the obstacle and far from the obstacle.

Remark 1.7. For $\alpha > 0$, using these coefficients, negative values of the function $F_{\alpha,\beta}(\cdot,\cdot)$ defined by the formula (2.13) can be compensated by the viscous Dirichlet integral; this analysis was not required in [**2**] because $F_{\alpha,\beta}(\cdot,\cdot)$ is positive when $\alpha < 0$.

Remark 1.8. The previous compensation cannot be associated with a large interval of positive values for α: So, we receive the technical condition $\alpha/\beta < y_1$. Using other type of weight functions characterized by some parameters, one can get another technical condition on these parameters.

Remark 1.9. We can improve the result from Theorem 1.4 removing the assumptions on g relative to its compact support and to its zero mean value: This will be the partial subject of a forthcoming paper. In the present paper, we have decided to use simply the approach by Girault-Raviart (see Subsection 6.1) and the standard Bogovski's lemma in bounded domains, to get finally Corollary 6.7.

2. Friedrichs-Poincaré inequality.

In this section we derive an inequality of the Friedrichs-Poincaré type in weighted Sobolev spaces. We also recall some necessary technical assertions, for more details see Kračmar and Penel [**27**].

Proposition 2.1. *For arbitrary $\alpha, \beta \geq 0$ and $\boldsymbol{x} \in \boldsymbol{R}^3$, $\boldsymbol{x} \neq \boldsymbol{0}$:*

$$\Delta\eta^{\alpha}_{\beta}(\boldsymbol{x}) \geq 2\beta\min(1,\beta)\varepsilon\,\delta\,\eta^{\alpha-1}_{\beta-1}(\boldsymbol{x}).$$

Proof. We introduce $\beta^* = \min(\beta,\,1)$ in an explicit expression of $\Delta\eta^{\alpha}_{\beta}$:

$$\begin{aligned}\Delta\eta^{\alpha}_{\beta} = \Bigg\{ &\left(\alpha^2\delta^2\,\frac{1+\varepsilon s}{1+\delta r} - \alpha\delta^2\,\frac{1+\varepsilon s}{1+\delta r}\right) + 2\alpha\beta\delta\varepsilon\,\frac{s}{r}\\ &+ 2\beta(\beta-1)\,\frac{\varepsilon}{r}\,(1+\delta r)\,\frac{\varepsilon s}{1+\varepsilon s}\\ &+ 2\alpha\,\delta^2(1+\varepsilon s)\,\frac{1}{\delta r} + (1-\beta^*+\beta^*)2\beta\,\frac{\varepsilon}{r}\,(1+\delta r)\Bigg\}\eta^{\alpha-1}_{\beta-1},\end{aligned}$$

for $r > 0$. We denote the five terms in $\{\ \}$ by T_1, $T_2, \ldots, T_5$, and overwrite the previous relation as $\Delta\eta^\alpha_\beta = \{[T_1 + T_4] + T_2 + [T_3 + (1-\beta^*)T_5] + \beta^* T_5\}\,\eta^{\alpha-1}_{\beta-1}$. Observing that $T_5 \geq 2\beta\varepsilon\delta$, the proposition is trivial. □

PROPOSITION 2.2. *Let* $\alpha \geq 0$, $\beta \geq 0$, $\delta > 0$, $\varepsilon > 0$ *and* $\kappa > 1$. *Then for* $\boldsymbol{x} \in \boldsymbol{R}^3$, $|\boldsymbol{x}| \geq |\delta^{-1} - (2\varepsilon)^{-1}|(\kappa-1)^{-1}$:

$$\Big|\nabla\eta^\alpha_\beta(\boldsymbol{x})\Big|^2 \leq 2\,\kappa\,\delta\,\varepsilon\,(\alpha+\beta)^2\Big(\eta^{\alpha-1/2}_{\beta-1/2}(\boldsymbol{x})\Big)^2. \tag{2.8}$$

Let $\alpha \geq 0$, $\beta \geq 0$, $\delta > 0$, $\varepsilon > 0$ *and* $(\beta-\alpha)(2\varepsilon-\delta) \geq 0$. *Then for* $\boldsymbol{x} \in \boldsymbol{R}^3$, $\boldsymbol{x} \neq 0$:

$$\Big|\nabla\eta^\alpha_\beta(\boldsymbol{x})\Big|^2 \leq (\alpha\delta + 2\beta\varepsilon)^2\Big(\eta^{\alpha-1/2}_{\beta-1/2}(\boldsymbol{x})\Big)^2. \tag{2.9}$$

PROOF. If $\beta = 0$ and $\alpha = 0$ then both inequalities (2.8) and (2.9) are valid. Let us concentrate on the nontrivial cases:

For $r > 0$, $s \in [0, 2r]$, we have that $\partial g/\partial s > 0$, where g is a function defined by relations:

$$\begin{aligned}\Big|\nabla\eta^\alpha_\beta(\boldsymbol{x})\Big|^2 &= g(s(\boldsymbol{x}), r(\boldsymbol{x}))\Big(\eta^{\alpha-1/2}_{\beta-1/2}(\boldsymbol{x})\Big)^2,\\ g(s,r) &\equiv \alpha^2\delta^2\left(\frac{1+\varepsilon s}{1+\delta r}\right) + 2\alpha\beta\delta\varepsilon\,\frac{s}{r} + 2\beta^2\varepsilon^2\left(\frac{1+\delta r}{1+\varepsilon s}\right)\frac{s}{r}.\end{aligned}$$

So, $g(s,r)$ is increasing as a function of s and

$$\begin{aligned}G(r) &\equiv \max_{s\in[0,2r]} g(s,r) = g(2r,r)\\ &= \alpha^2\delta^2\,\frac{1+2\varepsilon r}{1+\delta r} + 4\alpha\beta\delta\varepsilon + 4\beta^2\varepsilon^2\,\frac{1+\delta r}{1+2\varepsilon r} \leq 2\kappa(\alpha+\beta)^2\delta\varepsilon\end{aligned} \tag{2.10}$$

for $\kappa > 1$ and $r \geq |\delta^{-1} - (2\varepsilon)^{-1}|(\kappa-1)^{-1}$. So, inequality (2.8) is proved.

To justify the second inequality (2.9), we observe that for the given values of α, β, δ, ε and for $r > 0$, $G(r) \leq G(0)$. □

Next we derive an inequality of the Friedrichs-Poincaré type in the space $\overset{\circ}{\boldsymbol{H}}{}^1_{\alpha,\beta}$. It is necessary for our aim to get expressions of constants in this inequality. It follows from Proposition 2.1.

LEMMA 2.3. *Let $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta < 3$, $\kappa > 1$. Let δ and ε be arbitrary positive constants, such that $(\beta - \alpha)(2\varepsilon - \delta) \geq 0$. Then for all $\boldsymbol{u} \in \overset{\circ}{\boldsymbol{H}}{}^1_{\alpha,\beta}$*

$$\|\boldsymbol{u}\|^2_{2,\alpha-1,\beta-1} \leq c_0 \|\nabla \boldsymbol{u}| B_{R_0}\|^2_{2,\alpha,\beta} + c_1 \left\|\nabla \boldsymbol{u}| B^{R_0}\right\|^2_{2,\alpha,\beta}, \tag{2.11}$$

where $c_0 = [(\alpha\delta + 2\beta\varepsilon)/(\beta\beta^*\delta\varepsilon)]^2$, $c_1 = [(2\kappa)/(\delta\varepsilon)] \cdot [(\alpha+\beta)/(\beta\beta^*)]^2$ *and* $R_0 \geq |\delta^{-1} - (2\varepsilon)^{-1}|(\kappa - 1)^{-1}$.

REMARK 2.4. Let us observe that if additionally $\delta < 2\varepsilon$ and $1 < \kappa \leq 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$ then $c_0 \geq c_1$.

PROOF OF LEMMA 2.3. Due to the density of $\boldsymbol{C}_0^\infty$ in $\overset{\circ}{\boldsymbol{H}}{}^1_{\alpha,\beta}$ it is sufficient to prove the inequality for all $\boldsymbol{u} \in \boldsymbol{C}_0^\infty$. From Proposition 2.1 it follows that for $\boldsymbol{v} \in \boldsymbol{C}_0^\infty$

$$\begin{aligned}
2\beta\beta^*\delta\varepsilon \int_{\boldsymbol{R}^3\setminus B_\rho} \boldsymbol{v}^2 \eta^{\alpha-1}_{\beta-1}\, d\boldsymbol{x} &\leq \int_{\boldsymbol{R}^3\setminus B_\rho} \boldsymbol{v}^2 \Delta \eta^\alpha_\beta \, d\boldsymbol{x} \\
&= -2 \int_{\boldsymbol{R}^3\setminus B_\rho} (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\nabla\eta^\alpha_\beta\, d\boldsymbol{x} + \int_{\partial B_\rho} \boldsymbol{v}^2 \nabla\eta^\alpha_\beta \cdot \boldsymbol{n}\, dS \\
&\leq \beta\beta^*\delta\varepsilon \int_{\boldsymbol{R}^3\setminus B_\rho} \boldsymbol{v}^2\eta^{\alpha-1}_{\beta-1}\, d\boldsymbol{x} + \frac{1}{\beta\beta^*\delta\varepsilon}\int_{\boldsymbol{R}^3\setminus B_\rho} |\nabla \boldsymbol{v}|^2 \left|\nabla\eta^\alpha_\beta\right|^2 \eta^{-\alpha+1}_{-\beta+1}\, d\boldsymbol{x} \\
&\quad + \int_{\partial B_\rho} \boldsymbol{v}^2\, \nabla\eta^\alpha_\beta\cdot\boldsymbol{n}\, dS.
\end{aligned}$$

Hence, because the surface integral is a value of the order $O(\rho^2)$, we have:

$$\beta\beta^*\delta\varepsilon \int_{\boldsymbol{R}^3} \boldsymbol{v}^2 \eta^{\alpha-1}_{\beta-1}\, d\boldsymbol{x} \leq \frac{1}{\beta\beta^*\delta\varepsilon} \int_{\boldsymbol{R}^3} |\nabla\boldsymbol{v}|^2 \left|\nabla\eta^\alpha_\beta\right|^2 \eta^{-\alpha+1}_{-\beta+1}\, d\boldsymbol{x}. \tag{2.12}$$

By means of the Cauchy-Schwarz inequality and from Proposition 2.2 with $R_0 \geq |\delta^{-1} - (2\varepsilon)^{-1}|/(\kappa - 1)$ we finally get (2.11). □

We will need some technical lemmas. Let us define $F_{\alpha,\beta}(s,r)$ by the relation:

$$F_{\alpha,\beta}(s,r)\cdot\eta^{\alpha-1}_{\beta-1} \equiv -\nu\left|\nabla\eta^\alpha_\beta\right|^2/\eta^\alpha_\beta - k\,\partial_1\,\eta^\alpha_\beta. \tag{2.13}$$

The following lemma gives the evaluation of $F_{\alpha,\beta}(s,r)$ from below.

LEMMA 2.5. *Let $0 \leq \alpha < \beta$, $\kappa > 1$, $0 < \varepsilon \leq (1/(2\kappa))\cdot(k/\nu)\cdot((\beta-\alpha)/\beta^2)$ and δ, ν, $k > 0$. Then*

$$F_{\alpha,\beta}(s,r) - \big(1-\kappa^{-1}\big)k\delta\varepsilon(\beta-\alpha)s \geq -\alpha\delta k\big(1+\nu k^{-1}\alpha\delta\big) \tag{2.14}$$

for all $r > 0$ *and* $s \in [0, 2r]$.

Proof. Expressing the function $F_{\alpha,\beta}(s,r)$ explicitly we get:

$$F_{\alpha,\beta}(s,r) = -\nu\alpha^2\delta^2\left(\frac{1+\varepsilon s}{1+\delta r}\right) - 2\nu\alpha\beta\delta\varepsilon\,\frac{s}{r} - 2\nu\beta^2\varepsilon^2\left(\frac{1+\delta r}{1+\varepsilon s}\right)\frac{s}{r}$$
$$- k\alpha\delta(1+\varepsilon s)\,\frac{r-s}{r} + k\beta\varepsilon(1+\delta r)\,\frac{s}{r}\,.$$

For convenient use we subtract $(1-\kappa^{-1})k\delta\varepsilon(\beta-\alpha)s$ from $F_{\alpha,\beta}(s,r)$. We observe (see Appendix A) that, for the given α, β, ε, κ, for all δ, ν, $k > 0$ and for $r > 0$, $F_{\alpha,\beta}(s,r) - (1-\kappa^{-1})k\delta\varepsilon(\beta-\alpha)s \geq F_{\alpha,\beta}(0,r)$, which immediately gives inequality (2.14). □

3. Uniqueness in $\boldsymbol{R}^3$.

In this section, we will start with the question about the unique weak solvability of the problem (1.1)–(1.3) in $\Omega = \boldsymbol{R}^3$. The presented approach will be also used in the next section, in the proof of existence of a solution verifying solenoidality of the constructed function $\boldsymbol{u}$.

Theorem 3.1 (Uniqueness in $\boldsymbol{R}^3$). *Let* $\{\boldsymbol{u}, p\}$ *be a distributional solution of the problem* (1.1)–(1.3) *with* $\boldsymbol{f} = \boldsymbol{0}$, $g = 0$ *such that* $\boldsymbol{u} \in \widehat{\boldsymbol{H}}_0^{1,2}$ *and* $p \in L^2_{loc}$. *Then* $\boldsymbol{u} = \boldsymbol{0}$ *and* $p = const.$

Proof. From the condition $\boldsymbol{u} \in \widehat{\boldsymbol{H}}_0^{1,2}$ we get $\nabla\boldsymbol{u} \in \boldsymbol{L}^2$, $\boldsymbol{u} \in \boldsymbol{L}^6$, $\boldsymbol{u} \in \mathscr{S}'$. Because $\operatorname{div}((\omega\times\boldsymbol{x})\cdot\nabla\boldsymbol{u} - \omega\times\boldsymbol{u}) = (\omega\times\boldsymbol{x})\cdot\nabla\operatorname{div}\boldsymbol{u} = 0$, we have $\triangle p = 0$. Hence, applying Laplacian and the Fourier transform we get

$$\triangle(-\nu\,\Delta\boldsymbol{u} + k\,\partial_1\boldsymbol{u} - (\omega\times\boldsymbol{x})\cdot\nabla\boldsymbol{u} + \omega\times\boldsymbol{u}) = \boldsymbol{0},$$
$$|\xi|^2\Big(\nu\,|\xi|^2\widehat{\boldsymbol{u}} + i\,k\,\xi_1\widehat{\boldsymbol{u}} - (\omega\times\xi)\cdot\nabla_\xi\widehat{\boldsymbol{u}} + \omega\times\widehat{\boldsymbol{u}}\Big) = \boldsymbol{0} \quad \text{in } \mathscr{S}'.$$

Assuming the equation in cylindrical coordinates (ξ_1, ρ, φ), and denoting $T(\varphi)\,\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{u}}(\xi_1, \rho, \varphi)$, where

$$T(\varphi) = \begin{bmatrix} 1, & 0, & 0 \\ 0, & \cos(\varphi), & -\sin(\varphi) \\ 0, & \sin(\varphi), & \cos(\varphi) \end{bmatrix},$$

we get

$$|\xi|^2\Big\{-\partial_\varphi\widehat{\boldsymbol{v}}+[(\nu/\widetilde{\omega})|\xi|^2+i(k/\widetilde{\omega})\xi_1]\widehat{\boldsymbol{v}}\Big\}=\mathbf{0}\quad\text{in }\mathscr{S}'. \tag{3.15}$$

We will show that from this equation it follows that $\operatorname{supp}\widehat{\boldsymbol{v}}\subset\{0\}$, and due to the definition of $\widehat{\boldsymbol{v}}$ we will have also $\operatorname{supp}\widehat{\boldsymbol{u}}\subset\{\mathbf{0}\}$. This means that $\boldsymbol{u}$ is a polynomial of x_1, x_2, x_3. Because $\boldsymbol{u}\in\boldsymbol{L}^6$ we get $\boldsymbol{u}=0$. Substituting into (1.1) we get $\nabla p=0$ and $p=const.$

So, we have to prove that for an arbitrary real vector function $\Psi\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^3\setminus\{\mathbf{0}\})$ defined for $[\xi_1,\xi_2,\xi_3]\in\boldsymbol{R}^3$ we have $\langle\widehat{\boldsymbol{v}},\Psi\rangle=0$. If for each $\Psi\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^3\setminus\{\mathbf{0}\})$ there is a function $\Phi\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^3\setminus\{\mathbf{0}\})$ such that

$$\partial_\varphi\Big(|\xi|^2\Phi\Big)+\Big[(\nu/\widetilde{\omega})|\xi|^2+i\,(k/\widetilde{\omega})\xi_1\Big]\Big(|\xi|^2\Phi\Big)=\Psi \tag{3.16}$$

then from (3.15) it follows:

$$\begin{aligned}0&=\Big\langle|\xi|^2\{-\partial_\varphi\widehat{\boldsymbol{v}}+[(\nu/\widetilde{\omega})|\xi|^2+i\,(k/\widetilde{\omega})\xi_1]\widehat{\boldsymbol{v}}\},\ \Phi\Big\rangle\\&=\Big\langle\widehat{\boldsymbol{v}},\ \partial_\varphi(|\xi|^2\Phi)+[(\nu/\widetilde{\omega})|\xi|^2+i\,(k/\widetilde{\omega})\xi_1](|\xi|^2\Phi)\Big\rangle=\langle\widehat{\boldsymbol{v}},\ \Psi\rangle.\end{aligned}$$

Hence, the proof of $\operatorname{supp}\widehat{\boldsymbol{v}}\subset\{0\}$ is reduced to the solvability of (3.16). First we note that it is sufficient to solve the equation

$$\partial_\varphi\zeta+\Big((\nu/\widetilde{\omega})\,|\xi|^2+i\,(k/\widetilde{\omega})\,\xi_1\Big)\zeta=\Psi \tag{3.17}$$

because the division on the expression $|\xi|^2$ defines the one-to-one correspondence of the space $\boldsymbol{C}_0^\infty(\boldsymbol{R}^3\setminus\{\mathbf{0}\})$ onto $\boldsymbol{C}_0^\infty(\boldsymbol{R}^3\setminus\{\mathbf{0}\})$.

Let us analyze the equation (3.17) in cylindrical coordinates $[\xi_1,\rho,\varphi]$, where $\rho=(\xi_2^2+\xi_3^2)^{1/2}$. For an arbitrary real vector function $\Psi\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^3\setminus\{\mathbf{0}\})$ defined for $[\xi_1,\xi_2,\xi_3]\in\boldsymbol{R}^3$ we define $f(t):=\Psi(\xi_1,\,\rho\cos t,\,\rho\sin t)$, $a:=(\nu/\widetilde{\omega})|\xi|^2+i(k/\widetilde{\omega})\xi_1$, assuming $\widetilde{\omega}>0$.

Now, we will use the following technical proposition about the existence of a solution of an ordinary differential equation in a space of periodical functions (and later also in the proof of existence of a solution of the problem for checking solenoidality of a constructed solution, see the proof of Theorem 4.4):

Proposition 3.2. *Let $a\in\boldsymbol{C}$, $Re\,a>0$. Let $f\in C^\infty(\boldsymbol{R})$ be a 2π-periodical complex function. Then there is unique 2π-periodical solution $g\in C^\infty(\boldsymbol{R})$ of the equation*

$$g' + a\,g = f$$

and the solution g can be expressed in the following form:

$$g(\varphi) = \big(\mathrm{e}^{2\pi a} - 1\big)^{-1} \int_0^{2\pi} e^{at} f(\varphi + t)\,dt = e^{-a\varphi} \int_{-\infty}^{\varphi} \mathrm{e}^{at} f(t)\,dt.$$

Proof of the proposition follows from standard computations.

Using the Proposition 3.2 we get the solution of (3.17) in the form

$$\zeta(\xi_1, \rho, \varphi) = \left\{ \exp\left[2\pi \left(\frac{\nu}{\tilde{\tilde{\omega}}} |\xi|^2 + i\,\frac{k}{\tilde{\tilde{\omega}}} \right) \right] - 1 \right\}^{-1}$$
$$\cdot \int_0^{2\pi} \exp\left[\left(\frac{\nu}{\tilde{\tilde{\omega}}} |\xi|^2 + i\,\frac{k}{\tilde{\tilde{\omega}}}\xi_1 \right) t \right] \Psi(\xi_1,\, \rho\,\cos(t+\varphi),\, \rho\,\sin(t+\varphi))\,dt.$$

It is easy to see that function ζ as the function of $[\xi_1, \xi_2, \xi_3]$ is infinitely differentiable with respect to these variables and $\zeta \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^3 \setminus \{\boldsymbol{0}\})$. Finally we put $\Phi = \zeta/|\xi|^2$. □

4. Existence of a solution in $\boldsymbol{R}^3$.

In this section we will construct a weak solution of the problem (1.1)–(1.3).

4.1. Existence of the pressure in $\boldsymbol{R}^3$.

If there exist distributions $\boldsymbol{u}, p$ satisfying

$$-\nu\,\Delta \boldsymbol{u} + k\,\partial_1 \boldsymbol{u} - (\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} + \omega \times \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \boldsymbol{R}^3$$
$$\operatorname{div} \boldsymbol{u} = g \quad \text{in } \boldsymbol{R}^3$$

then pressure p satisfies the equation

$$\triangle\, p = \operatorname{div} \boldsymbol{F}, \quad \text{where } \boldsymbol{F} = \boldsymbol{f} + \nu \nabla g - kg\,\boldsymbol{e}_1 + g(\omega \times \boldsymbol{x}), \tag{4.18}$$

because $\operatorname{div}((\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \omega \times \boldsymbol{u}) = (\omega \times \boldsymbol{x}) \cdot \nabla \operatorname{div} \boldsymbol{u} = \operatorname{div}[g(\omega \times \boldsymbol{x})]$.

Let $\mathscr{E}$ be the fundamental solution of the Laplace equation, i.e. $\mathscr{E} = -1/(4\pi r)$. Assuming firstly $\boldsymbol{F} \in \boldsymbol{C}_0^\infty$ we have $p = \mathscr{E} \star \operatorname{div} \boldsymbol{F}$ and $\nabla p = \nabla \mathscr{E} \star \operatorname{div} \boldsymbol{F}$ and so, $p = \nabla \mathscr{E} \star \boldsymbol{F}$ and $\nabla p = \nabla^2 \mathscr{E} \star \boldsymbol{F}$. It is well known that both formulas can be extended for $\boldsymbol{F} \in \boldsymbol{L}^2_{\alpha+1,\beta}$ with $0 < \beta < 1$ and $-2 < \alpha + \beta < 2$ (the last convolution $\nabla p = \nabla^2 \mathscr{E} \star \boldsymbol{F}$ due to the fact that $\nabla^2 \mathscr{E}$ is the singular kernel of the Calderon-Zygmund type and that $\eta_\beta^{\alpha+1}$ belongs to the Muckenhoupt class of weights A_2), see [**2**, Theorem 3.2, Theorem 5.5], [**26**, Theorem 4.4, Theorem 5.4], where the theorems are formulated for the pressure part $\mathscr{P}$ of the fundamental solution of the classical Oseen problem, so $\mathscr{P} = \nabla\mathscr{E}$ and $\nabla\mathscr{P} = \nabla^2\mathscr{E}$. For $\boldsymbol{F} \in \boldsymbol{L}^2_{\alpha+1,\beta}$ we get $p \in L^2_{\alpha,\beta-1}$ and $\nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}$, and there are positive constants C_1, C_2 such that the following estimates are satisfied:

$$\|p\|_{2,\alpha,\beta-1} \le C_1\|\boldsymbol{F}\|_{2,\alpha+1,\beta}, \quad \|\nabla p\|_{2,\alpha+1,\beta} \le C_2\|\boldsymbol{F}\|_{2,\alpha+1,\beta} \tag{4.19}$$

REMARK. Another possibility of construction of the pressure is the use of Hörmander-Michlin multiplier theorem. Both techniques can be used in L^2- as well as in L^q-framework to get an estimate of ∇p.

4.2. The problem in B_R.

We will study in this section the existence of a weak solution of the following problem in a bounded domain B_R, pressure p is assumed here to be known, the right hand side $\boldsymbol{f} - \nabla p = \widetilde{\boldsymbol{f}} \in \boldsymbol{L}^2_{\alpha+1,\beta}$:

$$-\nu\Delta\boldsymbol{u} + k\,\partial_1\boldsymbol{u} - (\omega\times\boldsymbol{x})\cdot\nabla\boldsymbol{u} + \omega\times\boldsymbol{u} = \widetilde{\boldsymbol{f}} \quad \text{in } B_R \tag{4.20}$$

$$\boldsymbol{u} = 0 \quad \text{on } \partial B_R. \tag{4.21}$$

We show the existence of a weak solution $\boldsymbol{u}_R \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$. Following (1.5), (1.6) again with $w = \eta^0_{\beta_0}$, $\beta_0 \in (0,1]$, using notation (2.13), let us introduce a continuous bilinear form $\widetilde{Q}(\cdot,\cdot)$ on $\overset{\circ}{\boldsymbol{H}}{}^1(B_R) \times \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$:

$$\widetilde{Q}(\boldsymbol{u},\boldsymbol{v}) = \nu\int_{B_R}\nabla\boldsymbol{u} : \nabla\Big(\boldsymbol{v}\cdot\eta^0_{\beta_0}\Big)\,d\boldsymbol{x} + k\int_{B_R}\partial_1\boldsymbol{u}\cdot\Big(\boldsymbol{v}\eta^0_{\beta_0}\Big)\,d\boldsymbol{x}$$
$$+ \int_{B_R}(\omega\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}\Big(\boldsymbol{v}\eta^0_{\beta_0}\Big)\,d\boldsymbol{x} + \int_{B_R}(\omega\times\boldsymbol{u})\cdot\Big(\boldsymbol{v}\eta^0_{\beta_0}\Big)\,d\boldsymbol{x},$$

$$\widetilde{Q}(\boldsymbol{v},\boldsymbol{v}) \ge 2^{-1}\nu\int_{B_R}|\nabla\boldsymbol{v}|^2\eta^0_{\beta_0}\,d\boldsymbol{x} + 2^{-1}\int_{B_R}\boldsymbol{v}^2F_{0,\beta_0}(s,r;\,\nu)\eta^{-1}_{\beta_0-1}\,d\boldsymbol{x}. \tag{4.22}$$

LEMMA 4.1. *Let $0 < \beta_0 \le 1$. Then, for all $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^2_{1,\beta_0}(B_R)$, $\varepsilon_0 < (1/2)\cdot(k/\nu)\cdot(1/\beta_0)$, $\eta^\alpha_{\beta_0} \equiv \eta^{\alpha,\varepsilon_0}_{\beta_0,\varepsilon_0}$, there exists unique $\boldsymbol{u}_R \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$ such that for all $\boldsymbol{v} \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$.*

$$\widetilde{Q}(\boldsymbol{u}_R,\boldsymbol{v}) = \int_{B_R}\widetilde{\boldsymbol{f}}\cdot\boldsymbol{v}\eta^0_{\beta_0}\,d\boldsymbol{x}. \tag{4.23}$$

PROOF. Bilinear form $\widetilde{Q}$ is coercive, i.e. there exists a constant $C_R > 0$ such that $\widetilde{Q}(\boldsymbol{v},\boldsymbol{v}) \ge C_R\,\|\boldsymbol{v}\|^2$, where $\|\cdot\|$ is the norm in the space $\overset{\circ}{\boldsymbol{H}}{}^1(B_R)$. Indeed, we get

$$\widetilde{Q}(\boldsymbol{v},\boldsymbol{v}) \geq \frac{\nu}{2}\int_{B_R} |\nabla \boldsymbol{v}|^2 \eta^0_{\beta_0}\, d\boldsymbol{x} + \frac{1}{2}\int_{B_R} \boldsymbol{v}^2 F_{0,\beta_0}(s,r)\, \eta^{-1}_{\beta_0-1}\, d\boldsymbol{x}$$

Because $\varepsilon_0 < (1/2)\cdot(k/\nu)\cdot(1/\beta_0)$ there is a constant κ satisfying all previous conditions and additionally $\varepsilon_0 \leq (1/2\kappa)\cdot(k/\nu)\cdot(1/\beta_0)$. Because $\alpha = 0$ we get from Lemma 2.5

$$\int_{B_R} \boldsymbol{v}^2 F_{0,\beta_0}(s,r)\, \eta^{-1}_{\beta_0-1}\, d\boldsymbol{x} \geq \big(1-\kappa^{-1}\big) k\varepsilon_0^2\beta_0 \int_{B_R} \boldsymbol{v}^2 \eta^{-1}_{\beta_0-1} s\, d\boldsymbol{x},$$

$$\widetilde{Q}(\boldsymbol{v},\boldsymbol{v}) \geq \frac{\nu}{2}\int_{B_R} |\nabla \boldsymbol{v}|^2 \eta^0_{\beta_0}\, d\boldsymbol{x} + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\varepsilon_0\beta_0 \int_{B_R} \boldsymbol{v}^2 \eta^{-1}_{\beta_0-1}(\varepsilon_0 s)\, d\boldsymbol{x}.$$

Using Lemma 2.3 and Remark 2.4 we derive:

$$\begin{aligned}
\widetilde{Q}(\boldsymbol{v},\boldsymbol{v}) &\geq \frac{\nu}{4}\int_{B_R} |\nabla \boldsymbol{v}|^2 \eta^0_{\beta_0}\, d\boldsymbol{x} + \frac{\nu}{16}\varepsilon_0^2\beta_0^2 \int_{B_R} \boldsymbol{v}^2 \eta^{-1}_{\beta_0-1}\, d\boldsymbol{x} \\
&\quad + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\varepsilon_0\beta_0 \int_{B_R} \boldsymbol{v}^2 \eta^{-1}_{\beta_0-1}(\varepsilon_0 s)\, d\boldsymbol{x} \\
&\geq \left(1-\frac{1}{\kappa}\right)\frac{\nu}{4}\min\left\{1,\, \frac{1}{4}\varepsilon_0^2\beta_0^2,\, 2\,\frac{k}{\nu}\,\beta_0\varepsilon_0\right\} \\
&\qquad \cdot \left(\int_{B_R} |\nabla v|^2 \eta^0_{\beta_0}\, d\boldsymbol{x} + \int_{B_R} \boldsymbol{v}^2 \eta^{-1}_{\beta_0}\, d\boldsymbol{x}\right)
\end{aligned} \tag{4.24}$$

$$\widetilde{Q}(\boldsymbol{v},\boldsymbol{v}) \geq C_R\left(\int_{B_R} |\nabla \boldsymbol{v}|^2\, d\boldsymbol{x} + \int_{B_R} \boldsymbol{v}^2\, d\boldsymbol{x}\right) = C_R \|\boldsymbol{v}\|^2, \tag{4.25}$$

where $C_R = (\nu/4)\cdot(1-\kappa^{-1})\cdot\min\{1,\, \varepsilon_0^2\beta_0^2/4,\, 2(k/\nu)\beta\varepsilon_0\}\cdot(1+\varepsilon_0 R)$. Using Lax-Milgram theorem we get that there is $\boldsymbol{u}_R \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$ such that (4.23) is satisfied. □

Remark 4.2. An arbitrary function $\boldsymbol{\Phi} \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$ can be expressed in the form $\boldsymbol{\Phi} = \boldsymbol{v}\,\eta^0_{\beta_0}$, where $\boldsymbol{v} \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$. Therefore for all $\boldsymbol{\Phi} \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$

$$Q(\boldsymbol{u}_R,\boldsymbol{\Phi}) = \int_{B_R} \widetilde{\boldsymbol{f}}\cdot\boldsymbol{\Phi}\, d\boldsymbol{x}, \tag{4.26}$$

where by the definition $Q(\boldsymbol{u}_R,\boldsymbol{\Phi}) \equiv Q(\boldsymbol{u}_R,\boldsymbol{v}\eta^0_{\beta_0}) \equiv \widetilde{Q}(\boldsymbol{u}_R,\boldsymbol{v})$.

4.3. Uniform estimates of $\boldsymbol{u_R}$ in $\boldsymbol{R}^3$.

Our next aim is to prove that the weak solutions $\boldsymbol{u}_R$ of (4.23) are uniformly bounded in $\boldsymbol{V}_{\alpha,\beta}$ as $R \to +\infty$.

Let y_1 be the unique real solution of the algebraic equation $4y^3 + 8y^2 + 5y - 1 = 0$. It is easy to verify that $y_1 \in (0,1)$. We will explain later, why the control of α/β by y_1 is necessary.

LEMMA 4.3. *Let $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$, $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^2_{\alpha+1,\beta}$. Then, as $R \to +\infty$, the weak solutions $\boldsymbol{u}_R$ of (4.23) given by Lemma 4.1 are uniformly bounded in $\boldsymbol{V}_{\alpha,\beta}$. There is a constant $c > 0$, which does not depend on R such that*

$$\int_{\boldsymbol{R}^3} \tilde{\boldsymbol{u}}_R^2 \eta_\beta^{\alpha-1}\, d\boldsymbol{x} + \int_{\boldsymbol{R}^3} |\nabla \tilde{\boldsymbol{u}}_R|^2 \eta_\beta^\alpha\, d\boldsymbol{x} \le c \int_{\boldsymbol{R}^3} \left|\widetilde{\boldsymbol{f}}\right|^2 \eta_\beta^{\alpha+1}\, d\boldsymbol{x} \tag{4.27}$$

for all R greater than some $R_0 > 0$, $\tilde{\boldsymbol{u}}_R$ being extension by zero of $\boldsymbol{u}_R$ on $\boldsymbol{R}^3 \setminus B_R$.

PROOF. First, we derive estimate of $\boldsymbol{u}_R$ on a bounded subdomain $B_{R_0} \subset B_R$; The choice of R_0 will be given in the next part of the proof. Our aim is to get an estimate with a constant not depending on R. Let us substitute $\boldsymbol{v} = \boldsymbol{u}_R$ into (4.23). Hence, we get from (4.24):

$$\widetilde{Q}(\boldsymbol{u}_R, \boldsymbol{u}_R) = \int_{B_R} \widetilde{\boldsymbol{f}}\, \boldsymbol{u}_R \eta_{\beta_0}^0\, d\boldsymbol{x} \ge C_1 \left(\int_{B_R} |\nabla \boldsymbol{u}_R|^2 \eta_{\beta_0}^0\, d\boldsymbol{x} + \int_{B_R} \boldsymbol{u}_R^2 \eta_{\beta_0}^{-1}\, d\boldsymbol{x} \right),$$

with the constant $C_1 > 0$ stated in (4.24). Let R_0 be some fixed positive number such that $0 < R_0 < R$. We get

$$\int_{B_{R_0}} |\nabla \boldsymbol{u}_R|^2 \eta_\beta^\alpha\, d\boldsymbol{x} + \int_{B_{R_0}} \boldsymbol{u}_R^2 \eta_\beta^{\alpha-1}\, d\boldsymbol{x} \le C_2 \int_{B_R} \left|\widetilde{\boldsymbol{f}}\right| |\boldsymbol{u}_R| \eta_\beta^\alpha\, d\boldsymbol{x}, \tag{4.28}$$

where the constant $C_2 = C_1^{-1}(1 + \varepsilon_0\ R_0)^\alpha (1 + \varepsilon_0\ 2\,R_0)^{|\beta-\beta_0|}$ depends on k, ν, α, β, β_0, ε_0, R_0, κ, but does not depend on R.

Now, we are going to derive an estimate of $\boldsymbol{u}_R$ on domain B_R. Using the test function $\boldsymbol{\Phi} = \boldsymbol{u}_R \eta_\beta^\alpha = \boldsymbol{u}_R (1 + \delta r)^\alpha (1 + \varepsilon s)^\beta \in \overset{\circ}{\boldsymbol{H}}{}^1(B_R)$ in (4.26) we get after integration by parts:

$$\begin{aligned} \nu \int_{B_R} |\nabla \boldsymbol{u}_R|^2 \eta_\beta^\alpha\, d\boldsymbol{x} &+ \nu \int_{B_R} (\boldsymbol{u}_R \cdot \nabla \boldsymbol{u}_R) \cdot \nabla \eta_\beta^\alpha\, d\boldsymbol{x} - \frac{k}{2} \int_{B_R} \boldsymbol{u}_R^2\, \partial_1 \eta_\beta^\alpha\, d\boldsymbol{x} \\ &= \int_{B_R} \widetilde{\boldsymbol{f}} \boldsymbol{u}_R \eta_\beta^\alpha\, d\boldsymbol{x}. \end{aligned}$$

So, we get for some $\kappa > 1$:

$$\frac{\nu}{2}\int_{B_R}|\nabla \boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}+\frac{1}{2}\int_{B_R}\boldsymbol{u}_R^2F_{\alpha,\beta}(s,r)\eta_{\beta-1}^{\alpha-1}\,d\boldsymbol{x}\le\int_{B_R}\left|\widetilde{\boldsymbol{f}}\right||\boldsymbol{u}_R|\eta_\beta^\alpha\,d\boldsymbol{x}.$$

Let $R_0\ge|\delta^{-1}-(2\varepsilon)^{-1}|(\kappa-1)^{-1}$. Using Lemma 2.5 (with $0\le\alpha<\beta$, $\varepsilon\le(1/(2\kappa))(k/\nu)((\beta-\alpha)/\beta^2))$ and Lemma 2.3 (with $\delta<2\varepsilon$), the second term in the previous estimate can be evaluated from below:

$$\begin{aligned}&\int_{B_R}\boldsymbol{u}_R^2\,F_{\alpha,\beta}(s,r))\eta_{\beta-1}^{\alpha-1}\,d\boldsymbol{x}\\&\quad\ge-\alpha\delta k\left(1+\frac{\nu\kappa}{k}\alpha\delta\right)\frac{2\kappa}{\delta\varepsilon}\left(\frac{\alpha+\beta}{\beta\beta^*}\right)^2\int_{B_R^{R_0}}|\nabla\boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}\\&\qquad+\left(1-\kappa^{-1}\right)k\delta\varepsilon(\beta-\alpha)\int_{B_R^{R_0}}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}s\,d\boldsymbol{x}-2C_4\int_{B_{R_0}}|\nabla\boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}.\end{aligned}$$

Denote $C_5=\alpha\delta k(1+\nu\kappa\alpha\delta/k)(\kappa/(\delta\,\varepsilon))((\alpha+\beta)/(\beta\beta^*))^2$. It is clear that $C_5\le\nu/(2\,\kappa^2)<\nu/(2\,\kappa)$ if $1+\nu\kappa\alpha\delta/k\le\kappa$ (i.e. $\delta\le(k/\nu)\cdot((\kappa-1))/(\kappa\beta))$ and $\alpha\le(1/(2\kappa^4))\cdot(\nu/k)\cdot((\beta\,\beta^*)/(\alpha+\beta))^2\varepsilon$. We have

$$\begin{aligned}&\frac{\nu}{2\kappa}\int_{B_R}|\nabla\boldsymbol{u}_R|^2\;\eta_\beta^\alpha\;d\boldsymbol{x}+\frac{1}{2}\left(1-\frac{1}{\kappa}\right)k\delta\varepsilon(\beta-\alpha)\int_{B_R}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}\,s\,d\boldsymbol{x}\\&\quad-C_6\int_{B_{R_0}}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}\,d\boldsymbol{x}-C_7\int_{B_{R_0}}|\nabla\boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}\le\int_{B_R}\left|\widetilde{\boldsymbol{f}}\right||\boldsymbol{u}_R|\eta_\beta^\alpha\,d\boldsymbol{x}.\end{aligned}$$

We use now relation (4.28) in order to estimate the integrals computed on the domain B_{R_0}. Before using the mentioned inequality we should re-scale it with respect to new values ε,δ, see Remark 1.1. The new constant in (4.28) after rescaling we denote C_2'.

$$\frac{\nu}{\kappa}\int_{B_R}|\nabla\boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}+k\delta\varepsilon(\beta-\alpha)\int_{B_R}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}s\,d\boldsymbol{x}\le C_8\int_{B_R}\left|\widetilde{\boldsymbol{f}}\right||\boldsymbol{u}_R|\eta_\beta^\alpha\,d\boldsymbol{x},$$

where $C_8=\{1+C_2'\max(C_6,\,C_7)\}\cdot2\cdot(1-\kappa^{-1})^{-1}$. We use Lemma 2.3 and Remark 2.4. So, if $\delta<2\varepsilon$ and $1<\kappa\le2\varepsilon/\delta+\delta/(2\varepsilon)-1$ we get

$$\frac{\nu}{2\,\kappa}\left(\frac{\beta\,\beta^*\delta\,\varepsilon}{\alpha\delta+2\beta\varepsilon}\right)^2\int_{B_R}\boldsymbol{u}_R^2\,\eta_{\beta-1}^{\alpha-1}\,d\boldsymbol{x}\le\frac{\nu}{2\kappa}\int_{B_R}|\nabla\boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x},$$

$$\frac{\nu}{2\kappa}\int_{B_R}|\nabla \boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}+\frac{\nu}{2\kappa}\left(\frac{\beta\,\beta^*\delta\,\varepsilon}{\alpha\delta+2\beta\varepsilon}\right)^2\int_{B_R}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}\,d\boldsymbol{x}$$
$$+\,k\delta\varepsilon(\beta-\alpha)\int_{B_R}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}s\,d\boldsymbol{x}\le C_8\int_{B_R}\left|\widetilde{\boldsymbol{f}}\right||\boldsymbol{u}_R|\eta_\beta^\alpha\,d\boldsymbol{x}.$$

So we get

$$\int_{B_R}|\nabla \boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}+2\int_{B_R}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}\,d\boldsymbol{x}+2\varepsilon\int_{B_R}\boldsymbol{u}_R^2\eta_{\beta-1}^{\alpha-1}s\,d\boldsymbol{x}$$
$$=\int_{B_R}|\nabla \boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}+2\int_{B_R}\boldsymbol{u}_R^2\eta_\beta^{\alpha-1}\,d\boldsymbol{x}\le C_{10}\int_{B_R}\left|\widetilde{\boldsymbol{f}}\right||\boldsymbol{u}_R|\eta_\beta^\alpha\,d\boldsymbol{x},$$

$C_9=\min(\nu/(2\kappa),(\nu/(2\kappa))\ (\beta\beta^*\delta\varepsilon/(\alpha\delta+2\beta\varepsilon))^2,\ k\delta(\beta-\alpha)/2)$ and $C_{10}=C_8/C_9$. We have also:

$$\int_{B_R}\left|\widetilde{\boldsymbol{f}}\right||\boldsymbol{u}_R|\eta_\beta^\alpha\,d\boldsymbol{x}\le\frac{t}{2}\int_{B_R}\boldsymbol{u}_R^2\eta_\beta^{\alpha-1}\,d\boldsymbol{x}+\frac{1}{2t}\int_{B_R}\left|\widetilde{\boldsymbol{f}}\right|^2\eta_\beta^{\alpha+1}\,d\boldsymbol{x}$$

So, if we choose $t=2\cdot C_{10}^{-1}$ then we get:

$$\int_{B_R}|\nabla \boldsymbol{u}_R|^2\eta_\beta^\alpha\,d\boldsymbol{x}+\int_{B_R}\boldsymbol{u}_R^2\,\eta_\beta^{\alpha-1}\,d\boldsymbol{x}\le c\int_{\boldsymbol{R}^3}\left|\widetilde{\boldsymbol{f}}\right|^2\eta_\beta^{\alpha+1}\,d\boldsymbol{x}.$$

It can be easily shown that the all conditions on α, β, δ, ε, κ used in the proof are compatible if $0\le\alpha<y_1\beta$, see Appendix B. □

4.4. The problem in $\boldsymbol{R}^3$.

Let y_1 be the same as in Lemma 4.3.

THEOREM 4.4 (Existence and uniqueness in $\boldsymbol{R}^3$). *Let $0<\beta\le1$, $0\le\alpha<y_1\beta$, $\boldsymbol{f}\in\boldsymbol{L}^2_{\alpha+1,\beta}$, $g\in H^1_{loc}$ such that $\nu\nabla g-kg\,\boldsymbol{e}_1+g(\omega\times\boldsymbol{x})\in L^2_{\alpha+1,\beta}$. Then there exists a unique weak solution $\{\boldsymbol{u},\,p\}$ of the problem*

$$-\nu\,\Delta\boldsymbol{u}+k\partial_1\boldsymbol{u}-(\omega\times\boldsymbol{x})\cdot\nabla\boldsymbol{u}+\omega\times\boldsymbol{u}+\nabla\boldsymbol{p}=\boldsymbol{f}\quad\text{in }\boldsymbol{R}^3,\tag{4.29}$$

$$\operatorname{div}\boldsymbol{u}=g\quad\text{in }\boldsymbol{R}^3\tag{4.30}$$

such that $\boldsymbol{u}\in\boldsymbol{V}_{\alpha,\beta}$, $p\in L^2_{\alpha,\beta-1}$, $\nabla p\in\boldsymbol{L}^2_{\alpha+1,\beta}$ and

$$\|\boldsymbol{u}\|_{2,\alpha-1,\beta}+\|\nabla\boldsymbol{u}\|_{2,\alpha,\beta}+\|p\|_{2,\alpha,\beta-1}+\|\nabla p\|_{2,\alpha+1,\beta}$$
$$\le C\Big(\|\boldsymbol{f}\|_{2,\alpha+1,\beta}+\|\nu\nabla g-kg\,\boldsymbol{e}_1+g(\omega\times\boldsymbol{x})\|_{2,\alpha+1,\beta}\Big).\tag{4.31}$$

Proof. The uniqueness of the solution follows from Theorem 3.1, and we now justify the existence. Let p be the same as in Subsection 4.1. Let $\{R_n\}$ be a sequence of positive real numbers, converging to $+\infty$. Let $\boldsymbol{u}_{R_n}$ be the weak solution of (4.20), (4.21) on B_{R_n}. Extending $\boldsymbol{u}_{R_n}$ by zero on $\boldsymbol{R}^3 \setminus B_{R_n}$ to a function $\tilde{\boldsymbol{u}}_{\boldsymbol{n}} \in \boldsymbol{V}_{\alpha,\beta}$ we get a bounded sequence $\{\tilde{\boldsymbol{u}}_{\boldsymbol{n}}\}$ in $\boldsymbol{V}_{\alpha,\beta}$. Thus, there is a subsequence $\tilde{\boldsymbol{u}}_{\boldsymbol{n}_k}$ of $\tilde{\boldsymbol{u}}_{\boldsymbol{n}}$ with a weak limit $\boldsymbol{u}$ in $\boldsymbol{V}_{\alpha,\beta}$. Obviously, $\boldsymbol{u}$ is a weak solution of (4.29) and

$$\begin{aligned}\|\boldsymbol{u}\|^2_{2,\alpha-1,\beta} + \|\nabla \boldsymbol{u}\|^2_{2,\alpha,\beta} &\le \liminf_{k\in\boldsymbol{N}} \left(\int_{\boldsymbol{R}^3} \tilde{\boldsymbol{u}}^2_{n_k} \eta_\beta^{\alpha-1}\, d\boldsymbol{x} \; + \; \int_{\boldsymbol{R}^3} |\nabla \tilde{\boldsymbol{u}}_{n_k}|^2 \eta_\beta^{\alpha}\, d\boldsymbol{x} \right) \\ &\le c \left|\widetilde{\boldsymbol{f}}\right|^2 \eta_\beta^{\alpha+1}\, d\boldsymbol{x} = c \int_{\boldsymbol{R}^3} |\boldsymbol{f} - \nabla p|^2\, \eta_\beta^{\alpha+1}\, d\boldsymbol{x}.\end{aligned}$$

Taking into account also relation (4.19) we get (4.31).

Let us also check that for $\boldsymbol{u}$ the equation (4.30) is satisfied. Let us mention that $\boldsymbol{u} \in \boldsymbol{H}^2_{loc}$ because $\boldsymbol{f} - \nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}$. So, computing the divergence of (4.29), we get

$$-\,\nu\,\Delta(\operatorname{div}\boldsymbol{u}) + k\partial_1(\operatorname{div}\boldsymbol{u}) - (\omega \times \boldsymbol{x})\cdot\nabla(\operatorname{div}\boldsymbol{u}) = \operatorname{div}\boldsymbol{f} - \triangle p \tag{4.32}$$

in the distributional sense. From (4.18) we have

$$-\nu\,\Delta\gamma + k\partial_1\gamma - (\omega\times\boldsymbol{x})\cdot\nabla\gamma = 0$$

for $\gamma = \operatorname{div}\boldsymbol{u} - g \in L^2_{\alpha,\beta} \subset L^2$. Using Fourier transform we get

$$\Big(\nu\,|\xi|^2 + i\,k\,\xi_1\Big)\widehat{\gamma} - (\omega\times\xi)\cdot\nabla_\xi\widehat{\gamma} = 0 \quad \text{in } \mathscr{S}'.$$

Assuming $\widehat{\gamma}$ in cylindrical coordinates $[\xi_1, \rho, \varphi]$, $\rho = (\xi_2^2 + \xi_3^2)^{1/2}$, we can overwrite the equation in the form:

$$-\partial_\varphi\widehat{\gamma} + \Big[(\nu/\widetilde{\omega})\,|\xi|^2 + i\,(k/\widetilde{\omega})\,\xi_1\Big]\widehat{\gamma} = 0.$$

Using the same approach as in the proof of the uniqueness Theorem 3.1 we prove that $\operatorname{supp}\widehat{\gamma} \subset \{0\}$. The proof of this fact is reduced to the solvability of the equation (3.17) which was proved for arbitrary $\Psi \in C_0^\infty(\boldsymbol{R}^3 \setminus \{0\})$ in the proof of Theorem 3.1. So, by the same procedure we derive that γ is a polynomial in $\boldsymbol{R}^3$ and because $\gamma \in L^2$ we get $\gamma \equiv 0$, i.e. (4.30). □

5. Uniqueness in an exterior domain $\boldsymbol{\Omega \subset R^3}$.

The last two sections are devoted to the problem in an exterior domain. We start with the question of uniqueness. The uniqueness theorem proved in this section together with the uniqueness theorem in $\boldsymbol{R}^3$ from Section 3 will be used in the next section in the proof of the existence of a solution in an exterior domain, in the localization procedure. The homogenous Dirichlet boundary condition on $\partial\Omega$ for $\boldsymbol{u}$ in the next theorem follows from the assumption $\boldsymbol{u} \in \boldsymbol{V}_{0,0}(\Omega)$.

THEOREM 5.1. *Let $\{\boldsymbol{u}, p\}$ be a distributional solution of the problem* (1.1)–(1.3) *with $\boldsymbol{f} = \boldsymbol{0}$ and $g = 0$ such that $\boldsymbol{u} \in \boldsymbol{V}_{0,0}(\Omega)$ and $p \in L^2_{-1,0}(\Omega)$. Then $\boldsymbol{u} = \boldsymbol{0}$ and $p = 0$.*

PROOF. Let $\Phi = \Phi(z) \in C_0^\infty(\langle 0, +\infty))$ be a non-increasing cut-off function such that $\Phi(z) \equiv 1$ for $z < 1/2$ and $\Phi(z) \equiv 0$ for $z > 1$. Let $|\Phi'| \le 3$. Let $\Phi_R \equiv \Phi_R(\boldsymbol{x}) \equiv \Phi(|\boldsymbol{x}|/R)$. We have $|\nabla\Phi_R| \le 3/R$ and $|\partial_1\Phi_R| \le 3/R$ for $\boldsymbol{x} \in \boldsymbol{R}^3$, $R/2 \le |\boldsymbol{x}| \le R$. Let $\{R_j\} \in \boldsymbol{R}$ be an increasing sequence of radii with the limit $+\infty$. So we have that $\boldsymbol{u}_j \equiv \boldsymbol{u} \cdot \Phi_{R_j} \in \overset{\circ}{\boldsymbol{H}}{}^1(\Omega)$, and $\{\boldsymbol{u}_j\}$ is a sequence of functions with limit $\boldsymbol{u}$ in the space $\boldsymbol{V}_{0,0}(\Omega)$. Using the (non-solenoidal) test functions $\varphi = \boldsymbol{u}\, \Phi^2_{R_j} = \boldsymbol{u}_j\, \Phi_{R_j} \in \overset{\circ}{\boldsymbol{H}}{}^1(\Omega)$ for equation (1.1) we get:

$$\nu \int_\Omega \nabla\boldsymbol{u} : \nabla\Big(\boldsymbol{u}\, \Phi^2_{R_j}\Big)\, d\boldsymbol{x} + k\int_\Omega \partial_1\boldsymbol{u} \cdot \boldsymbol{u}\, \Phi^2_{R_j}\, d\boldsymbol{x} + \int_\Omega (\omega \times \boldsymbol{x}) \cdot \nabla\boldsymbol{u} \cdot \boldsymbol{u}\, \Phi^2_{R_j}\, d\boldsymbol{x} + \int_\Omega \nabla p \cdot \boldsymbol{u}\, \Phi^2_{R_j}\, d\boldsymbol{x} = 0. \tag{5.33}$$

Using in (5.33) relation $\nabla\boldsymbol{u} : \nabla(\boldsymbol{u}\,\Phi^2_{R_j}) = |\nabla\boldsymbol{u}_j|^2 - \nabla\Phi_{R_j} \cdot \nabla\Phi_{R_j}\, |\boldsymbol{u}|^2$, integrating by parts, we get after some evident rearrangements

$$\begin{aligned}
&\nu \int_\Omega \big|\nabla\boldsymbol{u}_j\big|^2\, d\boldsymbol{x} - \frac{1}{2}\int_\Omega \operatorname{div}(\omega \times \boldsymbol{x})\big|\boldsymbol{u}_j\big|^2\, d\boldsymbol{x} \\
&- \frac{k}{2}\int_\Omega |\boldsymbol{u}|^2\, \partial_1\Phi^2_{R_j} d\boldsymbol{x} - \frac{1}{2}\int_\Omega |\boldsymbol{u}|^2 (\omega \times \boldsymbol{x}) \cdot \nabla\Phi^2_{R_j} d\boldsymbol{x} \\
&- \nu \int_\Omega \big|\nabla\Phi_{R_j}\big|^2\, |\boldsymbol{u}|^2\, d\boldsymbol{x} - \int_\Omega p\boldsymbol{u} \cdot \nabla\Big(\Phi^2_{R_j}\Big)\, d\boldsymbol{x} = 0.
\end{aligned}$$

$$\nu \int_\Omega \big|\nabla\boldsymbol{u}_j\big|^2\, d\boldsymbol{x} \le C\left(\int_{\Omega^{R_j/2}_{R_j}} |\boldsymbol{u}|^2 r^{-1}\, d\boldsymbol{x} + \int_{\Omega^{R_j/2}_{R_j}} |p||\boldsymbol{u}| r^{-1}\, d\boldsymbol{x}\right).$$

$\boldsymbol{u} \in \boldsymbol{L}^2_{-1,0}(\Omega)$, $p \in L^2_{-1,0}(\Omega)$, $p\boldsymbol{u} \in \boldsymbol{L}^1_{-1,0}(\Omega)$. So, for $j \to \infty$ we get $\int_\Omega |\nabla\boldsymbol{u}|^2\, d\boldsymbol{x} \le 0$. Hence, the function $\nabla\boldsymbol{u} = 0$ a.e. in Ω, and this means $\boldsymbol{u}$ is a constant a.e. in Ω.

From $\boldsymbol{u} \in \boldsymbol{L}^2_{-1,0}(\Omega)$ it follows that $\boldsymbol{u} = \boldsymbol{0}$ a.e. in Ω. Using now an arbitrary test function ϕ for equation (1.1), we get $\int_\Omega \nabla p \, \phi \, d\boldsymbol{x} = 0$. So, the function $\nabla p = 0$ a.e. in Ω, and this means p is a constant a.e. in Ω. From $p \in L^2_{-1,0}(\Omega)$ it follows that $p = 0$ a.e. in Ω, and the uniqueness is proved. □

6. Existence of solution in exterior domains.

In this section we assume problem (1.1)–(1.4) in an exterior domain Ω. First we assume the case of the homogenous Dirichlet boundary condition on $\partial\Omega$.

6.1. Homogenous Dirichlet boundary conditions.

Function g is assumed to be zero, and $\boldsymbol{f} = \operatorname{div} \boldsymbol{F}$ with $\boldsymbol{F} \in C_0^\infty(\Omega)^9$. We will prove that the problem has a weak solution $\{\boldsymbol{u}, p\} \in \widehat{\boldsymbol{H}}^1_0(\Omega) \times L^2_{loc}(\Omega)$. So we assume the following sequence of problems on domains $\Omega_R = B_R \cap \Omega$:

$$-\nu \, \Delta \boldsymbol{u}_R + k \, \partial_1 \boldsymbol{u}_R + (\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u}_R - \omega \times \boldsymbol{u}_R + \nabla p_R = \operatorname{Div} \boldsymbol{F} \quad \text{in} \quad \Omega_R \tag{6.34}$$

$$\operatorname{div} \boldsymbol{u}_R = 0 \quad \text{in} \quad \Omega_R \tag{6.35}$$

$$\boldsymbol{u}_R = 0 \quad \text{on} \quad \partial\Omega_R \tag{6.36}$$

Following Girault-Raviart [**16**], we formulate each problem in the following mixed variational form: To find $\{\boldsymbol{u}_R, p_R\} \in \boldsymbol{W}_R \times \Pi_R$, such that for all $\boldsymbol{v} \in \boldsymbol{W}_R$, $\pi \in \Pi_R$:

$$a(\boldsymbol{u}_R, \, \boldsymbol{v}) + b(\boldsymbol{v}, \, p_R) = \langle \operatorname{Div} \boldsymbol{F}, \, \boldsymbol{v} \rangle \tag{6.37}$$

$$b(\boldsymbol{u}_R, \pi) = 0, \tag{6.38}$$

where $\boldsymbol{W}_R = \widehat{\boldsymbol{H}}^1_0(\Omega_R)$, $\Pi_R = \left\{ \pi \in L^2(\Omega_R); \, \int_{\Omega_R} \pi \, d\boldsymbol{x} = 0 \right\}$ with usual norms $\|\phi\|_{W_R} = \|\nabla \phi\|_2$, $\|\pi\|_{\Pi_R} = \|\pi\|_2$, and

$$a(\phi, \, \psi) = \nu \int_{\Omega_R} \nabla \phi \cdot \nabla \psi \, d\boldsymbol{x} + k \int_{\Omega_R} \partial_1 \phi \cdot \psi \, d\boldsymbol{x} + \int_{\Omega_R} [(\omega \times \boldsymbol{x}) \cdot \nabla \phi - \omega \times \phi] \cdot \psi \, d\boldsymbol{x}$$

$$b(\phi, \, \pi) = - \int_{\Omega_R} \pi \operatorname{div} \phi \, d\boldsymbol{x}.$$

These bilinear forms are continuous on $\boldsymbol{W}_R \times \boldsymbol{W}_R$ and $\boldsymbol{W}_R \times \Pi_R$, respectively. It is easy to see that $a(\phi, \, \phi) \geq \nu \, \|\phi\|^2_{\boldsymbol{W}_R}$, and it is known that

$$\sup_{v \in \boldsymbol{W}_R} \frac{(\pi, \, \operatorname{div} \boldsymbol{v})}{|\boldsymbol{v}|_{\boldsymbol{W}_R}} \geq C_0 \|\pi\|_{\Pi_R}$$

for some $C_0 = C_0(R) > 0$. Hence, there exists a weak solution $\{\boldsymbol{u}_R, p_R\}$ of the problem and $\|\boldsymbol{u}_R\|_{\boldsymbol{W}_R} + \|p_R\|_{\Pi_R} \le C_1 \|\mathrm{Div}\,\boldsymbol{F}\|_{-1}$ for some $C_1 = C_1(R) > 0$. Testing now (6.37) by $\boldsymbol{v} = \boldsymbol{u}_R$ we get:

$$\nu \int_{\Omega_R} |\nabla \boldsymbol{u}_R|^2\, d\boldsymbol{x} = \int_{\Omega_R} (\mathrm{Div}\,\boldsymbol{F}) \cdot \boldsymbol{u}_R\ d\boldsymbol{x} = \int_{\Omega_R} \boldsymbol{F} : \nabla \boldsymbol{u}_R\ d\boldsymbol{x} \le \|\boldsymbol{F}\|_2 \|\nabla \boldsymbol{u}_R\|_2$$

$$\|\nabla \boldsymbol{u}_R\|_2 \le \nu^{-1} \|\boldsymbol{F}\|_2. \tag{6.39}$$

Since the a priori estimate (6.39) is available, where $\boldsymbol{u}_R$ is understood as its extension by setting zero in $\Omega \setminus \Omega_R$, there exists $\boldsymbol{u} \in \widehat{\boldsymbol{H}}{}^1_0(\Omega)$ and a sequence $\{R_n\} \to \infty$ so that $\boldsymbol{u}_{R_n} \rightharpoonup \boldsymbol{u}$ weakly in $\widehat{\boldsymbol{H}}{}^1_0(\Omega)$ as $n \to \infty$.

Let us show that $\mathrm{div}\,\boldsymbol{u} = 0$ in $L^2(\Omega)$. From the same inequality follows the weak convergence of $\mathrm{div}\,\boldsymbol{u}_{R_n}$ in $L^2(\Omega)$. From (6.38) we get $\mathrm{div}\,\boldsymbol{u}_{R_n} \equiv C_n$ on Ω_{R_n} for some real constant C_n depending on n. In spite of (6.39) we get that the weak limit of $\mathrm{div}\,\boldsymbol{u}_{R_n}$ is zero in $L^2(\Omega)$.

Finally, for all $\phi \in \boldsymbol{C}_0^\infty(\Omega)$ with $\mathrm{div}\,\phi = 0$ we have from (6.37) after $R_n \to \infty$

$$\langle L\boldsymbol{u} - \mathrm{Div}\,\boldsymbol{F},\, \phi \rangle = 0,$$

$$L\boldsymbol{u} \equiv -\nu\, \Delta \boldsymbol{u} + k\, \partial_1 \boldsymbol{u} + (\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \omega \times \boldsymbol{u}.$$

By a result of de Rham, there is a distribution p such that $-\nabla p = L\boldsymbol{u} - \mathrm{Div}\,\boldsymbol{F}$ in $\mathscr{D}'(\Omega)$. Because the right-hand side belongs to $H^{-1}(\Omega_R)$ for every sufficiently large $R > 0$ we have that $p \in L^2(\Omega_R)$ and so, $p \in L^2_{loc}(\Omega)$.

Now we use the following

Lemma 6.1 (Kozono and Sohr [**22**, Lemma 2.2, Corollary 2.3]). *Let $\Omega \subset \boldsymbol{R}^n (n \ge 2)$ be any domain and let $1 < q < \infty$. For all $g \in \widehat{W}^{-1,q}(\Omega)$, there is $G \in L^q(\Omega)^n$ such that*

$$\mathrm{div}\, G = g, \qquad \|G\|_{q,\Omega} \le C \|g\|_{-1,q,\Omega}$$

with some $C > 0$. As a result, the space $\{\mathrm{div}\, G;\ G \in C_0^\infty(\Omega)^n\}$ is dense in $\widehat{W}^{-1,q}(\Omega)$.

Hence, we get the existence of solution $\{\boldsymbol{u}, p\} \in \widehat{\boldsymbol{H}}{}^1_0(\Omega) \times L^2_{loc}(\Omega)$ for an arbitrary function $\widetilde{\boldsymbol{f}} \in \widehat{\boldsymbol{H}}{}^{-1}(\Omega)$.

For the extension of Theorem 4.4 to the case of an exterior domain we use the localization procedure, see [**22**]. Let now $\boldsymbol{f} \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$. We define for an arbitrary $R > 0$:

$$\boldsymbol{f}_R = \begin{cases} \boldsymbol{f}, & \boldsymbol{x} \in \Omega_R \\ \boldsymbol{0}, & \boldsymbol{x} \in \Omega \setminus \Omega_R. \end{cases}$$

It can be shown that $\boldsymbol{f}_R$ belongs to $\widehat{\boldsymbol{H}}^{-1}(\Omega) \cap \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$. By use of cut-off function Ψ we decompose the solution $\{\boldsymbol{u}, p\}$ of the problem (1.1)–(1.4) (with the homogenous Dirichlet boundary condition) on the solution of a problem in $\boldsymbol{R}^3$ and the solution of a Stokes problem in a bounded domain:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{U} + \boldsymbol{V} \quad \text{where} \quad \boldsymbol{U} = (1-\Psi)\boldsymbol{u}, \quad \boldsymbol{V} = \Psi \boldsymbol{u} \\ p &= \sigma + \tau \quad \text{where} \quad \sigma = (1-\Psi)p, \quad \tau = \Psi p, \end{aligned}$$

where $\Psi \in C_0^\infty$, $\operatorname{supp} \Psi \subset\subset B_{\rho_1}$ such that $\Psi \equiv 1$ on B_{ρ_0}, $0 < \rho_0 < \rho_1 < \rho$ so that $\boldsymbol{R}^3 \setminus \Omega \subset B_{\rho_0}$. We get that $\{\boldsymbol{U}, \sigma\}$ is a weak solution of the modified Oseen problem in $\boldsymbol{R}^3$

$$-\nu \triangle \boldsymbol{U} + k\,\partial_1 \boldsymbol{U} - (\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{U} + \omega \times \boldsymbol{U} + \nabla \sigma = \boldsymbol{Z}_1 \tag{6.40}$$

$$\operatorname{div} \boldsymbol{U} = -\nabla \Psi \cdot \boldsymbol{u} \tag{6.41}$$

and $\{\boldsymbol{V}, \tau\}$ is weak solution of the Stokes problem in a bounded domain Ω_ρ

$$-\nu \triangle \boldsymbol{V} + \nabla \tau = \boldsymbol{Z}_2 \quad \text{in} \quad \Omega_\rho \tag{6.42}$$

$$\operatorname{div} \boldsymbol{V} = \nabla \Psi \cdot \boldsymbol{u} \quad \text{in} \quad \Omega_\rho \tag{6.43}$$

$$\boldsymbol{V}|_{\partial \Omega_\rho} = 0 \tag{6.44}$$

where the right-hand sides are given by $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$.

$$\begin{aligned} \boldsymbol{Z}_1 &= 2\nabla \Psi \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \triangle \Psi - k\partial_1 \Psi\, \boldsymbol{u} + (\nabla \Psi \cdot (\omega \times \boldsymbol{x}))\, \boldsymbol{u} - \nabla \Psi\, p \\ &\quad + (1-\Psi)\boldsymbol{f}_R, \\ \boldsymbol{Z}_2 &= -2\nabla \Psi \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \triangle \Psi + k\partial_1 \Psi\, \boldsymbol{u} + \Psi\, [(\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{u} - \omega \times \boldsymbol{u}] \\ &\quad + \nabla \Psi\, p + \Psi\, \boldsymbol{f}_R. \end{aligned}$$

Let us mention that $\boldsymbol{Z}_1 \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$. To solve the Stokes problem on the bounded domain we use the following lemma, see [**22**]:

Lemma 6.2 (The Stokes problem on a bounded domain). *Let Ω be a bounded domain of $\boldsymbol{R}^n$, $n \geq 2$, of class C^{m+2}, $m \geq 0$. For any*

$$\boldsymbol{f} \in \boldsymbol{W}^{m,q}(\Omega), \qquad g \in W^{m+1,q}(\Omega), \quad \boldsymbol{v}_* \in \boldsymbol{W}^{m+2-1/q,q}(\partial\Omega),$$

$1 < q < \infty$, *with*

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}\, dS = \int_{\Omega} g\, d\boldsymbol{x}, \tag{6.45}$$

there exists one and only one solution $\{\boldsymbol{V}, \tau\}$ *to the Stokes system*

$$\begin{aligned} -\triangle\, \boldsymbol{V} + \nabla\tau &= \boldsymbol{f} \quad && in \quad \Omega \\ \operatorname{div} \boldsymbol{V} &= g && in \quad \Omega \\ \boldsymbol{V} &= \boldsymbol{v}_* && on \quad \partial\Omega \end{aligned}$$

such that $\boldsymbol{V} \in \boldsymbol{W}^{m+2,q}(\Omega)$, $\tau \in W^{m+1,q}(\Omega)$ *and*

$$\|\boldsymbol{V}\|_{m+2,\,q} + \|\tau - \overline{\tau}\|_{m+1,\,q} \le c\Big(\|\boldsymbol{f}\|_{m,\,q} + \|\boldsymbol{v}_*\|_{m+2-1/q,\,q} + \|g\|_{m+1,q}\Big), \tag{6.46}$$

where $\overline{\tau} = |\Omega|^{-1} \int_\Omega \tau\, d\boldsymbol{x}$ *and* $c = c(m, n, q, \Omega)$.

Furthermore, for Ω *of class* C^2*, for every*

$$\boldsymbol{f} \in \boldsymbol{W}_0^{-1,q}(\Omega), \qquad g \in L^q(\Omega), \quad \boldsymbol{v}_* \in \boldsymbol{W}^{1-1/q,q}(\partial\Omega),$$

$1 < q < \infty$, *with* (6.45) *there exists one and only one q-generalized solution* $\{\boldsymbol{V}, \tau\}$ *to the Stokes system such that* $\boldsymbol{V} \in \boldsymbol{W}^{1,q}(\Omega)$, $\tau \in L^q(\Omega)$ *and the estimate* (6.46) *is valid with* $m = -1$.

From the results about the existence and uniqueness of solutions of the Oseen problem in $\boldsymbol{R}^3$ (6.40), (6.41), i.e. from Theorem 4.4 and Theorem 3.1 it follows, that a solution $\{\boldsymbol{U}, \sigma\}$ is subject of the estimate (4.31), with $\boldsymbol{f}$ and g replaced by $\boldsymbol{Z}_1$ and $-\nabla\Psi \cdot \boldsymbol{u}$, respectively. Using also the respective results in a bounded domain for (6.42)–(6.44), see Lemma 6.2 with $m = 0$ and bounded domain Ω_ρ, we get the following lemma for an exterior domain:

LEMMA 6.3. *Let* $\Omega \subset \boldsymbol{R}^3$ *be an exterior domain and* $0 < \beta \le 1$, $0 \le \alpha < y_1 \cdot \beta$; y_1 *is given in Lemma* 4.3*. Then there exists a weak solution* $\{\boldsymbol{u}, p\}$ *of the problem* (1.1)–(1.3) *with the homogenous Dirichlet boundary condition,* $\boldsymbol{f} := \boldsymbol{f}_R$ *and* $g = 0$*, such that* $\boldsymbol{u} \in \boldsymbol{V}_{\alpha,\beta}(\Omega)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$ *and*

$$\begin{aligned} &\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla\boldsymbol{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \\ &\quad \le C_1\Big(\|\boldsymbol{f}_R\|_{2,\alpha+1,\beta} + \|\boldsymbol{u}\|_{1,2;\,A_\rho} + \|p\|_{0,2;\,\Omega_\rho}\Big), \end{aligned} \tag{6.47}$$

where $A_\rho := B_\rho \setminus B_{\rho/2}$*, and constant* C_1 *does not depend on* R.

Now, we would like to show that the preceding estimate is valid (with another constant) also if we add to the left-hand side the L^2-norm of second gradient of $\boldsymbol{u}$ on some compact subset of Ω. Taking into account the assertion of Lemma 6.2 for $m=0$, we get that $\boldsymbol{u} \in \boldsymbol{W}^{2,2}_{loc}(\Omega)$, $p \in W^{1,2}_{loc}(\Omega)$. Multiplying the relation (1.1)–(1.4) in an exterior domain Ω (with $g=0$ and the homogenous Dirichlet boundary condition on $\partial\Omega$) by $\Delta\boldsymbol{u}$ and integrating over the compact set K_1 with $A_\rho \subset K_1 \subset \Omega$, we get

$$\|\Delta\boldsymbol{u}\|_{2;K_1} \le C_2\Big(\|\boldsymbol{u}\|_{2;K_1} + \|\nabla\boldsymbol{u}\|_{2;K_1} + \|p\|_{2;K_1} + \|\nabla p\|_{2;K_1} + \|\boldsymbol{f}_R\|_{2;K_1}\Big). \tag{6.48}$$

Using (6.47), (6.48) and the known relation

$$\|\nabla^2\boldsymbol{u}\|_{2;K} \le c\Big(\|\Delta\boldsymbol{u}\|_{2;K_1} + \|\nabla\boldsymbol{u}\|_{2;K_1}\Big)$$

with $A_\rho \subset K \subset K_1$, we get

Corollary 6.4. *In conditions of Lemma* 6.3 *the following estimate is valid and constant C does not depend on R*:

$$\begin{aligned}\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla\boldsymbol{u}\|_{2,\alpha,\beta} + \big\|\nabla^2\boldsymbol{u}\big\|_{2;\,A_\rho} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta}\\ \le C\Big(\|\boldsymbol{f}_R\|_{2,\alpha+1,\beta} + \|\boldsymbol{u}\|_{1,2;\,A_\rho} + \|p\|_{0,2;\Omega_\rho}\Big).\end{aligned} \tag{6.49}$$

Now, we will prove that the estimate (6.49) is valid without the right-hand side terms containing $\boldsymbol{u}$ and p with constant c which does not depend on R, i.e. we will prove:

$$\begin{aligned}\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla\boldsymbol{u}\|_{2,\alpha,\beta} + \big\|\nabla^2\boldsymbol{u}\big\|_{2;\,A_\rho} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta}\\ \le c\|\boldsymbol{f}_R\|_{2,\alpha+1,\beta}\end{aligned} \tag{6.50}$$

Let us define the norms:

$$\begin{aligned}\|(\boldsymbol{v},q)\|_{(1)} &:= \|\boldsymbol{v}\|_{1,2;A_\rho} + \|q\|_{0,2;\Omega_\rho}\\ \|(\boldsymbol{v},q)\|_{(2)} &:= \|\boldsymbol{v}\|_{2,\alpha-1,\beta} + \|\nabla\boldsymbol{v}\|_{2,\alpha,\beta} + \|\nabla^2\boldsymbol{v}\|_{2;\,A_\rho}\\ &\quad + \|q\|_{2,\alpha,\beta-1} + \|\nabla q\|_{2,\alpha+1,\beta}.\end{aligned}$$

For the corresponding Hilbert spaces H_1, H_2, we have $H_2 \hookrightarrow\hookrightarrow H_1$. Let us assume that the estimate (6.50) is not true. This means that there is a sequence of functions $\left\{ \boldsymbol{f}_{R_k}^{(k)} \right\}_{k=1}^{\infty}$ with $R_k \to +\infty$, a sequence of corresponding solutions $\{(\boldsymbol{u}_k, p_k)\}_{k=1}^{\infty}$ and a sequence of constants $\{c_k\}_{k=1}^{\infty} \to \infty$ such that:

$$\begin{aligned} 1 &\equiv \|\boldsymbol{u}_k\|_{2,\alpha-1,\beta} + \|\nabla \boldsymbol{u}_k\|_{2,\alpha,\beta} + \|\nabla^2 \boldsymbol{u}_k\|_{2;\, A_\rho} + \|p_k\|_{2,\alpha,\beta-1} + \|\nabla p_k\|_{2,\alpha+1,\beta} \\ &\equiv \|(\boldsymbol{u}_k, p_k)\|_{(2)} \geq c_k \left\| \boldsymbol{f}_{R_k}^{(k)} \right\|_{2,\alpha+1,\beta}. \end{aligned}$$

So we get $\left\{ \left\| \boldsymbol{f}_{R_k}^{(k)} \right\|_{2,\alpha+1,\beta} \right\}_k \to 0$. The sequence $\{(\boldsymbol{u}_k, p_k)\}_{k=1}^{\infty}$ is bounded in the norm $\|\cdot\|_{(2)}$, so there is a subsequence of this sequence (we will denote this subsequence using the same notation) with the weak limit $(\boldsymbol{u}, p)$ in the corresponding Hilbert space H_2. So, $(\boldsymbol{u}, p)$ is a solution of the problem with the zero right-hand side. Due to uniqueness given by Theorem 5.1 we conclude that $\|(\boldsymbol{u}, p)\|_{(2)} = 0$. Because $H_2 \hookrightarrow\hookrightarrow H_1$, we have $\|(\boldsymbol{u} - \boldsymbol{u}_k, p - p_k)\|_{(1)} \to 0$. From Corollary 6.4 we also get

$$\|(\boldsymbol{u} - \boldsymbol{u}_k, p - p_k)\|_{(2)} \to 0,$$

i.e. $\{(\boldsymbol{u}_k, p_k)\}_{k=1}^{\infty}$ converges strongly in H_2. Because $\|(\boldsymbol{u}_k, p_k)\|_{(2)} = 1$ for $k \in \boldsymbol{N}$, so we also get $\|(\boldsymbol{u}, p)\|_{(2)} = 1$. This is the contradiction.

So, we proved the following

THEOREM 6.5. *Let $\Omega \subset \boldsymbol{R}^3$ be an exterior domain and $0 < \beta \leq 1$, $0 \leq \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\boldsymbol{f} \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$. Then there exists a weak solution $\{\boldsymbol{u}, p\}$ of the problem (1.1)–(1.3) with the homogenous Dirichlet boundary condition on $\partial\Omega$, $g = 0$, such that $\boldsymbol{u} \in \boldsymbol{V}_{\alpha,\beta}(\Omega)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla \boldsymbol{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \leq C\|\boldsymbol{f}\|_{2,\alpha+1,\beta}.$$

REMARK 6.6. The used contradiction argument is based on a subtle choice of the sequence $\left\{ \boldsymbol{f}_{R_k}^{(k)} \right\}_k$ with $R_k \to +\infty$. We cannot construct a contradiction separately for $\boldsymbol{f}_R$ with fixed R because then the constant c in (6.50) may depend on R.

6.2. Non-homogenous cases.

In this subsection we take into account the non-homogenous Dirichlet boundary condition and the non-homogenous continuity equation.

We can prove the following extension of Theorem 6.5 for the case $g \neq 0$:

COROLLARY 6.7. *Let $\Omega \subset \boldsymbol{R}^3$ be an exterior domain and $0 < \beta \leq 1$, $0 \leq \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\boldsymbol{f} \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$, $g \in W^{1,2}_0(\Omega)$, with* $\operatorname{supp} g =$

$K \subset\subset \Omega$ and $\int_\Omega g\, d\boldsymbol{x} = 0$. Then there exists a weak solution $\{\boldsymbol{u}, p\}$ of the problem (1.1)–(1.3) with the homogenous boundary condition on $\partial\Omega$ such that $\boldsymbol{u} \in \boldsymbol{V}_{\alpha,\beta}(\Omega)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$ and

$$\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla \boldsymbol{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C\Big(\|\boldsymbol{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2}\Big).$$

First of all let us recall the lemma which will be used for the extension of our results to the case with nonzero divergence:

Lemma 6.8 (M. E. Bogovski, G. P. Galdi, H. Sohr). *Let $\Omega \subseteq \boldsymbol{R}^n$, $n \ge 2$, be a bounded Lipschitz domain, and $1 < q < \infty$, $n \in \boldsymbol{N}$. Then for each $g \in W^{k,q}_0(\Omega)$ with $\int_\Omega g\, d\boldsymbol{x} = 0$, there exists $\boldsymbol{G} \in \Big(W^{k+1,q}_0(\Omega)\Big)^n$ satisfying*

$$\operatorname{div} \boldsymbol{G} = g, \quad \|\boldsymbol{G}\|_{(W^{k+1,q}_0(\Omega))^n} \le C\|g\|_{W^{k,q}_0(\Omega)}$$

with some constant $C = C(q, k, \Omega) > 0$.

For the proof and further references see e.g. [**31**, Lemma 2.3.1].

Proof of Corollary 6.7. Using Lemma 6.8 we find $\boldsymbol{G} \in \boldsymbol{W}^{2,2}_0(\Omega)$, $\operatorname{supp} \boldsymbol{G} \subset \mathscr{K}$, where $\mathscr{K}$ is a bounded Lipschitz domain being contained in ε-neighbourhood $\mathscr{K}_\varepsilon$ of compact set K for an arbitrary $\varepsilon > 0$, $\operatorname{div} \boldsymbol{G} = g$, $\|\boldsymbol{G}\|_{2,2} \le C\|g\|_{1,2}$. We choose ε such that $\mathscr{K}_\varepsilon \subset \Omega$. Let us assume the following problem

$$\begin{aligned} -\nu\,\Delta \boldsymbol{U} + k\,\partial_1 \boldsymbol{U} - (\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{U} + \omega \times \boldsymbol{U} + \nabla p = \boldsymbol{F} \quad & \text{in } \Omega \\ \operatorname{div} \boldsymbol{U} = 0 \quad & \text{in } \Omega \end{aligned}$$

with the homogenous Dirichlet boundary condition for $\boldsymbol{U}$, where $\boldsymbol{U} = \boldsymbol{u} - \boldsymbol{G}$, $\boldsymbol{F} = \boldsymbol{f} + \nu\,\Delta \boldsymbol{G} - k\,\partial_1 \boldsymbol{G} + (\omega \times \boldsymbol{x}) \cdot \nabla \boldsymbol{G} - \omega \times \boldsymbol{G}$. The assertion of Corollary 6.7 follows from Theorem 6.5. □

Now we justify our third main theorem.

Theorem 6.9. *Let $\Omega \subset \boldsymbol{R}^3$ be an exterior domain and $0 < \beta \le 1$, $0 \le \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\boldsymbol{f} \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$, $g \in W^{1,2}_0(\Omega)$, with $\operatorname{supp} g = K \subset\subset \Omega$ and $\int_\Omega g\, d\boldsymbol{x} = 0$. Then there exists a weak solution $\{\boldsymbol{u}, p\}$ of the problem (1.1)–(1.4) such that $\boldsymbol{u} \in \boldsymbol{V}_{\alpha,\beta}(\overline{\Omega})$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{2,\alpha-1,\beta} + \|\nabla \boldsymbol{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \\
&\quad \le C\Big(\|\boldsymbol{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2} + \omega^2 + \omega + k^2 + k\Big).
\end{aligned}
$$

PROOF. Let $\rho > 0$ be such that $\boldsymbol{R}^3 \setminus B_{\rho/2} \subset \Omega$. Let $\Phi = \Phi(z) \in C_0^\infty(\langle 0, +\infty))$ be a non-increasing cut-off function such that $\Phi(z) \equiv 1$ for $z < 1/2$ and $\Phi(z) \equiv 0$ for $z > 1$. Let $|\Phi'| \le 3$. Let $\Phi_\rho \equiv \Phi_\rho(\boldsymbol{x}) \equiv \Phi(|\boldsymbol{x}|/\rho)$. We have $|\nabla \Phi_\rho| \le 3/\rho$ and $|\partial_1 \Phi_\rho| \le 3/\rho$ for $\boldsymbol{x} \in \boldsymbol{R}^3$, $\rho/2 \le |\boldsymbol{x}| \le \rho$. Let us define $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - [(\omega \times \boldsymbol{x}) - k\boldsymbol{e}_1] \cdot \Phi_\rho(\boldsymbol{x})$. Then function $(\widetilde{\boldsymbol{u}}, p)$ satisfies to (1.1)–(1.3) with the homogenous Dirichlet boundary condition, where $\boldsymbol{f} \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$ is replaced by some another function $\widetilde{\boldsymbol{f}} \in \boldsymbol{L}^2_{\alpha+1,\beta}(\Omega)$, and g by another function $\widetilde{g} \in C_0^\infty(\Omega)$ with $\operatorname{supp} \widetilde{g} = K \cup A_\rho$, $A_\rho := B_\rho \setminus \overline{B_{\rho/2}} \subset\subset \Omega$ and

$$\int_\Omega \widetilde{g}\, d\boldsymbol{x} = 0.$$

So, using now Corollary 6.7 we get the assertion of Theorem 6.9. □

Appendix A.

Relation (2.14) follows from an estimate of the derivative of F_1:

$$
\begin{aligned}
\frac{\partial}{\partial s} F_1(s,r) &\equiv \frac{\partial}{\partial s}\big\{F_{\alpha,\beta}(s,r) - \big(1-\kappa^{-1}\big)k\delta\varepsilon(\beta-\alpha)s\big\} \\
&= -\nu\alpha^2\delta^2\varepsilon\,\frac{1}{1+\delta r} - 2\nu\alpha\beta\delta\varepsilon\,\frac{1}{r} - 2\nu\beta^2\varepsilon^2\,\frac{1+\delta r}{r}\,\frac{1}{(1+\varepsilon s)^2} \\
&\quad - k\alpha\delta\varepsilon + k\alpha\delta\,\frac{1}{r}\,(1+2\varepsilon s) + k\beta\varepsilon(1+\delta r)\,\frac{1}{r} \\
&\quad - \big(1-\kappa^{-1}\big)k\delta\varepsilon(\beta-\alpha) \\
&\ge \delta\varepsilon\Big\{r^{-1}\big[k(\alpha/\varepsilon + \beta/\delta) - \nu\alpha^2 - 2\nu\alpha\beta - 2\nu\beta^2\varepsilon/\delta\big] \\
&\quad + \big[-2\nu\beta^2\varepsilon + k(\beta-\alpha)/\kappa\big]\Big\} \ge 0.
\end{aligned}
$$

The last inequality follows from the fact that we have $k\alpha/\varepsilon \ge \nu\,\alpha^2 + 2\,\nu\,\alpha\,\beta$, $k\beta/\delta \ge 2\,\nu\,\beta^2\varepsilon/\delta$, $k(\beta-\alpha)/\kappa \ge 2\nu\beta^2\varepsilon$ if $\varepsilon \le (1/(2\kappa))(k/\nu)((\beta-\alpha)/\beta^2)$. Hence, if the last inequality (which is included in the conditions of Lemma 2.5) is satisfied then $(\partial/\partial s)F_1(s,r) \ge 0$. So, we get immediately:

$$F_1(s,r) \ge F_1(0,r) \equiv -k\alpha\delta - \nu\alpha^2\delta^2(1+\delta r)^{-1} \ge -\alpha\delta k\big(1 + \nu k^{-1}\alpha\delta\big).$$

Appendix B.

Let us show that all conditions on α, β, δ, ε, κ used in the proof of Lemma 4.3 are compatible if $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$. Let us collect these assumptions: $0 < \delta < 2\varepsilon$, $1 < \kappa \le 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$, $0 \le \alpha < \beta$, $\varepsilon \le (1/(2\kappa^2)) \cdot (k/\nu) \cdot ((\beta - \alpha)/\beta^2)$, $\delta \le (k/\nu) \cdot (\kappa - 1)/(\kappa\beta)$, $\alpha \le (1/(2\,\kappa^4)) \cdot (k/\nu) \cdot (\beta\,\beta^*/(\alpha+\beta))^2\varepsilon$.

From $\alpha \le (1/(2\,\kappa^4)) \cdot (k/\nu) \cdot (\beta\,\beta^*/(\alpha+\beta))^2\varepsilon$, and $\varepsilon \le (1/(2\kappa^2)) \cdot (k/\nu) \cdot ((\beta-\alpha)/\beta^2)$ we get $\alpha \le (1/(4\kappa^6)) \cdot (\beta^*)^2(\beta-\alpha)/(\alpha+\beta)^2$. So we get ($\kappa > 1$, $\beta \le 1$): $\alpha/\beta \le (1/(4\kappa^6))(1-\alpha/\beta)/(1+\alpha/\beta)^2$. By substitution $y = \alpha/\beta$ we get the inequality

$$4y^3 + 8y^2 + 4y + \kappa^{-6} \cdot (y-1) \le 0. \tag{6.51}$$

Taking into account the condition $0 \le \alpha < \beta$ we seek for solutions from $[0,1)$. It is clear that the equation $4y^3 + 8y^2 + y + \kappa^{-6}(y-1) = 0$ has a unique real solution $y_\kappa \in (0,1)$ for $\kappa > 1$. It is also clear that arbitrary $y \in [0, y_\kappa)$ solves (6.51). The value y_κ as a function of κ is decreasing. For $\kappa \to 1$ we get the inequality $4y^3 + 8y^2 + 5y - 1 \le 0$. This respective equation has a unique solution $y_1 = (\sqrt{13}/(6\sqrt{6}) + 53/216)^{1/3} + (1/30)(\sqrt{13}/(6\sqrt{6}) + 53/216)^{-1/3}$. Approximately, with an error less than 10^{-8} we have $y_1 \doteq 0.1582981$, ($y_1 > 1/7$). If $0 \le \alpha < y_1\beta$ then there is $\kappa > 1$ sufficiently close to number 1, such that $0 \le \alpha \le y_\kappa\beta$, so the relation $\alpha \le (1/(4\kappa^6)) \cdot (\beta^*)^2(\beta-\alpha)/(\alpha+\beta)^2$ is satisfied. Then we can define $\varepsilon = 1/(2\kappa^2) \cdot (k/\nu) \cdot ((\beta-\alpha)/(\beta^2))$. The relation $\varepsilon \le (1/(2\kappa)) \cdot (k/\nu) \cdot (1/\beta)$ is satisfied. Then we take sufficiently small $\delta > 0$ such that $0 < \delta < 2\varepsilon$ and $1 < \kappa \le 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$. Hence, all conditions which we assume in the proof of Lemma 4.3 are satisfied.

Acknowledgements. The authors gratefully acknowledge and appreciate the hospitality of the Mathematical Institute of the Academy of Sciences of the Czech Republic and of the University in Toulon. The authors also thank the reviewer who gives them opportunity to add comments and remarks through the present article. The first and second authors were supported by the Grant Agency of the Academy of Sciences No. IAA100190804. The research of the first author was supported by the research plans of the Ministry of Education of the Czech Republic No. 6840770010 and the research of the second author was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

References

[1] R. Farwig, A variational approach in weighted Sobolev spaces to the operator $-\triangle + \partial/\partial x_1$ in exterior domains of $\boldsymbol{R}^3$, Math. Z., **210** (1992), 449–464.

[2] R. Farwig, The stationary exterior 3-D problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces, Math. Z., **211** (1992), 409–447.

[3] R. Farwig, An L^q-analysis of viscous fluid flow past a rotating obstacle, Tôhoku Math. J. (2), **58** (2006), 129–147.

[4] R. Farwig, Estimates of lower order derivatives of viscous fluid flow past a rotating obstacle, Banach Center Publ., **70** (2005), 73–84.

[5] R. Farwig, T. Hishida and D. Müller, L^q-Theory of a singular "winding" integral operator arising from fluid dynamics, Pacific J. Math., **215** (2004), 297–312.

[6] R. Farwig and T. Hishida, Stationary Navier-Stokes flow around a rotating obstacles, Funkcial. Ekvac., **50** (2007), 371–403.

[7] R. Farwig, M. Krbec and Š. Nečasová, A weighted L^q-approach to Stokes flow around a rotating body, Ann. Univ. Ferrara Sez. VII Sci. Mat., **54** (2008), 61–84.

[8] R. Farwig, M. Krbec and Š. Nečasová, A weighted L^q-approach to Oseen flow around a rotating body, accepted in Math. Methods Appl. Sci.

[9] R. Finn, Estimates at infinity for stationary solution of Navier-Stokes equations, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.), 3, **51** (1959), 387–418.

[10] R. Finn, On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems, Arch. Rational Mech. Anal., **19** (1965), 363–406.

[11] G. P. Galdi, On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications, Handbook of Mathematical Fluid Dynamics, **1**, (eds. S. Friedlander and D. Serre), Elsevier, North-Holland, Amsterdam, 2002, pp. 653–791.

[12] G. P. Galdi, Steady flow of a Navier-Stokes fluid around a rotating obstacle, Essays and papers dedicated to the memory of Clifford Ambrose Truesdell III, **II**, J. Elasticity, **71** (2003), 1–31.

[13] G. P. Galdi and A. L. Silvestre, On the steady motion of a Navier-Stokes liquid around a rigid body, Arch. Ration. Mech. Anal., **184** (2007), 371–400.

[14] G. P. Galdi and A. L. Silvestre, Further results on steady-state flow of a Navier-Stokes liquid around a rigid body, Existence of the wake, RIMS Kôkyûroku Bessatsu, B1, Res. Inst. Math. Sci., Kyoto, 2007, pp. 127–143.

[15] M. Geissert, H. Heck and M. Hieber, L^p theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle, J. Reine Angew. Math., **596** (2006), 45–62.

[16] V. Girault and P.-A. Raviart, Finite element approximation of the Navier-Stokes equations, Lecture Notes in Mathematics, **749**, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[17] T. Hishida, The Stokes operator with rotation effect in exterior domains, Analysis (Munich), **19** (1999), 51–67.

[18] T. Hishida, An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle, Arch. Ration. Mech. Anal., **150** (1999), 307–348.

[19] T. Hishida, L^q-estimates of weak solutions to the stationary Stokes equations around a rotating body, J. Math. Soc. Japan, **58** (2006), 743–767.

[20] J.-L. Impagliazzo, Resolution des equations de Navier-Stokes compressibles a l'aide de la methode de decomposition, These de doctorat de l'Universite de Toulon et du Var, 1997.

[21] T. Kobayashi and Y. Shibata, On the Oseen equation in the three-dimensional exterior domains, Math. Ann., **310** (1998), 1–45.

[22] H. Kozono and H. Sohr, New a priori estimates for the Stokes equations in exterior domains, Indiana Univ. Math. J., **40** (1991), 1–27.

[23] S. Kračmar, Š. Nečasová and P. Penel, Anisotropic L^2-estimates of weak solutions to the stationary Oseen type equations in $\boldsymbol{R}^3$ for a rotating body, RIMS Kôkyûroku Bessatsu, B1, Res. Inst. Math. Sci., Kyoto, 2007, pp. 219–235.

[24] S. Kračmar, Š. Nečasová and P. Penel, Remarks on the non-homogeneous Oseen problem arising from modeling of the fluid around a rotating body, Proceedings of the International Conference "Hyperbolic equations", held in Lyon, France, 2006, in press.

[25] S. Kračmar, A. Novotný and M. Pokorný, Estimates of three dimensional Oseen kernels in weighted L^p spaces, In: Applied Nonlinear Analysis, London-New York, Kluwer Academic, 1999, pp. 281–316.

[26] S. Kračmar, A. Novotný and M. Pokorný, Estimates of Oseen kernels in weighted L^p spaces, J. Math. Soc. Japan, **53** (2001), 59–111.

[27] S. Kračmar and P. Penel, Variational properties of a generic model equation in exterior 3D domains, Funkcial. Ekvac., **47** (2004), 499–523.

[28] S. Kračmar and P. Penel, New regularity results for a generic model equation in exterior 3D domains, Banach Center Publ., **70** (2005), 139–155.

[29] Š. Nečasová, Asymptotic properties of the steady fall of a body in a viscous fluid, Math. Methods Appl. Sci., **27** (2004), 1969–1995.

[30] C. W. Oseen, Neuere Methoden und Ergebnisse in der Hydrodynamik, Leipzig, Akad. Verlagsges. M.B.H., 1927.

[31] H. Sohr, The Navier-Stokes Equations: An Elementary Functional Analytic Approach, Berlin, Birkhäuser, 2001.

[32] E. A. Thomann and R. B. Guenther, The fundamental solution of the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions — time dependent case, J. Math. Fluid Mech., **8** (2006), 77–98.

Stanislav KRAČMAR

Department of Technical Mathematics
Czech Technical University
Karlovo nám, 13, 121 35 Prague 2
Czech Republic

Šárka NEČASOVÁ

Mathematical Institute of
the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Prague 1
Czech Republic

Patrick PENEL

Université du Sud Toulon–Var
Département de Mathématique et Laboratoire
Systèmes Navals Complexes
B.P. 20132, 83957 La Garde Cedex
France

SIAM J. MATH. ANAL.
Vol. 43, No. 2, pp. 705–738

ON POINTWISE DECAY OF LINEARIZED STATIONARY INCOMPRESSIBLE VISCOUS FLOW AROUND ROTATING AND TRANSLATING BODIES*

PAUL DEURING†, STANISLAV KRAČMAR‡, AND ŠÁRKA NEČASOVÁ§

Abstract. We consider a system arising by linearization of a model for stationary viscous incompressible flow past a rotating and translating rigid body. Using a fundamental solution proposed by Guenther and Thomann [*J. Math. Fluid Mech.*, 8 (2006), pp. 77–98], we derive a representation formula for the velocity field. This formula is then used to obtain pointwise decay estimates and to identify a leading term with respect to this decay. In addition, we prove a representation theorem for weak solutions of the stationary Navier–Stokes system with Oseen and rotational terms.

Key words. viscous incompressible flow, rotating body, fundamental solution, decay, Navier–Stokes system

AMS subject classifications. 35Q30, 65N30, 76D05

DOI. 10.1137/100786198

1. Introduction.

We consider the system of equations

$$-\Delta u(z) - (U + \omega \times z) \cdot \nabla u(z) + \omega \times u(z) + \nabla \pi(z) = f(z), \ \ \operatorname{div} u(z) = 0 \quad \text{for } \ z \in \mathbb{R}^3 \backslash \overline{\mathfrak{D}}. \tag{1.1}$$

This system arises by linearization and normalization of a mathematical model describing the stationary flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating at a constant angular velocity, under the assumption that the velocity of the body and its angular velocity are parallel to each other. The open set $\mathfrak{D} \subset \mathbb{R}^3$ describes the rigid body, the vector $U \in \mathbb{R}^3 \backslash \{0\}$ represents the constant translational velocity of this body, and the vector $\omega \in \mathbb{R}^3 \backslash \{0\}$ represents its constant angular velocity. The given function $f : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ stands for an exterior force, and the unknowns $u : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ and $\pi : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}$ correspond respectively to the normalized velocity and pressure field of the fluid. More information on the physical background of (1.1) may be found in [22, Chapter 1]. Since (1.1) is related to the case that translational and angular velocities of the rigid body in question are parallel, we assume that the vectors U and ω point in the same or in the opposite direction. If the two types of velocities are not parallel, terms depending on time have to be included in a suitable mathematical model, and the corresponding problem has to be studied by different methods. We refer to [12] for more details.

*Received by the editors February 17, 2010; accepted for publication (in revised form) November 24, 2010; published electronically March 9, 2011.
http://www.siam.org/journals/sima/43-2/78619.html

†Université Lille Nord de France, 59000 Lille, France; ULCO, LMPA, 62228 Calais, France (paul.deuring@lmpa.univ-littoral.fr).

‡Department of Technical Mathematics, Czech Technical University, Karlovo nám. 13, 121 35 Prague 2, Czech Republic (stakr51@gmail.com). This author was supported by the research plans of the Ministry of Education of the Czech Republic through grant 6840770010 and by the Grant Agency of the Academy of Sciences through grant IAA100190804.

§Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic (matus@math.cas.cz). This author was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503, and by the Grant Agency of the Academy of Sciences through grant IAA100190804.

We are interested only in the case $U \neq 0$. Thus we may suppose without loss of generality that there is some $\tau > 0$ with $U = -\tau(1,0,0)$, and hence $\omega = \varrho(1,0,0)$ for some $\varrho \in \mathbb{R}\backslash\{0\}$. In this way we end up with the following variant of (1.1):

$$L(u) + \nabla\pi = f, \;\; \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\mathfrak{D}}, \tag{1.2}$$

where the differential operator L is defined by

$$\begin{aligned} &L(u)(z) := -\Delta u(z) + \tau\,\partial_1 u(z) - (\omega\times z)\cdot\nabla u(z) + \omega\times u(z) \\ &\text{for } \; u \in W^{2,1}_{loc}(U)^3, \; z \in U, \; U \subset \mathbb{R}^3 \text{ open.} \end{aligned} \tag{1.3}$$

The aim of the work at hand is twofold. First we want to represent suitably regular functions $u : \mathbb{R}^3\backslash\overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ in terms of $L(u) + \nabla\pi$, $\operatorname{div} u$, $u|\partial\mathfrak{D}$, $\nabla u|\partial\mathfrak{D}$, and $\pi|\partial\mathfrak{D}$. Note that we do not suppose $\operatorname{div} u$ to vanish. The second aim of this article consists in using our representation theorem in order to link the decay of $|u(x)|$ and $|\nabla u(x)|$ for $|x| \to \infty$ with that of $|(L(u)+\nabla\pi)(x)|$ and $|\operatorname{div} u(x)|$. In particular, for a solution (u,π) of (1.2), we obtain a link between the decay of $|u(x)|$ and $|\nabla u(x)|$ on the one hand and the asymptotic behavior of $|f(x)|$ for $|x| \to \infty$ on the other. In addition we derive an asymptotic profile of $u(x)$ for $|x| \to \infty$, and we extend our representation formula to weak solutions of the Navier–Stokes system with Oseen and rotational terms.

The starting point of our theory is a fundamental solution constructed by Guenther and Thomann [27] for the time-dependent variant of (1.1). At the end of their article, Guenther and Thomann indicate that by integrating their solution with respect to time, they obtain a fundamental solution to (1.1). In [7], we took up this hint in order to derive a representation formula of the type mentioned above (related to (1.1) instead of (1.2)); see [7, Theorem 4.3]. However, we assumed u to be C^2 and π to be C^1, we required a rather strong decay of $u(x)$ and $\pi(x)$, and we did not prove some crucial auxiliary results (see [7, inequality (3.6), Lemma 4.1, Theorem 4.1]; compare with the comments in section 2 before Lemma 2.16).

In the present article we consider (1.2) instead of (1.1) to simplify our presentation. This does not mean a loss of generality. We will fill the gaps left in [7] (see Lemma 2.16 and Theorems 2.17 and 2.18), and we will extend our representation formula to functions u and π with regularity and rate of decay corresponding to those of a weak solution to (1.2). More precisely, we will assume that u belongs to $L^6(\mathbb{R}^3\backslash\overline{\mathfrak{D}})^3$, and ∇u and π are L^2 in $\mathbb{R}^3\backslash\overline{\mathfrak{D}}$, and both u and π are locally L^p-regular for some $p > 1$ (Theorem 4.6). As a consequence of our representation formula, we will specify conditions on $L(u) + \nabla\pi$ and $\operatorname{div} u$ such that

$$\begin{aligned} &|u(x)| = O\Big[\Big(|x|\big(1+\tau(|x|-x_1)\big)\Big)^{-1}\Big], \\ &|\nabla u(x)| = O\Big[\Big(|x|\big(1+\tau(|x|-x_1)\big)\Big)^{-3/2}\Big] \quad \text{for } |x| \to \infty \end{aligned} \tag{1.4}$$

(Theorem 5.3). In the case that $L(u) + \nabla\pi$ and $\operatorname{div} u$ have compact support, we will identify an asymptotic profile of $u(x)$ for $|x| \to \infty$ (Theorem 5.4). Finally, in Theorem 5.5, we will present a representation formula for weak solutions to the nonlinear problem

$$\begin{aligned} &-\Delta u(z) + \tau\,\partial_1 u(z) - (\omega\times z)\cdot\nabla u(z) + \omega\times u(z) + \tau\,(u(z)\cdot\nabla)u(z) = f(z), \\ &\operatorname{div} u(z) = 0 \quad \text{for } \; z \in \mathbb{R}^3\backslash\overline{\mathfrak{D}} \end{aligned} \tag{1.5}$$

(stationary Navier–Stokes system with Oseen and rotational terms). The key element of our theory is the fundamental solution of (1.1) mentioned above (which we adapt to (1.2), of course). Since this solution is only very briefly discussed in [27], we will present detailed proofs of its key properties, except for some features already set out in [7].

Our results are the best possible in two respects. First, for the velocity part u of a solution (u, π) of the Oseen system

$$-\Delta u + \tau\,\partial_1 u + \nabla \pi = f, \;\; \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\mathfrak{D}}, \tag{1.6}$$

the decay rates stated in (1.4) cannot be improved in general. This follows from the asymptotic expansions in [20, (VII.6.18), (VII.6.20)], and from the behavior of the Oseen fundamental solution as exhibited in [35, (1.15)]. Since it cannot be expected that a solution of system (1.2) decays faster than an Oseen flow, the decay rates in (1.4) should be optimal. Of course, these relations hold only if the right-hand side f in (1.2) and (1.6), respectively, tends to zero sufficiently fast for $|x| \to \infty$. In this respect, in view of applications to the nonlinear problem (1.5), it is important to find decay conditions on f that are as weak as possible but still allow us to maintain (1.4). For solutions of the Oseen system, such conditions were derived in [35, section 3]. We obtain inequality (1.4) for solutions of the rotational problem (1.2) under these same conditions. This is the second optimal feature of our theory.

The work at hand was inspired by Galdi and Silvestre [24], [25], who proved existence, uniqueness, and decay results for solutions of the linear problem (1.2), and also for solutions of the nonlinear system (1.5), under Dirichlet boundary conditions. Concerning decay results pertaining to (1.2), the theory in [24], [25] states that if (u, π) is a solution to (1.2) with

$$\sup\{|u(x)|\,|x| \,:\, x \in \mathbb{R}^3 \backslash B_S\} \,<\, \infty \quad \text{for some } \; S > 0 \text{ with } \overline{\mathfrak{D}} \subset B_S \tag{1.7}$$

("physical reasonable solution"), if $\|\pi\|_2 < \infty$, $\int_{\partial\mathfrak{D}} u \cdot n^{(\mathfrak{D})} \, d\mathfrak{D} = 0$, if u and π are locally L^2-regular, and if

$$\sup\Big\{|f(x)|\Big(|x|\big(\,1 + \tau\,(|x| - x_1)\,\big)\Big)^{5/2} \,:\, x \in \mathbb{R}^3 \backslash B_S\Big\} \,<\, \infty, \tag{1.8}$$

then the decay relations in (1.4) hold (see [25, Theorem 3]). Theorem 5.3 below improves this result in several respects: Assumption (1.8), which is not the best possible, is replaced by optimal conditions on f, as explained above. Instead of condition (1.7), we require that $u \in L^6(\mathbb{R}^3 \backslash \overline{\mathfrak{D}})^3$ and $\nabla u \in L^2(\mathbb{R}^3 \backslash \overline{\mathfrak{D}})^9$. In other words, we consider weak solutions instead of physical reasonable ones. Moreover we do not assume the zero flux condition $\int_{\partial\mathfrak{D}} u \cdot n^{(\mathfrak{D})} \, d\mathfrak{D} = 0$, and we admit the case div $\neq 0$, although for the estimate of $|\nabla u(x)|$ indicated in (1.4), we have to require that the support of $\operatorname{div} u$ is compact (Theorem 5.3). Instead of local L^2-regularity, we suppose only local L^p-regularity for an arbitrary $p > 1$.

The relevance of the work at hand, however, goes beyond some technical improvements of the results in [24] and [25]. To explain this, let us return to the Oseen system (1.6) and its nonlinear counterpart

$$-\Delta u + \tau\,\partial_1 u + \tau\,(u \cdot \nabla)u + \nabla \pi = f, \;\; \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\mathfrak{D}}. \tag{1.9}$$

Since Finn's pioneering work [18], [19] at the beginning of the 1960s, a great number of papers have dealt with the asymptotic properties of solutions to (1.6) or (1.9);

see [2], [9], [44], [20, section VII.6], [21, section IX.8], [35], [5], [6], [3], for example. As a consequence of this research work, a rather complete theory is now available on the asymptotics of Oseen flows. But all the papers just mentioned are based on estimates of the Oseen fundamental solution introduced in [43]. On the other hand, concerning (1.2), although a fundamental solution has been known due to Guenther and Thomann [27], how to estimate this solution was an open problem. For this reason, the asymptotic behavior of solutions to (1.2) or its nonlinear version (1.5) had to be studied without making use of a fundamental solution. Therefore it is not astonishing that our knowledge on the asymptotics of these "rotational flows" is limited compared to the detailed theory on Oseen flows.

The work at hand should help to change this situation. In fact, our theory should make it possible to deal with rotational flows in the same way as with Oseen flows, as concerns the study of asymptotics. In fact, in Lemmas 2.12 and 2.16 and Theorems 2.17 and 2.19 below, we estimate the Guenther–Thomann fundamental solution in such a way that asymptotic properties of rotational flows become accessible via evaluation of this fundamental solution. This becomes apparent in the proofs of Theorems 5.3 and 5.4, where we derive decay rates and an asymptotic profile of solutions to the linear problem (1.2). Moreover, our representation formula for solutions to the nonlinear problem (1.5), combined with our estimates of the Guenther–Thomann fundamental solution, might allow one to adapt the theory of the decay of nonlinear Oseen flows (solutions to (1.9)), as presented in [21, section IX.8], for example, to nonlinear rotational flows (solutions to (1.5)). But this is a subject we do not take up here.

There is another aspect of our theory we deem interesting. Due to Lemma 2.16 and Theorem 2.17 (decomposition of the Guenther–Thomann fundamental solution into the usual Stokes fundamental solution and a less singular part), we may possibly provide an access to a potential-theoretic approach to (1.2). The starting point of such a theory would be to consider a boundary integral equation consisting of the same terms as in the well-known Stokes case, plus a compact perturbation. We refer to [8] for a theory on boundary integral equations related to the Stokes system, and to [5] for a way to adapt some elements of this theory to the Oseen system. Arguments similar to those in [5] may be used in the context of (1.2).

As for other previous articles besides [7], [22], [24], [25], [27] pertaining to (1.2), (1.5) or to the time-dependent counterparts of these equations, we mention [10], [11], [12], [13], [14], [15], [16], [17], [23], [26], [28], [29], [30], [31], [32], [33], [34], [40], [41], [42]. Additional references may be found in [22].

It is perhaps interesting to briefly indicate some of the various approaches used in the preceding references in order to tackle (1.2) or (1.5) or the corresponding time-dependent equations. In [24], [25], a main idea consists in reducing a boundary value problem for (1.2) to the Oseen system in the whole space $\mathbb{R}^3$. That latter system was then handled by using the well-known Oseen fundamental solution mentioned above and studied in [35], for example. As remarked before, the work at hand makes use of the Guenther–Thomann fundamental solution to (1.2). Other papers deal with (1.2) or (1.5) in a weighted Sobolev space setting. One may distinguish two variants of this approach. The first one uses variational calculus in L^2-spaces. This method has been applied in [9] by Farwig and in [36, 37] by Kračmar and Penel in order to solve the scalar model equations

$$-\nu\,\Delta u + k\partial_3 u = f \quad \text{in } \Omega$$

and

$$-\nu\,\Delta u + k\partial_3 u - a\cdot\nabla u = f \text{ in } \Omega,$$

respectively, under the boundary conditions $u = 0$ on $\partial\Omega$ and $u(x) \to 0$ as $|x| \to \infty$. Here Ω is an exterior domain, and a is a function that may be nonconstant and nonsolenoidal. By Kračmar, Nečasová, and Penel [34], this theory was extended to (1.2) in an L^2-framework with anisotropic weights, yielding a positive answer to the existence of wake. The second approach involves more general weights in L^q-spaces, weighted multiplier and Littlewood–Paley theory, as well as the theory of one-sided Muckenhoupt weights corresponding to one-sided maximal functions. This method was first introduced by Farwig, Hishida, and Müller [14] (zero velocity at infinity) and Farwig [10], [11] (nonzero velocity at infinity) for the case that no weight is present, and then extended to the weighted case by Farwig, Krbec, and Nečasová [15], [16] and Nečasová and Schumacher [42].

Pointwise estimates, even for solutions of the nonlinear Navier–Stokes equations, can be found in [23]. Indeed, according to this latter reference, there exists a stationary strong solution of the nonlinear problem with the velocity part u of this solution satisfying the estimate $|u(x)| \le \frac{c}{|x|}$. This result must be considered with regard to the fact that the corresponding fundamental solution of (1.2) cannot be dominated by $|x-y|^{-1}$; see [14]. Moreover, this pointwise estimate suggests discussing (1.2) in weak L^q-spaces ($L^{3/2,\infty}$ and $L^{3,\infty}$) as done in [13], [30]. Stability estimates in the L^2-setting are proved in [25], and in the $L^{3,\infty}$-setting in [31].

2. Notation, definitions, and auxiliary results. If $A \subset \mathbb{R}^3$, we write A^c for the complement $\mathbb{R}^3\backslash A$ of A. The symbol $|\ |$ denotes the Euclidean norm of $\mathbb{R}^3$ and also the length of a multi-index from $\mathbb{N}_0^3$, that is, $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$ for $\alpha \in \mathbb{N}_0^3$. The open ball centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. Put $e_1 := (1,0,0)$. Let $x \times y$ denote the usual vector product of $x, y \in \mathbb{R}^3$. Set $p' := (1-1/p)^{-1}$ for $p \in (1,\infty)$.

We fix parameters $\tau \in (0,\infty)$, $\varrho \in \mathbb{R}\backslash\{0\}$, and we set $\omega := \varrho e_1$ and

$$s_\tau(x) := 1 + \tau\,(|x| - x_1) \quad \text{for } x \in \mathbb{R}^3.$$

Define the matrix $\Omega \in \mathbb{R}^{3\times 3}$ by

$$\Omega := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

so that $\omega \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$. By the symbol $\mathfrak{C}$, we denote constants depending only on τ or ω. We write $\mathfrak{C}(\gamma_1, \ldots, \gamma_n)$ for constants that additionally depend on parameters $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$ for some $n \in \mathbb{N}$.

Let $\mathfrak{D}$ be an open bounded set in $\mathbb{R}^3$ with C^2-boundary $\partial\mathfrak{D}$. This set will be kept fixed throughout. We denote its outward unit normal by $n^{(\mathfrak{D})}$. For $T \in (0,\infty)$, put $\mathfrak{D}_T := B_T\backslash\overline{\mathfrak{D}}$ ("truncated exterior domain").

For $p \in [1,\infty)$, $k \in \mathbb{N}$, and for open sets $A \subset \mathbb{R}^3$, we write $W^{k,p}(A)$ for the usual Sobolev space of order k and exponent p. Its standard norm will be denoted by $\|\ \|_{k,p}$. If $B \subset \mathbb{R}^3$ is open, define $W^{k,p}_{loc}(B)$ as the set of all functions $g : B \mapsto \mathbb{R}$ such that $g|U \in W^{k,p}(U)$ for any open bounded set $U \subset \mathbb{R}^3$ with $\overline{U} \subset B$. Also we will need the fractional order Sobolev space $W^{2-1/p,p}(\partial\mathfrak{D})$ equipped with its intrinsic norm, which we denote by $\|\ \|_{2-1/p,\,p}$ $\big(p \in (1,\infty)\big)$; see [39] for the corresponding definitions. If $\mathfrak{H}$ is a normed space whose norm is denoted by $\|\ \|_{\mathfrak{H}}$, and if $n \in \mathbb{N}$, we equip the product space $\mathfrak{H}^n$ with a norm $\|\ \|^{(n)}_{\mathfrak{H}}$ defined by $\|v\|^{(n)}_{\mathfrak{H}} := \big(\sum_{j=1}^n \|v_j\|^2_{\mathfrak{H}}\big)^{1/2}$ for $v \in \mathfrak{H}^n$. But for simplicity we will write $\| \,\|_{\mathfrak{H}}$ instead of $\| \,\|^{(n)}_{\mathfrak{H}}$. Concerning the term $s_\tau(x)$, we will need the following estimates.

LEMMA 2.1 (see [9, Lemma 4.3]). *Let $\beta \in (1,\infty)$. Then*

$$\int_{\partial B_r} s_\tau(x)^{-\beta}\, do_x \le \mathfrak{C}(\beta)\, r \quad \textit{for } r \in (0,\infty).$$

LEMMA 2.2 (see [6, Lemma 4.8]). *For $x, y \in \mathbb{R}^3$, we have*

$$s_\tau(x-y)^{-1} \;\le\; \mathfrak{C}(1+|y|)\, s_\tau(x)^{-1}.$$

LEMMA 2.3 (see [4, Lemma 2]). *Let $S \in (0,\infty),\ x \in B_S,\ t \in (0,\infty)$. Then*

$$|x - \tau\, t\, e_1|^2 + t \ge \mathfrak{C}(S)\,(|x|^2 + t).$$

LEMMA 2.4. *Let $S \in (0,\infty),\ x \in B_S^c$. Then $|x| \ge \mathfrak{C}(S)\, s_\tau(x)$.*

Proof. $|x| \ge S/2 + |x|/2 \ge S/2 + (|x| - x_1)/4 \ge \min\{S/2,\, 1/(4\tau)\}\, s_\tau(x)$. ◻

Let K denote the usual fundamental solution to the heat equation, that is,

$$K(x,t) := (4\pi t)^{-3/2}\, e^{-|x|^2/(4t)} \quad \text{for } x \in \mathbb{R}^3,\ t \in (0,\infty).$$

We recall the definition of the Kummer function ${}_1F_1(1,c,u)$, which is given by

$${}_1F_1(1,c,u) := \sum_{n=0}^{\infty} \big(\,\Gamma(c)/\Gamma(n+c)\,\big)\, u^n \quad \text{for } u \in \mathbb{R},\ c \in (0,\infty),$$

where the letter Γ denotes the usual gamma function. We will need the following estimates of ${}_1F_1(1,5/2,u)$ and K.

THEOREM 2.5 (see [38]). *Let $S \in (0,\infty)$. Then there is $C(S) > 0$ such that for $k \in \{0,\,1,\,2\}$,*

$$\begin{aligned} &\big|d^k/du^k\big(\,e^{-u}\ {}_1F_1(1,5/2,u)\,\big)\big| \le C(S)\, u^{-3/2-k} \quad \textit{for } u \in [S,\infty),\\ &\big|d^k/du^k\, {}_1F_1(1,5/2,u)\big| \le C(S) \quad \textit{for } u \in [-S,S]. \end{aligned}$$

LEMMA 2.6 (see [45]). *For $\alpha \in \mathbb{N}_0^3,\ l \in \mathbb{N}_0$ with $|\alpha| + 2l \le 2$, there is $C > 0$ such that*

$$\big|\partial_x^\alpha \partial_t^l K(x,t)\big| \le C\,(|x|^2+t)^{-3/2-|\alpha|/2-l} \quad \textit{for } x \in \mathbb{R}^3,\ t \in (0,\infty).$$

Of course, analogous estimates hold for $\partial_x^\alpha \partial_t^l K(x,t)$ with $|\alpha| + 2l > 2$ (with a constant depending on $|\alpha| + 2l$), but the inequality stated in Lemma 2.6 is sufficient for our purposes. A similar remark may be made with respect to the inequalities in Theorem 2.5. Next we put

$$\begin{aligned} &\mathfrak{H}_{jk}(x) := x_j\, x_k\, |x|^{-2} \quad \text{for } x \in \mathbb{R}^3\backslash\{0\},\\ &\Lambda_{jk}(x,t) := K(x,t)\Big(\delta_{jk} - \mathfrak{H}_{jk}(x) -\ {}_1F_1\big(\,1, 5/2,\, |x|^2/(4t)\,\big)\,\big(\,\delta_{jk}/3 - \mathfrak{H}_{jk}(x)\,\big)\Big) \end{aligned}$$

for $x \in \mathbb{R}^3\backslash\{0\},\ t \in (0,\infty),\ j,k \in \{1,\,2,\,3\}$. Further put

$$\big(\,\Gamma_{jk}(y,z,t)\,\big)_{1\le j,k\le 3} := \big(\,\Lambda_{rs}(y - \tau\, t\, e_1 - e^{-t\Omega}\cdot z,\ t)\,\big)_{1\le r,s\le 3} \cdot e^{-t\Omega}$$

for $y, z \in \mathbb{R}^3,\ t \in (0,\infty)$ with $y - \tau t e_1 - e^{-t\Omega} \cdot z \neq 0$. The function $(\Gamma_{jk})_{1\le j,k\le 3}$ is the velocity part of the fundamental solution introduced by Guenther and Thomann for the time-dependent variant of (1.1), here adapted to the time-dependent variant of (1.2). As explained in [7], the functions Γ_{jk} may be considered as smooth functions in $\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)$.

LEMMA 2.7 (see [7, Corollary 3.1]). *The functions Λ_{jk} and Γ_{jk} may be extended continuously to $\mathbb{R}^3 \times (0,\infty)$ and $\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)$, respectively, and these extensions are C^∞-functions ($1 \le j,k \le 3$).*

In particular we will always consider Λ_{jk} and Γ_{jk} as functions defined on $\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)$. We further set

$$E_{4j}(x) := (4\pi)^{-1} x_j\, |x|^{-3} \quad (1 \le j \le 3,\ x \in \mathbb{R}^3 \backslash \{0\}).$$

Among the properties of Γ_{jk} proved in [27], we will use the following ones.

THEOREM 2.8 (see [27, Theorem 1.3, Proposition 4.1]). *Let $j,k \in \{1, 2, 3\}$, $y,z \in \mathbb{R}^3$. Then*

$$\begin{aligned}(2.1)\quad &\partial_t \Gamma_{jk}(y,z,t) - \Delta_z \Gamma_{jk}(y,z,t) - \tau\, \partial z_1 \Gamma_{jk}(y,z,t) + (\omega \times z)\cdot \nabla_z \Gamma_{jk}(y,z,t) \\ &\qquad - \big[\, \omega \times \big(\Gamma_{js}(y,z,t) \big)_{1\le s\le 3} \big]_k = 0 \quad \big(t \in (0,\infty) \big),\end{aligned}$$

$$(2.2)\quad \Gamma_{jk}(y,z,t) \to -\partial_k E_{4j}(y-z) \quad \textit{for } t \downarrow 0 \ \textit{ if } y \neq z.$$

Concerning the matrix Ω, we observe

LEMMA 2.9. *Let $x \in \mathbb{R}^3,\ t \in \mathbb{R}$. Then*

$$|e^{t\Omega} \cdot x| = |x|, \quad (e^{t\Omega} \cdot x)_1 = x_1, \quad e^{t\Omega} \cdot e_1 = e_1.$$

Proof. For the first equation, we refer to [7, Lemma 2.3]. The second and third immediately follow from the relation

$$\Omega = \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \qquad \square$$

Due to Lemma 2.9, we get the following.

LEMMA 2.10.

$$(2.3)\qquad \big(\Gamma_{jk}(y,z,t) \big)_{1\le j,k\le 3} = e^{-t\Omega} \cdot \big(\Lambda_{rs}(e^{t\Omega}\cdot y - \tau t e_1 - z,\ t) \big)_{1\le r,s\le 3}$$

for $y,z \in \mathbb{R}^3,\ t \in (0,\infty)$.

The ensuing lemma, proved in [7], is crucial for estimating $\int_0^\infty |\Gamma_{jk}(y,z,t)|\, dt$ when y and z are close to each other.

LEMMA 2.11 (see [7, Lemma 2.3]). *Let $R \in (0,\infty)$. Then there are constants $C_1, C_2 \in (0,\infty)$, depending on R, τ, and ω, such that for $y,z \in B_R$ with $y \neq z$, $t \in (0, C_2]$ with $t \le C_1 |y-z|$, we have*

$$|y - \tau t e_1 - e^{-t\Omega}\cdot z| \ge |y-z|/12.$$

Note that in [7, Lemma 2.3], constants C_1, C_2 with the above properties were given explicitly in terms of R, τ, and ω. The ensuing Lemmas 2.12 to 2.14 were proved in [7], except inequalities (2.4) and (2.6), which are obvious consequences of Lemma 2.12.

LEMMA 2.12 (see [7, Lemma 3.2]). *For* $j,k \in \{1,2,3\}$, $x,y,z \in \mathbb{R}^3$, $t \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *the inequalities*

$$|\partial_x^\alpha \Lambda_{jk}(x,t)| \le \mathfrak{C}(|x|^2+t)^{-3/2-|\alpha|/2},$$
$$|\partial_y^\alpha \Gamma_{jk}(y,z,t)| + |\partial_z^\alpha \Gamma_{jk}(y,z,t)| \le \mathfrak{C}(|y-\tau t e_1 - e^{-t\Omega}\cdot z|^2+t)^{-3/2-|\alpha|/2}$$

hold.

LEMMA 2.13 (see [7, Theorem 3.1]). *Let* $k \in \{0,1\}$, $R \in (0,\infty)$, $y,z \in B_R$ *with* $y \ne z$. *Then*

$$\int_0^\infty \big(\,|y-\tau t e_1 - e^{-t\Omega}\cdot z|^2 + t\,\big)^{-3/2-k/2}\,dt \le \mathfrak{C}(R)\,|y-z|^{-1-k}.$$

Due to Lemma 2.12, *this means for* y,z *as above and for* $j,k \in \{1,2,3\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$ *that*

$$\int_0^\infty \big(\,|\partial_y^\alpha \Gamma_{jk}(y,z,t)| + |\partial_z^\alpha \Gamma_{jk}(y,z,t)|\,\big)\,dt \le \mathfrak{C}(R)\,|y-z|^{-1-|\alpha|}.$$

Let $x \in \mathbb{R}^3\backslash\{0\}$, *and take* j,k,α *as in the preceding inequality. Then*

$$\int_0^\infty |\partial_x^\alpha \Lambda_{jk}(x,t)|\,dt \le \mathfrak{C}|x|^{-1-|\alpha|}. \tag{2.4}$$

LEMMA 2.14 (see [7, Lemma 3.3]). *Let* $R \in (0,\infty)$, $y \in B_R$, $\epsilon \in (0,\infty)$ *with* $\overline{B_\epsilon(y)} \subset B_R$, $z \in B_R\backslash B_\epsilon(y)$, $x \in B_\epsilon^c$, $t \in (0,\infty)$, $j,k \in \{1,2,3\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$. *Then*

$$|\partial_y^\alpha \Gamma_{jk}(y,z,t)| + |\partial_z^\alpha \Gamma_{jk}(y,z,t)| \le \mathfrak{C}(R,\epsilon)\,(\chi_{(0,1]}(t) + \chi_{(1,\infty)}(t)\,t^{-3/2}), \tag{2.5}$$

$$|\partial_x^\alpha \Lambda_{jk}(x,t)| \le \mathfrak{C}(\epsilon)\,(\chi_{(0,1]}(t) + \chi_{(1,\infty)}(t)\,t^{-3/2}). \tag{2.6}$$

In view of Lemma 2.13, we may define

$$\mathfrak{Z}_{jk}(y,z) := \int_0^\infty \Gamma_{jk}(y,z,t)\,dt, \quad \mathfrak{Y}_{jk}(x) := \int_0^\infty \Lambda_{jk}(x,t)\,dt \tag{2.7}$$

for $x \in \mathbb{R}^3\backslash\{0\}$, $y,z \in \mathbb{R}^3$ with $y \ne z$, $j,k \in \{1,2,3\}$. The function $(\mathfrak{Z}_{jk})_{1\le j,k\le 3}$ is the fundamental solution of (1.2) proposed by Guenther and Thomann in [27].

LEMMA 2.15. *Let* $j,k \in \{1,2,3\}$. *Then* $\mathfrak{Z}_{jk} \in C^1\big((\mathbb{R}^3\times\mathbb{R}^3)\backslash\{(x,x) : x \in \mathbb{R}^3\}\big)$, $\mathfrak{Y}_{jk} \in C^1(\mathbb{R}^3\backslash\{0\})$,

$$\partial y_n \mathfrak{Z}_{jk}(y,z) = \int_0^\infty \partial y_n \Gamma_{jk}(y,z,t)\,dt, \tag{2.8}$$
$$\partial z_n \mathfrak{Z}_{jk}(y,z) = \int_0^\infty \partial z_n \Gamma_{jk}(y,z,t)\,dt,$$
$$\partial_n \mathfrak{Y}_{jk}(x) = \int_0^\infty \partial x_n \Lambda_{jk}(x,t)\,dt$$

for $y,z \in \mathbb{R}^3$ *with* $y \ne z$, $x \in \mathbb{R}^3\backslash\{0\}$, $n \in \{1,2,3\}$.

If $R \in (0,\infty)$, $y,z \in B_R$ *with* $y \ne z$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *we have*

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)| + |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z)| \le \mathfrak{C}(R)\,|y-z|^{-1-|\alpha|}. \tag{2.9}$$

Proof. Let $U,\ V \subset \mathbb{R}^3$ be open and bounded, with $\overline{U} \cap \overline{V} \neq \emptyset$. Then $\epsilon := dist(U,V) > 0$, and there is $R>0$ with $\overline{U} \cup \overline{V} \subset B_R$. Therefore inequality (2.5) holds for $y \in U,\ z \in V,\ t \in (0,\infty)$. Since $\int_0^\infty \big(\chi_{(0,1)}(t) + \chi_{(1,\infty)}(t)\, t^{-3/2}\big)\, dt < \infty$, and in view of Lemma 2.7, the continuous differentiability of $\mathfrak{Z}_{jk}$ as well as the first two equations in (2.8) follow by Lebesgue's theorem on dominated convergence. Estimate (2.9) is a consequence of (2.8) and Lemma 2.13. Analogous arguments hold for $\mathfrak{Y}_{jk}$. ☐

Next, in Lemma 2.16 and Theorems 2.17 and 2.18, we prove some technical points that were only stated but not shown in [7]. They constituted a major obstacle in the proof of a representation formula for smooth functions $u : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}^3$ in terms of $L(u) + \nabla \pi,\ \operatorname{div} u$, and $u|\partial\mathfrak{D}$ (Theorem 4.3). This obstacle consisted in finding a leading term in a decomposition of $\partial z_n \mathfrak{Z}_{jk}(y,z)$ such that the remainder term is weakly singular with respect to surface integrals in $\mathbb{R}^3$. The interest of such a decomposition will become apparent in the proof of Theorem 2.18. The leading term in question is in fact the function $\mathfrak{Y}_{jk}(y-z)$, which turns out to coincide with the usual fundamental solution of the Stokes system.

LEMMA 2.16. *Let* $j,k \in \{1,\,2,\,3\},\ x \in \mathbb{R}^3\backslash\{0\}$. *Then*

$$\mathfrak{Y}_{jk}(x) = (8\pi|x|)^{-1}(\delta_{jk} + x_j x_k |x|^{-2}).$$

Proof. Abbreviate $\mathfrak{F}(u) := {}_1F_1(1,5/2,u)$ for $u \in \mathbb{R}$. Then

$$\begin{aligned}
(2.10)\ \ &\mathfrak{Y}_{jk}(x)\\
&= \big(\delta_{jk} - \mathfrak{H}_{jk}(x)\big) \int_0^\infty K(x,t)\, dt\\
&\qquad + \big(-\delta_{jk}/3 + \mathfrak{H}_{jk}(x)\big)(4\pi)^{-3/2} \int_0^\infty t^{-3/2} e^{-|x|^2/(4t)}\, \mathfrak{F}\big(|x|^2/(4t)\big)\, dt\\
&= (4|x|)^{-1}\pi^{-3/2} \Big(\big(\delta_{jk} - \mathfrak{H}_{jk}(x)\big) \int_0^\infty s^{-3/2} e^{-1/s}\, ds\\
&\qquad\qquad + \big(-\delta_{jk}/3 + \mathfrak{H}_{jk}(x)\big) \int_0^\infty s^{-3/2} e^{-1/s}\, \mathfrak{F}(1/s)\, ds\Big)\\
&= (4|x|)^{-1}\pi^{-3/2} \Big(\big(\delta_{jk} - \mathfrak{H}_{jk}(x)\big) \int_0^\infty t^{-1/2} e^{-t}\, dt\\
&\qquad\qquad + \big(-\delta_{jk}/3 + \mathfrak{H}_{jk}(x)\big) \int_0^\infty t^{-1/2} e^{-t}\, \mathfrak{F}(t)\, dt\Big).
\end{aligned}$$

But $\int_0^\infty t^{-1/2} e^{-t}\, dt = \pi^{1/2}$ by a result about the gamma function. Therefore, using the abbreviation

$$A := (1/4)\pi^{-3/2} \int_0^\infty t^{-1/2} e^{-t}\, \mathfrak{F}(t)\, dt,$$

we conclude from (2.10) that

$$(2.11)\ \ \mathfrak{Y}_{jk}(x) = (4\pi|x|)^{-1}\big(\delta_{jk} - \mathfrak{H}_{jk}(x)\big) + A|x|^{-1}\big(-\delta_{jk}/3 + \mathfrak{H}_{jk}(x)\big).$$

But $\int_0^\infty t^{-1/2} e^{-t}\, \mathfrak{F}(t)\, dt = 3\pi^{1/2}/2$, as follows by some standard properties of the gamma function and by the equation $\sum_{n=1}^\infty \big((2n-1)(2n+1)\big)^{-1} = 1/2$. Therefore $A = 3(8\pi)^{-1}$, so the lemma may be deduced from (2.11). ☐

The ensuing theorem will imply that $\nabla(\mathfrak{Z}_{jk} - \mathfrak{Y}_{jk})$ is indeed weakly singular with respect to surface integrals in $\mathbb{R}^3$.

THEOREM 2.17. *Let* $R \in (0,\infty)$, $y, z \in B_R$ *with* $y \neq z$, $j,k,n \in \{1, 2, 3\}$. *Then*

$$\int_0^\infty |\partial z_n \Gamma_{jk}(y,z,t) - \partial z_n \Lambda_{jk}(y-z,t)|\, dt \le \mathfrak{C}(R)\,|y-z|^{-3/2}. \tag{2.12}$$

Proof. Abbreviate $\epsilon := \min\{C_1\,|y-z|,\, C_2\}$, with C_1, C_2 from Lemma 2.11. Further abbreviate

$$\psi(y,z,t) := e^{t\Omega}\cdot y - \tau\, t\, e_1 - z \quad \text{for } t\in(0,\infty), \qquad \mathfrak{F}(u) := {}_1F_1(1,5/2,u) \quad \text{for } u \in \mathbb{R}.$$

Recalling the choice of ϵ and referring to Lemmas 2.9 and 2.11, we find for $t \in (0,\epsilon)$, $\vartheta \in [0,1]$ that

$$|\psi(y,z,\vartheta t)| = |y - \tau\,\vartheta\, t\, e_1 - e^{-\vartheta\, t\,\Omega}\cdot z| \ge \mathfrak{C}|y-z| \ge \mathfrak{C}\epsilon. \tag{2.13}$$

(Note that in the corresponding inequality [7, (3.7)], the term $y + tU - e^{-t\Omega}\cdot z$ was mistakenly replaced by the letter x.) Starting from (2.3), we split the left-hand side of (2.12) in the following way:

$$\begin{aligned}(2.14)\quad &\int_0^\infty |\partial z_n \Gamma_{jk}(y,z,t) - \partial z_n \Lambda_{jk}(y-z,t)|\, dt \\ &\le \sum_{\nu=1}^{9}\int_0^\epsilon \mathfrak{N}_\nu(t)\,dt + \int_\epsilon^\infty |\partial z_n \Gamma_{jk}(y,z,t)|\,dt + \int_\epsilon^\infty |\partial z_n \Lambda_{jk}(y-z,t)|\,dt,\end{aligned}$$

with

$$\begin{aligned}
\mathfrak{N}_1(t) &:= \Big|\sum_{l=1}^{3}\big(\,(e^{-t\Omega})_{jl} - \delta_{jl}\,\big)\,\partial z_n\Big(\Lambda_{lk}\big(\,\psi(y,z,t),\, t\,\big)\Big)\Big|,\\
\mathfrak{N}_2(t) &:= \Big|\partial z_n\Big(K\big(\,\psi(y,z,t),\, t\,\big) - K(y-z,t)\Big)\Big(\delta_{jk} - \mathfrak{H}_{jk}\big(\,\psi(y,z,t)\,\big)\Big)\Big|,\\
\mathfrak{N}_3(t) &:= \Big|\partial z_n\Big(-K\big(\,\psi(y,z,t),\, t\,\big)\mathfrak{F}\big(\,|\psi(y,z,t)|^2/(4t)\,\big)\\
&\qquad\qquad + K(y-z,t)\mathfrak{F}\big(\,|y-z|^2/(4t)\,\big)\Big)\Big(\delta_{jk}/3 - \mathfrak{H}_{jk}\big(\,\psi(y,z,t)\,\big)\Big)\Big|,\\
\mathfrak{N}_4(t) &:= \Big|\partial z_n\big(\,K(y-z,t)\,\big)\Big(\mathfrak{H}_{jk}\big(\,\psi(y,z,t)\,\big) - \mathfrak{H}_{jk}(y-z)\Big)\Big|,\\
\mathfrak{N}_5(t) &:= \Big|\partial z_n\Big(K(y-z,t)\mathfrak{F}\big(\,|y-z|^2/(4t)\,\big)\Big)\Big(\mathfrak{H}_{jk}\big(\,\psi(y,z,t)\,\big) - \mathfrak{H}_{jk}(y-z)\Big)\Big|.
\end{aligned}$$

The terms $\mathfrak{N}_6(t)$ to $\mathfrak{N}_9(t)$ are defined in the same way as $\mathfrak{N}_2(t)$ to $\mathfrak{N}_5(t)$, respectively, but with the derivative ∂z_n acting on the second factor instead of the first. For example, in the definition of the term $\mathfrak{N}_6(t)$, the derivative is applied to the factor $\delta_{jk} - \mathfrak{H}_{jk}\big(\,\psi(y,z,t)\,\big)$, instead of $K\big(\,\psi(y,z,t),\,t\,\big) - K(y-z,t)$ as in the definition of $\mathfrak{N}_2(t)$.

In order to estimate $\mathfrak{N}_1(t)$, we observe that the eigenvalues of the matrix Ω are 0, $i\,|\omega|$, and $-i\,|\omega|$. Therefore there is an invertible matrix $A \in \mathbb{C}^{3\times 3}$ such that

$$\Omega = A\cdot\begin{pmatrix} 0 & 0 & 0\\ 0 & i\,|\omega| & 0\\ 0 & 0 & -i\,|\omega|\end{pmatrix}\cdot A^{-1},$$

and hence

$$e^{-t\Omega} = A\cdot\begin{pmatrix} 1 & 0 & 0\\ 0 & e^{-i\,t\,|\omega|} & 0\\ 0 & 0 & e^{i\,t\,|\omega|}\end{pmatrix}\cdot A^{-1},$$

so for $r, s \in \{1, 2, 3\}$,

$$|(e^{-t\Omega})_{rs} - \delta_{rs}| \le \mathfrak{C}\big(\,|1-\cos(|\omega|t)| + |\sin(|\omega|t)|\,\big) \le \mathfrak{C}t.$$

Therefore, with Lemma 2.12 and (2.13),

$$\mathfrak{N}_1(t) \le \mathfrak{C}t\big(\,|\psi(y,z,t)|^2 + t\,\big)^{-2} \le \mathfrak{C}|\psi(y,z,t)|^{-2} \le \mathfrak{C}\epsilon^{-2},$$

and hence $\int_0^\epsilon \mathfrak{N}_1(t)\,dt \le \mathfrak{C}\epsilon^{-1}$. In view of estimating $\mathfrak{N}_2(t)$ to $\mathfrak{N}_9(t)$, we observe that

$$|\partial^\beta \mathfrak{H}_{jk}(x)| \le \mathfrak{C}|x|^{-|\beta|} \quad \text{for } x \in \mathbb{R}^3\backslash\{0\},\ \beta \in \mathbb{N}_0^3 \text{ with } |\beta| \le 2; \tag{2.15}$$

$$\begin{aligned} |\partial\vartheta\,\big(\,|\psi(y,z,\vartheta t)|^2\,\big)| &= \Big|\sum_{m=1}^{3} 2\,\psi(y,z,\vartheta t)_m\,t\,(\Omega\cdot e^{\vartheta t\Omega}\cdot y - \tau\,e_1)_m\Big| \\ &\le \mathfrak{C}|\psi(y,z,\vartheta t)|\,t\,(1+|y|) \le \mathfrak{C}(R)|\psi(y,z,\vartheta t)|\,t \quad \text{for } t\in(0,\epsilon),\ \vartheta\in[0,1]. \end{aligned} \tag{2.16}$$

Similarly,

$$|\partial\vartheta\big(\,\psi(y,z,\vartheta t)_s\,\big)| \le \mathfrak{C}(R)t \tag{2.17}$$

for t, ϑ as before and for $s \in \{1, 2, 3\}$. In order to obtain an estimate of $\mathfrak{N}_2(t)$, we apply (2.17), (2.15), and Lemma 2.6 to get

$$\begin{aligned} \mathfrak{N}_2(t) &\le \mathfrak{C}\Big|\int_0^1 \sum_{s=1}^{3} \partial z_s \partial z_n\Big(K\big(\,\psi(y,z,\vartheta t),\,t\,\big)\Big)\,\partial\vartheta\,\big(\,\psi(y,z,\vartheta t)_s\,\big)\,d\vartheta\Big| \\ &\le \mathfrak{C}(R)\int_0^1 \big(\,|\psi(y,z,\vartheta t)|^2 + t\,\big)^{-5/2} t\,d\vartheta \quad \text{for } t\in(0,\epsilon). \end{aligned}$$

By referring to (2.13), we may conclude that $\mathfrak{N}_2(t) \le \mathfrak{C}(R)\,(\epsilon^2+t)^{-3/2}$ for $t\in(0,\epsilon)$, so $\int_0^\epsilon \mathfrak{N}_2(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-1}$. Similar arguments yield that $\int_0^\epsilon \mathfrak{N}_6(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-3/2}$. Turning to $\mathfrak{N}_3(t)$, we find that

$$\begin{aligned} &\mathfrak{N}_3(t) \\ &\le \mathfrak{C}t^{-3/2}\Big|\int_0^1 \partial\vartheta\,\partial z_n\Big(e^{-|\psi(y,z,\vartheta t)|^2/(4t)}\,\mathfrak{F}\big(\,|\psi(y,z,\vartheta t)|^2/(4t)\,\big)\Big)\,d\vartheta\Big| \\ &= \mathfrak{C}t^{-3/2}\Big|\int_0^1 \Big([e^{-u}\mathfrak{F}(u)]''_{|u=|\psi(y,z,\vartheta t)|^2/(4t)}\,\psi(y,z,\vartheta t)_n\,(2t)^{-1} \\ &\qquad\qquad \times \partial\vartheta\,\big(\,|\psi(y,z,\vartheta t)|^2\,\big)\,(4t)^{-1} \\ &\qquad + [e^{-u}\mathfrak{F}(u)]'_{|u=|\psi(y,z,\vartheta t)|^2/(4t)}\,\partial\vartheta\big(\,\psi(y,z,\vartheta t)_n\,\big)\,(2t)^{-1}\Big)\,d\vartheta\Big| \\ &\le \mathfrak{C}(R)\,t^{-3/2}\int_0^1 \Big(\big|\,[e^{-u}\mathfrak{F}(u)]''\,\big|_{|u=|\psi(y,z,\vartheta t)|^2/(4t)}\,|\psi(y,z,\vartheta t)|^2\,t^{-1} \\ &\qquad + \big|\,[e^{-u}\mathfrak{F}(u)]'\,\big|_{|u=|\psi(y,z,\vartheta t)|^2/(4t)}\Big)\,d\vartheta \\ &\le \mathfrak{C}(R)\,t^{-3/2}\int_0^1 \big(\,\chi_{(0,1]}(u)\,(u+1) + \chi_{(1,\infty)}(u)\,u^{-5/2}\,\big)_{|u=|\psi(y,z,\vartheta t)|^2/(4t)}\,d\vartheta \\ &\le \mathfrak{C}(R)\,t^{-3/2}\int_0^1 u^{-1}\ {}_{|u=|\psi(y,z,\vartheta t)|^2/(4t)}\,d\vartheta \le \mathfrak{C}(R)\,t^{-1/2}\int_0^1 |\psi(y,z,\vartheta t)|^{-2}\,d\vartheta. \end{aligned} \tag{2.18}$$

Note that we applied (2.15) in the first inequality. In the second, we used (2.16) and (2.17), whereas in the third, we applied Theorem 2.5. Concerning the next-to-last inequality, we chose the upper bound u^{-1} in order to obtain suitable negative powers of t and $|\psi(y, z, \vartheta t)|$. Making use of (2.13), we may conclude that

$$\int_0^\epsilon \mathfrak{N}_3(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-2}\int_0^\epsilon t^{-1/2}\,dt \le \mathfrak{C}(R)\,\epsilon^{-3/2}.$$

By exactly the same references and techniques, one may show that

$$\int_0^\epsilon \mathfrak{N}_7(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-3/2}.$$

Next we observe that by (2.15), (2.17), and (2.13),

$$\begin{aligned}(2.19)\qquad &\Big|\partial z_n\Big(\mathfrak{H}_{jk}\big(\,\psi(y,z,t)\,\big) - \mathfrak{H}_{jk}(y-z)\Big)\Big| \\ &= \Big|\int_0^1 \sum_{s=1}^3 \partial z_s \partial z_n\Big(\mathfrak{H}_{jk}\big(\,\psi(y,z,\vartheta t)\,\big)\Big)\,\partial\vartheta\,\big(\,\psi(y,z,\vartheta t)_s\,\big)\,d\vartheta\Big| \\ &\le \mathfrak{C}(R)\int_0^1 |\psi(y,z,\vartheta t)|^{-2}\,t\,d\vartheta \le \mathfrak{C}(R)\,\epsilon^{-2}\,t.\end{aligned}$$

Now we get with Lemma 2.6 that

$$\mathfrak{N}_8(t) \le \mathfrak{C}(R)\,(|y-z|^2+t)^{-3/2}\,\epsilon^{-2}\,t \le \mathfrak{C}(R)\,\epsilon^{-2}\,t^{-1/2} \quad \text{for } \ t\in(0,\epsilon),$$

so that $\int_0^\epsilon \mathfrak{N}_8(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-3/2}$. A similar reasoning yields for $t\in(0,\epsilon)$ that

$$\mathfrak{N}_4(t) \le \mathfrak{C}(R)\,(\epsilon^2+t)^{-2}\,\epsilon^{-1}\,t \le \mathfrak{C}(R)\,(\epsilon^2+t)^{-3/2}\,\epsilon^{-1/2},$$

and hence $\int_0^\epsilon \mathfrak{N}_4(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-3/2}$. We find with Theorem 2.5 and (2.19) that

$$\begin{aligned}&\mathfrak{N}_9(t) \le \mathfrak{C}(R)\,t^{-3/2}\,\big|\,e^{-u}\,\mathfrak{F}(u)\,\big|_{|u=|y-z|^2/(4\,t)}\,\epsilon^{-2}\,t \\ &\le \mathfrak{C}(R)\,\epsilon^{-2}\,t^{-1/2}\,\big(\,\chi_{(0,1]}(u) + \chi_{(1,\infty)}(u)\,u^{-3/2}\,\big)_{|u=|y-z|^2/(4\,t)} \le \mathfrak{C}(R)\,\epsilon^{-2}\,t^{-1/2}\end{aligned}$$

for $t\in(0,\epsilon)$, and hence $\int_0^\epsilon \mathfrak{N}_9(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-3/2}$. In the same way we get $\int_0^\epsilon \mathfrak{N}_5(t)\,dt \le \mathfrak{C}(R)\,\epsilon^{-3/2}$. It is an immediate consequence of Lemma 2.12 that

$$\int_\epsilon^\infty |\partial z_n \Gamma_{jk}(y,z,t)|\,dt + \int_\epsilon^\infty |\partial z_n \Lambda_{jk}(y-z,t)|\,dt \le \int_\epsilon^\infty t^{-2}\,dt \le \mathfrak{C}\,\epsilon^{-1}.$$

Thus, in view of (2.14), we have shown that the left-hand side of (2.12) is bounded by $\mathfrak{C}(R)\,\epsilon^{-3/2}$. But since $|y-z|\le 2\,R$, and by the choice of ϵ, we have $\epsilon \ge \mathfrak{C}(R)\,|y-z|$, so inequality (2.12) follows. □

THEOREM 2.18. *Let* $j,k\in\{1,2,3\}$, $y\in\mathbb{R}^3$, $\epsilon_0>0$, $\mu\in(0,1)$, *and* $w\in C^\mu\big(\overline{B_{\epsilon_0}(y)}\big)$. *Then*

$$(2.20)\quad \int_{\partial B_\epsilon(y)} \sum_{m=1}^3 \partial z_m \mathfrak{Z}_{jk}(y,z)\,(y-z)_m/\epsilon\, w(z)\,do_z \to 2\,\delta_{jk}\,w(y)/3 \quad (\epsilon\downarrow 0).$$

Proof. We choose $R > 0$ with $\overline{B_{\epsilon_0}(y)} \subset B_R$. For $\epsilon \in (0, \epsilon_0]$, we observe that the difference of the left- and right-hand sides of (2.20) is bounded by $\sum_{\nu=1}^{3} \mathfrak{N}_\nu(\epsilon)$, with

$$\mathfrak{N}_1(\epsilon) := \int_{\partial B_\epsilon(y)} \sum_{m=1}^{3} |\partial z_m \mathfrak{Z}_{jk}(y,z)| |w(z) - w(y)| \, do_z,$$

$$\mathfrak{N}_2(\epsilon) := |w(y)| \sum_{m=1}^{3} \int_{\partial B_\epsilon(y)} |\partial z_m \mathfrak{Z}_{jk}(y,z) - \partial z_m \mathfrak{Y}_{jk}(y-z)| \, do_z,$$

$$\mathfrak{N}_3(\epsilon) := \Big| w(y) \int_{\partial B_\epsilon(y)} \sum_{m=1}^{3} \partial z_m \mathfrak{Y}_{jk}(y-z)\,(y-z)_m/\epsilon \, do_z - 2\,\delta_{jk}\, w(y)/3 \Big|.$$

Put

$$[w]_\mu := \sup\{|w(z) - w(z')|\,|z - z'|^{-\mu} \;:\; z, z' \in \overline{B_{\epsilon_0}(y)},\; z \neq z'\}.$$

Let $\epsilon \in (0, \epsilon_0]$. Then with (2.9) we find

$$\mathfrak{N}_1(\epsilon) \le \mathfrak{C}(R)\,[w]_\mu \int_{\partial B_\epsilon(y)} |y - z|^{-2+\mu} \, do_z \le \mathfrak{C}(R)\,[w]_\mu\, \epsilon^\mu.$$

Moreover, referring to (2.8) and to Theorem 2.17, we get

$$\mathfrak{N}_2(\epsilon) \le \mathfrak{C}(R)\,|w(y)| \int_{\partial B_\epsilon(y)} |y - z|^{-3/2} \, do_z \le \mathfrak{C}(R)\,|w(y)|\,\epsilon^{1/2}.$$

Using Lemma 2.16 and noting that $\int_{\partial B_1} \eta_r\, \eta_s \; do_\eta = 4\pi\,\delta_{rs}/3$ for $r, s \in \{1,\,2,\,3\}$, we find

$$\begin{aligned}
&\int_{\partial B_\epsilon(y)} \sum_{m=1}^{3} \partial z_m \mathfrak{Y}_{jk}(y-z)\,(y-z)_m/\epsilon \, do_z \\
&= (8\pi)^{-1} \int_{\partial B_1} \sum_{m=1}^{3} \big(\delta_{jk}\, \eta_m^2 - \delta_{jm}\, \eta_k\, \eta_m - \delta_{km}\, \eta_j\, \eta_m + 3\,\eta_j\, \eta_k\, \eta_m^2 \big) \, do_\eta \\
&= 2\,\delta_{jk}/3,
\end{aligned}$$

so that $\mathfrak{N}_3(\epsilon) = 0$. Letting ϵ tend to zero, we obtain the theorem. □

To end this chapter, we estimate $\mathfrak{Z}_{jk}(y,z)$ in the case that $|z| \le S,\ |y| \ge (1+\delta)\,S$, with $\delta, S > 0$ considered as given quantities. This estimate will play a crucial role in the following.

THEOREM 2.19. *Let* $S, \delta \in (0, \infty),\ \nu \in (1, \infty)$. *Then*

$$\int_0^\infty (|y - \tau\, t\, e_1 - e^{-t\,\Omega} \cdot z|^2 + t)^{-\nu} \, dt \le \mathfrak{C}(S, \delta, \nu)\, \big(\, |y|\, s_\tau(y)\, \big)^{-\nu+1/2} \tag{2.21}$$

for $y \in B^c_{(1+\delta)\,S},\ z \in \overline{B_S}$. *In particular,*

$$|\partial_y^\alpha \mathfrak{Z}(y,z)| + |\partial_z^\alpha \mathfrak{Z}(y,z)| \le \mathfrak{C}(S, \delta)\, \big(\, |y|\, s_\tau(y)\, \big)^{-1-|\alpha|/2} \tag{2.22}$$

for y, z *as above,* $j, k \in \{1,\,2,\,3\},\ \alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$. *Moreover*

$$|\partial_y^\alpha \mathfrak{Z}(y,z)| + |\partial_z^\alpha \mathfrak{Z}(y,z)| \le \mathfrak{C}(S, \delta)\, \big(\, |z|\, s_\tau(z)\, \big)^{-1-|\alpha|/2} \tag{2.23}$$

for $z \in B^c_{(1+\delta)\,S},\ y \in \overline{B_S}$, *and for* j, k, α *as in* (2.22).

Proof. Take $y \in B^c_{(1+\delta)\,S}$, $z \in \overline{B_S}$. We abbreviate $y' := (y_2, y_3)$. In what follows, we will make frequent use of the equation $|e^{-t\,\Omega} \cdot z| = |z|$; see Lemma 2.9. We will distinguish several cases. To begin with, we suppose that $|y| \le 8\,S$. Then, for $t \in (0,\infty)$, we get

$$|y - \tau\, t\, e_1 - e^{-t\,\Omega} \cdot z|^2 + t \ge \mathfrak{C}(S)\,(|y - e^{-t\,\Omega} \cdot z|^2 + t), \tag{2.24}$$

where we used Lemma 2.3 with $9\,S$ instead of S. But since $|y| \ge (1+\delta)\,S$, $|z| \le S$, we have $|y - e^{-t\,\Omega} \cdot z| \ge |y| - |z| \ge \delta\,S$, so that from (2.24),

$$|y - \tau\, t\, e_1 - e^{-t\,\Omega} \cdot z|^2 + t \ge \mathfrak{C}(S,\delta)\,(1+t) \quad \text{for } \ t \in (0,\infty),$$

and hence

$$\begin{aligned} \int_0^\infty (|y - \tau\, t\, e_1 - e^{-t\,\Omega} \cdot z|^2 + t)^{-\nu}\, dt &\le \mathfrak{C}(S,\delta,\nu) \int_0^\infty (1+t)^{-\nu}\, dt \\ \le \mathfrak{C}(S,\delta,\nu) \le \mathfrak{C}(S,\delta,\nu)\,|y|^{-2\,\nu+1} &\le \mathfrak{C}(S,\delta,\nu)\,\big(\,|y|\,s_\tau(y)\,\big)^{-\nu+1/2}, \end{aligned} \tag{2.25}$$

with the third inequality following from the assumption $|y| \le 8\,S$, and the last one from Lemma 2.4. In the rest of this proof, we suppose that $|y| \ge 8\,S$. We note that

$$\int_0^\infty (|y - \tau\, t\, e_1 - e^{-t\,\Omega} \cdot z|^2 + t)^{-\nu}\, dt = \tau^{-1} \int_0^\infty \big(\,\gamma(y,z,r)^2 + r/\tau\,\big)^{-\nu}\, dr, \tag{2.26}$$

where we used the abbreviation $\gamma(y,z,r) := |y - r\, e_1 - e^{-(r/\tau)\,\Omega} \cdot z|$ for $r \in (0,\infty)$. In view of the assumption $|y| \ge 8\,S$, another easy case arises if $y_1 \le 0$. In fact, we then have

$$\gamma(y,z,r) \ge |y - r\, e_1| - |z| \ge (|y|^2 + r^2)^{1/2} - S \ge |y|/2 + r/2 - S \ge |y|/4 + r/2$$

for $r \in (0,\infty)$, so that $\gamma(y,z,r)^2 \ge \mathfrak{C}\,(|y| + r)^2$, and hence

$$\begin{aligned} \int_0^\infty \big(\,\gamma(y,z,r)^2 + r/\tau\,\big)^{-\nu}\, dr &\le \mathfrak{C}(\nu) \int_0^\infty (|y| + r)^{-2\,\nu}\, dr \\ &\le \mathfrak{C}(\nu)\,|y|^{-2\,\nu+1} \le \mathfrak{C}(S,\nu)\,\big(\,|y|\,s_\tau(y)\,\big)^{-\nu+1/2}, \end{aligned} \tag{2.27}$$

where the last inequality is a consequence of Lemma 2.4. A similar argument holds if $0 \le y_1 \le |y|/2$. In fact, since $|y| = (y_1^2 + |y'|^2)^{1/2}$, we then have $|y'| = (|y|^2 - y_1^2)^{1/2} \ge (3\,|y|^2/4)^{1/2} \ge |y|/2$, so we get for $r \in (0,\infty)$ that

$$\begin{aligned} \gamma(y,z,t) &\ge |y - r\,e_1|/2 + |y - r\,e_1|/2 - |z| \ge |y - r\,e_1|/2 + |y'|/2 - S \\ &\ge |y - r\,e_1|/2 + |y|/4 - S \ge |y_1 - r|/2 + |y|/8, \end{aligned}$$

where the last inequality follows from the assumption $|y| \ge 8\,S$. We thus get

$$\begin{aligned} \int_0^\infty \big(\,\gamma(y,z,r)^2 + r/\tau\,\big)^{-\nu}\, dr &\le \mathfrak{C}(\nu) \int_0^\infty (|y| + |y_1 - r|)^{-2\,\nu}\, dr \\ \le \mathfrak{C}(\nu) \int_{y_1}^\infty (|y| + r - y_1)^{-2\,\nu}\, dr &\le \mathfrak{C}(\nu)\,|y|^{-2\,\nu+1} \le \mathfrak{C}(S,\nu)\,\big(\,|y|\,s_\tau(y)\,\big)^{-\nu+1/2}. \end{aligned} \tag{2.28}$$

The last of the preceding inequalities follows from Lemma 2.4. From now on we suppose that $y_1 \ge |y|/2$. We thus work under the assumption that $y_1 \ge |y|/2 \ge 4\,S$. Then we note

$$\int_0^\infty \big(\,\gamma(y,z,r)^2 + r/\tau\,\big)^{-\nu}\, dr \le \mathfrak{A}_1 + \mathfrak{A}_2, \tag{2.29}$$

with

$$\mathfrak{A}_1 := \int_{y_1-2S}^{y_1+2S} \big(\gamma(y,z,r)^2 + r/\tau\big)^{-\nu}\, dr,$$

and with $\mathfrak{A}_2$ defined in the same way as $\mathfrak{A}_1$, but with the domain of integration $(y_1 - 2S,\, y_1 + 2S)$ replaced by $(0,\infty)\backslash(y_1 - 2S,\, y_1 + 2S)$. We observe that for $r \in (y_1 - 2S,\, y_1 + 2S)$,

$$r \ge y_1 - 2S \ge |y|/2 - 2S \ge |y|/4,$$

because $y_1 \ge |y|/2,\ |y| \ge 8S$. Therefore

$$\mathfrak{A}_1 \le \int_{y_1-2S}^{y_1+2S} (r/\tau)^{-\nu}\, dr \le \mathfrak{C}(\nu)\,|y|^{-\nu} \int_{y_1-2S}^{y_1+2S} dr \le \mathfrak{C}(S,\nu)\,|y|^{-\nu}. \tag{2.30}$$

On the other hand, for $r \in (0,\infty)\backslash(y_1 - 2S,\, y_1 + 2S)$, we have

$$\gamma(y,z,r) \ge |y - r\,e_1| - |z| \ge |y_1 - r| - S \ge |y_1 - r|/2 + |y_1 - r|/2 - S \ge |y_1 - r|/2,$$

and hence

$$\begin{aligned}
\mathfrak{A}_2 &\le \int_{(0,\infty)\backslash(y_1-2S,\, y_1+2S)} \big((|y_1 - r|/2)^2 + r/\tau\big)^{-\nu}\, dr \\
&\le \mathfrak{C}(\nu) \int_0^\infty (|y_1 - r| + r^{1/2})^{-2\nu}\, dr \\
&\le \mathfrak{C}(\nu) \Big(\int_0^{y_1/2} (y_1/2)^{-2\nu}\, dr + \int_{y_1/2}^\infty \big(\,|y_1 - r| + (y_1/2)^{1/2}\big)^{-2\nu}\, dr \Big) \\
&\le \mathfrak{C}(\nu) \Big(y_1^{-2\nu+1} + \int_{\mathbb{R}} (|y_1 - r| + y_1^{1/2})^{-2\nu}\, dr \Big) \le \mathfrak{C}(\nu)\,(y_1^{-2\nu+1} + y_1^{-\nu+1/2}) \\
&\le \mathfrak{C}(S,\nu)\,|y|^{-\nu+1/2},
\end{aligned} \tag{2.31}$$

with the last inequality following from the assumption $y_1 \ge |y|/2 \ge 4S$. Combining (2.29)–(2.31) yields

$$\int_0^\infty \big(\gamma(y,z,r)^2 + r/\tau\big)^{-\nu}\, dr \le \mathfrak{C}(S,\nu)\,|y|^{-\nu+1/2}.$$

Therefore, if $\tau\,(|y| - y_1) \le \max\{1,\, 2\tau S\}$, we have

$$\int_0^\infty \big(\gamma(y,z,r)^2 + r/\tau\big)^{-\nu}\, dr \le \mathfrak{C}(S,\nu)\big(\,|y|\,s_\tau(y)\big)^{-\nu+1/2}. \tag{2.32}$$

Thus we are reduced to the case

$$\tau\,(|y| - y_1) \ge \max\{1,\, 2\tau S\}, \quad y_1 \ge |y|/2 \ge 4S.$$

Using the relations $\tau\,(|y| - y_1) \ge 1,\ y_1 \ge 0$, we observe that

$$|y|\,s_\tau(y) \le |y|\,2\tau\,(|y| - y_1) = 2\tau\,|y|\,|y'|^2/(|y| + y_1) \le 2\tau\,|y'|^2. \tag{2.33}$$

We further observe that for $r \in (0,\infty)\backslash(y_1 - 2S,\, y_1 + 2S)$,

$$\begin{aligned}
&\gamma(y,z,r) \ge |y - r\,e_1| - |z| \ge |y - r\,e_1|/2 + |y_1 - r|/2 - S \ge |y - r\,e_1|/2 \\
&\ge |y_1 - r|/4 + |y'|/4,
\end{aligned}$$

so that

$$\mathfrak{A}_2 \le \mathfrak{C}(\nu) \int_{\mathbb{R}} (|y_1 - r| + |y'|)^{-2\nu} \le \mathfrak{C}(\nu)\, |y'|^{-2\nu+1} \le \mathfrak{C}(\nu) \big(|y|\, s_\tau(y) \big)^{-\nu+1/2}, \tag{2.34}$$

with the last inequality following from (2.33). Using (2.33) again, and recalling that $\tau(|y| - y_1) \ge 2\tau S,\ |y| \ge 4S$, we find for $r \in (0, \infty)$ that

$$\gamma(y, z, r) \ge |y - r\,e_1| - |z| \ge |y'| - S \ge |y'|/2 + \big((2\tau)^{-1} |y|\, s_\tau(y) \big)^{1/2}/2 - S$$
$$\ge |y'|/2 + (|y|\, S)^{1/2}/2 - S \ge |y'|/2 + (4S^2)^{1/2}/2 - S = |y'|/2.$$

It follows that

$$\mathfrak{A}_1 \le \mathfrak{C}(\nu)\, |y'|^{-2\nu} \int_{y_1 - 2S}^{y_1 + 2S} dr \le \mathfrak{C}(S, \nu)\, |y'|^{-2\nu} \le \mathfrak{C}(S, \nu) \big(|y|\, s_\tau(y) \big)^{-\nu} \le \mathfrak{C}(S, \nu) \big(|y|\, s_\tau(y) \big)^{-\nu+1/2}, \tag{2.35}$$

where inequality (2.33) was used once more. By (2.29), (2.34), and (2.35), we see that inequality (2.32) holds also in the case $\tau(|y| - y_1) \ge \max\{1, 2\tau S\},\ y_1 \ge |y|/2 \ge 4S$. Inequality (2.21) follows with (2.25)–(2.28) and (2.32). As concerns estimate (2.22), it is an immediate consequence of (2.8), Lemma 2.12, and (2.21) with $\nu = -3/2 - |\alpha|/2$. This leaves us to deal with (2.23). In this respect, we remark that the only property of Ω we used in the preceding proof is the relation $|e^{-t\Omega} \cdot x| = |x|$ for $x \in \mathbb{R}^3,\ t \in (0, \infty)$ (Lemma 2.9). Since this relation holds, of course, for any $t \in \mathbb{R}$, and because by Lemma 2.9,

$$|y - t\tau\, e_1 - e^{-\tau\Omega} \cdot z| = | - z - t\tau\, e_1 - e^{t\Omega} \cdot (-y)| \quad (y, z \in \mathbb{R}^3,\ t \in \mathbb{R}),$$

we see that we have proved (2.21) also for $z \in B^c_{(1+\delta)S}$ and $y \in \overline{B_S}$, but with y replaced by z on the right-hand side. Now inequality (2.23) follows with (2.8) and Lemma 2.12. ∎

3. Some volume potentials. The representation formula we have in mind contains volume and surface potentials (Theorem 4.6). In the present section, we study the volume potentials which will arise. There are two types of such potentials, involving the kernels $\mathfrak{Z}_{jk}$ and E_{4j}, respectively. We begin by considering the potential related to $\mathfrak{Z}_{jk}$.

LEMMA 3.1. *Let $p \in (1, \infty),\ q \in (1, 2),\ f \in L^p_{loc}(\mathbb{R}^3)^3$ with $f|B^c_S \in L^q(B^c_S)^3$ for some $S \in (0, \infty)$. Then, for $j, k \in \{1, 2, 3\},\ \alpha \in \mathbb{N}^3_0$ with $|\alpha| \le 1$, we have*

$$\int_{\mathbb{R}^3} |\partial^\alpha_y \mathfrak{Z}_{jk}(y, z)|\, |f_k(z)|\, dy < \infty \quad \text{for a.e. } y \in \mathbb{R}^3. \tag{3.1}$$

We define $\mathfrak{R}(f) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by

$$\mathfrak{R}_j(f)(y) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} \mathfrak{Z}_{jk}(y, z)\, f_k(z)\, dz$$

for $y \in \mathbb{R}^3$ such that (3.1) holds; otherwise we set $\mathfrak{R}_j(f)(y) := 0$ $(1 \le j \le 3)$. Then $\mathfrak{R}(f) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$ and

$$\partial_l \mathfrak{R}_j(f)(y) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} \partial y_l \mathfrak{Z}_{jk}(y, z)\, f_k(z)\, dz \tag{3.2}$$

for $j, l \in \{1, 2, 3\}$ *and for a.e.* $y \in \mathbb{R}^3$. *Moreover, for* $R \in (0, \infty)$ *we have*

$$\|\mathfrak{R}(f|B_R)\,|\,B_R\|_p \le \mathfrak{C}(R,p)\,\|f|B_R\|_p. \tag{3.3}$$

Proof. Take j, k, α as in (3.1). Let $R \in (0, \infty)$. Then we find with (2.9) that

$$\int_{B_R} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,dz \le \mathfrak{C}(R) \int_{B_R} |y-z|^{-1-|\alpha|}\,dz \le \mathfrak{C}(R) \int_{B_{2\,R}(y)} |y-z|^{-1-|\alpha|}\,dz \le \mathfrak{C}(R)$$

for $y \in B_R$, and analogously $\int_{B_R} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,dy \le \mathfrak{C}(R)$ for $z \in B_R$. It follows by Hölder's inequality that

$$\begin{aligned}(3.4)\quad &\Big(\int_{B_R}\Big(\int_{B_R} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,|f_k(z)|\,dz\Big)^p dy\Big)^{1/p} \\ &\le \Big(\int_{B_R}\Big(\int_{B_R} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,dz\Big)^{p-1}\Big(\int_{B_R} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,|f(z)|^p\,dz\Big)dy\Big)^{1/p} \\ &\le \mathfrak{C}(R,p)\Big(\int_{B_R}\int_{B_R} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,|f(z)|^p\,dz\,dy\Big)^{1/p} \le \mathfrak{C}(R,p)\,\|f|B_R\|_p.\end{aligned}$$

This means in particular that the integral $\int_{B_n} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,|f_k(z)|\,dz$ is finite for a.e. $y \in B_n,\ n \in \mathbb{N}$, and that inequality (3.3) is proved.

Once again take j, k, α as in (3.1), and let $n \in \mathbb{N}$ with $n \ge S$. Then, using (2.23) with S replaced by $n/2$ and with $\delta = 1/2$, we find for $y \in B_{n/2}$ that

$$\begin{aligned}(3.5)\quad &\int_{B_n^c} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z)|\,|f_k(z)|\,dz \le \mathfrak{C}(n) \int_{B_n^c} \big(\,|z|\,s_\tau(z)\,\big)^{-1-|\alpha|/2} |f(z)|\,dz \\ &\le \mathfrak{C}(n)\Big(\int_{B_n^c} \big(\,|z|\,s_\tau(z)\,\big)^{-q'}\,dz\Big)^{1/q'} \|f|B_n^c\|_q \le \mathfrak{C}(n,q)\,\|f|B_S^c\|_q,\end{aligned}$$

where the last inequality holds due to Theorem 2.1 and the assumption $q < 2$ (hence $q' > 2$). We thus have shown that the relation in (3.1) holds for a.e. $y \in B_{n/2}$. Since this is true for any $n \in \mathbb{N}$ with $n \ge S$, (3.1) is proved. We deduce from (3.4) and (3.5) that

$$\int_{B_{n/2}} \int_{\mathbb{R}^3} \Big|\sum_{k=1}^3 \partial_y^\alpha \mathfrak{Z}_{jk}(y,z)\,f_k(z)\Big|\,dz\,dy \le \mathfrak{C}(n,p,q)\,(\|f|B_n\|_p + \|f|B_S^c\|_q) \tag{3.6}$$

for $n \in \mathbb{N}$ with $n \ge S$. This means that $\mathfrak{R}_j(f) \in L_{1,loc}(\mathbb{R}^3)$ and that the function associating a.e. $y \in \mathbb{R}^3$ with the integral $\int_{\mathbb{R}^3} \sum_{k=1}^3 \partial y_l \mathfrak{Z}_{jk}(y,z)\,f_k(z)\,dz$ also belongs to $L_{1,loc}(\mathbb{R}^3)$ for $1 \le l \le 3$. Now take $\Phi \in C_0^\infty(\mathbb{R}^3)^3$. Then, by (3.6) and because the support of Φ is compact,

$$\int_{\mathbb{R}^3} \partial_l \Phi(y) \mathfrak{R}_j(f)(y)\,dy = \sum_{k=1}^3 \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \setminus B_\epsilon(z)} \partial_l \Phi(y)\,\mathfrak{Z}_{jk}(y,z)\,dy\,f_k(z)\,dz. \tag{3.7}$$

But for any $\epsilon > 0$, we may perform a partial integration in the inner integral on the right-hand side of (3.7) (first statement in Lemma 2.15). Due to (2.9), the term with a surface integral on $\partial B_\epsilon(z)$ arising in this way tends to zero for $\epsilon \downarrow 0$. (Note that for

$\epsilon \in (0,1]$, say, and for $y \in \partial B_\epsilon(z)$, the integral with respect to z extends over B_{n+1} only if $n \in \mathbb{N}$ is chosen so large that $\mathrm{supp}(\Phi) \subset B_n$.) After letting ϵ tend to zero, we obtain an equation which implies that $\mathfrak{R}_j(f) \in W^{1,1}_{loc}(\mathbb{R}^3)$ and (3.2) holds. ∎

LEMMA 3.2. *Take p, q, f as in Lemma 3.1, and suppose in addition that $p > 3/2$. Then the relation in (3.1) holds for any $y \in \mathbb{R}^3$ (without the restriction "a.e."), and the function $\mathfrak{R}(f)$ is continuous.*

The general approach for proving this lemma seems to be well known, but we cannot give a reference (although similar results were shown in [20, section II.9]). So, for the convenience of the reader, we provide a proof.

Proof. We show that $\mathfrak{R}(f)$ is continuous. The relation in (3.1) for any $y \in \mathbb{R}^3$ may be established by a similar but simpler argument.

Let $j \in \{1, 2, 3\}$, $R \in (S, \infty)$. It suffices to prove that $\mathfrak{R}_j(f) \,|\, B_R$ is continuous. But for $z \in B^c_{2R}$, $y \in B_R$, we get by (2.23) that

$$\Big|\sum_{k=1}^{3} \mathfrak{Z}_{jk}(y,z)\, f_k(z)\Big| \le \mathfrak{C}(R)\big(\,|z|\, s_\tau(z)\,\big)^{-1} |f(z)|.$$

Since by a computation as in (3.5) the function

$$\mathbb{R}^3 \ni z \mapsto \chi_{B^c_{2R}}(z)\big(\,|z|\, s_\tau(z)\,\big)^{-1} |f(z)| \in [0,\infty)$$

is integrable, we may conclude in view of the first statement of Lemma 2.15 that the integral $\int_{B^c_{2R}} \sum_{k=1}^{3} \mathfrak{Z}_{jk}(y,z)\, f_k(z)\, dz$ as a function of $y \in B_R$ is continuous. Thus we still have to show that the function

$$I(y) := \int_{B_{2R}} \sum_{k=1}^{3} \mathfrak{Z}_{jk}(y,z)\, f_k(z)\, dz \quad (y \in B_R)$$

is continuous as well. So take y, $y' \in B_R$ with $y \ne y'$. Then

$$|I(y) - I(y')| \le \mathfrak{N}_1 + \mathfrak{N}_2, \tag{3.8}$$

with

$$\mathfrak{N}_1 := \sum_{x \in \{y,\, y'\}} \int_{B_R \cap A} \sum_{k=1}^{3} |\mathfrak{Z}_{jk}(x,z)\, f_k(z)|\, dz,$$

$$\mathfrak{N}_2 := \int_{B_R \setminus A} \Big| \int_0^1 \sum_{k,l=1}^{3} \partial x_l \mathfrak{Z}_{jk}(x,z)_{|x=y'+\vartheta\,(y-y')}\,(y-y')_l\, d\vartheta \Big|\, |f_k(z)|\, dz,$$

with $A := B_{2\,|y-y'|}(y)$. We get with (2.9) that

$$\begin{aligned}\mathfrak{N}_1 &\le \mathfrak{C}(R) \sum_{x \in \{y,y'\}} \int_{B_R \cap A} |x-z|^{-1} |f(z)|\, dz \\ &\le \mathfrak{C}(R) \sum_{x \in \{y,y'\}} \Big(\int_{B_{3\,|y-y'|}(x)} |x-z|^{-p'}\, dz \Big)^{1/p'} \|f|B_R\|_p.\end{aligned}$$

Since $p > 3/2$, hence $p' < 3$, we may conclude that $\mathfrak{N}_1 \le \mathfrak{C}(R)\,|y-y'|^{-1+3/p'}\, \|f|B_R\|_p$, with $-1+3/p' > 0$. In order to estimate $\mathfrak{N}_2$, we note that

$$|y' + \vartheta\,(y-y') - z| \ge |y-z| - |y-y'| \ge |y-z|/2 \ge |y-y'|$$

for $z \in \mathbb{R}^3 \backslash A,\ \vartheta \in [0,1]$. Therefore with (2.9), if $2p' > 3$,

$$\mathfrak{N}_2 \le \mathfrak{C}(R)\,|y-y'|\Big(\int_{B_R \backslash A} |y-z|^{-2\,p'}\,dz\Big)^{1/p'}\,\|f|B_R\|_p$$
$$\le \mathfrak{C}(R)\,|y-y'|^{-1+3/p'}\,\|f|B_R\|_p.$$

In the case $2p' < 3$, the factor $|y-y'|^{-1+3/p'}$ on the right-hand side of the preceding inequality may be replaced by $|y-y'|$, and in the case $2p' = 3$ by $|y-y'|\ln\big(|y-y'|/(2R)\big)$. In view of (3.8), we have thus shown that $I(y)$ is a continuous function of $y \in B_R$. This completes the proof of Lemma 3.2. ∎

The crucial idea of the proof of the next theorem consists in reducing an estimate of $\mathfrak{R}(f)$ to an estimate of a convolution integral involving an upper bound of an Oseen fundamental solution. This latter integral may be handled by a reference to [35].

THEOREM 3.3. *Let* $S,\ S_1,\ \gamma \in (0,\infty)$ *with* $S_1 < S,\ p \in (1,\infty),\ A \in [2,\infty),\ B \in \mathbb{R},\ f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *measurable with*

$$f|B_{S_1} \in L^p(B_{S_1})^3, \quad |f(z)| \le \gamma\,|z|^{-A}\,s_\tau(z)^{-B} \quad \textit{for } z \in B^c_{S_1}, \quad A + \min\{1,B\} \ge 3.$$

Let $i,j \in \{1,\,2,\,3\},\ y \in B^c_S$. *Then*

$$|\mathfrak{R}_j(f)(y)| \le \mathfrak{C}(S,S_1,A,B)\,(\|f|B_{S_1}\|_1 + \gamma)\,\big(\,|y|\,s_\tau(y)\,\big)^{-1}\,l_{A,B}(y), \tag{3.9}$$

$$|\partial_{y_i}\mathfrak{R}_j(f)(y)| \le \mathfrak{C}(S,S_1,A,B)\,(\|f|B_{S_1}\|_1 + \gamma)\,\big(\,|y|\,s_\tau(y)\,\big)^{-3/2}\,s_\tau(y)^{\max(0,\,7/2-A-B)}\,l_{A,B}(y), \tag{3.10}$$

where

$$l_{A,B}(y) = \begin{cases} 1 & \textit{if} \quad A + \min\{1,B\} > 3, \\ \max(1,\ln|y|) & \textit{if} \quad A + \min\{1,B\} = 3. \end{cases}$$

Proof. By (2.22) with $S,\ \delta$ replaced by $S_1,\ S/S_1 - 1$, respectively, we find for $k \in \{1,\,2,\,3\},\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$ that

$$\int_{B_{S_1}} |\partial^\alpha_y \mathfrak{Z}_{jk}(y,z)|\,|f(z)|\,dz \le \mathfrak{C}(S,S_1)\,\big(\,|y|\,s_\tau(y)\,\big)^{-1-|\alpha|/2}\,\|f|B_{S_1}\|_1. \tag{3.11}$$

Recalling Lemmas 2.15, 2.12, and 2.9, we see that

$$\mathfrak{A}_\alpha := \int_{B^c_{S_1}} |\partial^\alpha_y \mathfrak{Z}_{jk}(y,z)|\,|f(z)|\,dz$$
$$\le \mathfrak{C}\gamma \int_0^\infty \int_{B^c_{S_1}} (|y - \tau\,t\,e_1 - e^{-t\,\Omega}\cdot z|^2 + t)^{-3/2-|\alpha|/2}\,|z|^{-A}\,s_\tau(z)^{-B}\,dz\,dt$$
$$= \mathfrak{C}\gamma \int_0^\infty \int_{B^c_{S_1}} (|y - \tau\,t\,e_1 - x|^2 + t)^{-3/2-|\alpha|/2}\,|x|^{-A}\,s_\tau(e^{t\,\Omega}\cdot x)^{-B}\,dx\,dt$$
$$= \mathfrak{C}\gamma \int_{B^c_{S_1}} \int_0^\infty (|y - \tau\,t\,e_1 - x|^2 + t)^{-3/2-|\alpha|/2}\,dt\,|x|^{-A}\,s_\tau(x)^{-B}\,dx,$$

where the last equation holds due to the first and second equations in Lemma 2.9. Now we apply (2.21) with y replaced by $y - x$ and with $z = 0$. Moreover we use

Lemma 2.13. It follows that

$$\mathfrak{A}_\alpha \le \mathfrak{C}(S)\,\gamma\Big(\int_{B^c_{S_1}\cap B_{S/2}(y)} |y-x|^{-1-|\alpha|}\,|x|^{-A}\,s_\tau(x)^{-B}\,dx + \int_{B^c_{S_1}\setminus B_{S/2}(y)} \big(\,|y-x|\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2}|x|^{-A}\,s_\tau(x)^{-B}\,dx\Big). \tag{3.12}$$

Next we observe that for $x \in B_{S/2}(y)$, we have $|x| \ge |y| - |y-x| \ge |y| - S/2 \ge |y|/2$,

$$s_\tau(x)^{-1} \le \mathfrak{C}(1+|y-x|)\,s_\tau(y)^{-1} \le \mathfrak{C}(S)\,s_\tau(y)^{-1}$$

(see Lemma 2.2), and similarly $s_\tau(y)^{-1} \le \mathfrak{C}(S)\,s_\tau(x)^{-1}$. For $x \in B_{S/2}(y)^c$, we find

$$|y-x| = |y-x|/2 + |y-x|/2 \ge S/4 + |y-x|/2 \ge \min\{S/4,\,1/2\}\,(1+|y-x|).$$

Thus, independently of the sign of B, we may conclude from (3.12) that

$$\begin{aligned}\mathfrak{A}_\alpha &\le \mathfrak{C}(S,S_1,A,B)\,\gamma\Big(|y|^{-A}\,s_\tau(y)^{-B}\int_{B_{S/2}(y)} |y-x|^{-1-|\alpha|}\,dx \\ &\quad + \int_{B^c_{S_1}\setminus B_{S/2}(y)} \big(\,(1+|y-x|)\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2}(1+|x|)^{-A}\,s_\tau(x)^{-B}\,dx\Big) \\ &\le \mathfrak{C}(S,S_1,A,B)\,\gamma\Big(|y|^{-A}\,s_\tau(y)^{-B} \\ &\quad + \int_{\mathbb{R}^3} \big(\,(1+|y-x|)\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2}(1+|x|)^{-A}\,s_\tau(x)^{-B}\,dx\Big).\end{aligned} \tag{3.13}$$

In the case $\alpha = 0$, we refer to the proof of [35, Theorem 3.1] and our assumptions on A and B to deduce from (3.13) that

$$\mathfrak{A}_0 \le \mathfrak{C}(S,S_1,A,B)\,\gamma\Big(|y|^{-A}\,s_\tau(y)^{-B} + \big(\,|y|\,s_\tau(y)\,\big)^{-1}\,l_{A,B}(y)\Big). \tag{3.14}$$

But by Lemma 2.4 and because $A - 3/2 > 0,\ A + B \ge A + \min\{1, B\} \ge 3$, we have

$$\begin{aligned}|y|^{-A}\,s_\tau(y)^{-B} &\le \mathfrak{C}(S,A)\,|y|^{-3/2}\,s_\tau(y)^{-A+3/2-B} \\ &\le \mathfrak{C}(S,A)\,|y|^{-3/2}\,s_\tau(y)^{-3/2},\end{aligned} \tag{3.15}$$

so we may conclude from (3.14) that

$$\mathfrak{A}_0 \le \mathfrak{C}(S,S_1,A,B)\,\gamma\,\big(\,|y|\,s_\tau(y)\,\big)^{-1}\,l_{A,B}(y).$$

Inequality (3.9) follows from (3.11) and the preceding estimate. If $|\alpha| = 1$, then (3.13) and the proof of [35, Theorem 3.2] yield

$$\begin{aligned}\mathfrak{A}_\alpha \;&\le\; \mathfrak{C}(S,S_1,A,B)\,\gamma \\ &\quad\Big(|y|^{-A}\,s_\tau(y)^{-B} + \big(\,|y|\,s_\tau(y)\,\big)^{-3/2}\,s_\tau(y)^{\max(0,\,7/2-A-B)}\,l_{A,B}(y)\Big).\end{aligned}$$

Hence with (3.15),

$$\mathfrak{A}_\alpha \le \mathfrak{C}(S,S_1,A,B)\,\gamma\,\big(\,|y|\,s_\tau(y)\,\big)^{-3/2}\,s_\tau(y)^{\max(0,\,7/2-A-B)}\,l_{A,B}(y).$$

This estimate together with (3.11) implies (3.10). $\square$

Now we turn to volume integrals involving the kernel E_{4j}.

LEMMA 3.4. *Let $p \in (1,\infty)$, $q \in (1,3)$, $g \in L^p_{loc}(\mathbb{R}^3)$ with $g|B_S^c \in L^q(B_S^c)$ for some $S \in (0,\infty)$. Then, for $j \in \{1, 2, 3\}$,*

$$\int_{\mathbb{R}^3} |E_{4j}(y-z)|\,|g(z)|\,dy < \infty \quad \textit{for a.e. } y \in \mathbb{R}^3. \tag{3.16}$$

Thus we may define $\mathfrak{S}(g) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by

$$\mathfrak{S}_j(g)(y) := \int_{\mathbb{R}^3} E_{4j}(y-z)\,g(z)\,dz$$

for $y \in \mathbb{R}^3$ such that (3.16) holds, otherwise $\mathfrak{S}_j(g)(y) := 0$ $(1 \le j \le 3)$. Then $\mathfrak{S}(g) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$. For $R \in (0,\infty)$ we have

$$\|\mathfrak{S}(g|B_R)\,|\,B_R\|_p \le \mathfrak{C}(R,p)\,\|g|B_R\|_p. \tag{3.17}$$

If $p > 3$, the relation in (3.16) holds for any $y \in \mathbb{R}^3$ (without the restriction "a.e."), and $\mathfrak{S}(g)$ is continuous.

Proof. Lemma 3.4 may be shown by arguments analogous to those we used to prove Lemmas 3.1 and 3.2, except as concerns the claim $\mathfrak{S}(g) \in W^{1,1}_{loc}(\mathbb{R}^3)^3$. To establish this latter point, a different reasoning based on the Calderón–Zygmund inequality is needed because the derivative $\partial_l E_{4j}$ is a singular kernel in $\mathbb{R}^3$. We refer to [20, section IV.2] for details. □

THEOREM 3.5. *Let $S, S_1, \widetilde{\gamma} \in (0,\infty)$ with $S_1 < S$, $p \in (1,\infty)$, $C \in (5/2,\infty)$, $D \in \mathbb{R}$, $g : \mathbb{R}^3 \mapsto \mathbb{R}$ measurable with*

$$g|B_{S_1} \in L^p(B_{S_1}), \quad |g(z)| \le \widetilde{\gamma}\,|z|^{-C}\,s_\tau(z)^{-D} \quad \textit{for } z \in B^c_{S_1}, \quad C + \min\{1, D\} > 3.$$

Let $j \in \{1, 2, 3\}$, $y \in B_S^c$. Then

$$|\mathfrak{S}_j(g)(y)| \le \mathfrak{C}(S, S_1, C, D)\,(\|g|B_{S_1}\|_1 + \widetilde{\gamma})\,|y|^{-2}. \tag{3.18}$$

If $\operatorname{supp}(g) \subset B_{S_1}$, *we further have*

$$|\partial_n \mathfrak{S}_j(g)(y)| \le \mathfrak{C}(S, S_1)\,\|g\|_1\,|y|^{-3} \quad (1 \le n \le 3). \tag{3.19}$$

Proof. Inequality (3.18) may be proved in the same way as Theorem 3.3, except that the reference to [35, Theorems 3.1 and 3.2] is replaced by [35, Theorem 3.4], and that the argument becomes simpler due to the much simpler structure of the kernel E_{4j} compared to $\mathfrak{Z}_{jk}$. As concerns (3.19), observe that $|y - z| \ge (1 - S_1/S)\,|y|$ for $z \in B_{S_1}$, so if $\operatorname{supp}(g) \subset B_{S_1}$, it is obvious that

$$\mathfrak{S}_j(g)|B_S^c \in C^1(B_S^c), \quad \int_{B_{S_1}} |\partial_l E_{4j}(y-z)|\,|g(z)|\,dz < \infty,$$

$$\partial_l \mathfrak{S}_j(g)(y) = \int_{B_{S_1}} \partial_l E_{4j}(y-z)\,g(z)\,dz \quad (1 \le l \le 3).$$

Inequality (3.19) now follows. □

In the rest of this paper, we will use the following notational convention. If $A \subset \mathbb{R}^3$ is a measurable set and $f : A \mapsto \mathbb{R}^3$ is a measurable function, if $\widetilde{f}$ denotes the zero extension of f to $\mathbb{R}^3$, and if $\widetilde{f}$ satisfies the assumptions of Lemma 3.1, we will write $\mathfrak{R}(f)$ instead of $\mathfrak{R}(\widetilde{f})$. A similar convention is to hold with respect to $\mathfrak{S}(g)$ if $g : A \mapsto \mathbb{R}$ is a measurable function such that its zero extension to $\mathbb{R}^3$ verifies the assumptions of Lemma 3.4.

4. A representation formula. In this section, we will present (Theorem 4.6) and prove the representation formula announced in section 1. We begin by two simple observations related to surface integrals on $\partial\mathfrak{D}_R$ and $\partial\mathfrak{D}$, respectively.

LEMMA 4.1. *Let $R \in (0, \infty)$ with $\overline{\mathfrak{D}} \subset B_R$, $f \in L^1(\partial\mathfrak{D}_R)$, $j, k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Define*

$$F(y) := \int_{\partial\mathfrak{D}_R} \partial_z^\alpha \mathfrak{Z}_{jk}(y, z) f(z)\, do_z, \quad H(y) := \int_{\partial\mathfrak{D}_R} E_{4j}(y - z) f(z)\, do_z$$

for $y \in \mathfrak{D}_R$. Then F and H are continuous. Moreover, let $x \in \mathfrak{D}_R$, and put $\delta_x := dist(\partial\mathfrak{D}_R, x)$. Then

$$|F(x)| + |H(x)| \le \mathfrak{C}(\delta_x, R)\, \|f\|_1. \tag{4.1}$$

Proof. Let $U \subset \mathbb{R}^3$ be open, with $\overline{U} \subset \mathfrak{D}_R$. Then $\delta_U := dist(\overline{U}, \partial\mathfrak{D}_R) > 0$, so we get with (2.9) that

$$|\partial_z^\alpha \mathfrak{Z}_{jk}(y, z) f(z)| \le \mathfrak{C}(R)\, \delta_U^{-1-|\alpha|}\, |f(z)| \quad \text{for } z \in \partial\mathfrak{D}_R.$$

In view of the first statement of Lemma 2.15, we may conclude that F is continuous. From (2.9), we get that $|F(x)| \le \mathfrak{C}(\delta_x, R)\|f\|_1$. Obviously $E_{4j} \in C^\infty(\mathbb{R}^3\backslash\{0\})$ and $|E_{4j}(x)| \le |x|^{-2}$ for $x \in \mathbb{R}^3\backslash\{0\}$, so the function H may be handled in the same way (and even belongs to $C^\infty(\mathfrak{D}_R)$). ∎

LEMMA 4.2. *Let $S \in (0, \infty)$ with $\overline{\mathfrak{D}} \subset B_S$. Let $f \in L^1(\partial\mathfrak{D})$, $g \in L^1(\mathfrak{D})$, $j, k \in \{1, 2, 3\}$, and define*

$$F^{(1)}(y) := \int_{\partial\mathfrak{D}} \mathfrak{Z}_{jk}(y, z) f(z)\, do_z, \quad F^{(2)}(y) := \int_{\mathfrak{D}} \mathfrak{Z}_{jk}(y, z) g(z)\, dz,$$

$$F^{(3)}(y) := \int_{\partial\mathfrak{D}} E_{4j}(y - z) f(z)\, do_z, \quad F^{(4)}(y) := \int_{\mathfrak{D}} \partial_k E_{4j}(y - z) g(z)\, dz$$

for $y \in \overline{\mathfrak{D}}^c$. Then $F^{(i)} \in C^1(\overline{\mathfrak{D}}^c)$ for $1 \le i \le 4$. Put $\delta := dist(\overline{\mathfrak{D}}, \partial B_S)$. Then

$$|\partial^\alpha F^{(i)}(y)| \le \mathfrak{C}(\delta, S)\big(|y|\, s_\tau(y)\big)^{-1-|\alpha|/2}\, \|f\|_1, \tag{4.2}$$

$$|\partial^\alpha F^{(j)}(y)| \le \mathfrak{C}(\delta, S)\big(|y|\, s_\tau(y)\big)^{-1-|\alpha|/2}\, \|g\|_1 \tag{4.3}$$

for $y \in B_S^c$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, $i \in \{1, 3\}$, $j \in \{2, 4\}$.

Proof. Let $U \subset \mathbb{R}^3$ be open and bounded, with $\overline{U} \subset \overline{\mathfrak{D}}^c$. Let $R \in (0, \infty)$ with $\overline{\mathfrak{D}} \cup \overline{U} \subset B_R$. Then an argument as in the proof of Lemma 4.1, based on (2.9) and Lemma 2.15, yields that $F^{(1)}|U \in C^1(U)$, and

$$\partial_l F^{(1)}(y) = \int_{\partial\mathfrak{D}} \partial y_l \mathfrak{Z}_{jk}(y, z) f(z)\, do_z \quad \text{for } y \in U,\ 1 \le l \le 3. \tag{4.4}$$

It follows that $F^{(1)} \in C^1(\overline{\mathfrak{D}}^c)$, and that (4.4) holds for $y \in \overline{\mathfrak{D}}^c$. Put $S_1 := S - \delta/2$. Then $S_1 \in (0, S)$ and $\overline{\mathfrak{D}} \subset B_{S_1}$, so inequality (2.22), with S, δ replaced by S_1, $S/S_1 - 1$, yields

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y, z) f(z)| \le \mathfrak{C}(S, S_1)\big(|y|\, s_\tau(y)\big)^{-1-|\alpha|/2} |f(z)|$$

for $z \in \partial\mathfrak{D},\ y \in B_S^c,\ \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Now we get with (4.4) that

$$|\partial^\alpha F^{(1)}(y)| \le \mathfrak{C}(\delta, S)\,\big(\,|y|\,s_\tau(y)\,\big)^{-1-|\alpha|/2}\,\|f\|_1$$

for y, α as before. The function $F^{(2)}$ may be dealt with in a similar way. As for $F^{(3)}$ and $F^{(4)}$, we note that for $y \in B_S^c$ and $z \in \overline{\mathfrak{D}}$, we have $|y-z| \ge (1 - S_1/S)\,|y|$. This observation and Lemma 2.4 yield the estimates of $F^{(3)}$ and $F^{(4)}$ stated in (4.2) and (4.3), respectively. ∎

In [7, Theorem 4.2], we showed how a smooth function u on a truncated exterior domain $\mathfrak{D}_R$ may be represented in terms of $L(u) + \nabla\pi$, $\operatorname{div} u$, $u|\partial\mathfrak{D}_R$, $\pi|\partial\mathfrak{D}$, and $\nabla u|\partial\mathfrak{D}$ with $\pi : \mathfrak{D}_R \mapsto \mathbb{R}$ also smooth. For the convenience of the reader, we state this result in the ensuing Theorem 4.3 and very briefly indicate its proof, which makes use of Theorem 2.17.

THEOREM 4.3. *Let $R \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset \mathbb{R}^3$, and let $n^{(R)} : \partial B_R \cup \partial\mathfrak{D} \mapsto \mathbb{R}^3$ denote the outward unit normal to $\mathfrak{D}_R$. Suppose that $u \in C^2(\overline{\mathfrak{D}_R})^3$, $\pi \in C^1(\overline{\mathfrak{D}_R})$, and put $F := L(u) + \nabla\pi$. Let $y \in \mathfrak{D}_R$ and $j \in \{1, 2, 3\}$. Then*

$$u_j(y) = \mathfrak{R}_j(F)(y) + \mathfrak{S}_j(\operatorname{div} u)(y) + \int_{\partial\mathfrak{D}_R} \mathfrak{A}_j^{(R)}(u,\pi)(y,z)\, do_z, \tag{4.5}$$

where

$$\begin{aligned} &\mathfrak{A}_j^{(R)}(u,\pi)(y,z) \\ &:= \sum_{k=1}^{3}\Big[\sum_{l=1}^{3}\Big(\mathfrak{Z}_{jk}(y,z)\,\big(\,\partial_l u_k(z) - \delta_{kl}\,\pi(z) + u_k(z)\,(-\tau\, e_1 + \omega\times z)_l\,\big) \\ &\qquad - \partial z_l \mathfrak{Z}_{jk}(y,z)\,u_k(z)\Big)\,n_l^{(R)}(z) \;-\; E_{4j}(y-z)\,u_k(z)\,n_k^{(R)}(z)\Big] \end{aligned} \tag{4.6}$$

for $y \in \mathfrak{D}_R,\ z \in \partial\mathfrak{D}_R$.

Indication of a proof. Let $\epsilon \in (0,\infty)$ with $\overline{B_\epsilon(y)} \subset \mathfrak{D}_R$, and consider the integral

$$A_{j,\epsilon} := \int_{\mathfrak{D}_R\backslash B_\epsilon(y)} \sum_{k=1}^{3} \mathfrak{Z}_{jk}(y,z)\,\big(\,L(u) + \nabla\pi)_k(z)\; dz.$$

By performing some integrations by parts, using (2.1), integrating with respect to t, and then exploiting (2.2), we obtain

$$A_{j,\epsilon} = \int_{\mathfrak{D}_R\backslash B_\epsilon(y)} -E_{4j}(y-z)\operatorname{div} u(z)\; dz - S_{j,\epsilon}(y),$$

where $S_{j,\epsilon}(y)$ denotes a surface integral defined in the same way as the surface integral on the right-hand side of (4.5), but with $\partial B_R \cup \partial\mathfrak{D} \cup \partial B_\epsilon(y)$ as domain of integration instead of $\partial B_R \cup \partial\mathfrak{D}$, and with $n^{(R)}$ replaced by the outward unit normal to $\mathfrak{D}_R\backslash B_\epsilon(y)$. Equation (4.5) then follows by a passage to the limit $\epsilon \downarrow 0$, with the calculation of $\lim_{\epsilon\downarrow 0} S_{j,\epsilon}(y)$ based on Theorem 2.18. This reasoning requires some applications of Fubini's and Lebesgue's theorems, all of which is made possible by Lemma 2.14. ∎

Our next aim consists in extending (4.5) to functions u and π, which are less regular than C^2 and C^1, respectively. We begin by specifying the type of functions we will consider. From now on we need that $\partial\mathfrak{D}$ is of class C^2. (Theorem 4.3 also holds if $\mathfrak{D}$ is only Lipschitz bounded.)

THEOREM 4.4. *Let $p \in (1,\infty)$. Define $\mathfrak{M}_p$ as the space of all pairs of functions (u,π) such that $u \in W^{2,p}_{loc}(\overline{\mathfrak{D}}^c)^3$, $\pi \in W^{1,p}_{loc}(\overline{\mathfrak{D}}^c)$,*

$$(4.7) \qquad u|\mathfrak{D}_T \in W^{1,p}(\mathfrak{D}_T)^3, \quad \pi|\mathfrak{D}_T \in L^p(\mathfrak{D}_T), \quad u|\partial\mathfrak{D} \in W^{2-1/p,\,p}(\partial\mathfrak{D})^3,$$
$$\operatorname{div} u|\mathfrak{D}_T \in W^{1,p}(\mathfrak{D}_T), \quad L(u) + \nabla\pi|\mathfrak{D}_T \in L^p(\mathfrak{D}_T)^3$$

for some $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$. Then $u|\mathfrak{D}_T \in W^{2,p}(\mathfrak{D}_T)^3$, $\pi|\mathfrak{D}_T \in W^{1,p}(\mathfrak{D}_T)$ for any $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$.

Proof. The theorem follows from the regularity theory for the Stokes system. To be more specific, we first note that our assumptions imply that the relations in (4.7) hold for all $T \in (0,\infty)$ with $\overline{\mathfrak{D}} \subset B_T$. Take such a number T. Let $S \in (T,\infty)$, and choose $\zeta \in C_0^\infty(\mathbb{R}^3)$ with $\zeta|B_T = 1$, $\zeta|B_S^c = 0$. Then

$$(4.8) \qquad \zeta u \,|\, \mathfrak{D}_S \in W^{2,p}_{loc}(\mathfrak{D}_S)^3 \cap W^{1,p}(\mathfrak{D}_S)^3, \quad \zeta\pi \,|\, \mathfrak{D}_S \in W^{1,p}_{loc}(\mathfrak{D}_S) \cap L^p(\mathfrak{D}_S),$$
$$\operatorname{div}(\zeta u) \,|\, \mathfrak{D}_S \in W^{1,p}(\mathfrak{D}_S),$$
$$\zeta u \,|\, \partial\mathfrak{D} = u|\partial\mathfrak{D} \in W^{2-1/p,p}(\partial\mathfrak{D})^3, \text{ and hence } \zeta u \,|\, \partial\mathfrak{D}_S \in W^{2-1/p,p}(\partial\mathfrak{D}_S)^3.$$

Moreover, since $u|\mathfrak{D}_S \in W^{1,p}(\mathfrak{D}_S)^3$, $L(u) + \nabla\pi \,|\, \mathfrak{D}_S \in L^p(\mathfrak{D}_S)^3$, we have $-\Delta u + \nabla\pi \,|\, \mathfrak{D}_S \in L^p(\mathfrak{D}_S)^3$. Once more observing that $u|\mathfrak{D}_S \in W^{1,p}(\mathfrak{D}_S)^3$, $\pi|\mathfrak{D}_S \in L^p(\mathfrak{D}_S)$, we may conclude that

$$(4.9) \qquad -\Delta(\zeta u) + \nabla(\zeta\pi) \,|\, \mathfrak{D}_S \in L^p(\mathfrak{D}_S)^3.$$

Obviously the function ζu is a weak solution of the Stokes system in $\mathfrak{D}_S$ with right-hand side $-\Delta(\zeta u) + \nabla(\zeta\pi) \,|\, \mathfrak{D}_S$, where "weak solution" is meant in the sense of [20, (IV.1.3)]. In view of (4.8) and (4.9), it follows from [20, Lemma IV.6.1, Exercise IV.6.2] that $\zeta u \,|\, \mathfrak{D}_S \in W^{2,p}(\mathfrak{D}_S)^3$, $\zeta\pi \,|\, \mathfrak{D}_S \in W^{1,p}(\mathfrak{D}_S)$. This implies that $u|\mathfrak{D}_T \in W^{2,p}(\mathfrak{D}_T)^3$ and $\pi|\mathfrak{D}_T \in W^{1,p}(\mathfrak{D}_T)$. ☐

Now we are in a position to generalize Theorem 4.3 to pairs of functions $(u,\pi) \in \mathfrak{M}_p$.

THEOREM 4.5. *Let $p \in (1,\infty)$, $(u,\pi) \in \mathfrak{M}_p$, $j \in \{1, 2, 3\}$. Put $F := L(u) + \nabla\pi$. Take R and $n^{(R)}$ as in Theorem 4.3. Then, for a.e. $y \in \mathfrak{D}_R$,*

$$(4.10) \quad u_j(y) = \mathfrak{R}_j(F|\mathfrak{D}_R)(y) + \mathfrak{S}_j(\operatorname{div} u|\mathfrak{D}_R)(y) + \int_{\partial\mathfrak{D}_R} \mathfrak{A}_j^{(R)}(u,\pi)(y,z)\, do_z,$$

with $\mathfrak{A}_j^{(R)}(u,\pi)(y,z)$ defined as in (4.6).

If $p > 3/2$, (4.10) holds for any $y \in \mathfrak{D}_R$ (without the restriction "a.e.").

Proof. By Theorem 4.4, we have $u|\mathfrak{D}_R \in W^{2,p}(\mathfrak{D}_R)^3$ and $\pi|\mathfrak{D}_R \in W^{1,p}(\mathfrak{D}_R)$. Therefore (see [1, (3.18)]) there are sequences (u_n) in $C^\infty(\mathbb{R}^3)^3$ and (π_n) in $C^\infty(\mathbb{R}^3)$ with

$$(4.11) \qquad \|(u - u_n)|\mathfrak{D}_R\|_{2,p} + \|(\pi - \pi_n)|\mathfrak{D}_R\|_{1,p} \to 0.$$

By a standard trace theorem, it follows that $u_k|\partial\mathfrak{D}_R$, $\partial_l u_k|\partial\mathfrak{D}_R$, and $\pi|\partial\mathfrak{D}_R$ belong to $L^1(\partial\mathfrak{D}_R)$, and

$$(4.12) \qquad \|(u - u_n)|\partial\mathfrak{D}_R\|_1 + \|(\partial_l u - \partial_l u_n)|\partial\mathfrak{D}_R\|_1 + \|(\pi - \pi_n)|\partial\mathfrak{D}_R\|_1 \to 0$$

for $n \to \infty$ $(1 \le k, l \le 3)$. Let $y \in \mathfrak{D}_R$. We may conclude from (4.1) and (4.12) that

$$\int_{\partial \mathfrak{D}_R} \mathfrak{A}_j^{(R)}(u_n, \pi_n)(y, z)\, do_z \to \int_{\partial \mathfrak{D}_R} \mathfrak{A}_j^{(R)}(u, \pi)(y, z)\, do_z \quad (n \to \infty), \tag{4.13}$$

where the definition of $\mathfrak{A}_j^{(R)}(u_n, \pi_n)(y, z)$ should be obvious by (4.6). For $n \in \mathbb{N}$, we set $F_n := L(u_n) + \nabla \pi_n$. By (4.11), we have

$$\|(F_n - F)|\mathfrak{D}_R\|_p \to 0, \quad \|\operatorname{div}(u - u_n)|\mathfrak{D}_R\|_p \to 0 \quad (n \to \infty).$$

These relations combined with (3.3) and (3.17) imply

$$\|\mathfrak{R}_j\big((F_n - F)|\mathfrak{D}_R\big)\,|\,\mathfrak{D}_R\|_p + \|\mathfrak{S}_j\big(\operatorname{div}(u_n - u)|\mathfrak{D}_R\big)\,|\,\mathfrak{D}_R\|_p \to 0 \quad (n \to \infty).$$

Passing from L^p-convergence to pointwise convergence of subsequences, and recalling (4.11), we see there is a strictly increasing function $\sigma : \mathbb{N} \mapsto \mathbb{N}$ such that

$$\begin{aligned} &\mathfrak{R}_j(F_{\sigma(n)}|\mathfrak{D}_R)(y) \to \mathfrak{R}_j(F|\mathfrak{D}_R)(y), \\ &\mathfrak{S}_j(\operatorname{div} u_{\sigma(n)}|\mathfrak{D}_R)(y) \to \mathfrak{S}_j(\operatorname{div} u|\mathfrak{D}_R)(y), \quad u_{\sigma(n)}(y) \to u(y) \quad (n \to \infty) \end{aligned} \tag{4.14}$$

for a.e. $y \in \mathfrak{D}_R$. On the other hand, by Theorem 4.3, (4.10) holds with u, π replaced by u_n, π_n, respectively, for $n \in \mathbb{N}$. Therefore we may conclude from (4.13) and (4.14) that (4.10) holds for a.e. $y \in \mathfrak{D}_R$.

Now suppose that $p > 3/2$. Since $(u, \pi) \in \mathfrak{M}_p$ and because of a Sobolev inequality (in the case $p \le 3$), we may conclude that $\operatorname{div} u|\mathfrak{D}_R \in L^q(\mathfrak{D}_R)$ for some $q \in (3, \infty)$. Recalling the relation $F|\mathfrak{D}_R \in L^p(\mathfrak{D}_R)^3$, we thus see by Lemmas 3.2 and 3.4 with $S = R$ that $\mathfrak{R}(F|\mathfrak{D}_R)$ and $\mathfrak{S}(\operatorname{div} u|\mathfrak{D}_R)$ are continuous. Moreover, since $p > 3/2$ and $u|\mathfrak{D}_R \in W^{2,p}(\mathfrak{D}_R)^3$, a Sobolev lemma implies that u may be considered as a continuous function on $\overline{\mathfrak{D}_R}$. According to Lemma 4.1, the function associating the integral $\int_{\partial \mathfrak{D}_R} \mathfrak{A}_j^{(R)}(y, z)\, do_z$ with each $y \in \mathfrak{D}_R$ is also continuous. Thus we may conclude that (4.10) is valid for any $y \in \mathfrak{D}_R$, without the restriction "a.e." ∎

Next we perform the transition from a representation formula on $\mathfrak{D}_R$ to one on $\overline{\mathfrak{D}}^c$. For this step, we only need the decay properties given implicitly by the relations in (4.15).

THEOREM 4.6. *Let $p \in (1, \infty)$, $(u, \pi) \in \mathfrak{M}_p$. Put $F := L(u) + \nabla \pi$, and suppose there are numbers $p_1, p_2 \in (1, 2)$, $S \in (0, \infty)$ such that $\overline{\mathfrak{D}} \subset B_S$,*

$$\begin{aligned} &u|B_S^c \in L^6(B_S^c)^3, \;\; \nabla u|B_S^c \in L^2(B_S^c)^9, \;\; \pi|B_S^c \in L^2(B_s^c), \\ &\qquad\qquad F|B_S^c \in L^{p_1}(B_S^c)^3 + L^{p_2}(B_S^c)^3. \end{aligned} \tag{4.15}$$

Let $j \in \{1, 2, 3\}$, and put

$$\begin{aligned} \mathfrak{B}_j(y) &:= \mathfrak{B}_j(u, \pi)(y) \\ &:= \int_{\partial \mathfrak{D}} \sum_{k=1}^{3} \Big[\sum_{l=1}^{3} \Big(\mathfrak{Z}_{jk}(y, z)\big(-\partial_l u_k(z) + \delta_{kl}\, \pi(z) + u_k(z)\,(\tau\, e_1 - \omega \times z)_l\big) \\ &\qquad + \partial z_l \mathfrak{Z}_{jk}(y, z)\, u_k(z)\Big) n_l^{(\mathfrak{D})}(z) + E_{4j}(y - z)\, u_k(z)\, n_k^{(\mathfrak{D})}(z)\Big]\, do_z \end{aligned} \tag{4.16}$$

for $y \in \overline{\mathfrak{D}}^c$. Then

$$u_j(y) = \mathfrak{R}_j(F)(y) + \mathfrak{S}_j(\operatorname{div} u)(y) + \mathfrak{B}_j(y) \tag{4.17}$$

for a.e. $y \in \overline{\mathfrak{D}}^c$. *If* $p > 3/2$, (4.17) *holds for any* $y \in \overline{\mathfrak{D}}^c$, *without the restriction "a.e."*

Proof. The assumptions on u and π yield that

$$\int_S^\infty \int_{\partial B_r} \big(|u(z)|^6 + |\nabla u(z)|^2 + |\pi(z)|^2 \big)\, do_z\, dr < \infty. \tag{4.18}$$

Therefore there is an increasing sequence (R_n) in (S, ∞) with $R_n \to \infty$ and

$$\int_{\partial B_{R_n}} \big(|u(z)|^6 + |\nabla u(z)|^2 + |\pi(z)|^2 \big)\, do_z \le R_n^{-1} \quad \text{for } n \in \mathbb{N}. \tag{4.19}$$

Otherwise there would be a constant $C \in [S, \infty)$ such that

$$\int_{\partial B_r} \big(|u(z)|^6 + |\nabla u(z)|^2 + |\pi(z)|^2 \big)\, do_z \ge r^{-1} \quad \text{for } r \in [C, \infty),$$

in contradiction to (4.18). (Here we used a standard convention from the theory of Lebesgue integration, which states that the integral of every measurable nonnegative function is defined, but may take the value ∞.) By our assumptions on F, there are functions $G^{(i)} \in L^{p_i}(B_S^c)^3$ for $i \in \{1, 2\}$ such that $F|B_S^c = G^{(1)} + G^{(2)}$. Thus, by Lemma 3.1,

$$\begin{aligned}\int_{\mathbb{R}^3} \sum_{k=1}^3 |\mathfrak{Z}_{jk}(y,z)| \Big(&\chi_{(0,S]}(|z|)\, |F_k(z)| \\ &+ \chi_{(S,\infty)}(|z|) \big(|G_k^{(1)}(z)| + |G_k^{(2)}(z)| \big) \Big)\, dz < \infty\end{aligned} \tag{4.20}$$

for a.e. $y \in \overline{\mathfrak{D}}^c$. Moreover, by Lemma 3.4 with $q = 2$,

$$\int_{\mathbb{R}^3} |E_{4j}(y-z)|\, |\operatorname{div} u(z)|\, dz < \infty \tag{4.21}$$

for a.e. $y \in \overline{\mathfrak{D}}^c$. Due to these observations and Theorem 4.5, we see there is a subset N of $\overline{\mathfrak{D}}^c$ with measure zero such that the relations in (4.20) and (4.21) hold for $y \in \overline{\mathfrak{D}}^c \backslash N$, and such that (4.10) with R replaced by R_n holds for $n \in \mathbb{N}$ and $y \in \mathfrak{D}_R \backslash N$. In the case $p > 3/2$, Lemma 3.2 yields that (4.20) is valid for any $y \in \overline{\mathfrak{D}}^c$, and Theorem 4.5 implies that (4.10) with R replaced by R_n is true for $n \in \mathbb{N}$ and any $y \in \overline{\mathfrak{D}}^c$. Moreover, if $p > 3/2$, the assumption $(u, \pi) \in \mathfrak{M}_p$, Lemma 3.4, and a Sobolev inequality (in the case $p \le 3$) allow us to drop the restriction "a.e." in (4.21).

Take $y \in \overline{\mathfrak{D}}^c$ in the case $p > 3/2$, and $y \in \overline{\mathfrak{D}}^c \backslash N$ otherwise. Let $n \in \mathbb{N}$ with $R_n > |y|$ (hence $y \in \mathfrak{D}_{R_n}$). Then, by (4.10) with R replaced by R_n, we get

$$u_j(y) = \mathfrak{R}_j(F|\mathfrak{D}_{R_n})(y) + \mathfrak{S}_j(\operatorname{div} u|\mathfrak{D}_{R_n})(y) + \mathfrak{A}_{j,n}(y) + \mathfrak{B}_j(y), \tag{4.22}$$

with

$$\begin{aligned}\mathfrak{A}_{j,n}(y) := \int_{\partial B_{R_n}} \sum_{k=1}^3 \Big[\sum_{l=1}^3 \Big(&\mathfrak{Z}_{jk}(y,z) \big(\partial_l u_k(z) - \delta_{kl}\, \pi(z) - \tau\, \delta_{1l}\, u_k(z) \big) \\ &- \partial z_l \mathfrak{Z}_{jk}(y,z)\, u_k(z) \Big) z_l/R_n \; - \; E_{4j}(y-z)\, u_k(z)\, z_k/R_n \Big]\, do_z.\end{aligned}$$

Note that in (4.22) we used the relation $\sum_{l=1}^{3}(\omega\times z)_l\,z_l/R_n=0$ for $z\in\partial B_R$. The term $\mathfrak{B}_j(y)$ was defined in (4.16). Let $n\in\mathbb{N}$ with $R_n/4\geq|y|$. Observe that

$$|\mathfrak{A}_{j,n}(y)|\leq\mathfrak{C}\sum_{\nu=1}^{4}\sum_{k=1}^{3}\mathfrak{V}_{\nu,k}(y), \tag{4.23}$$

with

$$\mathfrak{V}_{1,k}(y):=\Big(\int_{\partial B_{R_n}}|\mathfrak{Z}_{jk}(y,z)|^{6/5}\,do_z\Big)^{5/6}\|u|\partial B_{R_n}\|_6,$$

$$\mathfrak{V}_{2,k}(y):=\Big(\int_{\partial B_{R_n}}|\mathfrak{Z}_{jk}(y,z)|^{2}\,do_z\Big)^{1/2}(\|\nabla u|\partial B_{R_n}\|_2+\|\pi|\partial B_{R_n}\|_2),$$

$$\mathfrak{V}_{3,k}(y):=\sum_{l=1}^{3}\Big(\int_{\partial B_{R_n}}|\partial z_l\mathfrak{Z}_{jk}(y,z)|^{6/5}\,do_z\Big)^{5/6}\|u|\partial B_{R_n}\|_6,$$

$$\mathfrak{V}_{4,k}(y):=\Big(\int_{\partial B_{R_n}}|y-z|^{-12/5}\,do_z\Big)^{5/6}\|u|\partial B_{R_n}\|_6$$

for $k\in\{1,2,3\}$. Since $|y|\leq R_n/4$, we may use inequality (2.23) with $S=2|y|$ in order to estimate $|\partial_z^\alpha\mathfrak{Z}_{jk}(y,z)|$ for $z\in\partial B_{R_n}$, $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\leq 1$. We get by (4.19) and (2.23) that

$$\begin{aligned}\mathfrak{V}_{1,k}(y)&\leq\mathfrak{C}(|y|)\Big(\int_{\partial B_{R_n}}\big(|z|\,s_\tau(z)\big)^{-6/5}\,do_z\Big)^{5/6}R_n^{-1/6}\\&\leq\mathfrak{C}(|y|)\Big(\int_{\partial B_{R_n}}s_\tau(z)^{-6/5}\,do_z\Big)^{5/6}R_n^{-7/6}\leq\mathfrak{C}(|y|)\,R_n^{-1/3},\end{aligned}\tag{4.24}$$

where the last inequality follows from Lemma 2.1. The same references yield

$$|\mathfrak{V}_{2,k}(y)|\leq\mathfrak{C}(|y|)\,R_n^{-1},\quad|\mathfrak{V}_{3,k}(y)|\leq\mathfrak{C}(|y|)\,R_n^{-5/6}\quad(1\leq k\leq 3).\tag{4.25}$$

Moreover, since $|y-z|\geq|z|/2$ for ∂B_{R_n}, we find with (4.19) that $|\mathfrak{V}_{4,k}(y)|\leq\mathfrak{C}(|y|)\,R_n^{-1/2}$. From (4.23)–(4.25) and the preceding inequality we may conclude that $\mathfrak{A}_{n,j}(y)\to 0$ for $n\to\infty$. Turning to $\mathfrak{R}_j(F|\mathfrak{D}_{R_n})(y)$, we observe that by (4.20), our choice of y, and Lebesgue's theorem on dominated convergence, we have $\mathfrak{R}_j(F|\mathfrak{D}_{R_n})(y)\to\mathfrak{R}_j(F)(y)$ for $n\to\infty$. Moreover, by (4.21) and again by the choice of y and Lebesgue's theorem, $\mathfrak{S}_j(\operatorname{div}u|\mathfrak{D}_{R_n})(y)\to\mathfrak{S}_j(\operatorname{div}u)(y)$ for $n\to\infty$. Recalling (4.22), we thus have proved (4.17). ▯

5. Applications. In our first application of our representation formula (4.17), we state conditions on $L(u)+\nabla\pi$ and $\operatorname{div}u$ such that u decays as described in (1.4). Since in the proof of this result we want to avoid estimates of the second derivatives of $\mathfrak{Z}_{jk}$, we have to transform the integral $\int_{\partial\mathfrak{D}}\partial z_l\mathfrak{Z}_{jk}(y,z)\,u_k(z)\,n_l^{(\mathfrak{D})}(z)\,do_z$ appearing in the definition of $\mathfrak{B}_j(y)$ (see (4.16)) into a term where no differential operator acts on $\mathfrak{Z}_{jk}$. This is done in the following lemma.

LEMMA 5.1. *Let $p\in(1,\infty)$, $(u,\pi)\in\mathfrak{M}_p$, $j\in\{1,2,3\}$. Define*

$$\mathfrak{U}_j(y):=\mathfrak{U}_j(u)(y):=\int_{\partial\mathfrak{D}}\sum_{k,l=1}^{3}\partial z_l\mathfrak{Z}_{jk}(y,z)\,u_k(z)\,n_l^{(\mathfrak{D})}(z)\,do_z\tag{5.1}$$

for $y \in \overline{\mathfrak{D}}^c$. *Let* $\mathfrak{E}_p : W^{2-1/p,p}(\partial\mathfrak{D}) \mapsto W^{2,p}(\mathfrak{D})$ *denote a continuous extension operator (see* [39]*). Then, for* $y \in \overline{\mathfrak{D}}^c$,

$$\begin{aligned}(5.2)\quad \mathfrak{U}_j(y) = &\int_{\mathfrak{D}} \sum_{k=1}^{3} \Big[\partial_k E_{4j}(y-z)\,\mathfrak{E}_p(u_k)(z) + \mathfrak{Z}_{jk}(y,z) \\ &\times \Big((\tau e_1 - \omega\times z)\cdot\nabla\mathfrak{E}_p(u_k)(z) + \big[\omega\times\big(\mathfrak{E}_p(u_s)(z)\big)_{1\le s\le 3}\big]_k - \Delta\mathfrak{E}_p(u_k)(z)\Big)\Big]\,dz \\ &+ \int_{\partial\mathfrak{D}} \sum_{k,l=1}^{3} \mathfrak{Z}_{jk}(y,z)\Big((-\tau e_1 + \omega\times z)_l\,u_k(z) + \partial_l\mathfrak{E}_p(u_k)(z)\Big)\,n_l^{(\mathfrak{D})}(z)\,do_z.\end{aligned}$$

Proof. Let $y \in \overline{\mathfrak{D}}^c$. Starting with (2.8), we may refer to Lemma 2.14 in order to apply Fubini's theorem and Lebesgue's theorem on dominated convergence, to obtain

$$\mathfrak{U}_j(y) = \lim_{\delta\downarrow 0,\, T\to\infty} \int_\delta^T \int_{\partial\mathfrak{D}} \sum_{k,l=1}^{3} \partial z_l \Gamma_{jk}(y,z,t)\,u_k(z)\,n_l^{(\mathfrak{D})}(z)\,do_z\,dt.$$

Next we apply the divergence theorem and then use (2.1). It follows that

$$\begin{aligned}\mathfrak{U}_j(y) = &\lim_{\delta\downarrow 0,\, T\to\infty} \int_\delta^T \int_{\mathfrak{D}} \sum_{k=1}^{3} \Big(\Delta_z\Gamma_{jk}(y,z,t)\,\mathfrak{E}_p(u_k)(z) \\ (5.3)\qquad &+\nabla_z\Gamma(y,z,t)\cdot\nabla\mathfrak{E}_p(u_k)(z)\Big)\,dz\,dt \\ = &\lim_{\delta\downarrow 0,\, T\to\infty} \Big[\int_\delta^T \int_{\mathfrak{D}} \sum_{k=1}^{3} \Big(\Big(\partial_t\Gamma_{jk}(y,z,t) + (-\tau e_1 + \omega\times z)\cdot\nabla_z\Gamma_{jk}(y,z,t) \\ &- \big[\omega\times\big(\Gamma_{js}(y,z,t)\big)_{1\le s\le 3}\big]_k\Big)\,\mathfrak{E}_p(u_k)(z) \;-\; \Gamma_{jk}(y,z,t)\,\Delta\mathfrak{E}_p(u_k)(z)\Big)\,dz\,dt \\ &+ \int_\delta^T \int_{\partial\mathfrak{D}} \sum_{k,l=1}^{3} \Gamma_{jk}(y,z,t)\,\partial_l\mathfrak{E}_p(u_k)(z)\,n_l^{(\mathfrak{D})}(z)\,do_z\,dt\Big].\end{aligned}$$

As explained in the proof of [7, Theorem 4.2], the relation in (2.2) and Lemma 2.14 yield

$$\begin{aligned}(5.4)\qquad &\lim_{\delta\downarrow 0,\, T\to\infty} \int_\delta^T \int_{\mathfrak{D}} \sum_{k=1}^{3} \partial_t\Gamma_{jk}(y,z,t)\,\mathfrak{E}_p(u_k)(z)\,dz\,dt \\ &\qquad = \int_{\mathfrak{D}} \sum_{k=1}^{3} \partial_k E_{4j}(y-z)\,\mathfrak{E}_p(u_k)(z)\,dz.\end{aligned}$$

For the other terms on the right-hand side of (5.3), the passage to the limit $\delta\downarrow 0$ and $T\to\infty$ presents no difficulty because due to Lemma 2.14 we may directly apply Fubini's and Lebesgue's theorems. We further use the formula $(a\times b)\cdot c = -(a\times c)\cdot b$ for vectors a, b, c in $\mathbb{R}^3$. In this way, letting δ tend to zero and T to infinity, and taking account of (5.4), we may deduce (5.2) from (5.3). ∎

Now we may prove a decay estimate for $\mathfrak{V}_j(u,\pi)$.

LEMMA 5.2. *Let* $p \in (1,\infty)$, $(u,\pi) \in \mathfrak{M}_p$, $j \in \{1, 2, 3\}$. *Define* $\mathfrak{V}_j = \mathfrak{V}_j(u,\pi)$ *as in* (4.16). *Then* $\mathfrak{V}_j \in C^1(\overline{\mathfrak{D}}^c)$.

Let $S \in (0,\infty)$ *with* $\overline{\mathfrak{D}} \subset B_S$*. Put* $\delta := dist(\overline{\mathfrak{D}}, \partial B_S)$*. Let* $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, $y \in B_S^c$*. Then*

$$(5.5)\quad |\partial^\alpha \mathfrak{B}_j(y)| \le \mathfrak{C}(S,\delta)\big(\,\|\nabla u\,|\,\partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 + \widetilde{C}(\mathfrak{D},p)\,\|u|\partial\mathfrak{D}\|_{2-1/p,p}\,\big)\big(\,|y|\,s_\tau(y)\,\big)^{-1-|\alpha|/2},$$

where $\widetilde{C}(\mathfrak{D},p)$ *is a constant depending only on* $\mathfrak{D}$ *and* p*.*

Proof. We use the decomposition $\mathfrak{B}_j(y) = \big(\mathfrak{B}_j(y) - \mathfrak{U}_j(y)\big) + \mathfrak{U}_j(y)$, with $\mathfrak{U}_j = \mathfrak{U}_j(u,\pi)$ defined in (5.1). Equation (5.2) and Lemma 4.2 yield that $\mathfrak{B}_j - \mathfrak{U}_j$ and $\mathfrak{U}_j$ belong to $C^1(\overline{\mathfrak{D}}^c)$. Therefore we have $\mathfrak{B}_j \in C^1(\overline{\mathfrak{D}}^c)$. Moreover, by (4.2), (4.3), (4.16), and (5.2),

$$(5.6)\quad |\partial^\alpha(\mathfrak{B}_j - \mathfrak{U}_j)(y)| + |\partial^\alpha \mathfrak{U}_j(y)| \le \mathfrak{C}(S,\delta)\big(\,|y|\,s_\tau(y)\,\big)^{-1-|\alpha|/2}\Big(\|\nabla u\,|\,\partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 + \|u|\partial\mathfrak{D}\|_1 + \sum_{k=1}^{3}\big(\,\|\mathfrak{E}_p(u_k)\|_{2,1} + \|\nabla\mathfrak{E}_p(u_k)\,|\,\partial\mathfrak{D}\|_1\,\big)\Big),$$

where the extension operator $\mathfrak{E}_p$ was introduced in Lemma 5.1. On the other hand, by a standard trace theorem and by the choice of $\mathfrak{E}_p$,

$$(5.7)\quad \|\nabla\mathfrak{E}_p(u_k)\,|\,\partial\mathfrak{D}\|_1 \le \mathfrak{C}\|\nabla\mathfrak{E}_p(u_k)\,|\,\partial\mathfrak{D}\|_p \le \mathfrak{C}(p)\,\|\mathfrak{E}_p(u_k)\|_{2,p} \le \mathfrak{C}(p)\,\|u|\partial\mathfrak{D}\|_{2-1/p,p},$$

$$(5.8)\quad \|\mathfrak{E}_p(u_k)\|_{2,1} \le \mathfrak{C}\|\mathfrak{E}_p(u_k)\|_{2,p} \le \mathfrak{C}(p)\,\|u|\partial\mathfrak{D}\|_{2-1/p,p}$$

for $k \in \{1,\,2,\,3\}$. Inequality (5.5) is a consequence of (5.6)–(5.8). ∎

At this point, we are in a position to derive the decay relations (1.4) for u if $L(u) + \nabla\pi$ and $\operatorname{div} u$ decay sufficiently fast.

THEOREM 5.3. *Let* $p \in (1,\infty)$, $(u,\pi) \in \mathfrak{M}_p$*. Put* $F := L(u) + \nabla\pi$*. Suppose there are numbers* $S_1, S, \gamma \in (0,\infty)$, $A \in [2,\infty)$, $B \in \mathbb{R}$ *such that* $S_1 < S$, $\overline{\mathfrak{D}} \subset B_{S_1}$,

$$u|B_S^c \in L^6(B_S^c)^3,\ \ \nabla u|B_S^c \in L^2(B_S^c)^9,\ \ \pi|B_S^c \in L^2(B_S^c),\ \ \operatorname{supp}(\operatorname{div} u) \subset B_{S_1},$$

$$A + \min\{1,B\} \ge 3,\ |F(z)| \le \gamma|z|^{-A}\,s_\tau(z)^{-B}\ \text{for}\ \ z \in B_{S_1}^c.$$

Put $\delta := dist(\overline{\mathfrak{D}}, \partial B_S)$*. Let* $i,j \in \{1,\,2,\,3\}$, $y \in B_S^c$*. Then*

$$(5.9)\quad |u_j(y)| \le \mathfrak{C}(S,S_1,A,B,\delta)\big(\,\gamma + \|F|B_{S_1}\|_1 + \|\operatorname{div} u\|_1 + \|\nabla u\,|\,\partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 + \widetilde{C}(\mathfrak{D},p)\,\|u|\partial\mathfrak{D}\|_{2-1/p,p}\,\big)\big(\,|y|\,s_\tau(y)\,\big)^{-1} l_{A,B}(y),$$

$$(5.10)\quad |\partial_i u_j(y)| \le \mathfrak{C}(S,S_1,A,B,\delta)\big(\,\gamma + \|F|B_{S_1}\|_1 + \|\operatorname{div} u\|_1 + \|\nabla u\,|\,\partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 + \widetilde{C}(\mathfrak{D},p)\,\|u|\partial\mathfrak{D}\|_{2-1/p,p}\,\big)\big(\,|y|\,s_\tau(y)\,\big)^{-3/2} s_\tau(y)^{\max(0,\,7/2-A-B)}\,l_{A,B}(y),$$

where $\widetilde{C}(\mathfrak{D},p)$ *was introduced in Lemma* 5.2 *and function* $l_{A,B}(y)$ *in Theorem* 3.3*. If* *the assumption* $\operatorname{supp}(\operatorname{div} u) \subset B_{S_1}$ *is replaced by the condition*

$$|\operatorname{div} u(z)| \le \widetilde{\gamma}\,|z|^{-C}\, s_\tau(z)^{-D} \quad \textit{for } z \in B^c_{S_1}$$

for some $\widetilde{\gamma} \in (0,\infty),\ C \in (5/2,\infty),\ D \in \mathbb{R}$ *with* $C + \min\{1, D\} > 3$, *then inequality* (5.9) *remains valid if the term* $\|\operatorname{div} u\|_1$ *on the right-hand side of* (5.9) *is replaced by* $\widetilde{\gamma} + \|\operatorname{div} u|B_{S_1}\|_1$. *Of course, in that case the constant in* (5.9) *additionally depends on* C *and* D.

Note that if $A + \min\{1, B\} > 3,\ A + B \ge 7/2$ in Theorem 5.3, then $l_{A,B}(y) = 1$ in (5.9) and $s_\tau(y)^{\max(0,\,7/2-A-B)}\, l_{A,B}(y) = 1$ in (5.10). The preceding conditions on A and B are verified if, for example, $A = 5/2,\ B = 1$, or $B = 3/2$ and $A = 2 + \epsilon$ for some $\epsilon \in (0, 1/2)$.

Proof of Theorem 5.3. By Lemma 2.1, we see that $\int_{B^c_{S_1}} |F(z)|^r\, dz < \infty$ for any $r \in (1,\infty)$. Thus Theorem 4.6 yields that the representation formula (4.17) holds for a.e. $y \in \overline{\mathfrak{D}}^c$. Therefore Theorem 5.3 follows from Theorems 3.3 and 3.5 and Lemma 5.2. ▯

In the next theorem, we present an asymptotic profile of u for the case that $L(u) + \nabla\pi$ and $\operatorname{div} u$ have compact support.

THEOREM 5.4. *Let* $p \in (1,\infty),\ (u,\pi) \in \mathfrak{M}_p,\ S,\ S_1 \in (0,\infty)$ *with* $S_1 < S$, *and put* $F := L(u) + \nabla\pi$. *Suppose that*

$$\overline{\mathfrak{D}} \cup \operatorname{supp}(F) \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_1},$$

$$u|B^c_S \in L^6(B^c_S)^3,\ \ \nabla u|B^c_S \in L^2(B^c_S)^9,\ \ \pi|B^c_S \in L^2(B^c_s).$$

Then there are coefficients $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ *and functions* $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \in C^0(B^c_S)$ *such that for* $j \in \{1, 2, 3\},\ y \in B^c_S$,

$$u_j(y) = \sum_{k=1}^{3} \beta_k\, \mathfrak{Z}_{jk}(y, 0) + \Big(\int_{\partial\mathfrak{D}} u \cdot n^{(\mathfrak{D})}\, do_z + \int_{B_{S_1}} \operatorname{div} u\, dz\Big) E_{4j}(y) + \mathfrak{F}_j(y) \tag{5.11}$$

and

$$|\mathfrak{F}_j(y)| \le \mathfrak{C}(S, S_1)\big(\,\|F\|_1 + \|\operatorname{div} u\|_1 + \|\nabla u\,|\,\partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 + C(\mathfrak{D}, p)\,\|u|\partial\mathfrak{D}\|_{2-1/p,p}\big)\big(\,|y|\, s_\tau(y)\,\big)^{-3/2}, \tag{5.12}$$

where $C(\mathfrak{D}, p) > 0$ *depends only on* $\mathfrak{D}$ *and* p. *(Note that* $|E_{4j}(y)| \le \mathfrak{C}|y|^{-2}$ *and* $|y|^{-2} \le \mathfrak{C}(S)\big(\,|y|\, s_\tau(y)\,\big)^{-1}$ *for* $y \in B^c_S$*; see Lemma* 2.4.*)*

Proof. Take $j \in \{1, 2, 3\},\ y \in B^c_S$. Observe that

$$|y - \vartheta z| \ge |y| - S_1 \ge (1 - S_1/S)\,|y| > 0 \quad \text{for } z \in B_{S_1},\ \vartheta \in [0, 1]. \tag{5.13}$$

In view of Lemma 2.15, we may conclude that the term $\mathfrak{Z}_{jk}(y,\, \vartheta z)$ is continuously differentiable with respect to $\vartheta \in [0, 1]$, for any $z \in B_{S_1}$ and $k \in \{1, 2, 3\}$, with obvious derivatives. Therefore we may define

$$\begin{aligned}
&\mathfrak{F}_j(y) \\
&:= \int_{B_{S_1}} \Big(\sum_{k,s=1}^{3} \int_0^1 \partial x_s \mathfrak{Z}_{jk}(y,x)_{|x=\vartheta\, z}\, d\vartheta\, z_s\, F_k(z) \\
&\qquad - \sum_{s=1}^{3} \int_0^1 \partial_s E_{4j}(y-\vartheta\, z)\, d\vartheta\, z_s \operatorname{div} u(z) \Big)\, dz \\
&\quad + \int_{\partial\mathfrak{D}} \Big(\sum_{k,s=1}^{3} \int_0^1 \partial x_s \mathfrak{Z}_{jk}(y,x)_{|x=\vartheta\, z}\, d\vartheta\, z_s \\
&\qquad \times \sum_{l=1}^{3} \big(-\partial_l u_k(z) + \delta_{kl}\, \pi(z) + \partial_l \mathfrak{E}_p(u_k)(z) \big)\, n_l^{(\mathfrak{D})}(z) \\
&\qquad - \sum_{s=1}^{3} \int_0^1 \partial_s E_{4j}(y-\vartheta\, z)\, d\vartheta\, z_s\, u_k(z)\, n_k^{(\mathfrak{D})}(z) \Big)\, do_z \\
&\quad + \int_{\mathfrak{D}} \sum_{k=1}^{3} \Big(\partial_k E_{4j}(y-z)\, \mathfrak{E}_p(u_k)(z) \\
&\qquad + \sum_{s=1}^{3} \int_0^1 \partial x_s \mathfrak{Z}_{jk}(y,x)_{|x=\vartheta\, z}\, d\vartheta\, z_s \Big((\tau\, e_1 - \omega \times z) \cdot \nabla \mathfrak{E}_p(u_k)(z) \\
&\qquad\qquad + \big[\, \omega \times \big(\mathfrak{E}_p(u_s)(z) \big)_{1 \le s \le 3} \big]_k - \Delta \mathfrak{E}_p(u_k)(z) \Big) \Big)\, dz,
\end{aligned}$$

where the extension operator $\mathfrak{E}_p$ was introduced in Lemma 5.1. We further set

$$\begin{aligned}
\beta_k :=& \int_{B_{S_1}} F_k(z)\, dz + \int_{\partial\mathfrak{D}} \sum_{l=1}^{3} \big(-\partial_l u_k(z) + \delta_{kl}\, \pi(z) + \partial_l \mathfrak{E}_p(u_k)(z) \big)\, n_l^{(\mathfrak{D})}(z)\, do_z \\
&+ \int_{\mathfrak{D}} \Big((\tau\, e_1 - \omega \times z) \cdot \nabla \mathfrak{E}_p(u_k)(z) \\
&\qquad + \big[\, \omega \times \big(\mathfrak{E}_p(u_s)(z) \big)_{1 \le s \le 3} \big]_k - \Delta \mathfrak{E}_p(u_k)(z) \Big)\, dz.
\end{aligned}$$

Then, referring to (4.17), (4.16), (5.1), and (5.2), we obtain (5.11). By (5.13), the choice of $\mathfrak{E}_p$ in Lemma 5.1, and (2.22), we further find

$$\begin{aligned}
(5.14)\quad |\mathfrak{F}_j(y)| \le\ & \mathfrak{C}(S,S_1) \big(|y|\, s_\tau(y) \big)^{-3/2} \Big(\|F\|_1 + \|\nabla u \,|\, \partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 \\
&\qquad + \sum_{k=1}^{3} \big(\|\nabla \mathfrak{E}_p(u_k) \,|\, \partial\mathfrak{D}\|_1 + \|\mathfrak{E}_p(u_k)\|_{2,1} \big) \Big) \\
&+ \mathfrak{C}(S,S_1)\, |y|^{-3} \Big(\|\operatorname{div} u\|_1 + \|u|\partial\mathfrak{D}\|_1 + \sum_{k=1}^{3} \|\mathfrak{E}_p(u_k)\|_1 \Big).
\end{aligned}$$

Inequality (5.14), Lemma 2.4, and (5.7) imply (5.11). ∎

Finally we use (4.17) in order to obtain a representation formula for weak solutions of the stationary Navier–Stokes system with Oseen and rotational terms.

THEOREM 5.5. *Let* $u \in W^{1,1}_{loc}(\overline{\mathfrak{D}}^c)^3 \cap L^6(\overline{\mathfrak{D}})^3$ *with* $\nabla u \in L^2(\overline{\mathfrak{D}})^9$. *Let* $\pi \in L^2(\overline{\mathfrak{D}})$, $p \in (1,\infty)$, $q \in (1,2)$, *and let* $f : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}^3$ *be a function with* $f|\mathfrak{D}_T \in L^p(\mathfrak{D}_T)^3$ *for* $T \in (0,\infty)$ *with* $\overline{\mathfrak{D}} \subset B_T$, $f|B_S^c \in L^q(B_S^c)^3$ *for some* $S \in (0,\infty)$ *with* $\overline{\mathfrak{D}} \subset B_S$.

Suppose that the pair (u, π) *is a weak solution of the Navier–Stokes system with Oseen and rotational terms, and with right-hand side* f*, that is,*

$$\int_{\overline{\mathfrak{D}}^c} \Big(\nabla u \cdot \nabla \varphi + \big(\tau (u \cdot \nabla) u + \tau \partial_1 u - (\omega \times z) \cdot \nabla u + \omega \times u \big) \cdot \varphi + \pi \operatorname{div} \varphi \Big) \, dz$$
$$= \int_{\overline{\mathfrak{D}}^c} f \cdot \varphi \, dz \quad \textit{for} \ \ \varphi \in C_0^\infty(\overline{\mathfrak{D}}^c)^3, \quad \operatorname{div} u = 0.$$

Then

$$u_j(y) = \mathfrak{R}_j\big(f - \tau (u \cdot \nabla) u \big)(y) + \mathfrak{B}_j(u, \pi)(y) \tag{5.15}$$

for $j \in \{1, 2, 3\}$ *and for a.e.* $y \in \overline{\mathfrak{D}}^c$*, where* $\mathfrak{B}_j(u, \pi)$ *was defined in* (4.16).

Proof. Since $u \in L^6(\overline{\mathfrak{D}})^3$ and $\nabla u \in L^2(\overline{\mathfrak{D}})^9$, Hölder's inequality yields $\tau (u \cdot \nabla) u \in L^{3/2}(\overline{\mathfrak{D}}^c)^3$. It further follows that the term $\tau \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z)$, considered as a function of $z \in \mathfrak{D}_T$, belongs to $L^2(\mathfrak{D}_T)^3$ for any $T \in (0, \infty)$ with $\mathfrak{D} \subset B_T$. Therefore, putting

$$F(z) := f(z) - \tau \big(u(z) \cdot \nabla \big) u(z) - \tau \partial_1 u(z) + (\omega \times z) \cdot \nabla u(z) - \omega \times u(z)$$

for $z \in \overline{\mathfrak{D}}^c$, we see that $F|\mathfrak{D}_T \in L^{\min\{p, 3/2\}}(\mathfrak{D}_T)^3$ for T as above. Thus, considering the pair (u, π) as a weak solution (in the sense of [20, (IV.1.3)]) of the Stokes system with right-hand side F, we may refer to [20, Theorem IV.4.1] (interior regularity for the Stokes system), to obtain that

$$u|\mathfrak{D}_T \in W^{2, \min\{p, 3/2\}}_{loc}(\mathfrak{D}_T)^3, \ \ \pi|\mathfrak{D}_T \in W^{1, \min\{p, 3/2\}}_{loc}(\mathfrak{D}_T) \ \ (T \text{ as above}),$$
$$-\Delta u + \nabla \pi = F, \quad \text{and hence} \quad L(u) + \nabla \pi = f - \tau (u \cdot \nabla) u.$$

As $\tau (u \cdot \nabla) u \in L^{3/2}(\overline{\mathfrak{D}}^c)^3$, we now conclude that $L(u) + \nabla \pi \,|\, \mathfrak{D}_T \in L^{\min\{p, 3/2\}}(\mathfrak{D}_T)^3$ for T as above, so $(u, \pi) \in \mathfrak{M}_{\min\{p, 3/2\}}$. The preceding observations mean that the assumptions of Theorem 4.6 are satisfied with p, p_1 replaced by, respectively, $\min\{p, 3/2\}$ and q, and with $p_2 = 3/2$. Thus (5.15) follows from Theorem 4.6. ∎

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] K. I. Babenko and M. M. Vasil'ev, *On the asymptotic behavior of a steady flow of viscous fluid at some distance from an immersed body*, Prikl. Mat. Meh., 37 (1973), pp. 690–705 (in Russian); English translation: J. Appl. Math. Mech., 37 (1973), pp. 651–665.

[3] P. Deuring, *Exterior stationary Navier-Stokes flows in 3D with nonzero velocity at infinity: Asymptotic behavior of the second derivatives of the velocity*, Comm. Partial Differential Equations, 30 (2005), pp. 987–1020.

[4] P. Deuring, *The single-layer potential associated with the time-dependent Oseen system*, in Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics, Chalkida, Greece, 2006, pp. 117–125.

[5] P. Deuring and S. Kračmar, *Artificial boundary conditions for the Oseen system in 3D exterior domains*, Analysis, 20 (2000), pp. 65–90.

[6] P. Deuring and S. Kračmar, *Exterior stationary Navier-Stokes flows in 3D with non-zero velocity at infinity: Approximation by flows in bounded domains*, Math. Nachr., 269-270 (2004), pp. 86–115.

[7] P. Deuring, S. Kračmar, and Š. Nečasová, *A representation formula for linearized stationary incompressible viscous flows around rotating and translating bodies*, Discrete Contin. Dynam. Syst. Ser. S, 3 (2010), pp. 237–253.

[8] E. B. Fabes, C. E. Kenig, and G. C. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J., 57 (1988), pp. 769–792.

[9] R. Farwig, *The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces*, Math. Z., 211 (1992), pp. 409–447.

[10] R. Farwig, *An L^q-analysis of viscous fluid flow past a rotating obstacle*, Tôhoku Math. J., 58 (2005), pp. 129–147.

[11] R. Farwig, *Estimates of lower order derivatives of viscous fluid flow past a rotating obstacle*, in Regularity and Other Aspects of the Navier-Stokes Equations, Banach Center Publ. 70, Polish Acad. Sci., Warsaw, 2005, pp. 73–84.

[12] R. Farwig, R. B. Guenther, Š. Nečasová, and E. A. Thomann, *The fundamental solution of the linearized instationary Navier-Stokes equations of motion around a rotating and translating body*, in preparation.

[13] R. Farwig and T. Hishida, *Stationary Navier-Stokes flow around a rotating obstacles*, Funkcial. Ekvac., 50 (2007), pp. 371–403.

[14] R. Farwig, T. Hishida, and D. Müller, *L^q-theory of a singular "winding" integral operator arising from fluid dynamics*, Pacific J. Math., 215 (2004), pp. 297–312.

[15] R. Farwig, M. Krbec, and Š. Nečasová, *A weighted L^q approach to Stokes flow around a rotating body*, Ann. Univ. Ferrara Sez. VII (N.S.), 54 (2008), pp. 61–84.

[16] R. Farwig, M. Krbec, and Š. Nečasová, *A weighted L^q-approach to Oseen flow around a rotating body*, Math. Methods Appl. Sci., 31 (2008), pp. 551–574.

[17] R. Farwig and J. Neustupa, *On the spectrum of a Stokes-type operator arising from flow around a rotating body*, Manuscripta Math., 122 (2007), pp. 419–437.

[18] R. Finn, *Estimates at infinity for stationary solutions of the Navier-Stokes equations*, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.), 3 (1959), pp. 387–418.

[19] R. Finn, *On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems*, Arch. Ration. Mech. Anal., 19 (1965), pp. 363–406.

[20] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol.* I. *Linearized Steady Problems*, revised ed., Springer, New York, 1998.

[21] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol.* II. *Nonlinear Steady Problems*, Springer, New York, 1994.

[22] G. P. Galdi, *On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications*, in Handbook of Mathematical Fluid Dynamics, Vol. 1, S. Friedlander and D. Serre, eds., North–Holland, Amsterdam, 2002, pp. 653–791.

[23] G. P. Galdi, *Steady flow of a Navier-Stokes fluid around a rotating obstacle*, J. Elasticity, 71 (2003), pp. 1–31.

[24] G. P. Galdi and A. L. Silvestre, *Strong solutions to the Navier-Stokes equations around a rotating obstacle*, Arch. Ration. Mech. Appl., 176 (2005), pp. 331–350.

[25] G. P. Galdi and A. L. Silvestre, *The steady motion of a Navier-Stokes liquid around a rigid body*, Arch. Ration. Mech. Appl., 184 (2007), pp. 371–400.

[26] M. Geissert, H. Heck, and M. Hieber, *L^p theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle*, J. Reine Angew. Math., 596 (2006), pp. 45–62.

[27] R. B. Guenther and E. A. Thomann, *The fundamental solution of the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions—time dependent case*, J. Math. Fluid Mech., 8 (2006), pp. 77–98.

[28] T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Ration. Mech. Anal., 150 (1999), pp. 307–348.

[29] T. Hishida, *The Stokes operator with rotating effect in exterior domains*, Analysis, 19 (1999), pp. 51–67.

[30] T. Hishida, *L^q estimates of weak solutions to the stationary Stokes equations around a rotating body*, J. Math. Soc. Japan, 58 (2006), pp. 744–767.

[31] T. Hishida and Y. Shibata, *$L_p - L_q$ estimate of the Stokes operator and Navier-Stokes flows in the exterior of a rotating obstacle*, Arch. Ration. Mech. Anal., 193 (2009), pp. 339–421.

[32] S. Kračmar, Š. Nečasová, and P. Penel, *Estimates of weak solutions in anisotropically weighted Sobolev spaces to the stationary rotating Oseen equations*, IASME Trans., 2 (2005), pp. 854–861.

[33] S. Kračmar, Š. Nečasová, and P. Penel, *Anisotropic L^2 estimates of weak solutions to the stationary Oseen type equations in R^3 for a rotating body*, RIMS Kokyuroku Bessatsu, B1 (2007), pp. 219–235.

[34] S. Kračmar, Š. Nečasová, and P. Penel, *Anisotropic L^2 estimates of weak solutions to the stationary Oseen type equations in 3D-exterior domain for a rotating body*, J. Math. Soc. Japan, 62 (2010), pp. 239–268.

[35] S. Kračmar, A. Novotný, and M. Pokorný, *Estimates of Oseen kernels in weighted L^p spaces*, J. Math. Soc. Japan, 53 (2001), pp. 59–111.

[36] S. Kračmar and P. Penel, *Variational properties of a generic model equation in exterior 3D domains*, Funkcial. Ekvac., 47 (2004), pp. 499–523.

[37] S. Kračmar and P. Penel, *New regularity results for a generic model equation in exterior 3D domains*, in Regularity and Other Aspects of the Navier-Stokes Equations, Banach Center Publ. 70, Polish Acad. Sci., Warsaw, 2005, pp. 139–155.
[38] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed., Grundlehren Math. Wiss. 52, Springer, New York, 1966.
[39] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.
[40] Š. Nečasová, *Asymptotic properties of the steady fall of a body in viscous fluids*, Math. Methods Appl. Sci., 27 (2004), pp. 1969–1995.
[41] Š. Nečasová, *On the problem of the Stokes flow and Oseen flow in $\mathbb{R}^3$ with Coriolis force arising from fluid dynamics*, IASME Trans., 2 (2005), pp. 1262–1270.
[42] Š. Nečasová and K. Schumacher, *Strong solution to the Stokes equations of a flow around a rotating body in weighted L^q spaces*, Math. Nachr., to appear.
[43] C. W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akademische Verlagsgesellschaft, Leipzig, 1927.
[44] L. I. Sazonov, *On the asymptotics of the solution to the three-dimensional problem of flow far from streamlined bodies*, Izv. Ross. Akad. Nauk Ser. Mat., 59 (1995), pp. 173–196 (in Russian); English translation: Izv. Math., 59 (1995), pp. 1051–1075.
[45] V. A. Solonnikov, *A priori estimates for second order parabolic equations*, Trudy Mat. Inst. Steklov., 70 (1964), pp. 133–212 (in Russian); English translation: Amer. Math. Soc. Transl., 65 (1967), pp. 51–137.

MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2004; **27**:1969–1995
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/mma.467
MOS subject classification: 35 Q; 76 D

Asymptotic properties of the steady fall of a body in viscous fluids

Š. Nečasová*,†

Mathematical Institute of Academy of Sciencies, Žitná 25, 11567 Prague 1, Czech Republic

Communicated by W. Wendland

SUMMARY

The paper deals with the properties of the steady fall of a body in linear fluids (Oseen and Stokes cases) and in nonlinear fluids (Navier–Stokes fluids) in L^q structure. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: steady fall of a body; motion of a rigid body; Stokes problem; Oseen problem; Navier–Stokes equations; Coriolis force; translational and rotational steady fall

1. INTRODUCTION

The problem of the motion of a rigid body through a liquid has attracted the attention of several scientists for over a century. We would like to mention works of Kirchhoff [1] and Lord Kelvin [2] regarding the motion of one or more bodies in a frictionless liquid. We wish to mention the work of Brenner [3] concerning the steady motion of one or more bodies in a linear viscous liquid in the Stokes approximation, further the works of Weinberger [4,5] and Serre [6] regarding the fall of a body in an incompressible Navier–Stokes fluid under the action of gravity. In the paper of Weinberger [4] an existence of weak solution in the Navier–Stokes fluids was proved and it was shown that the existence of the steady fall in the Stokes flow can be obtained as a limit of the Navier–Stokes solutions. Also the variational properties of the steady fall in the Stokes flow were studied in Reference [5].

The work of Galdi and Vaidya studied the translational steady fall of symmetric bodies in a Navier–Stokes liquid, see Reference [7]. We would like to mention the work of Childress [8] here. He studied the uniform, slow motion of a sphere in a viscous fluid and he examined the case where the undisturbed fluid rotates with the constant angular velocity ω and the axis

*Correspondence to: Š. Nečasová, Mathematical Institute of Academy of Sciences, Žitná 25, 11567 Prague 1, Czech Republic.
†E-mail: matus@math.cas.cz

Contract/grant sponsor: Grant Agency of the Czech Republic; contract/grant number: 201/99/0267, 201/02/0684

 Received 14 February 2003

of rotation is taken to coincide with the line of motion. The various modifications of classical problem for small Reynolds number are discussed. The analytical result is a correction to Stokes's drag formula, valid for small values of the Reynolds number and the Taylor number and tending to the classical Oseen correction as the last parameter tends to zero.

In the work of Saffmann [9] we find an estimate for the lift. In the work of Maslennikova [10] the uniform rate of decrease for large time with respect to $x \in E_2$, of the solution of the Cauchy problem for a linearized system governing the motion of a rotating viscous fluid is obtained for the case of two space variables. She obtained a time decay of $O(1/t^{3/2})$ for the velocity vector $v(x,t)$ and a time decay of $O(1/t)$ for the pressure function $P(x,t)$. We also mention the work of Hishida [11] here. He was interested in an existence theorem for the Navier–Stokes flow in the exterior of a rotating obstacle. He proved that a unique solution exists locally in time if the initial velocity possesses the regularity $H^{1/2}$.

In the work of Gunther *et al.* [12] we find formulas for computing hydrodynamic forces on a submerged rigid body under assumption that the governing equations for the fluid flow are the steady Navier–Stokes equations. Also of related interest is the case when the body moves by self-propulsion—a subject recently treated by Pukhnachev [13,14], Galdi [15–19]. Self-propulsion is a common means of locomotion of macroscopic bodies in fluids—motions performed by birds, fishes, airplanes, rockets and submarines. In microscopic realm, many minute organism like flagellates and ciliates move by self-propulsion. The main feature of this type of problems lies in the fact that the mechanical system is constituted by solid and liquid so that the motion of the one influences that of the other. Among the many applications that this study may lead to, an interesting one is sedimentation of particles in a quiescent viscous liquid. Here, a homogeneous, symmetric particle is dropped into the liquid under the action of gravity. After a certain time, the particle will eventually reach an equilibrium orientation with respect to gravity. This orientation depends on the physical properties of the liquid and the particle. For instance, if the liquid is viscous and Newtonian, and the particle is a homogeneous body of revolution around an axis a with fore-aft symmetry (like prolate or oblate spheroids of constant density), the particle will always reach an equilibrium orientation orthogonal or parallel to the gravity, depending upon whether the particle is prolate or oblate in shape—no matter what its initial orientation is. If the liquid is viscoelastic, the same particle will reach a different orientation, depending on the balance between the elastic and inertial properties of the fluid see Reference [20]. Mathematical models describing the sedimentation and consolidation of solid–liquid suspension under the influence of gravity are of great importance for a variety of application such as wastewater treatment, mineral processing, chemical and civil engineering see References [21,22].

Global existence of weak solutions for viscous non-steady incompressible flows was studied by many authors see References [23–27], etc. The problem with rotating fluid where Coriolis force plays important role was intensively studied by Babin *et al.* [28–30], etc.

We consider the fall under its own weight of a bounded connected rigid body in an infinite Navier–Stokes fluid which is at rest at infinity. We say that the falling motion is steady if the velocity and pressure of the fluid in a co-ordinate system which is attached to the body are independent of time. Such a motion may be a limit of a class of unsteady falling motions. We prescribe the shape and downward orientation of the body. We think of the body as a hollow one inside which we are free to move masses about. We seek a position of the centre of mass which will result in a steady falling motion with the given downward orientation. In general, the body must undergo a rotation about the vertical axis as well as a translation in this motion.

The velocity at infinity in this moving co-ordinate frame is $-[U+\omega\times r]$, where U and ω are linear and angular velocities of the body relative to a Galilean frame. Since the flow is to be independent of time, U and ω must be constant. These two vectors are to be determined by equating the viscous force and torque on the body to the force and torque due to gravity. The viscous force must be constant. The direction of the gravitational field g in the moving co-ordinates will rotate about the ω-axis, so that the force is not constant, unless ω is in the direction of g. We have $\omega=\lambda g$, where λ is a scalar constant and g is a vector which is fixed in the body.

Since U and ω are constant, the difference between the velocity and its limiting value $-[U+\omega\times r]$ will also be independent of time in the moving co-ordinates. We call this difference $u(x)$. Its components u_j are the components of velocity in Galilean co-ordinate system resolved along the co-ordinate directions of the moving co-ordinate frame.

The steady Navier–Stokes equations has the following form:

$$\begin{aligned}\rho[(u-U-\lambda g\times r)\cdot\nabla]u+\lambda\rho g\times u+\nabla p-\mu\Delta u&=\rho g\\ \nabla\cdot u&=0\end{aligned}\tag{1}$$

Equations (1) are to hold in a fixed domain Ω which is the exterior of a closed bounded connected set $B\subset R^3$. The constant vectors U and λg are to be determined from the equilibrium conditions:

$$\begin{aligned}\int_{\partial B} f\,\mathrm{d}S&=m'g\\ \int_{\partial B} f\times r\,\mathrm{d}S&=m'g\times r'\end{aligned}\tag{2}$$

and with a condition on boundary

$$u(x)=U(x)+\lambda g\times x,\quad x\in\partial\Omega \tag{$*$}$$

and with the behaviour at infinity

$$\lim_{|x|\to\infty} u(x)=0 \tag{$**$}$$

where m', r' are the mass and the position of the centre of mass of the mass distribution on B and $f_i=\sigma_{ij}n_j$ is the surface force per unit area on B. We are interested in the case given g, m', find r' such that there is a solution u of (1) with a condition on the boundary and behaviour at infinity given, respectively, by $(*)$, $(**)$ as well as the equilibrium conditions (2) hold.

First, we would like to mention some analytical results. Then we show asymptotic properties of the Stokes and the Oseen problem when we take into account the Coriolis force (we extend results from paper [31]). We obtained that the Coriolis force gives us more regularity in components of velocity if there is no rotation. Moreover, we prove the existence of the solution of the Navier–Stokes equations under smallness of data. Finally, we are interested in 2D and 3D cases when we also include term $(\omega\times x)\cdot\nabla u$, which the rotation effect brings.

Complementary comments: The mathematical study of the flow of viscous fluid around a three-dimensional obstacle and flow past the obstacle was the subject of many papers. The existence theory of solutions with a finite Dirichlet integral (D-solutions) for both problems is well known, see Reference [32]. In Reference [32] it was also proved that the solutions approach their limits at infinity pointwise. In 1965 Finn introduced the so-called PR-solutions (physically reasonable solution), i.e. solutions satisfying the relation

$$|v(x)| = O(|x|^{-1}) \quad \text{if } v_\infty = 0$$

$$|v(x) - v_\infty| = O(|x|^{-1/2-\varepsilon}) \quad \text{if } v_\infty \neq 0$$

where ε may be arbitrary small. In the case of the flow past the obstacle ($v_\infty \neq 0$) it was proved by Babenko [33] and Galdi [34] that every D-solution is PR-solution. Moreover, in References [33,34] the asymptotic behaviour of solutions was investigated and the existence of a wake region behind the obstacle was shown. The uniqueness of such solutions was studied under additional smallness assumptions, see Reference [34].

The first complete treatment of existence and uniqueness of the Oseen problem in an exterior domain is due to Faxén [35], who generalized the method introduced by Odqvist in his thesis for the problem [36]. Here we would like to mention work of Finn [37], where the existence of solution of steady Navier–Stokes equations and associated perturbation problems in an exterior domain were proved. The Oseen problem in the whole space was studied first by Babenko [33] and we can find the expanded version in works of Galdi [34]. Classical results on behaviour at infinity of the Oseen problem were described in the work of Chang and Finn [39]. Existence and uniqueness results for three-dimensional flows in weighted (anisotropic) Sobolev spaces with weights reflecting the decay properties of fundamental solution have been proved by Farwig [40,41]. A boundary integral approach is provided by Fischer *et al.* [42]. The first existence and uniqueness theorems for the Stokes problem in an exterior domain Ω is due to Borgio [53], for Ω a ball. In the same hypothesis on Ω, Oseen furnishes the explicit form of the Green's tensor. The first existence and uniqueness result in the general case can be found in the work of Odqvist. The exhaustive list of references concerning these problems can be found in the books of Galdi [34] and in the survey paper of Farwig [43]. In 2D case of the Stokes problem we would like to stress the classical paper by Finn and Noll [44] and Chang and Finn [39]. We refer the reader to works of Sequeira [45,46] and Hsiao and McCamy [47]. For the Stokes and the Navier–Stokes problem in weighted function spaces with detached asymptotic, we refer to work of Nazarov and Pileckas [48].

2. PRELIMINARIES

The Lebesgue spaces are denoted by $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, and equipped with the norms $\|\cdot\|_{0,p}$. By $W^{k,p}(\mathbb{R}^n)$, $k \geqslant 0$, an integer, $1 \leqslant p \leqslant \infty$, we denote the usual Sobolev spaces with the norms

$$\|\cdot\|_{k,p} = \sum_{|\alpha|=0}^{k} \|D^\alpha \cdot\|_{0,p} \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ denotes the standard multi-index, see References [52,54]. Further, we define the homogeneous Sobolev spaces $D^{m,q}(\mathbb{R}^n)$ as

$$D^{m,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\nabla\cdot\|_{m-1,q}}$$

equipped with the norm $\|\nabla\cdot\|_{m-1,q}$. Denote by $S(\mathbb{R}^n)$ the space of functions of rapid decrease consisting of element u from $C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n}(|x_1|^{\gamma_1}\dots|x_n|^{\gamma_n}|D^\beta u(x)|)<\infty$$

for all $\gamma_1,\dots,\gamma_n>0$ and $|\beta|\geqslant 0$. For $u\in S(\mathbb{R}^n)$ we denote by $\hat{u}$ its Fourier transform

$$\hat{u}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}x\xi}u(x)\,\mathrm{d}x$$

where i stands for the imaginary unit. It is well-known that $\hat{u}\in S(\mathbb{R}^n)$ and that, moreover,

$$u(x)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}x\xi}\hat{u}(\xi)\,\mathrm{d}\xi$$

Given a function $\Phi:\mathbb{R}^n\to\mathbb{R}$, let us consider the integral transform

$$Tu\equiv h(x)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}x\xi}\Phi(\xi)\hat{u}(\xi)\,\mathrm{d}x,\quad u\in S(\mathbb{R}^n)\tag{4}$$

We define the homogeneous Sobolev spaces

$$\hat{\mathscr{H}}^{2,q}(R^n)=\{u:R^n\to R\text{ measurable, } u,\partial_i u\in L^6_{\text{loc}}(R^n)\partial_i\partial_j u\in L^q(R^n)\text{ for } i,j=1,\dots,n\}$$

We choose any $\emptyset\neq G_0\subset\subset R^n$ and define

$$\hat{\mathscr{H}}^{2,q}(R^n;G_0)=\left\{u\in\hat{\mathscr{H}}^{2,q}(R^n):\int_{G_0}u\,\mathrm{d}x=\int_{G_0}\partial_i u\,\mathrm{d}x=0\text{ for } i=1,\dots,n\right\}$$

with norm

$$\|u\|_{2,q;G_0}=\left(\sum_{i,j=1}^n\|\partial_i\partial_j u\|_q^q\right)^{1/q}+\left|\int_{G_0}u\,\mathrm{d}x\right|+\sum_{i=1}^n\left|\int_{G_0}\partial_i u\,\mathrm{d}x\right|$$

We define $\hat{\mathscr{H}}^{2,q}(R^n,1,G_0)$ by

$$\hat{\mathscr{H}}^{2,q}(R^n,1,G_0)=\left\{u\in\hat{\mathscr{H}}^{2,q}(R^n):\ v\in L^{q^*}(R^n),\nabla v\in L^{q^*}(R^n)^n\text{ and }\int_{G_0}v\,\mathrm{d}x=0\right\}$$

where $q^*=nq/(n-q)$ with norm $\|u\|_{2,q}$ (for more details see Reference [49]).

Lemma 2.1
Let $\Phi:\mathbb{R}^n\to\mathbb{R}$ be continuous together with the derivative $\partial^n\Phi/\partial\xi_1\dots\partial\xi_n$ and all preceding derivatives for $|\xi_i|>0$, $i=1,\dots,n$. Then, if for some $\beta\in[0,1)$ and $M>0$

$$|\xi_1|^{k_1+\beta}\dots|\xi_n|^{k_n+\beta}\left|\frac{\partial^k\Phi}{\partial\xi_1^{k_1}\dots\partial\xi_n^{k_n}}\right|\leqslant M$$

where K_i is zero or one and $K=\sum_{i=1}^{n} k_i=0,1,\ldots,n$, the integral transform (4) defines a bounded linear operator from $L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$, $1<q<\infty$, with $1/r=1/q-\beta$ and further we have $\|Tu\|_r \leqslant c\|u\|_q$.

For more details see Reference [50].

Lemma 2.2
Let $\hat{G}\in C^{m+n-1}(R^n\setminus\{0\})$ $(n\geqslant 2)$ be such that

$$A := \sup_{|\alpha|\leqslant m+n-1,\, \xi\in R^n\setminus\{0\}} (1+|\xi|)^4 |\xi|^{|\alpha|-2} |D^\alpha \hat{G}(\xi)| \leqslant \infty$$

Then $G\in C^m(R^n\setminus\{0\})$ and there exists $c_{m,n}>0$ such that for all $0\neq x\in B_1$ (by B_1 we denote the ball with radius 1 and by B_1^c its complement)

$$|G(x)| \leqslant c_{m,n}A|x|^{2-n} \quad \text{if } n\geqslant 3$$
$$|D^\beta G(x)| \leqslant c_{m,n}A|x|^{2-n-|\beta|} \quad \text{if } n\geqslant 2,\ 0<|\beta|\leqslant m$$

Furthermore, for $x\in B_1^c$, $n\geqslant 2$ and $0\leqslant|\beta|\leqslant m$,

$$|D^\beta G(x)| \leqslant c_{m,n}A|x|^{1-n-|\beta|}$$

Proof
see Reference [51].

We are interested in the strong L^q solution of $-\Delta u=f$ in the whole space.

Lemma 2.3
Let $n\geqslant 2$ and let $1<q<\infty$. Let $\emptyset\neq G_0\subset\subset R^n$. Then

$$-\Delta : \hat{\mathscr{H}}^{2,q}(R^n, G_0) \rightarrow L^q(R^n)$$

is a continuous bijection and with c_q it holds

$$|u|_{2,q} \leqslant c_q\|\Delta u\|_q \quad \text{for } u\in \hat{H}^{2,q}(R^n; G_0)$$

Moreover, let $n\geqslant 2$. In case $n=2$ let $1<q<2$ and in case $n\geqslant 3$ let $n/2\leqslant q<n$. Let $\emptyset\neq G_0\subset\subset R^n$. Then

$$-\Delta : \hat{H}^{2,q}(R^n; 1, G_0) \rightarrow L^q(R^n)$$

is a continuous bijection. With $C(n,q)>0$ and $c_q>0$ it holds

$$\|\nabla u\|_{q^*} + |u|_{2,q} \leqslant (C+1)c_q\|\Delta u\|_q$$

for all $u\in\hat{H}^{2,q}(R^n;1;G_0)$. Finally, let $n\geqslant 3$ and let $1<q<n/2$. Then

$$-\Delta : \hat{H}^{2,q}(R^n) \rightarrow L^q(R^n)$$

is continuous bijection and it holds

$$\|u\|_{q^{**}} + \|\nabla u\|_{q^*} + |u|_{2,q} \leqslant (c+1)c_q \|\Delta u\|_q \quad \text{for all } u \in \hat{H}^{2,q}(R^n)$$

where $q^* = nq/(n-q)$ and $q^{**} = nq/(n-2q)$

Proof
See Reference [49].

3. ANALYTICAL RESULTS

Childress was interested in the slow motion of a sphere in a rotating, viscous fluid with the angular velocity $\tilde{\Omega}$ and he assumed that the axis of rotation is taken to coincide with the line of motion. A sphere of radius a moves with speed U along the axis of rotation and is free to rotate about the same axis. An approximation description of the flow pattern which is valid in the asymptotic sense for small values of the Reynolds number $Re = Ua/v$ and Taylor number $Ta = \tilde{\Omega}a^2/v$, is sought. Such an approximation can be obtained by expansion with respect to Re alone, with a new parameter

$$\alpha = 2(Ta/Re^2) = 2(\tilde{\Omega}v/U^2) \tag{5}$$

held fixed. The principal results of the present investigation may be summarized as follows.

If D is the drag experienced by the sphere, and if ω is the angular velocity of the sphere relative to the fluid at infinity, then the expansion with respect to Re are

$$D/6\pi\rho v U a = 1 + \lambda(\alpha)Re + o(Re) \tag{6}$$

$$\omega/\tilde{\Omega} = \chi(\alpha)Re + o(Re) \tag{7}$$

where the functions $\lambda(\alpha)$ and $\chi(\alpha)$ are given by the definite integrals

$$\begin{aligned} \lambda(\alpha) &= \frac{3}{4}\int_0^1 \{(t^2 + 4\mathrm{i}\alpha t)^{1/2} + (t^2 - 4\mathrm{i}\alpha t)^{1/2}\}(3t^2 - 1)\,\mathrm{d}t \\ \chi(\alpha) &= \frac{3\mathrm{i}}{8\alpha}\int_0^1 \{(t^2 + 4\mathrm{i}\alpha t)^{1/2} - (t^2 - 4\mathrm{i}\alpha t)^{1/2}\}(3t^3 - t)\,\mathrm{d}t \end{aligned} \tag{8}$$

It was found that the effect of rotation are two-fold. First, there is near sphere an added acceleration caused by the Coriolis force. This acceleration is $O(Ta)$ and therefore is here of higher order and the non-linear effect is $O(Re)$. Secondly, in the outer flow field, a distance $O(Re^{-1})$ from the sphere the Coriolis term is of the same order of magnitude as the viscous and convective terms and the perturbation caused by the sphere is consequently altered.

The equations appropriate to the physical problem in dimensionless notation with the reference velocity U and length a are

$$\begin{aligned} Re\, q\nabla q + \nabla p + 2Ta e_1 \times q - \nabla^2 q &= 0 \\ \nabla \cdot q &= 0 \end{aligned} \tag{9}$$

Since the sphere is free to rotate about the axis of symmetry, the boundary conditions are

$$\begin{aligned} q &= (a\omega/U)\,e_1 \times r \quad \text{when } r = (x^2 + y^2 + z^2)^{1/2} = 1 \\ q &= e_1, \quad p = 0 \quad \text{when } r = \infty \end{aligned} \tag{10}$$

In the above, the co-ordinate system moves with the sphere, and rotates with undisturbed fluid so that the term in $(9)_1$ containing Ta has its origin Coriolis force experienced by a fluid element.

The quantity p is defined by

$$p = (a/\rho\nu U)[p^* - \tfrac{1}{2}\rho\tilde{\Omega}^2 a^2 (y^2 + z^2)] \tag{11}$$

where p^* is the pressure. In $(10)_1$, ω is the dimensional angular velocity of sphere relative to the rotating co-ordinate system. This parameter is not known in advance but it will be determined (as a function of Re and Ta) by the requirement that the torque on the sphere is zero.

Childress considered the inner and outer expansion in Re, having the following forms:

$$\begin{aligned} q(r; Re) &= q_0(r) + Re\, q_1(r) + o(Re), \quad (1 \leqslant r < \infty) \\ q(r; Re) &= e_1 + Re\, q'(\tilde{r}) + o(Re) \quad (\tilde{r} > 0) \end{aligned} \tag{12}$$

where in $(12)_2$ the outer variables $\tilde{x} = xRe$, $\tilde{y} = yRe$, $\tilde{z} = zRe$ are used. Expansions for the pressure are similar. If $\alpha > 0$ inner and outer expansions of axial vorticity η, $\eta = e_1 \cdot \nabla \times q$, of the respective forms

$$\begin{aligned} \eta(r; Re) &= Re^2\, \eta_2(r) + o(Re^2) \quad (1 \leqslant r < \infty) \\ \eta(r; Re)\, Re &= Re^2\, \eta_2'(\tilde{r}) + o(Re^2), \quad (\tilde{r} > 0) \end{aligned} \tag{13}$$

must also be considered.

The leading terms q_0, p_0 can be shown to be solutions of Stokes's problem

$$\begin{aligned} \nabla p_0 - \nabla^2 q_0 = 0, \quad \nabla \cdot q_0 = 0 \\ q_0 = 0 \quad \text{when } r = 1 \\ p_0 = 0, \quad q_0 = e_1 \quad \text{when } r = \infty \end{aligned} \tag{14}$$

In particular, the matching conditions $(14)_3$ are unchanged by rotation to this order. The inner boundary condition $(14)_2$ states that the sphere does not rotate differentially to this order. The first-order outer terms q', p' satisfy the Oseen equations, containing now the Coriolis term

$$\begin{aligned} \frac{\partial q'}{\partial \tilde{x}} + \tilde{\nabla} p' + \alpha e_1 \times q' - \tilde{\nabla}^2 q' = 0 \\ \tilde{\nabla} \cdot q' = 0 \end{aligned} \tag{15}$$

The outer boundary conditions are

$$q' = 0, \quad p' = 0 \quad \text{when } \tilde{r} = \infty \tag{16}$$

and there is in addition a matching condition at $\tilde{r} = 0$.

They require that q' has at $\tilde{r}=0$ the singularity of a 'fundamental solution' of (15) and (16) corresponding to a force equal to the Stokes drag of the sphere. Such a solution may be obtained formally by solving (15) and (16) with the right-hand side of $(15)_1$ replaced by $-6\pi\delta(\tilde{r})e_1$. The precise matching condition states that a certain part of q_0, which dominates this term in some intermediate region where matching condition is applied (overlap domain), is cancelled there by a part of $e_1 + Re\, q'$. The common part of these two terms may be shown to be equal to a fundamental solution of the Stokes equations, corresponding as before to Stokes drag. If we denote this common part by A, the matching condition then states that $e_1 + Re\, q' - A$, which is bounded when $\tilde{r}$ is small, i.e. in a region where the Stokes equations approximate the Oseen equations. The physical meaning of this is that in the overlap domain the sphere has the same effect as a point disturbance.

The inner terms of order Re satisfy

$$\nabla p_1 - \nabla^2 q_1 = q_0 \cdot \nabla q_0, \quad \nabla \cdot q_1 = 0 \tag{17}$$

The matching condition is obtained by writting

$$\begin{aligned} e_1 + Re\, q' - A &= Re\,\{B(\alpha) + o(1)\} \quad \text{as } \tilde{r} \to 0 \\ A &= e_1 - \frac{3}{2}\left(\frac{e_1}{r} - \nabla \frac{x}{2r}\right) \end{aligned} \tag{18}$$

where $B(\alpha)$ is also dependent on the direction of $\tilde{r}$.

Then it is required that

$$\lim_{Re\to 0} q_1 = B(\alpha) \tag{19}$$

where the co-ordinates lie in some overlap domain. The condition that the torque on the sphere vanishes to order Re inclusive implies in the same way as before that the term q_1 satisfies the null condition

$$q_1 = 0 \quad \text{when } r = 1 \tag{20}$$

Then he solved it by the Fourier transformation. He got

$$\begin{aligned} q' &= \frac{1}{8\pi^3}\int e^{ik\tilde{r}}\,\Gamma(k)\,dk \\ p' &= \frac{1}{8\pi^3}\int e^{ik\tilde{r}}\,\Pi(k)\,dk \\ \Gamma(k) &= -6\pi\left[\frac{(k^2 e_1 - k_1 k)(k^2 + ik_1) - \alpha k_1(k\times e_1)}{(k^2+ik_1)^2 k^2 + \alpha^2 k_1^2}\right] \\ \Pi(k) &= \frac{i}{k^2[6\pi k_1 + \alpha(k\times e_1)\cdot\Gamma]} \end{aligned} \tag{21}$$

He showed that:

(i) q' plus 6π times the fundamental solution of the Stokes equations is bounded in a neighbourhood of origin;
(ii) the boundary conditions (16) are satisfied;
(iii) the terms in $(15)_1$, $(15)_2$ exist and the equations are satisfied when $\tilde{r}>0$.

Also he calculated the Stokes solution $(A-e_1)/Re$ by

$$\begin{aligned} ik\Pi_S + k^2\Gamma_S &= -6\pi e_1 \\ k\cdot\Gamma_S &= 0 \end{aligned} \tag{22}$$

We can find the constants M, N such that, for $\alpha>0$, $0<k\leqslant\infty$,

$$\begin{aligned} |\Gamma-\Gamma_S| &\leqslant \frac{M}{k^2(1+k)} \\ |\Pi-\Pi_S| &\leqslant \frac{N}{k^2(1+k)} \end{aligned} \tag{23}$$

Now, we define

$$\begin{aligned} \left(\frac{1}{8\pi^3}\right)\int_{\mathbb{R}^3}(\Gamma-\Gamma_S)\,e^{ik\tilde{r}}\,dk &= B(\alpha)+o(1) \\ \left(\frac{i}{8\pi^3}\right)\int_{\mathbb{R}^3} e_1\cdot(k\times\Gamma)\,e^{ik\tilde{r}}\,dk &= C(\alpha)+o(1) \end{aligned} \tag{24}$$

as $\tilde{r}\to 0$. In the limit there is obtained

$$\begin{aligned} B(\alpha) &= e_1\frac{3}{4\pi^2}\int\frac{(k^2-k_1)(ik_1k^4-k^2k^2+\alpha k_1^2)}{(k^2+ik_1)k^6+\alpha^2k_1^2k^4}\,dk+\cdots \\ C(\alpha) &= \frac{3i}{4\pi^2}\alpha\int\frac{k_1(k^2-k_1^2)}{(k^2+ik_1)^2+\alpha^2k_1^2}\,dk+\cdots \\ \lambda(\alpha) &= \frac{2\sqrt{2\alpha}}{7}\left(1+\frac{7}{40}\frac{1}{\alpha}+\frac{15}{1408}\frac{1}{\alpha^2}-\frac{49}{39\,936}\frac{1}{\alpha^3}-\cdots\right) \\ \chi(\alpha) &= -\frac{\sqrt{2}}{5\sqrt{\alpha}}\left(1-\frac{75}{616}\frac{1}{\alpha}+\frac{35}{4992}\frac{1}{\alpha^2}-\frac{45}{28\,160}\frac{1}{\alpha^3}+\cdots\right) \quad \text{for } 4\alpha>1 \end{aligned} \tag{25}$$

It is interesting to note that, for sufficiently small values of α, the effect of rotation is actually to decrease the drag, the minimum occurring for $\alpha=0.175$ approximately. The behaviour of $\lambda(\alpha)$ near $\alpha=0$ is given by the expansion

$$\lambda(\alpha)=\frac{3}{8}+3\alpha 62\log\alpha+O(\alpha^4\log\alpha) \tag{26}$$

The limit-process expansions (12) and (13) have been introduced there with the specific intension of finding how the rotation of the fluid affects the classical Stokes flow. Therefore, in a definite sense the present theory is of 'higher order' and the effects calculated are in the same sense 'small' effects. The relation between the Stokes and the Oseen flows is such that arbitrary small rotation alters the perturbation at large distance. The freedom which we have in the choice of $Ta(Re)$ and the stretching of the co-ordinates appear to offer more than one possible form for these results. They investigated this question in physical terms by reducing all small terms to the role of forcing terms in the Stokes's problem. In order to examine the effect of the Coriolis force we may assume that α is large. Then the approximation can be made in the inner problem as well, so that with no further restriction on Ta the governing equations may be taken to be

$$\nabla p + 2Tae_1 \times q - \nabla^2 q = 0, \quad \nabla \cdot q = 0 \tag{27}$$

For these 'Stokes' equations, the results are true for Ta small. In case of non-small Ta the effect of the Coriolis force is of order unity over the inner flow field and our equations must be solved with the boundary condition on the sphere.

4. ASYMPTOTIC PROPERTIES OF THE NAVIER–STOKES EQUATIONS IN THE PRESENCE OF THE CORIOLIS FORCE

First, we give the summary of results from paper [31]. We shall begin with the simplest situation, a steady, indefinitely slow motion occurring in the whole space and in an exterior domain where the Coriolis forces are presented.

4.1. Stokes problem in the whole space

We are interested in the Stokes problem with the Coriolis force in the whole space in R^3:

$$\left.\begin{aligned} -\mu\Delta u + \lambda g \times u = \nabla p + f \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } R^3 \tag{28}$$

Let for simplicity $g = e_2$.

Theorem 4.1
Given $f \in L^q(\mathbb{R}^3)$, $1 < q < \infty$, there exists a pair of functions u, p with $u_1, u_3 \in L^q(\mathbb{R}^3)$, $u \in D^{2,q}(\mathbb{R}^3)$, $\nabla p \in L^q(\mathbb{R}^3)$ satisfying the Stokes problem (28) and moreover

$$|u|_{2,q} + |p|_{1,q} + \|u_1\|_q + \|u_3\|_q \leqslant \|f\|_q \tag{29}$$

Also, if $1 < q < 3$

$$|u_i|_{1,q} + |u_2|_{1,3q/(3-q)} + |u|_{2,q} + |p|_{1,q} \leqslant \|f\|_q, \quad i = 1, 3 \tag{30}$$

and if $1 < q < \frac{3}{2}$

$$|u_i|_{1,q} + |u_i|_q + |u_2|_{3q/(3-2q)} + |u_2|_{1,3q/(3-q)} + |u|_{2,q} + |p|_{1,q} \leqslant \|f\|_q, \quad i = 1, 3 \tag{31}$$

Proof
Applying the Fourier transformation we obtain the system of algebraic equations

$$\begin{cases} \xi^2\hat{u}_1 + \mathrm{i}\xi_1\hat{p}(\xi) + \lambda\hat{u}_3 = \hat{f}_1 \\ \xi^2\hat{u}_2 + \mathrm{i}\xi_2\hat{p}(\xi) = \hat{f}_2 \\ \xi^2\hat{u}_3 + \mathrm{i}\xi_3\hat{p}(\xi) - \lambda\hat{u}_1 = \hat{f}_3 \\ \mathrm{i}\xi_m\hat{u}_m = 0, \quad m = 1,\dots,3 \end{cases} \tag{32}$$

Solving the system we get solution $(\hat{u}, \hat{p})$ in the following form:

$$\hat{u}_1 = \frac{(\xi^4 + \lambda\xi_1\xi_3)(\hat{f}_1\xi^2 - \xi_1\xi_m\hat{f}_m) - \lambda(\xi_2^2 + \xi_3^2)(\hat{f}_3\xi^2 - \xi_3\xi_m\hat{f}_m)}{\xi^8 + \lambda^2\xi_1^2\xi_3^2 + \lambda^2(\xi_1^2 + \xi_2^2)(\xi_2^2 + \xi_3^2)} \tag{33}$$

$$\hat{u}_3 = \frac{\lambda(\xi^2\hat{f}_1 - \xi_1\xi_m\hat{f}_m)(\xi_1^2 + \xi_2^2) + (\xi^4 + \lambda\xi_1\xi_3)(\xi^2\hat{f}_3 - \xi_3\xi_m\hat{f}_m)}{\xi^8 + \lambda^2\xi_1^2\xi_3^2 + \lambda^2(\xi_1^2 + \xi_2^2)(\xi_2^2 + \xi_3^2)} \tag{34}$$

$$\hat{p} = \frac{\xi_m\hat{f}_m}{\mathrm{i}\xi^2} - \frac{\lambda(\xi_1\hat{u}_3 - \xi_3\hat{u}_1)}{\mathrm{i}\xi^2} \tag{35}$$

$$\hat{u}_2 = \frac{\hat{f}_2}{\xi^2} - \frac{\mathrm{i}\xi_2}{\xi^2}\left(\frac{\xi_m\hat{f}_m}{\mathrm{i}\xi^2} - \frac{\lambda(\xi_1\hat{u}_3 - \xi_3\hat{u}_1)}{\mathrm{i}\xi^2}\right) \tag{36}$$

Applying the Lizorkin theorem we get the desired existence of solution. For more details see Reference [31]. □

4.2. The Oseen problem in the whole space in $\mathbb{R}^3$

Now, we investigate the problem

$$\left.\begin{aligned} \tilde{v}\cdot\nabla u + \lambda g \times u - \mu\Delta u + \nabla p = f \\ \nabla\cdot u = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \tag{36$'$}$$

Let $\tilde{v} = e_2$, $g = e_2$.

Theorem 4.2
Given $f \in L^q(\mathbb{R}^n)$, $1 < q < \infty$, there exists a pair of functions (u, p) with $u \in D^{2,q}(\mathbb{R}^3)$, $\nabla p \in L^q(\mathbb{R}^3)$, $u_1, u_3 \in L^q(\mathbb{R}^3)$, $\partial u_1/\partial x_2$, $\partial u_2/\partial x_2$, $\partial u_3/\partial x_2 \in L^q(\mathbb{R}^3)$, satisfying (36) and (36)′. Moreover

$$\left\|\frac{\partial u}{\partial x_2}\right\|_q + \left\|\frac{\partial u_1}{\partial x_l}\right\|_q + \left\|\frac{\partial u_3}{\partial x_l}\right\|_q + \|u_1\|_q + \|u_3\|_q + |p|_{1,q} + |u|_{2,q} \leqslant c\|f\|_q \tag{37}$$

Further, if $1 < q < 4$ then

$$\left\|\frac{\partial u}{\partial x_2}\right\|_q + \left\|\frac{\partial u_i}{\partial x_l}\right\|_q + \|u_i\|_q + |u_2|_{1,4q/(4-q)} + \|p\|_{1,q} \leqslant c\|f\|_q, \quad i = 1,3 \tag{38}$$

Also, if $1<q<2$, $i=1,3$

$$\left\|\frac{\partial u}{\partial x_2}\right\|_q + |u_i|_q + |u_2|_{2q/(2-q)} + |u_i|_{1,q} + |u_2|_{1,4q/(4-q)} + |p|_{1,q} + |u|_{2,q} \leqslant \|f\|_q \tag{39}$$

Proof

Applying the Fourier transformation we get the following systems of algebraic equations:

$$\begin{cases} (\xi^2 + \mathrm{i}\xi_2)\hat{u}_1(\xi) + \mathrm{i}\xi_1\hat{p}(\xi) + \lambda\hat{u}_3(\xi) = \hat{f}_1(\xi) \\ (\xi^2 + \mathrm{i}\xi_2)\hat{u}_2(\xi) + \mathrm{i}\xi_2\hat{p}(\xi) = \hat{f}_2(\xi) \\ (\xi^2 + \mathrm{i}\xi_2)\hat{u}_3(\xi) + \mathrm{i}\xi_3\hat{p}(\xi) - \lambda\hat{u}_1(\xi) = \hat{f}_3(\xi), \quad \mathrm{i}\xi_m\hat{u}_m = 0 \quad m=1,2,3 \end{cases} \tag{40}$$

which has the solution $(\hat{u}, \hat{p})$

$$\hat{u}_1 = \frac{(\xi^2(\xi^2+\mathrm{i}\xi_2) - \lambda\xi_1\xi_3)(\xi^2\hat{f}_1 - \xi_1\xi_m\hat{f}_m) - \lambda(\xi_2^2+\xi_3^2)(\hat{\xi}^2\hat{f}_3 - \xi_3\xi_m\hat{f}_m)}{\lambda^2(\xi_2^2\xi_1^2 + \xi_2^4 + \xi_3^2\xi_1^2 + \xi_3^2\xi_2^2) + (\xi^2+\mathrm{i}\xi_2)^2\xi^4 - \lambda\xi_1^2\xi_3^2} \tag{41}$$

$$\hat{u}_3 = \frac{(\xi^2+\mathrm{i}\xi_2)\xi^2(\hat{f}_3\xi^2 + \hat{f}_m\xi_3\xi_m) + (\xi_1^2+\xi_2^2)(\lambda\xi^2\hat{f}_1 - \lambda\xi_1\xi_m\hat{f}_m) + \lambda\xi_1\xi_3[\xi^2\hat{f}_3 - \lambda\xi_m\xi_3\hat{f}_m)}{\lambda^2(\xi_2^2\xi_1^2 + \xi_2^4 + \xi_3^2\xi_1^2 + \xi_3^2\xi_2^2) + (\xi^2+\mathrm{i}\xi_2)^2\xi^4 - \lambda^4\xi_1^2\xi_3^2} \tag{42}$$

$$\hat{p}(\xi) = \frac{\xi_m\hat{f}_m - \lambda(\xi_1\hat{u}_3 - \xi_3\hat{u}_1)}{\mathrm{i}\xi^2} \tag{43}$$

$$\hat{u}_2(\xi) = \frac{\xi^2\hat{f}_2 - \xi_3\xi_m\hat{f}_m}{\xi^2(\xi^2+\mathrm{i}\xi_2)} + \frac{\lambda(\xi_2(\xi_1\hat{u}_3 - \xi_3\hat{u}_1))}{\xi^2(\xi^2+\mathrm{i}\xi_2)} \tag{44}$$

Now, applying the Lizorkin theorem we get our theorem. □

4.3. *The Stokes problem in an exterior domain*

The aim of this section is to extend the theorems proved in Sections 4.1 and 4.2 to an exterior domain Ω.

Let us begin considering the Stokes problem in an exterior domain Ω of class C^{m+2}, $m \geqslant 0$, with data $f \in C_0^\infty(\bar{\Omega})$, $v_* \in W^{m+2-1/q}(\partial\Omega)$, $v_\infty \neq 0$. The governing equations are

$$\left.\begin{aligned} -\Delta v + \omega \times v &= \nabla p + f \\ \nabla \cdot v &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{45}$$

$$\begin{gathered} v|_{\partial\Omega} = v_* \\ \lim v_{|x|\to\infty} v(x) = v_\infty \end{gathered} \tag{45$'$}$$

Theorem 4.3

Let Ω be an exterior domain in $\mathbb{R}^3$ of class C^{m+2}, $m \geqslant 0$. Given $f \in W^{m,q}(\Omega)$, $v_* \in W^{m+2-1/q,q}(\partial\Omega)$, $1<q<\frac{3}{2}$, $v_\infty \in \mathbb{R}^3$ there exists one and only one solution v, p to the Stokes problem such that

$$v_i - v_{*_i} \in W^{m,q}(\Omega) \cap \left\{ \bigcap_{l=0}^{m} [D^{l+1,q}(\Omega) \cap D^{l+2,q}] \right\}, \quad i=1,3$$

$$v_2 - v_{*_2} \in W^{m,3q/(3-2q)}(\Omega) \cap \left\{ \bigcap_{l=0}^{m} [D^{l+1,3q/(3-q)}(\Omega) \cap D^{l+2,q}] \right\} \tag{46}$$

$$p \in \bigcap_{l=0}^{m} D^{l+1,q}(\Omega)$$

Moreover, v, p satisfy

$$\begin{aligned} &\|v_i - v_{i\infty}\|_{m,q} + \|v_2 - v_{2\infty}\|_{m,3q/(3-2q)} \\ &\quad + \sum_{l=0}^{m} \{|v_i|_{l+1,q} + \|v_2\|_{l+1,3q/(3-q)} + |v|_{l+2,q} + |p|_{l+1,q}\} \\ &\quad \leqslant c(\|f\|_{m,q} + \|v_* - v_\infty\|_{m+2-1/q,q,\partial\Omega}) \end{aligned} \tag{47}$$

where c depends on m, n, q, Ω. Moreover, let $f \in L^1(\Omega)$. Then for $x \in B_1$ (by B_1 we denote the ball with radius 1 and B_1^c its complement)

$$\begin{aligned} |v(x)| &\leqslant c_{m,n}|x|^{-1} \\ |D^\beta v(x)| &\leqslant c_{m,n}|x|^{-1-\beta}, \quad 0<|\beta|\leqslant 2 \end{aligned} \tag{48}$$

and for $x \in B_1^c$, $0 \leqslant \beta \leqslant 2$

$$|D^\beta v(x)| \leqslant c_{m,n}|x|^{-2-\beta} \tag{49}$$

Proof

The existence was proved in Reference [31]. Now, we investigate the asymptotic behaviour of solution. From (33) and (34) it follows that

$$|\hat{v}_i|_\infty \leqslant \frac{|\xi^2|+1}{(|\xi|^4+1)}|\hat{f}|_\infty, \quad i=1,3 \tag{49$'$}$$

From (36) and (36)$'$ we obtain

$$|\hat{v}_2|_\infty \leqslant \frac{|\xi^2|+1}{(|\xi|^4+1)}|\hat{f}|_\infty|\xi|^{-2/3} \tag{49$''$}$$

Applying Lemma 2.2 into (49)$'$ and modifying Lemma 2.2 and applying to (49)$''$ we get (48)–(50). By modification we mean that instead of term

$$A := \sup_{|\alpha| \leqslant m+n-1, \xi \in R^n \setminus \{0\}} (1+|\xi|)^4 |\xi|^{|\alpha|-2} |D^\alpha \hat{G}(\xi)| \leqslant \infty$$

we use term

$$A' := \sup_{|\alpha| \leqslant m+n-1, \xi \in R^n \setminus \{0\}} (1+|\xi|)^{4+2/3} |\xi|^{|\alpha|-2} |D^\alpha \hat{G}(\xi)| \leqslant \infty \qquad \square$$

4.4. The Oseen problem in an exterior domain

Let us begin considering the Oseen problem in an exterior domain Ω of class C^{m+2}, $m\geqslant 0$, with data $f\in C_0^\infty(\Omega)$, $v_\infty\neq 0$, $v_*\in W^{m+2-1/q}$. The governing equations are

$$\left.\begin{aligned} -\Delta v+\frac{\partial v}{\partial x_2}+\omega\times v=\nabla p+f\\ \nabla\cdot v=0\end{aligned}\right\}\quad \text{in } \Omega \tag{50}$$

$$\begin{gathered} v|_{\partial\Omega}=v_*\\ \lim_{|x|\to\infty} v=v_\infty\end{gathered} \tag{50$'$}$$

Theorem 4.4

Let Ω be an exterior domain in $\mathbb{R}^3$ of class C^{m+2}, $m\geqslant 0$. Given $f\in W^{m,q}(\Omega)$, $v_*\in W^{m+2-1/q,q}(\partial\Omega)$, $1<q<2$, $v_\infty\in\mathbb{R}^3$, there exists one and only one solution v,p to the Oseen problem such that

$$\begin{gathered} v_i-v_{*_i}\in W^{m,q}(\Omega)\cap\left\{\bigcap_{l=0}^{m}[D^{l+1,q}(\Omega)\cap D^{l+2,q}]\right\}\\ v_2-v_{*_2}\in W^{m,2q/(2-q)}(\Omega)\cap\left\{\bigcap_{l=0}^{m}[D^{l+2,q}]\right\}\\ \frac{\partial v}{\partial x_2}\in W^{m,q}(\Omega)\\ \frac{\partial v_2}{\partial x_l}\in W^{m,4q/(4-q)}(\Omega)\\ p\in\bigcap_{l=0}^{m} D^{l+1,q}(\Omega)\end{gathered} \tag{51}$$

Moreover, v,p satisfy

$$\begin{aligned} &\|v_i-v_{\infty_i}\|_{m,q}+\|v_2-v_{\infty_2}\|_{m,\,2q/(2-q)}\\ &\quad+\sum_{l=0}^{m}\left\{|v_i|_{l+1,q}+\left\|\frac{\partial v_2}{\partial x_e}\right\|_{l,4q/(4-q)}+|v|_{l+2,q}+|p|_{l+1,q}\right\}\\ &\leqslant c(\|f\|_{m,q}+\|v_*-v_\infty\|_{m+2-1/q,q,\partial\Omega}),\quad i,\ l=1,3\end{aligned} \tag{52}$$

where c depends on m,q,n,Ω. Further, suppose that $f\in L^1(\Omega)$. Then for $x\in B_1$ (as before by B_1 we denote the ball with radius 1, and B_1^c its complement)

$$\left.\begin{aligned} |v(x)|&\leqslant c_{m,n}|x|^{-1}\\ |D^\beta v(x)|&\leqslant c_{m,n}|x|^{-1-\beta},\quad 0<|\beta|\leqslant 2\end{aligned}\right\} \tag{53}$$

and for $x\in B_1^c$, $0\leqslant\beta\leqslant 2$

$$|D^\beta v(x)|\leqslant c_{m,n}|x|^{-2-\beta} \tag{54}$$

Proof
The existence was proved in Reference [31]. Now, we investigate the asymptotic behaviour of solution. From (41) and (42) it follows that

$$|\hat{v}_i|_\infty \leqslant \frac{|\xi^2|+1}{(|\xi|^4+1)}|\hat{f}|_\infty, \quad i=1,3 \tag{54$'$}$$

Equation (44) implies

$$|\hat{v}_2|_\infty \leqslant \frac{|\xi^2|+1}{(|\xi|^4+1)}|\hat{f}|_\infty|\xi|^{-2/3} \tag{54$''$}$$

Applying Lemma 2.2 to (54)′ and modified Lemma 2.2 to (54)″ we get (53) and (54). By modification we mean that instead of term

$$A := \sup_{|\alpha|\leqslant m+n-1, \xi\in R^n\setminus\{0\}} (1+|\xi|)^4|\xi|^{|\alpha|-2}|D^\alpha\hat{G}(\xi)| \leqslant \infty$$

we use the term

$$A' := \sup_{|\alpha|\leqslant m+n-1, \xi\in R^n\setminus\{0\}} (1+|\xi|)^{4+1/2}|\xi|^{|\alpha|-2}|D^\alpha\hat{G}(\xi)| \leqslant \infty$$

□

4.5. The Navier–Stokes equations

Now, we are interested in the Navier–Stokes equations with the Coriolis force. The problem reads

$$\left.\begin{aligned} -\Delta v + v\cdot\nabla v + \omega\times v &= \nabla p + f \\ \nabla\cdot v &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{55}$$

$$\begin{aligned} v|_{\partial\Omega} &= v_* \\ \lim v_{|x|\to\infty} &= v_\infty \end{aligned} \tag{55$'$}$$

Theorem 4.5
Let Ω be an exterior domain in $R^3$of class C^{m+2}, $m\geqslant 0$. Given $f\in W^{m,q}(\Omega)$, $v_*\in W^{m+2-1/q,q}(\partial\Omega)$, $1<q<2$, $v_\infty\in R^3$ there exist constant β,λ depending on $m,\partial\Omega$ such that if $0<|v_\infty|<\lambda$, $|f|_{m,q}\leqslant\beta$, then problem (55) and (55)′ admits a unique solution v, p to the Navier–Stokes equations such that

$$\begin{aligned} v_i - v_{*_i} &\in W^{m,q}(\Omega)\cap\left\{\bigcap_{l=0}^{m}[D^{l+1,q}(\Omega)\cap D^{l+2,q}]\right\} \\ v_2 - v_{*_2} &\in W^{m,2q/(2-q)}(\Omega)\cap\left\{\bigcap_{l=0}^{m}[D^{l+2,q}]\right\}, \quad i=1,3 \\ \frac{\partial v}{\partial x_2} &\in W^{m,q}(\Omega) \\ \frac{\partial v_2}{\partial x_l} &\in W^{m,4q/(4-q)}(\Omega) \\ p &\in \bigcap_{l=0}^{m} D^{l+1,q}(\Omega) \end{aligned} \tag{56}$$

Moreover, v, p satisfy

$$\|v_i - v_{\infty_i}\|_{m,q} + \|v_2 - v_{\infty_2}\|_{m,2q/(2-q)} + \sum_{l=0}^{m} \left\{ |v_i|_{l+1,q} + \left\| \frac{\partial v_2}{\partial x_e} \right\|_{l,4q/(4-q)} + |v|_{l+2,q} + |p|_{l+1,q} \right\} \leqslant c(\|f\|_{m,q} + \|v_* - v_\infty\|_{m+2-1/q,q,\partial\Omega}), \quad i,\ l = 1,3 \tag{57}$$

where c depends on m, q, n, Ω. Further, suppose that $f \in L^1(\Omega)$. Then for $x \in B_1$

$$\begin{aligned} |v(x)| &\leqslant c_{m,n}|x|^{-1} \\ |D^\beta v(x)| &\leqslant c_{m,n}|x|^{-1-\beta}, \quad 0 < |\beta| \leqslant 2 \end{aligned} \tag{58}$$

and for $x \in B_1^c$, $0 \leqslant \beta \leqslant 2$

$$|D^\beta v(x)| \leqslant c_{m,n}|x|^{-2-\beta} \tag{59}$$

Proof
Defining $u = v - v_\infty$, we rewrite our problem into the Oseen problem

$$\left.\begin{aligned} -\Delta u + v_\infty \nabla u + \omega \times u &= \nabla p + F \\ \nabla \cdot u &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{60}$$

$$\begin{aligned} u|_{\partial\Omega} &= u_* \\ \lim u_{|x| \to \infty} &= 0 \end{aligned} \tag{60$'$}$$

where

$$F = f - u\nabla u, \quad u_* = v_* - v_\infty$$

The solution of (60) and (61)$'$ satisfy the following estimate:

$$\left\| \frac{\partial u}{\partial x_2} \right\|_q + |u_i|_q + |u_2|_{2q/(2-q)} + |u_i|_{1,q} + |u_2|_{1,4q/(4-q)} + |p|_{1,q} + |u|_{2,q} \leqslant \|F\|_q, \quad i = 1,3 \tag{61}$$

Now linearizing problem (60) and (60)$'$ to the problem

$$\left.\begin{aligned} -\Delta u + v_\infty \nabla u + \omega \times u &= \nabla p + F \\ \nabla \cdot u &= 0 \end{aligned}\right\} \quad \text{in } \Omega \tag{60$''$}$$

$$\begin{aligned} u|_{\partial\Omega} &= u_* \\ \lim_{|x| \to \infty} u &= 0 \end{aligned} \tag{60$'''$}$$

where

$$F = f - w\nabla w, \quad u_* = v_* - v_\infty$$

We look for a solution (u, p) to the non-linear system (55) and (55)$'$ as a fixed point of map $\mathcal{N}: w \to u$ defined by (60)$''$ and (60)$'''$. Let us define the set $\mathcal{D} = \{u \in \mathcal{W}, \|u\|_{\mathcal{W}} \leqslant \infty\}$, $\mathcal{D}_\delta = \{u \in \mathcal{W}, \|u\|_{\mathcal{W}} \leqslant \delta\}$, where

$$\mathcal{W} = \left\{ u: \begin{array}{c} u_i - u_{*_i} \in W^{m,q}(\Omega) \cap \left\{ \bigcap_{l=0}^{m} [D^{l+1,q}(\Omega) \cap D^{l+2,q}] \right\} \\ u_2 - u_{*_2} \in W^{m,2q/(2-q)}(\Omega) \cap \left\{ \bigcap_{l=0}^{m} [D^{l+2,q}] \right\} \\ \dfrac{\partial u}{\partial x_2} \in W^{m,q}(\Omega) \\ \dfrac{\partial u_2}{\partial x_l} \in W^{m,4q/(4-q)}(\Omega) \\ |x| u \in L^\infty(\Omega) \end{array} \right\} \qquad (61)'$$

Note that $\mathcal{D}_\delta$ is a closed, convex and bounded subset of $\mathcal{D}$ and

$$\|u\|_{\mathcal{W}} = \left\| \frac{\partial u}{\partial x_2} \right\|_q + |u_i|_q + |u_2|_{2q/(2-q)} + |u_i|_{1,q} + |u_2|_{1,4q/(4-q)} + |p|_{1,q} + |u|_{2,q}, \quad i = 1, 3 \qquad (62)$$

First, we have to show that the map $\mathcal{N}$ introduced in (60) and (60)$'$ is well defined from D_δ to $\mathcal{D}$. We have to verify that $F \in W^{m,q}(\Omega)$.

$$|F|_{m,q} \leqslant |f|_{m,q} + |w \cdot \nabla w|_{m,q} \leqslant |f|_{m,q} + \| x| w|_\infty |D^2 w|_{m,q} \qquad (63)$$

Since $|w|_{D_\delta} \leqslant \delta$ it implies that $\mathcal{N}$ maps the set $\mathcal{D}_\delta$ into itself. It remains to show that $\mathcal{N}$ is a contraction in the topology of $\mathcal{W}$. Let us take w_1, w_2 and the corresponding images though the mapping u_1, u_2. Further, let us set $w = w_1 - w_2$, $u = u_1 - u_2$, $F = F_1 - F_2$. One obtains the following equations:

$$\left. \begin{array}{c} -\Delta u + v_\infty \nabla u + \omega \times u = \nabla p + F \\ \nabla \cdot u = 0 \end{array} \right\} \quad \text{in } \Omega \qquad (64)$$

$$\begin{array}{c} u|_{\partial\Omega} = u_* \\ \lim\limits_{|x| \to \infty} u = 0 \end{array} \qquad (64)'$$

where

$$F = f - w \nabla w$$

It follows that

$$\left\| \frac{\partial u}{\partial x_2} \right\|_q + |u_i|_q + |u_2|_{2q/(2-q)} + |u_i|_{1,q} + |u_2|_{1,4q/(4-q)} + |p|_{1,q} + |u|_{2,q} \leqslant \|F\|_q, \quad i = 1, 3 \qquad (65)$$

and

$$|u|_{\mathcal{W}} \leqslant |w|_{\mathcal{W}} \qquad (66)$$

Therefore, $\mathcal{N}$ is a contraction in $\mathcal{D}$, provided that the mapped elements belong to $\mathcal{D}_\delta$. It implies since $\mathcal{D}$ is closed also in topology of $\mathcal{W}$, the mapping $\mathcal{N}$ has a unique fixed point in the ball $\mathcal{D}_\delta$. Hence, we have found a unique solution to system (55) and (55)$'$. Applying Theorem 4.4 we get also the asymptotic behaviour. □

5. THE OSEEN PROBLEM WHICH INCLUDES ALSO TERM $(\omega \times x) \cdot \nabla u$ IN THE WHOLE SPACE

We consider the following problem:

$$\left.\begin{aligned} -\Delta u + \tfrac{\partial u}{\partial x_2} + \omega \times u - (\omega \times x) \cdot \nabla u = \nabla p + g \\ \nabla \cdot u = 0 \\ \lim_{|x| \to \infty} u = 0 \end{aligned}\right\} \tag{67}$$

Applying the divergence on $(67)_1$ and using the property that the term $\omega \times u - (\omega \times x) \cdot \nabla u$ is divergence free (see Reference [11]) we obtain the pressure

$$p = (\Delta)^{-1} \nabla \cdot g$$

Applying the standard elliptic regularity results, we have

$$\begin{aligned} |p|_{W^{1,s}(\Omega)} &\leqslant c(s)|f|_{L^s}, \quad 1 < s < \infty. \text{ In particular} \\ |p|_{L^q(\Omega)} &\leqslant c(s,q)|f|_{L^s}, \quad q \text{ finite, provided } 1/q \geqslant 1/s - 1/3 \\ |p|_{L^\infty(\Omega)} &\leqslant c(s)|f|_{L^s}, \quad \text{if } s > 3 \end{aligned}$$

It implies that we can consider the pressure as a known function. Let $f := \nabla p + g$. For the moment we omit the second two terms on the left-hand side. First, we are interested in 2D situation.

5.1. *Two-dimensional problem*

5.1.1. Particular problem (P1). We consider the situation in 2D which has the following form:

$$\begin{aligned} -\Delta u_1 &= -\omega y \frac{\partial u_1}{\partial x} + f_1 \\ -\Delta u_2 &= \omega x \frac{\partial u_2}{\partial y} + f_2 \\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} &= 0 \end{aligned} \tag{P1}$$

Applying the Fourier transformation we get the following system:

$$\begin{aligned} \frac{\partial \hat{u}_1}{\partial \xi_2} &= \frac{\xi^2}{\omega \xi_1} \hat{u}_1 + \frac{\hat{f}_1}{\omega \xi_1} \\ \frac{\partial \hat{u}_2}{\partial \xi_1} &= -\frac{\xi^2}{\omega \xi_2} \hat{u}_2 - \frac{\hat{f}_2}{\omega \xi_2} \end{aligned} \tag{68}$$

We rewrite our problem in the following way:

$$\begin{aligned}\frac{\partial}{\partial\xi_2}\left(\hat{u}_1 e^{-\int_{\xi_2}^{\infty}\frac{\xi^2}{\omega\xi_1}\,ds_2}\right) &= \frac{\hat{f}_1}{\omega\xi_1}\,e^{-\int_{\xi_2}^{\infty}\frac{\xi^2}{\omega\xi_1}\,ds_2}\\ \frac{\partial}{\partial\xi_1}\left(\hat{u}_2 e^{-\int_0^{\xi_1}\frac{\xi^2}{\omega\xi_2}\,ds_1}\right) &= \frac{\hat{f}_2}{\omega\xi_2}\,e^{-\int_0^{\xi_1}\frac{\xi^2}{\omega\xi_2}\,ds_1}\end{aligned} \tag{69}$$

Now, we integrate $(69)_1$ from ξ_2 to ∞ and $(69)_2$ from 0 to ξ_1. We get the following solution:

$$\begin{aligned}\hat{u}_1(\xi_1,\xi_2) &= e^{\int_{\xi_2}^{\infty}\frac{\xi^2}{\omega\xi_1}\,ds_2}\int_{\xi_2}^{\infty} e^{\int_{\xi_2}^{\infty}\frac{\xi^2}{\omega\xi_1}\,ds_2}\frac{\hat{f}_1}{\omega\xi_1}\,d\chi_2\\ \hat{u}_2(\xi_1,\xi_2) &= \hat{u}_2(0,\xi_2)e^{-\int_0^{\xi_1}\frac{\xi^2}{\omega\xi_1}\,ds_1} + e^{-\int_0^{\xi_1}\frac{\xi^2}{\omega\xi_1}\,ds_1}\int_0^{\xi_1} e^{\int\frac{\xi^2}{\omega\xi_2}\,ds_1}\frac{\hat{f}_2}{\omega\xi_2}\,d\chi_1\end{aligned} \tag{70}$$

Requiring $\lim_{\xi_1,\xi_2\to 0}\hat{u}(\xi_1,\xi_2)=0$, it implies that $\lim_{\xi_2\to\infty}\hat{u}_2(0,\xi_2)=0$. From Lizorkin theorem, see Lemma 2.1, we get the following lemma.

Lemma 5.1
Let $\hat{u}_1,\hat{u}_2$ be given by (70). Then the assumptions of Lemma 2.1 are satisfied and

(i) $\hat{u}_1,\hat{u}_2,\beta=1/2$,
(ii) $\xi_1\hat{u}_1,\beta=0$,
(iii) by $\xi_1^2\hat{u}_1,\beta=0$,
(iv) by $\xi_2\hat{u}_2,\beta=0$,
(v) by $\xi_2^2\hat{u}_2,\beta=0$.

Theorem 5.1
Given $g\in L^p(\mathbb{R}^2)$, $\nabla\cdot g\in L^p(\mathbb{R}^2)$, $1<p<2$, there exists a pair of functions u,p satisfying problem (P.1) and moreover

$$\begin{aligned}&\|u\|_{2p/(2-p)} + |p|_{2,p} + \left\|\frac{\partial u_1}{\partial x}\right\|_p + \left\|\frac{\partial u_2}{\partial y}\right\|_p\\ &\quad + |\Delta u|_p + \|\nabla p\|_{2p/(2-p)} \leqslant \|g\|_p + (c+1)c_p\|\nabla\cdot g\|_p\end{aligned} \tag{71}$$

with $c>0$, $c_p>0$ (see Lemma 2.3).

Proof
Equations (68)–(70) Lemmas 2.1, 2.3 and 5.1 imply (71). □

Now, we can go to the origin problem

$$u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\xi}\hat{u}(\xi)\,d\xi$$

From Lemma 5.1(i) and (ii) we get that $y\partial u_1/\partial x\in L^1(\mathbb{R}^2)$ and the Fourier transformation makes sense. Assuming that $\hat{f}=O(|\xi|^2)$ near origin it implies that $\lim_{x\to\infty} u=0$.

5.1.2. Particular problem (P2). Now, we also include the term $\partial u/\partial y$ and consider the equations

$$\begin{aligned} -\Delta u_1 &= -\omega y \frac{\partial u_1}{\partial x} - \frac{\partial u}{\partial y} + f_1 \\ -\Delta u_2 &= \omega x \frac{\partial u_2}{\partial y} - \frac{\partial u}{\partial y} + f_2 \\ \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} &= 0 \end{aligned} \tag{P2}$$

Applying the Fourier transformation we get the following system:

$$\begin{aligned} \frac{\partial \hat{u}_1}{\partial \xi_2} &= \frac{\xi^2 + \mathrm{i}\xi_2}{\omega \xi_1} \hat{u}_1 + \frac{\hat{f}_1}{\omega \xi_1} \\ \frac{\partial \hat{u}_2}{\partial \xi_1} &= -\frac{\xi^2 - \mathrm{i}\xi_2}{\omega \xi_2} \hat{u}_2 - \frac{\hat{f}_2}{\omega \xi_2} \end{aligned} \tag{72}$$

We rewrite our problem in the following way:

$$\begin{aligned} \frac{\partial}{\partial \xi_2}\left(\hat{u}_1 \mathrm{e}^{-\int_{\xi_2}^{\infty} \frac{\xi^2+\mathrm{i}\xi_2}{\omega\xi_1} \mathrm{d}s_2}\right) &= \frac{\hat{f}_1}{\omega \xi_1} \mathrm{e}^{-\int_{\xi_2}^{\infty} \frac{\xi^2+\mathrm{i}\xi_2}{\omega\xi_1} \mathrm{d}s_2} \\ \frac{\partial}{\partial \xi_1}\left(\hat{u}_2 \mathrm{e}^{-\int_0^{\xi_1} \frac{\xi^2-\mathrm{i}\xi_2}{\omega\xi_2} \mathrm{d}s_1}\right) &= \frac{\hat{f}_2}{\omega \xi_2} \mathrm{e}^{-\int_0^{\xi_1} \frac{\xi^2-\mathrm{i}\xi_2}{\omega\xi_2} \mathrm{d}s_1} \end{aligned} \tag{73}$$

Now, we integrate $(73)_1$ from ξ_2 to ∞ and $(73)_2$ from 0 to ξ_1. We get the following solution:

$$\begin{aligned} \hat{u}_1(\xi_1, \xi_2) &= \mathrm{e}^{\int_{\xi_2}^{\infty} \frac{\xi^2+\mathrm{i}\xi_2}{\omega\xi_1} \mathrm{d}s_2} \int_{\xi_2}^{\infty} \mathrm{e}^{\int_{\xi_2}^{\infty} \frac{\xi^2+\mathrm{i}\xi_2}{\omega\xi_1} \mathrm{d}s_2} \frac{\hat{f}_1}{\omega\xi_1} \, \mathrm{d}\chi_2 \\ \hat{u}_2(\xi_1, \xi_2) &= \hat{u}_2(0, \xi_2) \mathrm{e}^{-\int_0^{\xi_1} \frac{\xi^2-\mathrm{i}\xi_2}{\omega\xi_1} \mathrm{d}s_1} + \mathrm{e}^{-\int_0^{\xi_1} \frac{\xi^2-\mathrm{i}\xi_2}{\omega\xi_1} \mathrm{d}s_1} \int_0^{\xi_1} \mathrm{e}^{\int_0^{\xi_1} \frac{\xi^2-\xi_2}{\omega\xi_2} \mathrm{d}s_1} \frac{\hat{f}_2}{\omega\xi_2} \, \mathrm{d}\chi_1 \end{aligned} \tag{74}$$

Since we required that $\lim_{\xi_1,\xi_2 \to 0} \hat{u}(\xi_1, \xi_2) = 0$, it implies that $\lim_{\xi_2 \to \infty} \hat{u}_2(0, \xi_2) = 0$. Applying the theory of multiplier we get the following lemma.

Lemma 5.2
Let $\hat{u}_1, \hat{u}_2$ be given by (74). Then the assumptions of Lemma 2.1 are satisfied and

(i) $\hat{u}_1, \hat{u}_2, \beta = 1/2$,
(ii) $\xi_1 \hat{u}_1, \beta = 0$,
(iii) by $\xi_1^2 \hat{u}_1, \beta = 0$,
(iv) by $\xi_2 \hat{u}_2, \beta = 0$,
(v) by $\xi_2^2 \hat{u}_2, \beta = 0$.

Theorem 5.2
Given $g \in L^p(\mathbb{R}^2)$, $\nabla \cdot g \in (\mathbb{R}^2)$, $1 < p < 2$, there exists a pair of functions u, p satisfying problem (P.2) and moreover

$$|u|_{2p/(2-p)} + |p|_{2,p} + \left\|\frac{\partial u_1}{\partial x}\right\|_p + \left\|\frac{\partial u_2}{\partial y}\right\|_p$$
$$+|\Delta u|_p + |\nabla p|_{2p/(2-p)} \leqslant \|g\|_p + (c+1)c_p\|\nabla \cdot g\|_p \tag{74$'$}$$

Proof
Equations (72)–(74)′ and Lemmas 2.1, 2.3 and 5.2 imply. □

Now, we would like to go to the origin problem

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} \hat{u}(\xi)\, d\xi$$

From Lemma 5.2(i) and (ii) we get that $y\partial u_1/\partial x \in L^1(\mathbb{R}^2)$ which implies that the Fourier transformation makes sense. Assuming that $\hat{f} = O(|\xi|^2)$ near origin it implies that $\lim_{x\to\infty} u = 0$.

5.2. 3D situation

First, we give the formal proof by the Fourier transformation and then we have to verify that our formal proof is correct.

5.2.1. Particular problem ($3D_1$). We consider the following system of equations:

$$\begin{aligned} \Delta u + \nabla p - (\omega \times x)\cdot \nabla u &= g \\ \nabla \cdot u &= 0 \end{aligned} \tag{75}$$

As before we denote $f := \nabla p + g$. Applying the Fourier transformation we get the following system of equations:

$$\begin{aligned} \xi^2 \hat{u}_1 - \frac{\partial}{\partial \xi_3}(\xi_1 \hat{u}_1(\xi)) + \frac{\partial}{\partial \xi_1}(\xi_3 \hat{u}_1(\xi)) &= \hat{f}_1 \\ \xi^2 \hat{u}_2 - \frac{\partial}{\partial \xi_3}(\xi_1 \hat{u}_2(\xi)) + \frac{\partial}{\partial \xi_1}(\xi_3 \hat{u}_2(\xi)) &= \hat{f}_2 \\ \xi^2 \hat{u}_3 - \frac{\partial}{\partial \xi_3}(\xi_1 \hat{u}_3(\xi)) + \frac{\partial}{\partial \xi_1}(\xi_3 \hat{u}_3(\xi)) &= \hat{f}_3 \end{aligned} \tag{76}$$

which can be solved by the method of characteristic. Defining $d\xi_1/d\tau = \xi_3$, $d\xi_3/d\tau = -\xi_1$ we solve our ODE system as $\xi_1 = a\cos\tau + b\sin\tau$, $\xi_3 = c\cos\tau + d\sin\tau$ and $\xi_2(\tau) = \text{const.}$ with a, b, c, d are constants.

We can rewrite our system in the form

$$\hat{u}_r + (r + \xi_2^2)\hat{u} = \hat{f} \tag{77}$$

with $r = \xi_1^2 + \xi_3^2$, where $\hat{f}$ is periodic with period 2π. By the classical theory of the periodic problem we know that there exists a periodic solution

$$\hat{u}(\xi_1(\tau), \xi_2, \xi_3(\tau)) = \int_{\xi_1^2+\xi_3^2} \hat{f} \exp(h\tau)\, \mathrm{d}s$$

where $h = r + \xi_2^2$. Applying the maximum principle it is clear that this is the unique solution which is periodic with period 2π. Multiplying (76) by $\hat{u}$ we get

$$|\xi\hat{u}|_2 \leqslant |\hat{f}|_2 |\hat{u}|_2$$

Assuming that $\hat{f} \in L^2(\mathbb{R}^3)$ we get that $|u|_{1,2} \leqslant |f|_2$. From Equation (77) it follows that

$$|\xi^2\hat{u}|_\infty \leqslant |\hat{f}|_\infty$$

Let $\hat{f} \in L^\infty(\mathbb{R}^3)$. Then $\lim_{\xi\to\infty} \hat{u} = 0$ and assuming that $\hat{f}|\xi|^3 \in L^\infty(\mathbb{R}^3)$ we have $\lim_{|\xi|\to 0} \hat{u} = 0$. Moreover, assuming that also $\partial f/\partial\xi_1 \in L^\infty(\mathbb{R}^3)$ and $\partial f/\partial\xi_1|\xi|^3 \in L^\infty(\mathbb{R}^3)$, we have that behaviour of u is $O(|x|^{-2})$ at infinity and u is $O(|x|)$ at origin, which implies that the Fourier transformation of all terms of type $x_1(\partial u_j/\partial x_3)$, $x_3(\partial u_j/\partial x_1)$ make sense.

5.2.2. Particular problem ($3D_2$). We consider the following problem:

$$\begin{aligned} \Delta u + \nabla p - (\omega \times x)\cdot\nabla u + \frac{\partial u}{\partial x_2} &= f \\ \nabla\cdot u &= 0 \end{aligned} \tag{78}$$

Applying the Fourier transformation, we get the following system of equations:

$$\begin{aligned} (\xi^2 + \mathrm{i}\xi_2)\hat{u}_1 - \frac{\partial}{\partial\xi_3}(\xi_1\hat{u}_1(\xi)) + \frac{\partial}{\partial\xi_1}(\xi_3\hat{u}_1(\xi)) &= \hat{f}_1 \\ (\xi^2 + \mathrm{i}\xi_2)\hat{u}_2 - \frac{\partial}{\partial\xi_3}(\xi_1\hat{u}_2(\xi)) + \frac{\partial}{\partial\xi_1}(\xi_3\hat{u}_2(\xi)) &= \hat{f}_2 \\ (\xi^2 + \mathrm{i}\xi_2)\hat{u}_3 - \frac{\partial}{\partial\xi_3}(\xi_1\hat{u}_3(\xi)) + \frac{\partial}{\partial\xi_1}(\xi_3\hat{u}_3(\xi)) &= \hat{f}_3 \end{aligned} \tag{79}$$

which can be solved by method of characteristic. Defining $\mathrm{d}\xi_1/\mathrm{d}\tau = \xi_3$, $\mathrm{d}\xi_3/\mathrm{d}\tau = -\xi_1$, we solve our ODE system as $\xi_1 = a\cos\tau + b\sin\tau$, $\xi_3 = c\cos\tau + d\sin\tau$ and $\xi_2(\tau) = \text{const.}$, where a, b, c, d are constants.

We can rewrite our system as

$$\hat{u}_\tau + (r + \xi_2^2 + \mathrm{i}\xi_2)\hat{u} = \hat{f} \tag{80}$$

with $r = \xi_1^2 + \xi_3^2$, where $\hat{f}$ is periodic with period 2π. By the classical theory of the periodic problem we know that there exists a periodic solution

$$\hat{u}(\xi_1(\tau), \xi_2, \xi_3(\tau)) = \int_{\xi_1^2+\xi_3^2} \hat{f} \exp(h\tau)\, \mathrm{d}s$$

where $h = r + \xi_2^2 + \mathrm{i}\xi_2$. Applying the maximum principle it is clear that this is the unique solution which is periodic with period 2π.

Multiplying (80) by $\hat{u}$ we get

$$|\xi\hat{u}|_2 \leqslant |\hat{f}|_2 |\hat{u}|_2$$

Assuming that $\hat{f} \in L^2(\mathbb{R}^3)$ we get that $|u|_{1,2} \leqslant |f|_2$. From Equation (80) it follows that

$$|\xi^2\hat{u}|_\infty \leqslant |\hat{f}|_\infty$$

Let $\hat{f} \in L^\infty(\mathbb{R}^3)$. Then $\lim_{\xi\to\infty} \hat{u} = 0$ and assuming that $\hat{f}|\xi|^3 \in L^\infty(\mathbb{R}^3)$, we have $\lim_{\xi\to 0} \hat{u} = 0$. Moreover, assuming that also $\partial f/\partial\xi_1 \in L^\infty(\mathbb{R}^3)$ and $\partial f/\partial\xi_1 |\xi|^3 \in L^\infty(\mathbb{R}^3)$, behaviour of u is $O(|x|^{-2})$ at infinity and u is $O(|x|)$ at origin, which implies that Fourier transformation of all terms of type $x_1(\partial u_j/\partial x_3)$, $x_3(\partial u_j/\partial x_1)$ make sense.

5.2.3. Particular problem $(3D_3)$. We consider

$$\begin{aligned} \Delta u + \nabla p - (\omega \times x)\cdot \nabla u + \frac{\partial u}{\partial x_2} + \omega \times u &= f \\ \nabla \cdot u &= 0 \end{aligned} \tag{81}$$

As before we denote $f = \nabla p + g$. Applying the Fourier transformation we get the following system of equations:

$$\begin{aligned} (\xi^2 + \mathrm{i}\xi_2)\hat{u}_1 - \frac{\partial}{\partial\xi_3}(\xi_1\hat{u}_1(\xi)) + \frac{\partial}{\partial\xi_1}(\xi_3\hat{u}_1(\xi)) - \lambda\hat{u}_3 &= \hat{f}_1 \\ (\xi^2 + \mathrm{i}\xi_2)\hat{u}_2 - \frac{\partial}{\partial\xi_3}(\xi_1\hat{u}_2(\xi)) + \frac{\partial}{\partial\xi_1}(\xi_3\hat{u}_2(\xi)) &= \hat{f}_2 \\ (\xi^2 + \mathrm{i}\xi_2)\hat{u}_3 - \frac{\partial}{\partial\xi_3}(\xi_1\hat{u}_3(\xi)) + \frac{\partial}{\partial\xi_1}(\xi_3\hat{u}_3(\xi)) + \lambda\hat{u}_1 &= \hat{f}_3 \end{aligned} \tag{82}$$

Second equation of (82) we can rewrite in form

$$\hat{u}_{2,\tau} + (r + \xi_2^2 + \mathrm{i}\xi_2)\hat{u}_2 = \hat{f}_2 \tag{83}$$

with $r = \xi_1^2 + \xi_3^2$, which is linear system of ODE. By the classical theory we know that there exist a solution that we can express by following way:

$$\hat{u}_2(\xi_1(\tau), \xi_2, \xi_3(\tau)) = \int_{\xi_1^2+\xi_3^2} \hat{f}_2 \exp(h\tau)\, \mathrm{d}s$$

where $h = r + \xi_2^2 + i\xi_2$. Concerning first and third one of (82) it gives us the system of ODE, which we can solve by standard method then as before we can look for periodic solution with period 2π.

ACKNOWLEDGEMENTS

I am deeply indebted to Prof. Galdi who introduced me to this fascinating problem. Further I express my gratitude to Prof. Wendland, Prof. Jaeger, Prof. Novotný, Prof. Pokorný, Prof. Guenther and Prof. Nicolaenko for their valuable comments and remarks. I thank Grant Agency of the Czech Republic, No. 201/99/0267. Last part of this paper was supported by the Grant Agency of the Czech Republic, No. 201/02/0684. Also, I thank the referee for his stimulating remarks. Last but not least, it is a great privilege and pleasure to thank my husband, Prof. Feireisl for discussing the topic.

REFERENCES

1. Kirchoff G. Uber die Bewegung eines Rotationskorpers in einer Flussigkeit. *Crelle* 1869; **71**:237–281.
2. Thomson W (Lord Kelvin). *Mathematical and Physical Papers*, vol. 4. Cambridge University Press: Cambridge, 1882.
3. Brenner H. The Stokes resistance of an arbitrary particle II. *Chemical Engineering Science* 1959; **19**:599–624.
4. Weinberger HF. On the steady fall of a body in a Navier–Stokes fluid. *Proceedings and Symposium on Pure Mathematics*, vol. 23, 1973; 421–440.
5. Weinberger HF. Variational principles for a body falling in steady Stokes flow. *Journal of Fluid Mechanics* 1972; **52**:321–344.
6. Serre D. Chute Libre d'un Solide dans un Fluide Visquex Incompressible. Existence. *Japanese Journal of Applied Mathematics* 1987; **4**(1):99–110.
7. Galdi GP, Vaidya A. Translational steady fall of symmetric body in a Navier–Stokes liquid, with application to particle sedimentation. *Journal of Mathematics and Fluid Mechanics* 2001; **3**(2):183–211.
8. Childress S. The slow motion of a sphere in a rotating viscous fluid. *Journal of Fluid Mechanics* 1964; **20**(Part 2): 305–314.
9. Saffman PG. The lift on a small sphere in a slow shear flow. *Journal of Fluid Mechanics* 1965; **22**(part 2): 385–400.
10. Maslennikova VN. The rate of decrease for large time of the solution of a Sobolev system with viscosity. *Matematicheskii Sbornik* 1973; **21**(4):584–606.
11. Hishida T. An existence theorem for the Navier–Stokes flow in the exterior to a rotating obstacle. *Archive for Rational Mechanics and Analysis* 1999; **150**:307–348.
12. Guenther BR, Hudspeth RT, Thomann EA. Hydrodynamic forces on submerged rigid bodies-steady flow, preprint, 2000.
13. Pukhnachev VV. *Stokes Approximation in a Problem of the Flow Around a Self-Propelled Body*, Boundary Value Problem in Mathematical Physics. Nauka Dumka: Kiev, 1990; 65–73.
14. Pukhnachev VV. *The Problem of Momentumless Flow for the Navier–Stokes Equations*. Lecture Notes in Mathematics, vol. 1431. Springer: Berlin, 1990; 87–94.
15. Galdi GP. On the steady self-propelled motion of a body in a viscous incompressible fluid. *Archive for Rational Mechanics and Analysis* 1999; **148**:53–88.
16. Galdi GP. On the steady, translational self-propelled motion of a body in a Navier–Stokes equations. In *Proceedings and Conference Equadiff*, Agarwal, Neuman, Vosmansky (eds), vol. 9, Brno, Czech Republic, 1998; 63–80.
17. Galdi GP. On the steady, translational self-propelled motion of a symmetric body in a Navier–Stokes fluid. *Quaderni Matematica* 1998; **1**(1997):97–169.
18. Galdi GP. Steady fall of slow particles in a second-order fluid. *Journal of Non-Newtonian Fluid Mechanics* 2000; **90**:81–89.
19. Galdi GP. The motion of a rigid body in a viscous liquid: mathematical theory and applications. In *Lecture on Summer School: Theoretical Fluid Dynamic and Related Topics*, Sequeira A, Beraio da Veiga H (organizers), Lisbon, June 28–July 2, 1999.
20. Liu YJ, Joseph DD. Sedimentation of particles in polymers solutions. *Journal of Fluid Mechanics* 1993; **255**:565–595.
21. Burger R, Wendland WL, Concha F. Model equations for gravitational sedimentation–consolidation processes. *Zeitschrift fur Angewandte Mathemalik und Mechanik* 2000; **80**(2):79–92.

22. Burger R, Liu Ch, Wendland WL. Existence and stability for mathematical models of sedimentation–consolidation processes in several space dimensions. *Journal of Mathematical Analysis and Applications* 2001; **264**:288–310.
23. Gunzbuerger MD, Lee HC, Seregin GA. Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. *Journal of Mathematics and Fluid Mechanics* 2000; **2**:219–266.
24. Desjardin B, Esteban M. Existence of weak solutions for the motion of rigid bodies in viscous fluid. *Archive for Rational Mechanics and Analysis* 1999; **46**:59–71.
25. Conca C, Martin JS, Tucsnak M. Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Communications in Partial Differential Equations* 2000; **25**:1019–1042.
26. San Martin JA, Starovoitov V, Tucsnak M. Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. *Archive for Rational Mechanics and Analysis* 2002; **161**: 113–147.
27. Feireisl E. On the motion of the rigid bodies in a viscous incompressible fluid. *Journal of Evolution Equations* 2003; **3**:419–441.
28. Babin A, Mahalov A, Nicolaenko B. Global regularity and integrability of 3D Euler and Navier–Stokes equations for uniformly rotating fluids. *Asymptotics Analysis* 1997; **15**:103–150.
29. Babin A, Mahalov A, Nicolaenko B. Global regularity of 3D, rotating Navier–Stokes equations for resonant domains. *Indiana University Mathematics Journal* 1999; **48**:1133–1176.
30. Babin A, Mahalov A, Nicolaenko B. 3D Navier–Stokes and Euler equations, initial data characterized by uniformly large vorticity. *Indiana University Mathematics Journal* 2001; **50**:1–35.
31. Nečasová Š. Some remarks on the steady fall of body in Stokes and Oseen flow, submitted to *Journal of Mathematics and Fluid Mechanics*, 2001.
32. Ladyzhenskaya OA. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach: New York, London, Paris, 1969.
33. Babenko KI. On stationary solutions of the problem of flow past a body of a viscous incompressible fluid. *Mathematicheskn Sbornik* 1973; **91**(133):3–27 (English Translation *Mathematics of the USSR-Sbornik* 1973; **20**:1–25).
34. Galdi GP. *An Introduction to the Mathematical Theory of the Navier–Stokes Equations: Linearized Steady Problems*. Springer: Berlin, 1998.
35. Faxén H. Fredholm'sche Integraleichungen zu der Hydro-dynamik Zaher Flussigkeiten. *Arkiv for Matematik Astromi Fysik* 1928/1929; **14**(21):1–20.
36. Odqvist FKG. Uber die Randwertaugaben der Hydrodynamik Zaher Flussigkeiten. *Mathematische Zeitschrift* 1930; **32**:329–375.
37. Finn R. On the exterior stationary problem for the Navier–Stokes equations, and associated perturbation problems. *Archive for Rational Mechanics and Analysis* 1965; **19**:363–406.
38. Galdi GP. In *On the Oseen Boundary-Value Problem in Exterior Domains, Navier–Stokes Equations: Theory and Numerical Methods*, Heywood JG, Masuda K, Rautmann R, Solonnikov VA (eds), Lectures Notes in Mathematics, vol. 1530. Springer: Berlin 1991; 111–131.
39. Chang I-D, Finn R. On the solutions of a class of equations occurring in continuum mechanics, with applications to the Stokes Paradox. *Archive for Rational Mechanics* 1961; **7**:388–401.
40. Farwig R. A variational approach in weighted Sobolev spaces to the operator $-\Delta+\partial/\partial x_1$ in the exterior domains of R^3. *Mathematische Zeitschrift* 1992; **210**:449–464.
41. Farwig R. The stationary exterior 3-D problem of Oseen and Navier–Stokes equations in anisotropically weighted Sobolev spaces. *Mathematische Zeitschrift* 1992; **211**:409–447.
42. Fischer TM, Hsiao GC, Wendland WL. Singular perturbations for the exterior three-dimensional slow viscous flow problem. *Journal of Mathematical Analysis and Application* 1985; **110**:583–603.
43. Farwig R. *The stationary Navier–Stokes equations in a 3D-exterior domain*. Lecture Notes in Numerical Applied Analysis, vol. 16. Kinokuniya: Tokyo, 1998; 53–115.
44. Finn R, Noll W. On the uniqueness and non-existence of stokes flows. *Archive for Rational Mechanics and Analysis* 1957; **1**:97–106.
45. Sequeira A. Couplage Entre le Methode des Elements Finis et la methode des Equations Integrales, Applications au Probleme Exterieur de Stokes Stationnaire dans le Plane. These de Doctorat de 3-me Cycle, Universite P. et M. Curie, 1981.
46. Sequeira A. The coupling of boundary integral and finite element methods for the bidimensional exterior steady Stokes problem. *Mathematical Methods in Applied Science* 1983; **5**:356–375.
47. Hsiao GC, MacCamy RC. In *Singular Perturbations for Two-Dimensional Viscous Flow Problem, Theory and Applications of Singular Perturbations*, Eckhaus W, de Jager EM (eds), Lectures Notes in Mathematics, vol. 1530. Springer: Berlin, 1981; 229–244.
48. Nazarov SA, Pileckas K. On the steady Stokes and Navier–Stokes problems with zero velocity at infinity in a three-dimensional exterior domain. *Journal of Mathematics of Kyoto University* 2000; **40**(3):475–492.

49. Simader SChG, Sohr H. *The Dirichlet problem for the Laplacian in bounded and unbounded domains*. Pitman Research Notes in Mathematical Series, vol. 360. Addison Wesley: Longman Ltd., Harlow, Great Britain, 1996.
50. Lizorkin PI. (L^p, L^q) Multipliers of Fourier integrals. *Doklady Akademn Nauk SSSR* 1963; **152**:808–811 (English Translation: *Soviet Mathematics Doklady* 1963; **4**:1420–1424).
51. Farwig R, Novotný A, Pokorný M. The fundamental solution of a modified Oseen problem, Zeitschricht fur Analysis und ihre Anwendungen. *Journal for Non-Analysis and its Applications* 2000; **19**(3):713–728.
52. Adams RA. *Sobolev Spaces*. Academic Press: New York, 1975.
53. Boggio T. sul moto stazionario di una Sfera in un Liquido Viscoso. *Rendiconti del Circolo Matematico di Palermo* 1910; **30**:65–81.
54. Nečas J. Les Methodes Directes en Theorie des equations Elliptiques. *Masson et C^{ie}*, 1967.

Ann. Univ. Ferrara - Sez. VII - Sc. Mat.
Vol. XLVIII, 1-20 (2002)

Free Boundary Problem for the Equation of One-Dimensional Motion of Compressible Gas with Density-Dependent Viscosity.

MARI OKADA (*) - ŠÁRKA MATUŠŮ-NEČASOVÁ (**) - TETU MAKINO (*)

SUNTO - Si considera un problema di frontiera libera per l'equazione del moto unidimensionale isoentropico con viscosità dipendente dalla densità secondo la legge $\mu = b\varrho^{\beta}$, dove b e β sono costanti positive. Si dimostra che esiste un'unica soluzione debole globale nel tempo, purché $\beta < 1/3$.

ABSTRACT - We consider a free boundary problem for the equation of the one-dimensional isentropic motion with density-dependent viscosity $\mu = b\varrho^{\beta}$, where b and β are positive constants. We prove that there exists an unique weak solution globally in time, provided that $\beta < 1/3$.

Key words: isentropic gas, density-dependent viscosity, a free boundary problem, a global weak solution, existence, uniqueness.

1. – Introduction.

We investigate the equations

$$\begin{cases} \dfrac{\partial \varrho}{\partial t} + \dfrac{\partial (\varrho u)}{\partial \xi} = 0\,, \\ \dfrac{\partial}{\partial t}(\varrho u) + \dfrac{\partial}{\partial \xi}(\varrho u^2 + p) = \dfrac{\partial}{\partial \xi}\left(\mu \dfrac{\partial u}{\partial \xi}\right) - \varrho g\,, \end{cases}$$

(*) Indirizzo dell'autore: Department of Applied Science, Faculty of Engineering, Yamaguchi University, Ube 755-8611, Japan.
(**) Indirizzo dell'autore: Mathematical Institute, Academy of Sciences, Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic.

where $t > 0$, $0 < \xi < y(t)$. The unknown functions ϱ and u represent the density and velocity, respectively; $p = a\varrho^\gamma$ and $\mu = b\varrho^\beta$ are the pressure and the viscosity coefficient, respectively, where a, b are positive constants and $\gamma > 1$ and $0 < \beta < \gamma - 1$. The non-negative constant g is the gravitation constant; $\xi = 0$ is the fixed boundary

$$u(t, 0) = 0 ,$$

and $\xi = y(t)$ is the free boundary, i.e. the interface of the gas and the vacuum:

$$\frac{dy}{dt} = u(t, y(t)) \qquad \textit{and} \qquad \left(p - \mu \frac{\partial u}{\partial \xi}\right)(t, y(t)) = 0 .$$

The problem of this type was solved for the case of the constant viscosity by M. Okada [1], 1989. Also see M. Padula [10], M. Padula, H. F. Yashima and A. Novotný [11].

The global existence of an isentropic motion of a compressible flow was proven by P. L. Lions [7]. In a case of time-periodic force the existence of the weak solutions was showed by E. Feireisl, Š. M. Nečasová, H. Petzeltová, I. Straškraba [7]. The integrability up to the boundary was investigated by E. Feireisl, H. Petzeltová (see [8]). All results are with the constant viscosity in a three dimensional case.

The global existence in a 2-dimensional case was investigated by Kazhikov, Vaigant [12] in a case when μ is a constant and $\lambda = \varrho^\beta$ in a bounded domain. Concerning the non-constant viscosity, S. Jiang investigated a case where the viscosity depends on the density. He proved the global existence of the smooth solutions of the one-dimensional motion of a viscous heat-conducting gas in cases with fixed boundary and free boundary problem. The global existence with density dependent viscosity in 2 or 3-dimensional cases is open.

The aim of this article is to show the global existence of a weak solutions and uniqueness. To prove it, we shall adopt the method of [1], [6] and use also some of the tools of paper [5].

We rewrite the equations in the Lagrangean mass coordinate:

$$x = \int_0^\xi \varrho(t, \zeta)\, d\zeta .$$

Assuming that

$$\int_0^{y(t)} \varrho(t, \xi)\, d\xi = 1 ,$$

the above problem is transformed to the following fixed boundary problem;

$$\frac{\partial \varrho}{\partial t} + \varrho^2 \frac{\partial u}{\partial x} = 0, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\mu\varrho \frac{\partial u}{\partial x}\right) - g, \tag{1.2}$$

in $t > 0$ and $0 < x < 1$, where $p = a\varrho^\gamma$, $\mu = b\varrho^\beta$ with the boundary conditions

$$u(t, 0) = 0, \quad \left(p - \mu\varrho \frac{\partial u}{\partial x}\right)(t, 1) = 0 \tag{1.3}$$

and the initial condition

$$(\varrho, u)(0, x) = (\varrho_0, u_0)(x), \qquad 0 \leqslant x \leqslant 1. \tag{1.4}$$

In this paper we consider the following assumptions (A.1), (A.2) and (A.3) for the initial data and β

(A.1) $\varrho_0 \in \mathrm{Lip}[0, 1]$ and $\varrho_0(x) \geqslant \underline{\varrho}$ ($\underline{\varrho}$ is a positive constant),

(A.2) $$u_0 \in C^1[0, 1] \text{ and } \frac{du_0}{dx} \in \mathrm{Lip}[0, 1],$$

(A.3) $$0 < \beta < \frac{1}{3}.$$

DEFINITION. A couple (ϱ, u) is called a global weak solution for (1.1)-(1.4) if

$$\varrho,\ u \in L^\infty([0, T] \times [0, 1]) \cap C^1([0, T]; L^2(0, 1)), \tag{1.5}$$

$$\varrho^{\beta+1} u_x \in L^\infty([0, T] \times [0, 1]) \cap C^{1/2}([0, T]; L^2(0, 1)), \tag{1.6}$$

for any T, and the following equations hold:

$$\frac{\partial \varrho}{\partial t} + \varrho^2 \frac{\partial u}{\partial x} = 0, \tag{1.7}$$

for *a.e.* $x \in (0, 1)$ and for any $t \geqslant 0$, and

$$\int_0^1 [\phi u_t - \phi_x (p - \mu \varrho u_x) + \phi g] \, dx = 0 , \tag{1.8}$$

for any test function $\phi \in C_0^\infty((0, 1])$ and for *a.e.* $t \in [0, T]$.

We adopt the usual notation, namely $C^k(.)$ for space of *k-times* continuous differentiable functions, $L^q(.)$ for the Lebesque's spaces with the power q, $C^p(.)$ the Hölder spaces and Lip(.) for the spaces which functions satisfy the Lipschitz condition.

REMARK 1. *Physicists claim that the viscosity of gas is proportional to the square root of the temperature (e.g.* [2], *vol.* 1, *p. 336). In this case, the temperature is keeping with* $\varrho^{\gamma-1}$, *provided that the pressure p is proportional to the product of* ϱ *and the temperature, i.e., the perfect fluid. In this situation we have* $\beta = \frac{\gamma - 1}{2}$ *and* $\beta < \frac{1}{3}$ *says* $\gamma < \frac{5}{3}$.

REMARK 2. *The compressible and heat- conductive Navier-Stokes equations are obtained as the second approximation of the formal Chapman-Enskog expansion to the nonlinear Boltzmann equations for a rarefied simple gas. Here we assume the cut-off hard potentials (cf.* [3]) *and consider two important spacial cases: the hard sphere and the cut-off inverse power forces. Then the coefficient of viscosity is given explicitly, i.e. for the first case we have already mentioned in Remark 1, and for the second case, the viscosity is proportional to the power* $\frac{s+3}{2(s-1)}$ $(s \geqslant 5)$ *of the temperature (e.g.* [4] *p.* 103). *Therefore in the case of the cut-off inverse power forces, we have* $\beta < \gamma - 1$ *says* $s > 5$, *provided that the equation of state is that of ideal and polytropic gas. From the condition* $\beta < \gamma - 1$ *(i.e.,* $s > 5$) [4] *deduced a plausible result by a mathematical rigorous way.*

2. – Difference scheme and estimates.

Discretizing the derivaties with respect to x of the equations (1.1) and (1.2), we have the following scheme:

$$\frac{d}{dt} \varrho_{n-1} + \varrho_{n-1}^2 \frac{u_n - u_{n-1}}{\Delta} = 0 , \tag{2.1}$$

$$\frac{d}{dt} u_n + \frac{p_n - p_{n-1}}{\Delta} = \frac{1}{\Delta} \left[\mu_n \varrho_n \frac{u_{n+1} - u_n}{\Delta} - \mu_{n-1} \varrho_{n-1} \frac{u_n - u_{n-1}}{\Delta} \right] - g , \tag{2.2}$$

for $n = 1, 2, \dots, N$, where $\varDelta = \frac{1}{N}$, N being a large natural number which divides the interval [0,1] into N intervals with length $\varDelta$. We set

$$p_{n-1} = a\varrho_{n-1}^{\gamma}, \tag{2.3}$$

$$\mu_{n-1} = b\varrho_{n-1}^{\beta}. \tag{2.4}$$

The boundary conditions are

$$u_0(t) = 0, \quad \left(p_N - \mu_N \varrho_N \frac{u_{N+1} - u_N}{\varDelta}\right)(t) = 0 \tag{2.5}$$

and the initial conditions are

$$\varrho_{n-1}(0) = \varrho_0((n-1)\varDelta) \geqslant \underline{\varrho} > 0, \qquad u_n(0) = u_0(n\varDelta). \tag{2.6}$$

By the elementary theory of the ordinary differential equations, the Cauchy problem (2.1)-(2.6) admits a temporarily local solution in the domain $R^{2N} = \{(\varrho_{n-1}, u_n)_{n=1,\dots,N}\}$ in class of regularity $C(0, T) \times C(0, T)$. Let $[0, T_\infty)$ be the right maximal interval of existence of this solution. By the equation (2.1) and the initial condition (2.6), we see $\varrho_{n-1}(t) > 0$ for $0 < t < T_\infty$. We will prove that $T_\infty = +\infty$ after getting some a priori estimates.

First, we will show that the solution satisfies a priori estimates independent of $\varDelta$.

We set

$$y_n(t) = \sum_{k=1}^{n} \frac{\varDelta}{\varrho_{k-1}(t)}. \tag{2.7}$$

We get

PROPOSITION 1. *Let* (A.1)-(A.3) *be satisfied then*

$$\frac{d}{dt} y_n(t) = u_n(t)$$

holds.

PROOF. From the equation (2.1) and the boundary condition (2.5), we get

$$\dot{y}_n = -\sum_{k=1}^{n} \frac{\dot{\varrho}_{k-1}}{\varrho_{k-1}^2} \varDelta = \sum_{k=1}^{n} (u_k - u_{k-1}) = u_n.$$

Next, we show the energy inequality.

PROPOSITION 2. *There exists a constant C independent of t and Δ such that*

$$\sum_{n=1}^{N}\left(\frac{1}{2}u_n^2+\frac{a}{\gamma-1}\varrho_{n-1}{}^{\gamma-1}+gy_n\right)(t)\Delta+\int_0^t\sum_{n=1}^{N}\left[\mu_{n-1}\varrho_{n-1}\left(\frac{u_n-u_{n-1}}{\Delta}\right)^2\right](\tau)\Delta d\tau$$

$$=\sum_{n=1}^{N}\left(\frac{1}{2}u_n^2+\frac{a}{\gamma-1}\varrho_{n-1}{}^{\gamma-1}+gy_n\right)(0)\Delta\leqslant C\,.$$

PROOF. Multiplying the equation (2.2) by $u_n\Delta$, summing from $n=1$ to $n=N$, using the boundary condition (2.5) and Proposition 1 and integrating with respect to τ from 0 to t, we have the required expression. Applying (2.6) and since $\varrho_0^{\gamma-1}$, $u_0\in C[0,1]$ we obtain the bound of the right hand side. (2.7) is obtained by the theory of Riemann integral.

From the above a priori estimates we have the following.

LEMMA 1. $T_\infty=+\infty$, that is, the solution of (2.1), (2.2) and (2.6) exists for $0\leqslant t<+\infty$ and $\varrho_{n-1}>0$ for $0\leqslant t<+\infty$.

Hereafter we consider estimates in an interval $0\leqslant t\leqslant T$, where T is an arbitrarily fixed large number, and $C(T)$ denotes various constants depending on the parameters γ, β, g, a, b and the initial conditions ϱ_0 and u_0, which does not depend on Δ.

PROPOSITION 3. *The following inequality*

$$\varrho_{n-1}\leqslant C(T)$$

is satisfied.

PROOF. Multiplying the equation (2.2) by Δ, summing over $k=n,\ldots,N$ we have the following equation:

$$\sum_{k=n}^{N}\frac{d}{dt}u_n\Delta+\sum_{k=n}^{N}(p_n-p_{n-1})$$

$$=\sum_{k=n}^{N}\left[\mu_n\varrho_n\frac{u_{n+1}-u_n}{\Delta}-\mu_{n-1}\varrho_{n-1}\frac{u_n-u_{n-1}}{\Delta}\right]-\sum_{k=n}^{N}g\Delta\,.$$

It implies that we get

$$\sum_{k=n}^{N}\frac{d}{dt}u_n\Delta+p_N-p_{n-1}+g(1-n\Delta)-\mu_N\varrho_N\frac{u_{N+1}-u_N}{\Delta}=-\mu_{n-1}\varrho_{n-1}\frac{u_n-u_{n-1}}{\Delta}\,.$$

Using the boundary condition (2.5), the equation (2.1) and the relation (2.4), we get

$$\sum_{k=n}^{N} \dot{u}_k \Delta - p_{n-1} + g(1-n\Delta) = -\mu_{n-1}\varrho_{n-1} \frac{u_n - u_{n-1}}{\Delta} = \frac{d}{dt}\left(\frac{b}{\beta}\varrho_{n-1}{}^{\beta}\right). \tag{2.8}$$

Integrating (2.8) with respect to τ from 0 to t, we have

$$\frac{b}{\beta}\varrho_{n-1}{}^{\beta}(t) = \frac{b}{\beta}\varrho_{n-1}{}^{\beta}(0) - \int_0^t p_{n-1}(\tau)\,d\tau + \sum_{k=n}^{N}(u_k(t) - u_k(0))\,\Delta + g(1-n\Delta)\,t\,.$$

Applying the assumption (A.1), using the positivness of the density and Proposition 2, we obtain the required estimate.

Also we have the following Proposition

PROPOSITION 4. *Under the assumptions* (A.1)-(A.3) *the inequality*

$$\sum_{n=1}^{N} \varrho_{n-1}{}^{\beta-1}(t)\,\Delta \leqslant C(T)$$

holds.

PROOF. Dividing the relation (2.8) by ϱ_{n-1}, integrating with respcet to τ from 0 to t, multiplying by Δ, summing from $n=1$ to $n=N$ and using (2.3), we get

$$\frac{b}{1-\beta}\sum_{n=1}^{N}\varrho_{n-1}{}^{\beta-1}(t)\,\Delta = \frac{b}{1-\beta}\sum_{n=1}^{N}\varrho_{n-1}{}^{\beta-1}(0)\,\Delta - \int_0^t \sum_{n=1}^{N}\frac{\Delta}{\varrho_{n-1}(\tau)}\sum_{k=n}^{N}\dot{u}_k(\tau)\,\Delta d\tau$$
$$+ a\int_0^t \sum_{n=1}^{N}\varrho_{n-1}{}^{\gamma-1}(\tau)\,\Delta d\tau - g\int_0^t \sum_{n=1}^{N}\frac{1-n\Delta}{\varrho_{n-1}(\tau)}\,\Delta d\tau\,.$$

From (2.6) and Proposition 3 follow that the first term and the third one of the right hand side are bounded. Applying (2.7) we obtain that the forth one is negative.

Therefore if the second term of the right hand side can be estimated, we will obtain the required estimate. By (2.1), (2.5), (2.6) and Proposition 2, we have

$$\int_0^t \sum_{n=1}^{N}\frac{\Delta}{\varrho_{n-1}(\tau)}\sum_{k=n}^{N}\dot{u}_k(\tau)\,\Delta d\tau$$
$$= \int_0^t \left[\frac{d}{d\tau}\left(\sum_{n=1}^{N}\frac{\Delta}{\varrho_{n-1}(\tau)}\sum_{k=n}^{N}u_k(\tau)\,\Delta\right) - \sum_{n=1}^{N}\frac{d}{d\tau}\left(\frac{\Delta}{\varrho_{n-1}(\tau)}\right)\sum_{k=n}^{N}u_k(\tau)\,\Delta\right]d\tau$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \frac{\Delta}{\varrho_{n-1}(t)} \sum_{k=n}^{N} u_k(t)\,\Delta - \sum_{n=1}^{N} \frac{\Delta}{\varrho_{n-1}(0)} \sum_{k=n}^{N} u_k(0)\,\Delta \\
&\qquad - \int_0^t \sum_{n=1}^{N} (u_n - u_{n-1})(\tau) \sum_{k=n}^{N} u_k(\tau)\,\Delta d\tau \qquad \text{(by (2.1))} \\
&= \sum_{n=1}^{N} \left(\frac{\Delta}{\varrho_{n-1}(t)} - \frac{\Delta}{\varrho_{n-1}(0)} \right) \sum_{k=n}^{N} u_k(t)\Delta + \sum_{n=1}^{N} \frac{\Delta}{\varrho_{n-1}(0)} \sum_{k=n}^{N} (u_k(t) - u_k(0))\Delta \\
&\qquad - \int_0^t \sum_{n=1}^{N} u_n^2(\tau)\,\Delta d\tau \\
&\leqslant \sum_{n=1}^{N} \int_0^t \frac{d}{d\tau} \left(\frac{\Delta}{\varrho_{n-1}(\tau)} \right) d\tau \sum_{k=n}^{N} u_k(t)\Delta + \sum_{n=1}^{N} \frac{\Delta}{\varrho_{n-1}(0)} \frac{1}{2} \left[\left(\sum_{n=1}^{N} (u_n^2(t) + u_n^2(0))\Delta \right) + \Delta \right] \\
&\leqslant \int_0^t \sum_{n=1}^{N} (u_n - u_{n-1})(\tau) \sum_{n=k}^{N} u_k(t)\,\Delta d\tau + C_1 \\
&= \int_0^t \sum_{n=1}^{N} u_n(\tau)\, u_n(t)\,\Delta d\tau + C_1 \\
&\leqslant \int_0^t \left(\sum_{n=1}^{N} u_n^2(\tau)\,\Delta \right)^{1/2} d\tau \left(\sum_{n=1}^{N} u_n^2(t)\,\Delta \right)^{1/2} + C_2 \\
&\leqslant C(T)\,.
\end{aligned}
$$

PROPOSITION 5. *Assuming* (A.1)-(A.3)

$$\sum_{n=1}^{N} \left(\frac{\varrho_n{}^{\beta} - \varrho_{n-1}{}^{\beta}}{\Delta} \right)^2 (t)\,\Delta \leqslant C(T)\,.$$

is satisfied.

PROOF. , Denoting

$$V_n(t) = \left(\frac{b}{\beta} \frac{\varrho_n{}^{\beta} - \varrho_{n-1}{}^{\beta}}{\Delta} + u_n \right)(t) + gt\,,$$

we can rewrite the equation (2.2) as

$$\frac{d}{dt} V_n(t) = - \frac{(p_n - p_{n-1})(t)}{\Delta}\,.$$

Multiplying the previous relation by $V_n(t)\,\Delta$, summing over $n = 1, \ldots, N$

and integrating with respect to τ from 0 to t, we obtain:

$$\frac{1}{2}\sum_{n=1}^{N} V_n^2(t)\,\Delta = \frac{1}{2}\sum_{n=1}^{N} V_n^2(0)\,\Delta - \int_0^t \sum_{n=1}^{N} V_n(\tau)\,\frac{p_n - p_{n-1}}{\Delta}(\tau)\,\Delta d\tau .$$

The first term of the right hand side is bounded by (A.1) and (A.2). Using Propositions 2, 3 and the mean value theorem the second one is estimated as follows,

$$-\frac{b}{\beta}\int_0^t \sum_{n=1}^{N} \frac{p_n - p_{n-1}}{\Delta}\,\frac{\varrho_n^{\beta} - \varrho_{n-1}^{\beta}}{\Delta}\,\Delta d\tau$$

$$= -\frac{ab\gamma}{\beta^2}\int_0^t \sum_{n=1}^{N} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{\gamma-\beta}\left(\frac{\varrho_n^{\beta} - \varrho_{n-1}^{\beta}}{\Delta}\right)^2 \Delta d\tau ,$$

where $0 < \theta_n < 1$,

$$-\int_0^t \sum_{n=1}^{N} \frac{p_n - p_{n-1}}{\Delta}\,u_n \Delta d\tau$$

$$= -\frac{a\gamma}{\beta}\int_0^t \sum_{n=1}^{N} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{\gamma-\beta}\,\frac{\varrho_n^{\beta} - \varrho_{n-1}^{\beta}}{\Delta}\,u_n \Delta d\tau$$

$$\leqslant \frac{ab\gamma}{2\beta^2}\int_0^t \sum_{n=1}^{N} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{\gamma-\beta}\left(\frac{\varrho_n^{\beta} - \varrho_{n-1}^{\beta}}{\Delta}\right)^2 \Delta d\tau + C(T),$$

$$-g\int_0^t \sum_{n=1}^{N} \frac{p_n - p_{n-1}}{\Delta}\,\tau \Delta d\tau = -g\int_0^t (p_N - p_0)\,\tau d\tau \leqslant g\int_0^t p_0\,\tau d\tau \leqslant C(T).$$

Thus we get

$$\sum_{n=1}^{N} V_n^2(t)\Delta + \int_0^t \sum_{n=1}^{N}\left[(\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{\gamma-\beta}\left(\frac{\varrho_n^{\beta} - \varrho_{n-1}^{\beta}}{\Delta}\right)^2\right](\tau)\,\Delta d\tau \leqslant C(T).$$

From this and Proposition 2 we obtain the required estimate.
We are interested in the bound of the density from below.

PROPOSITION 6. *Let* (A.1)-(A.3) *be satisfied then*

$$\varrho_{n-1}(t) \geqslant \underline{\varrho}(T),$$

where $\underline{\varrho}(T)$ *is a posistive constant depending on* T.

PROOF. Putting

$$\varrho_{K-1} = \max_n \varrho_{n-1}, \qquad (1 \leq K \leq N),$$

and applying the Proposition 4 and since $\beta - 1 < 0$ we get

$$\varrho_{K-1}{}^{\beta-1} = \varrho_{K-1}{}^{\beta-1} \sum_{n=1}^{N} \Delta \leq \sum_{n=1}^{N} \varrho_{n-1}{}^{\beta-1} \Delta \leq C(T).$$

We have also

$$\varrho_{n-1}{}^{\beta-1} = \varrho_{K-1}{}^{\beta-1} + \sum_{k=K}^{n-1} \frac{\varrho_k{}^{\beta-1} - \varrho_{k-1}{}^{\beta-1}}{\Delta} \Delta$$

$$= \varrho_{K-1}{}^{\beta-1} + \sum_{k=K}^{n-1} \frac{\beta-1}{\beta} (\varrho_{k-1} + \theta_n(\varrho_k - \varrho_{k-1}))^{-1} \frac{\varrho_k{}^{\beta} - \varrho_{k-1}{}^{\beta}}{\Delta} \Delta$$

$$\leq C(T) + C\left(\sum_{n=1}^{N} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{-2} \Delta\right)^{1/2} \left(\sum_{n=1}^{N} \left(\frac{\varrho_n{}^{\beta} - \varrho_{n-1}{}^{\beta}}{\Delta}\right)^2 \Delta\right)^{1/2}.$$

Further, we would like to estimate the second term of the previous inequality. So,

$$\sum_{n=1}^{N} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{-2} \Delta$$

$$= \sum_{n=1}^{N} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{-\beta-1} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{\beta-1} \Delta$$

$$\leq \max_n [(\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{-\beta-1}] \sum_{n=1}^{N} (\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{\beta-1} \Delta$$

$$\leq \max_n [\varrho_{n-1}{}^{-\beta-1}] \cdot 2 \sum_{n=1}^{N} \varrho_{n-1}{}^{\beta-1} \Delta$$

$$\leq C(T) \max_n [\varrho_{n-1}{}^{-\beta-1}] \cdot \text{(by Proposition 4)}.$$

From the previous estimates and applying the Proposition 5 it follows that

$$\varrho_{n-1}{}^{\beta-1} \leq C(T)\left(1 + \max_n \left[\varrho_{n-1}{}^{-\frac{\beta+1}{2}}\right]\right),$$

From the assumption $\beta < \frac{1}{3}$, we have $\frac{\beta+1}{2} < 1 - \beta$. And then there is a positive constant $\underline{\varrho}(T)$ depending on T such that $\varrho_{n-1}(t) \geq \underline{\varrho}(T)$.

Moreover, we have the following proposition

PROPOSITION 7. *Under assumptions* (A.1)-(A.3)

$$\sum_{n=1}^{N} \dot{u}_n^2(t)\,\Delta + \int_0^t \sum_{n=1}^{N} \left[\mu_{n-1}\varrho_{n-1} \left(\frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \right)^2 \right](\tau)\,\Delta d\tau \leqslant C(T), \tag{2.9}$$

and

$$\left| \left(\mu_{n-1}\varrho_{n-1} \frac{u_n - u_{n-1}}{\Delta} \right)(t) \right| \leqslant C(T) \tag{2.10}$$

hold.

PROOF. Differentiating the equation (2.2) with respect to t, multiplying it by $\dot{u}_n \Delta$, summing over $n = 1, \ldots, N$ and using the equation (2.1) and the boundary condition (2.5), we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} \sum_{n=1}^{N} \dot{u}_n^2 \Delta \right) &= - \sum_{n=1}^{N} \mu_{n-1}\varrho_{n-1} \left(\frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \right)^2 \Delta \\ &\qquad - a\gamma \sum_{n=1}^{N} \varrho_{n-1}{}^{\gamma+1} \frac{u_n - u_{n-1}}{\Delta} \frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \Delta \\ &\qquad + b(\beta+1) \sum_{n=1}^{N} \varrho_{n-1}{}^{\beta+2} \left(\frac{u_n - u_{n-1}}{\Delta} \right)^2 \frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \Delta \\ &\leqslant - \frac{1}{2} \sum_{n=1}^{N} \mu_{n-1}\varrho_{n-1} \left(\frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \right)^2 \Delta \\ &\qquad + C(T) \sum_{n=1}^{N} \mu_{n-1}\varrho_{n-1} \left(\frac{u_n - u_{n-1}}{\Delta} \right)^2 \Delta \\ &\qquad + C \sum_{n=1}^{N} \varrho_{n-1}{}^{\beta+3} \left(\frac{u_n - u_{n-1}}{\Delta} \right)^4 \Delta . \end{aligned}$$

Using (2.8), we have

$$\left(\mu_{n-1}\varrho_{n-1} \frac{u_n - u_{n-1}}{\Delta} \right)^2 \leqslant C(T) \left(\sum_{n=1}^{N} \dot{u}_n^2 \Delta + 1 \right). \tag{2.11}$$

Therefore we can estimate the above inequality:

$$\begin{aligned} \frac{d}{dt}\left(\sum_{n=1}^{N} \dot{u}_n^2 \Delta \right) &+ \sum_{n=1}^{N} \mu_{n-1}\varrho_{n-1} \left(\frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \right)^2 \Delta \\ &\leqslant C(T) \sum_{n=1}^{N} \mu_{n-1}\varrho_{n-1} \left(\frac{u_n - u_{n-1}}{\Delta} \right)^2 \Delta \left(\sum_{n=1}^{N} \dot{u}_n^2 \Delta + 1 \right). \end{aligned}$$

Integrating this with respect to τ and using the assumption (A.2), we get that $\sum_{n=1}^{N} \dot{u}_n^2(0)\,\Delta \leqslant C$. This implies (2.9). Using (2.11) and (2.9), we obtain (2.10).

PROPOSITION 8. *Under assumptions* (A.1)-(A.3)

$$\sum_{n=1}^{N} |u_n(t) - u_{n-1}(t)| \leqslant C(T),$$

and

$$|u_n(t)| \leqslant C(T)$$

hold.

PROOF. From (2.10) and Proposition 6, we get

$$\sum_{n=1}^{N} \frac{|u_n - u_{n-1}|}{\Delta}\Delta \leqslant C_1(T) \sum_{n=1}^{N} \varrho_{n-1}{}^{-\beta-1}\Delta \leqslant C(T).$$

Note that

$$u_n = \sum_{k=1}^{n} (u_k - u_{k-1}).$$

Further the following properties are satisfied

PROPOSITION 9

$$\sum_{n=1}^{N} |\varrho_n(t) - \varrho_{n-1}(t)| \leqslant C(T).$$

PROOF. Using the mean value theorem and Proposition 3, we get

$$\left|\frac{\varrho_n - \varrho_{n-1}}{\Delta}\right| = \left|\frac{\varrho_n - \varrho_{n-1}}{\varrho_n{}^{\beta} - \varrho_{n-1}{}^{\beta}}\right| \left|\frac{\varrho_n{}^{\beta} - \varrho_{n-1}{}^{\beta}}{\Delta}\right|$$
$$= \frac{1}{\beta}(\varrho_{n-1} + \theta_n(\varrho_n - \varrho_{n-1}))^{1-\beta} \left|\frac{\varrho_n{}^{\beta} - \varrho_{n-1}{}^{\beta}}{\Delta}\right|$$
$$\leqslant C(T) \left|\frac{\varrho_n{}^{\beta} - \varrho_{n-1}{}^{\beta}}{\Delta}\right|.$$

Therefore by Proposition 5, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\left|\frac{\varrho_n-\varrho_{n-1}}{\Delta}\right|\Delta &\leqslant C_1(T)\sum_{n=1}^{N}\left|\frac{\varrho_n{}^{\beta}-\varrho_{n-1}{}^{\beta}}{\Delta}\right|\Delta\\ &\leqslant C_2(T)\left(\sum_{n=1}^{N}\left|\frac{\varrho_n{}^{\beta}-\varrho_{n-1}{}^{\beta}}{\Delta}\right|^2\Delta\sum_{n=1}^{N}\Delta\right)^{1/2}\\ &\leqslant C(T).\end{aligned}$$

PROPOSITION 10. *We have*

$$\sum_{n=1}^{N}\left|\mu_n\varrho_n\frac{u_{n+1}-u_n}{\Delta}(t)-\mu_{n-1}\varrho_{n-1}\frac{u_n-u_{n-1}}{\Delta}(t)\right|\leqslant C(T).$$

PROOF. From (2.2), (2.9) and Proposition 3 and 5 we get

$$\begin{aligned}&\sum_{n=1}^{N}\left|\mu_n\varrho_n\frac{u_{n+1}-u_n}{\Delta}-\mu_{n-1}\varrho_{n-1}\frac{u_n-u_{n-1}}{\Delta}\right|\\ &\leqslant\sum_{n=1}^{N}|\dot{u}_n|\Delta+\sum_{n=1}^{N}\left|\frac{p_n-p_{n-1}}{\Delta}\right|\Delta+g\sum_{n=1}^{N}\Delta\\ &\leqslant\left(\sum_{n=1}^{N}\dot{u}_n^2\Delta\sum_{n=1}^{N}\Delta\right)^{1/2}+a\sum_{n=1}^{N}\left|\frac{\varrho_n{}^{\gamma}-\varrho_{n-1}{}^{\gamma}}{\varrho_n{}^{\beta}-\varrho_{n-1}{}^{\beta}}\right|\left|\frac{\varrho_n{}^{\beta}-\varrho_{n-1}{}^{\beta}}{\Delta}\right|\Delta+g\\ &\leqslant C(T).\end{aligned}$$

This completes the proof.
Finally, we get

PROPOSITION 11. *Let* (A.1)-(A.3) *be satisfied then*

$$\sum_{n=1}^{N}|\varrho_{n-1}(t)-\varrho_{n-1}(s)|^2\Delta\leqslant C(T)|t-s|^2, \tag{2.12}$$

$$\sum_{n=1}^{N}|u_n(t)-u_n(s)|^2\Delta\leqslant C(T)|t-s|^2, \tag{2.13}$$

and

$$\sum_{n=1}^{N}\left|\mu_{n-1}\varrho_{n-1}\frac{u_n-u_{n-1}}{\Delta}(t)-\mu_{n-1}\varrho_{n-1}\frac{u_n-u_{n-1}}{\Delta}(s)\right|^2\Delta\leqslant C(T)|t-s|. \tag{2.14}$$

hold.

PROOF. From the equation (2.1), using the same procedure adopted to reach (2.8), and (2.10)

$$\left| \frac{1}{\varrho_{n-1}^2} \frac{d\varrho_{n-1}}{dt} \right| \leq \frac{C(T)}{\mu_{n-1}\varrho_{n-1}} . \tag{2.15}$$

Using Proposition 3, (2.12) it follows. From (2.9), we have $\sum_{n=1}^{N} \dot{u}_n^2 \Delta \leq C(T)$, which implies (2.13). From (2.15) and Proposition 3, we have

$$\left| \frac{d}{dt}\left(\mu_{n-1}\varrho_{n-1} \frac{u_n - u_{n-1}}{\Delta}\right) \right|$$

$$= \left| b(\beta+1)\,\varrho_{n-1}{}^{\beta}\dot{\varrho}_{n-1} \frac{u_n - u_{n-1}}{\Delta} + \mu_{n-1}\varrho_{n-1} \frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \right|$$

$$\leq C(T)\,\mu_{n-1}^{1/2}\varrho_{n-1}^{1/2}\left(\left| \frac{u_n - u_{n-1}}{\Delta} \right| + \left| \frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta} \right| \right).$$

Therefore applying Proposition 2 and (2.9) it yields

$$\int_0^T \sum_{n=1}^{N} \left[\frac{d}{dt}\left(\mu_{n-1}\varrho_{n-1} \frac{u_n - u_{n-1}}{\Delta}\right)\right]^2 \Delta dt$$

$$\leq C_1(T) \int_0^T \sum_{n=1}^{N} \mu_{n-1}\varrho_{n-1}\left[\left(\frac{u_n - u_{n-1}}{\Delta}\right)^2 + \left(\frac{\dot{u}_n - \dot{u}_{n-1}}{\Delta}\right)^2\right] \Delta dt$$

$$\leq C(T),$$

which implies (2.14).

3. – Convergence of approximate solutions to weak solutions.

We interpolate ϱ_{n-1}, u_n in the following way. Define $\varrho_\Delta(t, x)$ and $u_\Delta(t, x)$ by

$$\varrho_\Delta(t, x) = \varrho_{n-1}(t), \qquad u_\Delta(t, x) = \frac{1}{\Delta}\left[(x - (n-1)\,\Delta)\, u_n(t) + (n\Delta - x)\, u_{n-1}(t)\right],$$

for $(n-1)\,\Delta \leq x < n\Delta$. Then we have

$$\varrho_\Delta(t, x) \leq C(T), \qquad \varrho_\Delta(t, x)^{-1} \leq C(T), \qquad |u_\Delta(t, x)| \leq C(T),$$

and

$$u_{\Delta,x}(t,x) = \frac{1}{\Delta}(u_n(t) - u_{n-1}(t)),$$

for $(n-1)\Delta \leq x < n\Delta$ and any $t \in [0, T]$. Thus from (2.10) we have

$$\{(\mu_\Delta \varrho_\Delta u_{\Delta,x})(t,x)\,| \leq C(T).$$

On the other hand we define

$$u_\Delta^R(t,x) = u_n(t), \qquad u_\Delta^L(t,x) = u_{n-1}(t) \qquad \text{for} \qquad (n-1)\Delta \leq x < n\Delta .$$

Then we have

$$|u_\Delta - u_\Delta^R| = O(\Delta), \qquad |u_\Delta - u_\Delta^L| = O(\Delta),$$

thanks to (2.10).

Now from Proposition 9 it follows that the functions ϱ_Δ as functions of x have total variations uniformly bounded with respect to x for any fixed t. Let $\{t_m \,|\, m = 1, 2, \ldots\}$ be a countable dense set of $[0, T]$. By Helly's theorem and the diagonal process we can select a sequence ϱ_{Δ_j}, $j = 1, 2, \ldots$, which converges almost everywhere in $x \in [0, 1]$ for any t_m. Then ϱ_{Δ_j} tend to a function ϱ in $L^2(0, 1)$ for any t_m. This convergence in $L^2(0, 1)$ norm is uniform with respect to t thanks to (2.12), since t_m is dense in $[0, T]$.

The same argument can be applied to u_Δ and $\mu_\Delta \varrho_\Delta u_{\Delta,x}$ thanks to Propositions 8, 10 and 11. Therefore we can assume that u_Δ, $\mu_\Delta \varrho_\Delta u_{\Delta,x}$ tend to a limit function u, $\mu\varrho u_x$, respectively, almost every x for any t and the convergence in $L^2(0, 1)$ is niform with respect to t.

Now we show that the limit function is a weak solution. The equation (1.7) holds clearly by passing to the limit from (2.1). Let us show (1.8). Let $\phi \in C_0^\infty(0, 1]$ be an arbitrary test function. Multiplying (2.2) by $\phi_n(t)\Delta = \phi(t, n\Delta)\Delta$ and summing this from $n = 1$ to $n = N$, we get

$$\begin{aligned} 0 &= \sum_{n=1}^{N} \phi_n \dot{u}_n \Delta \\ &+ \sum_{n=1}^{N} \phi_n \frac{1}{\Delta}\left[(p_n - p_{n-1}) - \left(\mu_n \varrho_n \frac{u_{n+1} - u_n}{\Delta} - \mu_{n-1}\varrho_{n-1}\frac{u_n - u_{n-1}}{\Delta}\right)\right]\Delta \\ &+ \sum_{n=1}^{N} \phi_n g \Delta . \end{aligned}$$

Since ϕ is smooth and vanishing at $x = 0$, we have

$$\sum_{n=1}^{N} \phi_n \dot{u}_n \Delta = \int_0^1 \phi u_{\Delta,t}^R \, dx + o(1),$$

$$\sum_{n=1}^{N} \phi_n \frac{1}{\Delta}\left[(p_n - p_{n-1}) - \left(\mu_n \varrho_n \frac{u_{n+1} - u_n}{\Delta} - \mu_{n-1}\varrho_{n-1}\frac{u_n - u_{n-1}}{\Delta}\right)\right]\Delta$$

$$= - \int_0^1 \phi_x (p_\Delta - \mu_\Delta \varrho_\Delta u_{\Delta,x})\, dx + o(1),$$

$$\sum_{n=1}^{N} \phi_n g \Delta = \int_0^1 \phi g dx + o(1),$$

as $\Delta \to 0$. Therefore, passing to the limit and thanks to the Lebesgue's dominated convergence theorem, we see that (1.8) holds for the limit function. Thus we reach the following.

THEOREM 1. *If the assumptions* (A.1), (A.2) *and* (A.3) *hold, then the initial-boundary value problem* (1.1)-(1.4) *admits a global weak solution in the sense* (1.5)-(1.8).

4. – Uniqueness of the weak solution.

In the above section, under the assumptions (A.1), (A.2) and (A.3) we constructed global weak solution (ϱ, u) of (1.1)-(1.4) such that for any $T(\varrho, u)$ satisfies (1.5)-(1.8) and that there exists a constant $C(T)$ with

$$\frac{1}{C(T)} \leq \varrho(t, x) \leq C(T), \qquad |u_x(t, x)| \leq C(T). \tag{4.1}$$

In this section we will prove that such a solution is unique by using the method of [6].

THEOREM 2. *Assume* (A.1), (A.2) *and* (A.3). *Let* (ϱ_1, u_1) *and* (ϱ_2, u_2) *be solutions of* (1.1)-(1.4) *satisfying* (1.5)-(1.8) *and* (4.1) *for any* T. *Then we have* $\varrho_1 = \varrho_2$, $u_1 = u_2$.

PROOF. Taking the difference of (1.8), we get

$$\int_0^1 \phi(u_2-u_1)_t\,dx=\int_0^1 [\phi_x(p_2-p_1)-\phi_x(\mu_2\varrho_2-\mu_1\varrho_1)\,u_{2,x}-\phi_x\mu_1\varrho_1(u_2-u_1)_x]\,dx\,,$$

for any $\phi\in C_0^\infty(0,1]$. Now, since $(\varrho_2^{\beta+1}(u_2)_x-\varrho_1^{\beta+1}(u_1)_x)(t,.)\in L^\infty(0,1)$ and $1/\varrho_1^{\beta+1}(t,.)$, $1/\varrho_2^{\beta+1}(t,.)\in L^1(0,1)$, then $(u_2-u_1)_x(t,\cdot)\in L^1(0,1)$, $u_2(t,\cdot)-u_1(t,\cdot)\in C[0,1]$ and $u_2(t,0)-u_1(t,0)=0$, there exists a sequence $\phi_n\in C_0^\infty(0,1]$ such that $\phi_n\to u_2(t,\cdot)-u_1(t,\cdot)$ in $C[0,1]$ and $\phi_{n,x}\to(u_2-u_1)_x(t,\cdot)$ in $L^1(0,1)$. Therefore, passing to the limit, we can replace ϕ by u_2-u_1. It means that we have the same class of regularity as for existence. Thus by keeping in mind that $p=a\varrho^\gamma$ and $\mu=b\varrho^\beta$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 (u_2-u_1)^2\,dx &= \int_0^1 (u_2-u_1)_x(p_2-p_1)\,dx-\int_0^1 (u_2-u_1)_x(\mu_2\varrho_2-\mu_1\varrho_1)\,u_{2,x}\,dx\\ &\quad -\int_0^1 \mu_1\varrho_1(u_2-u_1)_x^2\,dx\\ &\leqslant -\frac{1}{2}\int_0^1 \mu_1\varrho_1(u_2-u_1)_x^2\,dx+\int_0^1 \mu_1\varrho_1(p_2-p_1)^2\,dx\\ &\quad +\int_0^1 \mu_1\varrho_1(\mu_2\varrho_2-\mu_1\varrho_1)^2u_{2,x}^2\,dx\\ &\leqslant -\frac{1}{2}\int_0^1 \mu_1\varrho_1(u_2-u_1)_x^2\,dx+C(T)\int_0^1 (\varrho_2-\varrho_1)^2\,dx\,,\end{aligned}$$

thanks to (4.1). Integrating this with respect to τ, we obtain

$$\int_0^1 (u_2-u_1)^2\,dx+\int_0^t\int_0^1 \mu_1\varrho_1(u_2-u_1)_x^2\,dx\,d\tau\leqslant C(T)\int_0^t\int_0^1 (\varrho_2-\varrho_1)^2\,dx\,d\tau\,. \tag{4.2}$$

We can rewrite the equation (1.1) as

$$\frac{\partial}{\partial t}\frac{1}{\varrho}=\frac{\partial u}{\partial x}. \tag{4.3}$$

Taking the difference of (4.3) and multiplying it by $\frac{1}{\varrho_2}-\frac{1}{\varrho_1}$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_1}\right)^2=(u_2-u_1)_x\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_1}\right).$$

Integrating this with respect to x from 0 to 1 and integrating it with respect to τ, we get

$$\begin{aligned}\frac{1}{2}\int_0^1\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_1}\right)^2dx&=\int_0^t\int_0^1(u_2-u_1)_x\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_1}\right)dx\,d\tau\\&\leqslant\int_0^t\int_0^1\mu_1\varrho_1(u_2-u_1)_x^2\,dx\,d\tau\\&\quad+\frac{1}{2}\int_0^t\int_0^1\frac{1}{\mu_1\varrho_1}\left(\frac{1}{\varrho_2}-\frac{1}{\varrho_1}\right)^2dx\,d\tau.\end{aligned}$$

Since

$$\frac{1}{\varrho_2}-\frac{1}{\varrho_1}=\frac{1}{\varrho_1\varrho_2}(\varrho_1-\varrho_2),$$

we have

$$\int_0^1(\varrho_2-\varrho_1)^2dx\leqslant C(T)\int_0^t\int_0^1(\varrho_2-\varrho_1)^2dx\,d\tau,$$

thanks to (4.2). So,

$$\int_0^1(\varrho_2-\varrho_1)^2dx\equiv 0,$$

and $\varrho_2\equiv\varrho_1$. Finally from (4.2)

$$\int_0^1(u_2-u_1)^2dx\equiv 0,$$

and it implies $u_2\equiv u_1$. This completes the proof.

Acknowledgement. *We would like to thank to the referee for his valuable comments to our article. One of the authors Š.M.-N. wants to acknowledge the Grant Agency of the Czech Republic N. 201/98/1450. This work has been concluded during her stay in Portugal with grants CMA/IST (Lisbon) and CIM/Fundação Calouste Gulbenkian (Coimbra)-Thematic Term on Theoretical and Computational Fluid Dynamics. The portuguese financial support is also deeply acknowledged.*

Recently our result was improved by Yang, Yao and Zhu (see [13]) to the case $0 < \beta < 1/2$ and by Jiang, Xin and Zhang (see [14]) to the case $0 < \beta < 1$.

REFERENCES

[1] M. OKADA, *Free boundary value problems for the equation of one-dimensionals motion of viscous gas*, Japan J. Indust. Appl. Math., **6** (1989), pp. 161-177.

[2] R. BALIAN, *From microphysics to macrophysics*, Texts and monographs in physics, Springer (1982).

[3] H. GRAD, *Asymptotic theory of the Boltzmann equation. II.* In: Rarefied gas dynamics, 1 (ed. J. Laurmann), New York Academic Press (1963), pp. 26-59.

[4] S. KAWASHIMA - A. MATSUMURA - T. NISHIDA, *On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equation*, Commun. Math. Phys., **70** (1979), pp. 97-124.

[5] S. JIANG, *Global smooth solutions of the equations of a viscous, heat-conducting, one-dimensional gas with density-dependent viscosity*, Math. Nachr., **190** (1998), pp. 169-183.

[6] Š. MATUŠŮ-NEČASOVÁ - M. OKADA - T. MAKINO, *Free boundary problem for the equation of spherically symmetric motion of viscous gas (II)*, Japan J. Indust. Appl. Math., **12** (1995), pp. 195-203.

[7] P. L. LIONS, *Mathematical topics in fluid dynamics*, Vol. 2, Compressible models, Oxford Science Publication, Oxford.

[8] E. FEIREISL - H. PETZELTOVÁ, *On the integrability up to the boundary of the weak solutions of the Navier-Stokes equations of compressible flow*, Preprint '99.

[9] E. FEIREISL - Š. MATUŠŮ-NEČASOVÁ - H. PETZELTOVÁ - I. STRAŠKRABA, *On the motion of a viscous compressible flow driven by a time-periodic external force*, accepted by ARMA.

[10] M. PADULA, *Existence and continuous dependence for solutions to the equations of a one-dimensional model in gas-dynamics*, Mecanica J. of the A.I.ME.T.A., **17** (1981), p. 128.

[11] M. PADULA - H. F. YASHIMA - A. NOVOTNÝ, *Existence of global solutions to one-dimensional flow of a compressible heat-conducting fluid having general initial densities*, Ricerche Math., **42** (1993), pp. 199-248.

[12] V. A. VAIGANT - A. V. KAZHIKOV, *On the existence of global solutions to the two-dimensional Navier-Stokes equations for a compressible viscous fluid*, Sib. Math. J., **36** (1995), pp. 1108-1141 in Russian translated to Sib. Math. Zh., **36**, No. 6, pp. 1283-1316.

[13] T. YANG - Z. YAO - C. ZHU, *Compressible Navier-Stokes equations with density-dependent viscosity and vacuum*, preprint, 1999.

[14] S. JIANG - Z. XIN - P. ZHANG, *Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity*, preprint, 1999.

Pervenuto in Redazione il 2 maggio 2001.

Free Boundary Problem for the Equation of Spherically Symmetric Motion of Viscous Gas (II)

Šárka MATUŠŮ-NEČASOVÁ[†] Mari OKADA[††] and Tetu MAKINO[†††]

[†]*Department of Technical Mathematics,*
Faculty of Mechanical Engineering, Czech Technical University,
Karlovo nám.13, 121 35 Praha 2, Czech Republic
[††]*Applied Science, Faculty of Engineering,*
Yamaguchi University, Ube 755, Japan
[†††]*Department of Liberal Arts, Osaka Sangyo University,*
3-1-1, Nakagaito, Daito, Osaka 574, Japan

Reprinted from the JAPAN JOURNAL OF INDUSTRIAL AND APPLIED MATHEMATICS
Vol. 12, No. 2, pp. 195–203, June 1995

Japan J. Indust. Appl. Math., 12 (1995), 195–203

Free Boundary Problem for the Equation of Spherically Symmetric Motion of Viscous Gas (II)

Šárka MATUŠŮ-NEČASOVÁ[†] *, Mari OKADA[††] and Tetu MAKINO[†††]

[†] *Department of Technical Mathematics,*
Faculty of Mechanical Engineering, Czech Technical University,
Karlovo nám.13, 121 35 Praha 2, Czech Republic
[††] *Applied Science, Faculty of Engineering,*
Yamaguchi University, Ube 755, Japan
[†††] *Department of Liberal Arts, Osaka Sangyo University,*
3-1-1, Nakagaito, Daito, Osaka 574, Japan

Received Feburary 14, 1994
Revised May 16, 1994

We study the spherically symmetric motion of viscous barotropic gas surrounding a solid ball. We are interested in the density distribution which contacts with the vacuum at a finite radius. This is a free boundary problem. We obtained the existence of a global weak solution with some regular properties. We can show that such a solution is unique.

Key words: Navier-Stokes equation, uniqueness of global weak solutions, free boundary problem, spherically symmetric motion

We are investigating the equations

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + u\dfrac{\partial\rho}{\partial r} + \rho\dfrac{\partial u}{\partial r} + \frac{2}{r}\rho u = 0, \\ \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial r}\right) + \dfrac{\partial p}{\partial r} = \nu\left(\dfrac{\partial^2 u}{\partial r^2} + \frac{2}{r}\,\dfrac{\partial u}{\partial r} - \dfrac{2}{r^2}u\right) - \dfrac{\rho M}{r^2}, \\ p = a\rho^\gamma, \end{cases}$$

where ν, a, γ are positive constants and $1 < \gamma \leqq 2$. These equations govern the spherically symmetric motion of viscous barotropic gas. We consider these equations in $r \geqq 1$ with the boundary condition

$$u\,|_{r=1} = 0$$

and the initial conditions

$$\rho\,|_{t=0} = \rho^0(r), \quad u\,|_{t=0} = u^0(r).$$

Since we are interested in the class of initial data which includes the stationary solutions

$$\rho = \begin{cases} \left[\dfrac{(\gamma-1)M}{a\gamma}\left(\dfrac{1}{r} - \dfrac{1}{R}\right)\right]^{1/(\gamma-1)} & (r \leqq R) \\ 0 & (R < r) \end{cases}, \quad u = 0,$$

* This work was completed during her stay at Università degli studi di Ferrara which was supported by GNFM of Italian CNR.

we introduce the Lagrange coordinates

$$x = 4\pi \int_1^r \rho(t,s) s^2 ds.$$

Then the equations turn out to be

(1) $$\frac{\partial \rho}{\partial t} + 4\pi\rho^2 \frac{\partial}{\partial x}(r^2 u) = 0,$$

(2) $$\frac{\partial u}{\partial t} + 4\pi r^2 \frac{\partial p}{\partial x} = 16\pi^2 \nu \frac{\partial}{\partial x}\left(r^4 \rho \frac{\partial u}{\partial x}\right) - 2\nu \frac{u}{r^2 \rho} - \frac{M}{r^2},$$

(3) $$p = a\rho^\gamma,$$

where

(4) $$r = \left[1 + \frac{3}{4\pi}\int_0^x \frac{d\xi}{\rho(t,\xi)}\right]^{1/3}.$$

By normalizing the total mass, we consider the equations (1)(2)(3)(4) in $0 \leqq x \leqq 1$ with the boundary conditions

(5) $$u\,|_{x=0} = 0, \quad \rho\,|_{x=1} = 0$$

and the initial conditions

(6) $$\rho\,|_{t=0} = \rho_0(x), \quad u\,|_{t=0} = u_0(x).$$

In the preceding paper [1], we constructed global weak solutions assuming that the initial condition satisfies the following assumptions:

(A.1) $\rho_0 \in C[0,1]$, $\rho_0(x) > 0$ for $0 \leqq x < 1$, $\rho_0(1) = 0$, Total Variation $[\rho_0] < +\infty$ and there exists a monotone decreasing function $\lambda(x)$ such that $0 \leqq \lambda(x) \leqq \rho(x)$ and $\int_0^1 \frac{dx}{\lambda(x)} < +\infty$.

(A.2) $u_0 \in C[0,1]$.

(A.3) This assumption is little bit complicated. See [1] for the details. Any way this assumption is satisfied at least if $\rho_0 = a\rho_0^\gamma \in C^1[0,1]$ and $u_0 = 0$.

Under these assumptions we constructed global weak solution (ρ, u) of (1)–(6) such that for any T

(7) $$\rho, u \in L^\infty([0,T]\times[0,1]) \cap C^1([0,T]; L^2(0,1)),$$

(8) $$\rho u_x \in L^\infty([0,T]\times[0,1]) \cap C^{1/2}([0,T]; L^2(0,1)),$$

(9) there exists a constant $C(T)$ with

$$\frac{1}{C(T)}\rho_0(x) \leqq \rho(t,x) \leqq C(T)\rho_0(x).$$

(ρ, u) is the weak solution of the initial-boundary problem in the following sence;

(E1) $\frac{\partial \rho}{\partial t} + 4\pi r^2 \rho^2 u_x + \frac{2u\rho}{r} = 0$ for a.e. $x \in (0,1)$ and for any $t \geqq 0$,

(E2) $\int_0^1 \left[\phi u_t - (4\pi r^2 \phi_x + \frac{2\phi}{r\rho})p + 16\pi^2 \nu \phi_x r^4 \rho u_x + 2\nu\phi \frac{u}{r^2\rho} + \phi \frac{M}{r^2}\right] dx = 0$ for any $\phi \in C_0^\infty(0,1)$ and for any $t \geqq 0$,

(I) $\rho(0,x) = \rho_0(x)$ and $u(0,x) = u_0(x)$ for any $x \in [0,1]$,

and

(B) $u(t,0) = 0$ for any $t \geqq 0$.

REMARK 1. Since $\rho u_x(t,\cdot) \in L^\infty(0,1)$ and $1/\rho(t,\cdot) \in L^1(0,1)$, we have $u_x(t,\cdot) \in L^1(0,1)$. Thus $u(t,\cdot) \in C[0,1]$ and the trace $u|_{x=0}$ has the meaning for any t. Note that we have $u \in C([0,T] \times [0,1))$ from (7)(8)(9)(E1), and that (A.1)(A.2) (A.3) imply the compatibility that $u_0(0) = 0$. On the other hand we cannot claim that the trace $\rho u_x|_{x=1}$ exists. So the boundary condition at $x = 1$ implied by (E2) is that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{1-\epsilon}^1 (-4\pi r^2 p + 16\pi^2 \nu r^4 \rho u_x) dx = 0.$$

Here, since $\lim_{x \to 1} p(t,x) = 0$ by (9) and (A.1), we get

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{1-\epsilon}^1 \rho u_x dx = 0.$$

REMARK 2. If $\frac{1}{C}(1-x)^{1/\gamma} \leqq \rho_0(x) \leqq C(1-x)^{1/\gamma}$, it can be proved that $\rho(t,x)$ satisfies (9) in a more general context. For the proof see [3].

In this paper we will prove that such a solution is unique. As for the uniqueness of solutions of a free boundary problem of this kind, we have the result of H.F.-Yashima and R. Benabidallah, [2]. The equation they considered includes the heat conduction, and our proof for the barotropic case may be more elementary.

THEOREM. *Assume* (A.1)(A.2)(A.3). *Let* (ρ_1, u_1) *and* (ρ_2, u_2) *be solutions of* (E1)(E2)(I)(B) *satisfying* (7)(8)(9) *for any* T. *Then we have* $\rho_1 = \rho_2$, $u_1 = u_2$.

To prove that, we consider the following quantities:

$$\alpha(t) = \int_0^1 \left| \frac{1}{\rho_2(t,x)} - \frac{1}{\rho_1(t,x)} \right| dx, \tag{10}$$

$$Q(t) = \int_0^1 \rho_0(x) \left(\frac{1}{\rho_2(t,x)} - \frac{1}{\rho_1(t,x)} \right)^2 dx, \tag{11}$$

$$G(t) = \int_0^1 \left[16\pi^2 \nu (u_2 - u_1)_x^2 r_2^4 \rho_2 + 2\nu \frac{(u_2 - u_1)^2}{r_1^2 \rho_1} \right] dx. \tag{12}$$

Here, of course, we denote

$$r_k = \left[1 + \frac{3}{4\pi}\int_0^x \frac{d\xi}{\rho_k(t,\xi)}\right]^{1/3}, \quad k = 1, 2. \tag{13}$$

Note that, since $\int_0^1 \frac{dx}{\rho_0(x)} < +\infty$, (9) implies the finiteness of (10) and (11). We cannot expect $\frac{1}{\rho} \in L^2(0,1)$, since $\rho \sim C(1-x)^{1/\gamma}$ for the stationary solution. Thus we multiply by $\rho_0(x)$ the integrand of $Q(t)$. By (7)(8)(9), $G(t)$ is also finite.

Hereafter C will denote various constants depending upon T, where T is arbitrarily fixed and we show $\rho_1 = \rho_2$ and $u_1 = u_2$ for $0 \leqq t \leqq T$.

PROPOSITION 1. *For any m we have*

$$| r_2^m - r_1^m | \leqq C\alpha(t). \tag{14}$$

Proof. By definition (13), we have

$$r_2^m - r_1^m = (r_2^3)^{m/3} - (r_1^3)^{m/3} = \frac{m}{3} r_*^{m-3}(r_2^3 - r_1^3) = \frac{1}{4\pi} r_*^{m-3} \int_0^x \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) dx,$$

where $r_* = r_1 + \theta(r_2 - r_1)$ with $0 < \theta < 1$. Since $1 \leqq r_k \leqq C$, we get (14) by the definition (10) of $\alpha(t)$. This completes the proof.

PROPOSITION 2. *We have*

$$\alpha(t)^2 \leqq CQ(t). \tag{15}$$

Proof. By the Schwarz's inequality, it is easy to see that

$$\alpha(t) = \int_0^1 \frac{1}{\rho_0^{1/2}} \rho_0^{1/2} \left| \frac{1}{\rho_2} - \frac{1}{\rho_1} \right| dx$$
$$\leqq \left[\int_0^1 \frac{dx}{\rho_0}\right]^{1/2} \left[\int_0^1 \rho_0 \left| \frac{1}{\rho_2} - \frac{1}{\rho_1} \right|^2 dx\right]^{1/2}.$$

Since $\int_0^1 \frac{dx}{\rho_0(x)} \leqq C$, we get (15). This completes the proof.

PROPOSITION 3. *We have*

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (u_2 - u_1)^2 dx + \frac{1}{2}G(t) \leqq C[\alpha(t)^2 + Q(t)]. \tag{16}$$

Proof. Taking the difference of (E2) by keeping in mind that $p = a\rho^\gamma$, we have

$$\int_0^1 \phi(u_2 - u_1)_t dx$$

$$
\begin{aligned}
&= 4\pi \int_0^1 \phi_x[(r_2^2 - r_1^2)p_2 + r_1^2(p_2 - p_1)]dx \\
&+2a \int_0^1 \phi \left[\left(\frac{1}{r_2} - \frac{1}{r_1} \right) \rho_2^{\gamma-1} + \frac{1}{r_1}(\rho_2^{\gamma-1} - \rho_1^{\gamma-1}) \right] dx \\
&-16\pi^2\nu \int_0^1 \phi_x(u_2 - u_1)_x r_2^4 \rho_2 dx - 16\pi^2\nu \int_0^1 \phi_x(r_2^4 - r_1^4)\rho_2 u_{1,x} dx \\
&-16\pi^2\nu \int_0^1 \phi_x r_1^4(\rho_2 - \rho_1)u_{1,x} dx \\
&-2\nu \int_0^1 \phi \left[\left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) \frac{u_2}{\rho_2} + \frac{1}{r_1^2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) u_2 \right] dx \\
&-2\nu \int_0^1 \frac{\phi(u_2 - u_1)}{r_1^2 \rho_1} dx - M \int_0^1 \phi \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) dx
\end{aligned}
$$

for any $\phi \in C_0^\infty(0,1]$. Now, since $(u_2 - u_1)_x(t,\cdot) \in L^1(0,1)$, $u_2(t,\cdot) - u_1(t,\cdot) \in C[0,1]$ and $u_2(t,0) - u_1(t,0) = 0$, there exists a sequence $\phi_n \in C_0^\infty(0,1]$ such that $\phi_n \to u_2(t,\cdot) - u_1(t,\cdot)$ in $C[0,1]$ and $\phi_{n,x} \to (u_2 - u_1)_x(t,\cdot)$ in $L^1(0,1)$. (In fact, take $\phi_n' \in C_0^\infty(0,1]$ such that $\phi_n' \to (u_2 - u_1)_x$ in $L^1(0,1)$ and put $\phi_n(x) = \int_0^x \phi_n'$.) Therefore, passing to the limit, we can replace ϕ by $u_2 - u_1$. Thus we get

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^1 (u_2 - u_1)^2 dx \\
&= 4\pi \int_0^1 (u_2 - u_1)_x[(r_2^2 - r_1^2)p_2 + r_1^2(p_2 - p_1)]dx \\
&+2a \int_0^1 (u_2 - u_1) \left[\left(\frac{1}{r_2} - \frac{1}{r_1} \right) \rho_2^{\gamma-1} + \frac{1}{r_1}(\rho_2^{\gamma-1} - \rho_1^{\gamma-1}) \right] dx \\
&-16\pi^2\nu \int_0^1 (u_2 - u_1)_x^2 r_2^4 \rho_2 dx - 16\pi^2\nu \int_0^1 (u_2 - u_1)_x(r_2^4 - r_1^4)\rho_2 u_{1,x} dx \\
&-16\pi^2\nu \int_0^1 (u_2 - u_1)_x r_1^4(\rho_2 - \rho_1)u_{1,x} dx \\
&-2\nu \int_0^1 (u_2 - u_1) \left[\left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) \frac{u_2}{\rho_2} + \frac{1}{r_1^2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) u_2 \right] dx \\
&-2\nu \int_0^1 \frac{1}{r_1^2 \rho_1}(u_2 - u_1)^2 dx - M \int_0^1 (u_2 - u_1) \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) dx.
\end{aligned}
$$

The each term can be estimated as below, using (14).

$$
\begin{aligned}
&4\pi \int_0^1 (u_2 - u_1)_x(r_2^2 - r_1^2)p_2 dx \leqq C \int_0^1 \mid (u_2 - u_1)_x \mid \rho_2^{1/2} \alpha(t) dx \\
&\leqq \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C'}{\epsilon} \alpha(t)^2.
\end{aligned}
$$

$$4\pi \int_0^1 (u_2 - u_1)_x r_1^2 (p_2 - p_1) dx \leqq C \int_0^1 \mid (u_2 - u_1)_x \mid \rho_2^{1/2} \rho_2^{-1/2} (\rho_2^\gamma - \rho_1^\gamma) dx$$

$$\leqq \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C'}{\epsilon} \int_0^1 \rho_0^{-1} (\rho_2^\gamma - \rho_1^\gamma)^2 dx$$

$$\leqq \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C''}{\epsilon} \int_0^1 \rho_0^{2\gamma+1} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right)^2 dx$$

$$\leqq \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C'''}{\epsilon} Q(t).$$

$$2a \int_0^1 (u_2 - u_1) \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \rho_2^{\gamma-1} dx \leqq C \int_0^1 (u_2 - u_1) \rho_1^{-1/2} \alpha(t) \rho_0^{\gamma-1/2} dx$$

$$\leqq \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C'}{\epsilon} \alpha(t)^2.$$

$$2a \int_0^1 (u_2 - u_1) \frac{1}{r_1} (\rho_2^{\gamma-1} - \rho_1^{\gamma-1}) dx$$

$$\leqq C \int_0^1 (u_2 - u_1) \rho_1^{-1/2} \rho_0^{1/2} (\rho_2^{\gamma-1} - \rho_1^{\gamma-1}) dx$$

$$\leqq \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C'}{\epsilon} \int_0^1 \rho_0 (\rho_2^{\gamma-1} - \rho_1^{\gamma-1}) dx$$

$$\leqq \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C''}{\epsilon} \int_0^1 \rho_0^{2\gamma+1} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right)^2 dx$$

$$\leqq \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C'''}{\epsilon} Q(t).$$

$$-16\pi^2 \nu \int_0^1 (u_2 - u_1)_x (r_2^4 - r_1^4) \rho_2 u_{1,x} dx$$

$$\leqq C \int_0^1 \mid (u_2 - u_1)_x | \rho_2^{1/2} \alpha(t) \rho_2^{1/2} u_{1,x} dx$$

$$\leqq \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C'}{\epsilon} \alpha(t)^2.$$

$$-16\pi^2 \nu \int_0^1 (u_2 - u_1)_x r_1^4 (\rho_2 - \rho_1) u_{1,x} dx$$

$$\leqq C \int_0^1 (u_2 - u_1)_x \rho_2^{1/2} \rho_2^{-3/2} (\rho_2 - \rho_1) \rho_2 u_{1,x} dx$$

$$\leqq \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C'}{\epsilon} \int_0^1 \rho_0^{-3} (\rho_2 - \rho_1)^2 dx$$

$$\leqq \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C''}{\epsilon} \int_0^1 \rho_0 \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right)^2 dx$$

$$= \epsilon \int_0^1 (u_2 - u_1)_x^2 \rho_2 dx + \frac{C''}{\epsilon} Q(t).$$

$$-2\nu \int_0^1 (u_2 - u_1) \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) \frac{u_2}{\rho_2} dx \leqq C \int_0^1 (u_2 - u_1) \rho_1^{-1/2} \alpha(t) \rho_1^{1/2} u_2 dx$$

$$\leqq \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C'}{\epsilon} \alpha(t)^2.$$

$$-2\nu \int_0^1 (u_2 - u_1) \frac{1}{r_1^2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) u_2 dx \leqq C \int_0^1 (u_2 - u_1) \rho_1^{-1/2} \rho_1^{1/2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) dx$$

$$\leqq \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C'}{\epsilon} \int_0^1 \rho_0 \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right)^2 dx$$

$$= \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C'}{\epsilon} Q(t).$$

$$-M \int_0^1 (u_2 - u_1) \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right) dx \leqq C \int_0^1 (u_2 - u_1) \rho_1^{-1/2} \rho_1^{1/2} \alpha(t) dx$$

$$\leqq \epsilon \int_0^1 \frac{(u_2 - u_1)^2}{\rho_1} dx + \frac{C'}{\epsilon} \alpha(t)^2.$$

Therefore, taking ϵ sufficiently small, we get (16). This completes the proof.

PROPOSITION 4. *We have*

$$Q(t) \leqq C \int_0^t G(\tau) d\tau. \tag{17}$$

Proof. We can rewrite the equation (E1) as

$$\frac{\partial}{\partial t} \frac{1}{\rho} = \frac{2u}{\rho r} + 4\pi r^2 \frac{\partial u}{\partial x}. \tag{E1$'$}$$

Taking the difference of (E1)$'$ and multiplying it by $\frac{1}{\rho_2} - \frac{1}{\rho_1}$, we have

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right)^2 = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \left[2 \left\{ \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \frac{u_2}{r_2} + \frac{1}{\rho_1} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) u_2 \right.\right.$$
$$\left.\left. + \frac{1}{\rho_1 r_1} (u_2 - u_1) \right\} + 4\pi \{ (r_2^2 - r_1^2) u_{2,x} + r_1^2 (u_2 - u_1)_x \} \right].$$

Integrating this with respect to x from 0 to 1 after multiplying it by ρ_0, and integrating it with respect to t, we get an expression of $Q(t)$. Each term of the integrand for t can be estimated as follows.

$$\int_0^1 \rho_0 \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)^2 \frac{|u_2|}{r_2} dx \leqq CQ(t).$$

$$\int_0^1 \rho_0 \left|\frac{1}{\rho_2} - \frac{1}{\rho_1}\right| \frac{1}{\rho_1} \left|\frac{1}{r_2} - \frac{1}{r_1}\right| |u_2|\, dx \leqq C \int_0^1 \rho_0^{1/2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) \alpha(t) \frac{u_2}{\rho_0^{1/2}} dx$$
$$\leqq C'[Q(t) + \alpha(t)^2].$$

$$\int_0^1 \rho_0 \left|\frac{1}{\rho_2} - \frac{1}{\rho_1}\right| \frac{1}{\rho_1 r_1} |u_2 - u_1|\, dx \leqq C \int_0^1 \rho_0^{1/2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) \frac{u_2 - u_1}{\rho_0^{1/2}} dx$$
$$\leqq C'[Q(t) + G(t)].$$

$$4\pi \int_0^1 \rho_0 \left|\frac{1}{\rho_2} - \frac{1}{\rho_1}\right| |r_2^2 - r_1^2|\, |u_{2,x}|\, dx \leqq C \int_0^1 \rho_0^{1/2} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) \alpha(t) \rho_0^{1/2} u_{2,x} dx$$
$$\leqq C'[Q(t) + \alpha(t)^2].$$

$$4\pi \int_0^1 \rho_0 \left|\frac{1}{\rho_2} - \frac{1}{\rho_1}\right| r_1^2 \,|(u_2 - u_1)_x|\, dx \leqq C \int_0^1 \rho_0^{1/2} \left|\frac{1}{\rho_2} - \frac{1}{\rho_1}\right| \rho_0^{1/2} \,|(u_2 - u_1)_x|\, dx$$
$$\leqq C[Q(t) + G(t)].$$

Therefore

$$Q(t) \leqq C \int_0^t [Q(\tau) + \alpha(\tau)^2 + G(\tau)] d\tau.$$

Using (15), we get

$$Q(t) \leqq C \int_0^t [Q(\tau) + G(\tau)] d\tau,$$

from which (17) follows. This completes the proof.

Proof of Theorem. Combining (15)(16)(17), we get

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (u_2 - u_1)^2 dx + \frac{1}{2} G(t) \leqq C \int_0^t G(\tau) d\tau.$$

Since $\int_0^1 (u_2 - u_1)^2 dx = 0$ at $t = 0$, $\widetilde{G}(t) = \int_0^t G(\tau) d\tau$ satisfies

$$\frac{1}{2} \widetilde{G}(t) \leqq C \int_0^t \widetilde{G}(\tau) d\tau.$$

Therefore $\widetilde{G}(t) \equiv 0$, so $G(t) \equiv 0$. This implies

$$\int_0^1 (u_2 - u_1)^2 dx = 0,$$

that is $u_2 \equiv u_1$. $G(t) \equiv 0$ implies $\rho_2 \equiv \rho_1$. This completes the proof.

References

[1] M. Okada and T. Makino, Free boundary problem for the equation of spherically symmetric motion of viscous gas. Japan J. Indust. Appl. Math., **10** (1993), 219–235.

[2] H. Fujita-Yashima et R. Benabidallah, Unicité de la solution de l'équation mono-dimensionnelle ou à symétrie sphérique d'un gaz visqueux et calorifère. Rend. Circ. Mat. Palermo, (2), **42** (1993), 195–218.

[3] Š. Matušû-Nečasová, Some results of the uniqueness of compressible motions (in preparation).

Free Boundary Problem for the Equation of Spherically Symmetric Motion of Viscous Gas (III)

Šárka MATUŠŮ-NEČASOVÁ[†], Mari OKADA[††] and Tetu MAKINO[††]

[†] *Mathematical Institute, Academy of Sciences,*
Czech Republic, Žitná 25, 115 67 Praha 1,
Czech Republic
[††] *Department of Applied Science,*
Faculty of Engineering, Yamaguchi University,
Ube 755, Japan

Reprinted from the JAPAN JOURNAL OF INDUSTRIAL AND APPLIED MATHEMATICS
Vol. 14, No. 2, pp. 199–213, June 1997

Japan J. Indust. Appl. Math., 14 (1997), 199–213

Free Boundary Problem for the Equation of Spherically Symmetric Motion of Viscous Gas (III)

Šárka MATUŠŮ-NEČASOVÁ*†, Mari OKADA†† and Tetu MAKINO††

† *Mathematical Institute, Academy of Sciences, Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic*

†† *Department of Applied Science, Faculty of Engineering, Yamaguchi University, Ube 755, Japan*

Received June 26, 1995
Revised October 19, 1995

We study the spherically symmetric motion of viscous barotropic gas surrounding a solid ball. We are interested in the density distribution which contacts the vacuum at a finite radius. The equilibrium is asymptotically stable with respect to small perturbation, provided that $\gamma > \frac{4}{3}$ and a is sufficiently small, when the equation of state is $p = a\rho^\gamma$, p being the pressure and ρ the density.

Key words: Navier-Stokes equation, asymptotic stability of equilibria, free boundary problem, spherically symmetric motion

We are investigating the equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial r} + \rho\dfrac{\partial u}{\partial r} + \dfrac{2}{r}\rho u = 0, \\ \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial r}\right) + \dfrac{\partial p}{\partial r} = \nu\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r} - \dfrac{2}{r^2}u\right) - \dfrac{\rho M}{r^2}, \\ p = a\rho^\gamma, \end{cases}$$

where ν, a, γ are positive constants and $1 < \gamma \leq 2$. These equations govern the spherically symmetric motion of a viscous barotropic gas. We consider these equations in $r \geq 1$ with the boundary condition

$$u|_{r=1} = 0$$

and the initial conditions

$$\rho|_{t=0} = \rho^0(r), \quad u|_{t=0} = u^0(r).$$

* The first author was supported by Grant of Czech Academy n° 201.93.2177.

These equations admit the equilibria

$$\rho = \begin{cases} \left[\dfrac{(\gamma-1)M}{a\gamma}\left(\dfrac{1}{r}-\dfrac{1}{R}\right)\right]^{1/(\gamma-1)} & (r \le R) \\ 0 & (R < r), \end{cases} \qquad u = 0.$$

Here $R > 1$ is arbitrary. Since we are interested in the class of initial data which includes these equilibria, we introduce the Lagrange coordinates

$$x = 4\pi \int_1^r \rho(s,t)s^2 ds.$$

Then the equations turn out to be

$$\frac{\partial \rho}{\partial t} + 4\pi\rho^2 \frac{\partial}{\partial x}(r^2 u) = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + 4\pi r^2 \frac{\partial p}{\partial x} = 16\pi^2 \nu \frac{\partial}{\partial x}\left(r^4 \rho \frac{\partial u}{\partial x}\right) - 2\nu \frac{u}{r^2 \rho} - \frac{M}{r^2}, \tag{2}$$

$$p = a\rho^\gamma, \tag{3}$$

where

$$r = \left[1 + \frac{3}{4\pi}\int_0^x \frac{d\xi}{\rho(\xi,t)}\right]^{1/3}. \tag{4}$$

Normalizing the total mass, we consider the equations (1) (2) (3) (4) in $0 \le x \le 1$ with the boundary conditions

$$u|_{x=0} = 0, \quad 4\pi r^2 p - 16\pi^2 \nu r^4 \rho \frac{\partial u}{\partial x}\bigg|_{x=1} = 0 \tag{5}$$

and the initial conditions

$$\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x). \tag{6}$$

In this case the equilibrium is unique. We denote it by $\rho = \overline{\rho}(x)$, $p = \overline{p}(x) = a\overline{\rho}(x)^\gamma$ and $r = \overline{r}(x)$, who satisfy

$$4\pi \overline{r}^2 \frac{\partial \overline{p}}{\partial x} = -\frac{M}{\overline{r}^2}, \tag{7}$$

$$\frac{1}{C}(1-x) \le \overline{p}(x) \le C(1-x). \tag{8}$$

In the paper [2], we constructed global solutions under the following assumptions:

(A.0) $\rho_0 \in C[0,1]$, $\rho_0(x) > 0$ for $0 \le x < 1$, $\rho_0(0) = 0$, total variation $[\rho_0] < +\infty$;

(A.1) There exists a monotone decreasing function $\lambda(x)$ such that $0 \le \lambda(x) \le \rho_0(x)$ and $\displaystyle\int_0^1 \frac{dx}{\lambda(x)} < +\infty$;

(A.2) $u_0 \in C[0,1]$;

(A.3) This is a slightly complicated assumption concerning ρ_0 and u_0. But it is satisfied at least if $p_0 = a\rho^\gamma \in C^1[0,1]$ and $u_0 = 0$. See [2] for the details.

The global solution (ρ, u) constructed in [2] satisfies that, for any T,

$$\rho, u \in L^\infty([0,T] \times [0,1]) \cap C^1([0,T]; L^2(0,1)), \tag{9}$$

$$\rho u_x \in L^\infty([0,T] \times [0,1]) \cap C^{1/2}([0,T]; L^2(0,1)), \tag{10}$$

and there exists a constant $C(T)$ such that

$$\frac{1}{C(T)}\rho_0(x) \le \rho(x,t) \le C(T)\rho_0(x) \quad \text{for} \quad 0 \le t \le T, \quad 0 \le x \le 1. \tag{11}$$

In the last paper [3], we showed that such a solution is unique.

In this paper we will show that the solution tends to the equilibrium as $t \to +\infty$ under some additional assumptions.

First we prepare some preliminary estimates. Here we apply the argument of I. Straskraba [4].

PROPOSITION 1. *There exists a constant C such that*

$$\rho(x,t) \le C \quad \textit{for} \quad 0 \le x \le 1 \quad \textit{and} \quad 0 \le t < +\infty. \tag{12}$$

Here and hereafter C denotes various constants depending on the parameters γ, ν, M, a and the initial conditions ρ_0 and u_0.

Proof. We rewrite the equation (2) as

$$u_t = 4\pi r^2 (4\pi\nu\rho(r^2 u)_x - p)_x - \frac{M}{r^2}. \tag{2$'$}$$

Integrating $(2)'$ with respect to x from x to 1 and using the boundary condition (5), we get

$$\int_x^1 \frac{u_t}{r^2} dx = -4\pi(4\pi\nu\rho(r^2 u)_x - p) - \int_x^1 \frac{M}{r^4} dx.$$

But, since $4\pi\rho(r^2u)_x = -(\log\rho)_t$ from (1), this can be rewritten

$$\frac{\partial}{\partial t}\log\rho = \frac{1}{\nu}\left[\frac{1}{4\pi}\frac{\partial}{\partial t}\int_x^1 \frac{u}{r^2}dx - p + H\right], \tag{13}$$

where

$$H = \frac{1}{2\pi}\int_x^1 \frac{u^2}{r^3}dx + \frac{1}{4\pi}\int_x^1 \frac{M}{r^4}dx. \tag{14}$$

Here we use the fact that $r_t = u$.

Now we use the energy equality

$$\frac{d}{dt}\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{\gamma-1}\frac{p}{\rho} - \frac{M}{r}\right)dx + Y(t) = 0, \tag{15}$$

where

$$Y(t) = \int_0^1 \left(16\pi^2\nu r^4\rho u_x^2 + 2\nu\frac{u^2}{r^2\rho}\right)dx. \tag{16}$$

Let us denote

$$E_0 = \int_0^1 \left(\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}\frac{p_0}{\rho_0} - \frac{M}{r_0}\right)dx. \tag{17}$$

Since $r \geq 1$, the energy equality (15) implies the boundedness of H, say $H \leq H^*$.

Suppose $p(x_0,T) > H^*$ for some T. Then there exists $t_1 < T$ such that i) $t_1 > 0$, $p(x_0,t) \geq H^*$ for $t \in [t_1,T]$, $p(x_0,t_1) = H^*$ or ii) $t_1 = 0$ and $p(x_0,t) \geq H^*$ for $t \in [0,T]$. Integrating (13) with respect to t from t_1 to T, we see

$$\log\rho(x_0,T) = \log\rho(x_0,t_1) + \frac{1}{\nu}\left[\frac{1}{4\pi}\int_{x_0}^1 \frac{u}{r^2}\Big|_{t_1}^T dx + \int_{t_1}^T (H-p)dt\right].$$

But, since $H - p \leq 0$ along $x = x_0$, $t_1 \leq t \leq T$, and since

$$\left|\int_{x_0}^1 \frac{u}{r^2}dx\right| \leq \left[\int_{x_0}^1 u^2 dx\right]^{1/2} \leq [2(E_0+M)]^{1/2},$$

we get

$$\log\rho(x_0,T) \leq \max\left(\log\rho_0(x_0), \log(H^*/a)^{1/\gamma}\right) + \frac{1}{2\pi\nu}[2(E_0+M)]^{1/2}.$$

This completes the proof, since $\rho_0 \in C[0,1]$ is bounded.

PROPOSITION 2. *We have*

$$\int_0^1 u(x,t)^2 dx \longrightarrow 0 \qquad \textit{as} \quad t \to +\infty. \tag{18}$$

Proof. We start with the energy equality (15), which can be rewritten as

$$\frac{d}{dt}\int_0^1 \left(\frac{1}{2}u^2 dx\right) dx + Y(t) - \int_0^1 4\pi p(r^2u)_x dx + \int \frac{Mu}{r^2} dx = 0. \tag{19}$$

Put

$$\epsilon(t) = \left[\int_{t-1}^t Y(s)ds\right]^{1/2}.$$

Since $\int_0^{+\infty} Y(t)dt \le E_0 + M < +\infty$, we see $\epsilon(t) \to 0$ as $t \to +\infty$. Now we integrate (19) with respect to t from s to t, and then integrate it with respect to s from $t-1$ to t. The result is

$$\begin{aligned}\int_0^1 \frac{1}{2}u^2(t)dx - \int_{t-1}^t \int_0^1 \frac{1}{2}u^2(s)dxds + \int_{t-1}^t \int_s^t Y(\tau)d\tau ds \\ = \int_{t-1}^t ds \int_s^t d\tau \left[\int_0^1 4\pi p(r^2u)_x dx - \int_0^1 \frac{M}{r^2}dx\right](\tau).\end{aligned}$$

We see

$$0 \le \int_{t-1}^t \int_s^t Y(\tau)d\tau ds = \int_{t-1}^t Y(\tau)(\tau-(t-1))d\tau \le \int_{t-1}^t Y(\tau)d\tau = \epsilon(t)^2 \to 0.$$

On the other hand, since

$$u^2 = \int_0^x 2uu_x dx \le CY, \tag{20}$$

we see

$$\int_{t-1}^t \int_0^1 \frac{1}{2}u^2(s)dxds \le C\int_{t-1}^t Y(s)ds \le C\epsilon(t)^2 \to 0.$$

Noting

$$p(r^2u)_x = \frac{1}{2\pi}\frac{1}{r}\frac{p}{\rho}u + pr^2u_x,$$

we easily see

$$\left|\int_{t-1}^t \int_s^t \int_0^1 p(r^2u)_x dxd\tau ds\right| \le C\left[\int_{t-1}^t \int_0^1 \left(\frac{u^2}{r^2\rho} + r^4\rho u_x^2\right) dxds\right]^{1/2} \le C'\epsilon(t).$$

Here we use the result of Proposition 1, say $\rho \le C$. Similarly

$$\left|\int_{t-1}^t \int_s^t \int_0^1 \frac{Mu}{r^2} dxd\tau ds\right| \le C\left[\int_{t-1}^t \int_0^1 \frac{u^2}{r^2\rho}dxds\right]^{1/2} \le C'\epsilon(t).$$

This completes the proof.

Now we are going to estimate ρ from below to show $\rho \to \overline{\rho}$. Then we must leave the argument of I. Straskraba [4]. The reason is that [4] supposes that the external force $f(r)$ satisfies $f'(r) \leq 0$ and this plays an important role in his argument, but in our case $f(r) = -\frac{M}{r^2}$ satisfies $f'(r) = \frac{2M}{r^3} > 0$.

We assume

$$\gamma > \frac{4}{3} \quad \text{and} \quad M > 0. \tag{A.4}$$

Moreover we assume temporarily

$$p_0(x) \leq M(1-x). \tag{a.1}$$

PROPOSITION 3. *There exists a constant $B(\gamma, \nu, E^*, M^*, R^*)$ such that*

$$B = \sup_{\substack{0 \leq t < +\infty \\ 0 \leq x < 1}} \frac{\rho_0(x)}{\rho(x,t)} \leq B(\gamma, \nu, E^*, M^*, R^*), \tag{21}$$

provided that $E_0 \leq E^*$, $M \leq M^*$ and $R_0 \leq R^*$. Here

$$R_0 = r(1,0) = \left[1 + \frac{3}{4\pi} \int_0^1 \frac{dx}{\rho_0(x)}\right]^{1/3}.$$

Proof. We write (13) (14) as

$$\frac{\partial}{\partial t} \log p + \frac{\gamma}{\nu} p = A(x,t), \tag{22}$$

where

$$A(x,t) = \frac{\gamma}{4\pi\nu} \frac{\partial}{\partial t} \int_x^1 \frac{u}{r^2} dx + \frac{\gamma}{2\pi\nu} \int_x^1 \frac{u^2}{r^3} dx + \frac{\gamma}{4\pi\nu} \int_x^1 \frac{M}{r^4} dx. \tag{23}$$

Solving (22), we get

$$\frac{1}{p(x,t)} = \frac{1}{p_0(x)} \left[\exp\left(-\int_0^t A ds\right) + \frac{\gamma}{\nu} p_0(x) \int_0^t \exp\left(-\int_s^t A d\tau\right) ds\right]. \tag{24}$$

We put

$$\beta(T) = \sup_{\substack{0 \leq t \leq T \\ 0 \leq x < 1}} \frac{p_0(x)}{p(x,t)}.$$

This is finite by (11). Then we have

$$R(t) = r(1,t) = \left[1 + \frac{3}{4\pi} \int_0^1 \frac{dx}{\rho(x,t)}\right]^{1/3} \leq K_1 \left(1 + \beta(T)^{1/\gamma}\right)^{1/3},$$

where $K_1 = K_1(R^*)$ and

$$-\int_s^t A d\tau \leq -\frac{\gamma}{4\pi\nu}\int_x^1 \frac{u}{r^2}\Big|_s^t dx - \frac{\gamma}{4\pi\nu}\int_s^t d\tau \int_x^1 \frac{M}{r^4}dx$$
$$\leq K_2 - \frac{\gamma M}{4\pi\nu {K_1}^4}(1-x)\left(1+\beta(T)^{1/\gamma}\right)^{-4/3}(t-s)$$

for $0 \leq s \leq t \leq T$, where $K_2 = \frac{\gamma}{2\pi\nu}\sqrt{2(E^*+M^*)}$. Applying this estimate to (24), we see

$$\frac{1}{p(x,t)} \leq \frac{e^{K_2}}{p_0(x)}\left(1+\frac{4\pi\nu {K_1}^4}{\gamma}\left(1+\beta(T)^{1/\gamma}\right)^{4/3}\right).$$

Here we use (a.1). Thus we get

$$\beta(T) \leq K_3\left(1+\left(1+\beta(T)^{1/\gamma}\right)^{4/3}\right), \quad \text{with} \quad K_3 = K_3(\gamma,\nu,E^*,M^*,R^*).$$

Consider the function

$$\varphi(\beta) = \frac{\beta}{K_3\left(1+\left(1+\beta^{1/\gamma}\right)^{4/3}\right)}.$$

Then $\varphi(\beta) \to +\infty$ as $\beta \to +\infty$, since $\gamma > \frac{4}{3}$ by (A.4). Thus $\varphi(\beta) \leq 1$ implies $\beta \leq \beta^* = \beta^*(\gamma,\nu,E^*,M^*,R^*)$. Putting $B(\gamma,\nu,E^*,M^*,R^*) = (\beta^*)^{1/\gamma}$ we get (21). This completes the proof.

Now we write (13) (14) as

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t} + \frac{1}{\nu}p = \frac{1}{4\pi\nu}\frac{\partial}{\partial t}\int_x^1 \frac{u}{r^2}dx + \frac{1}{2\pi\nu}\int_x^1 \frac{u^2}{r^3}dx + \frac{M}{4\pi\nu}\int_x^1 \frac{dx}{r^4},$$

or

$$\text{(25)} \qquad \frac{\partial}{\partial t}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right) - \frac{1}{\nu}\frac{p}{\rho} = -\frac{1}{4\pi\nu}\frac{1}{\rho}\frac{\partial}{\partial t}\int_x^1 \frac{u}{r^2}dx - \frac{1}{2\pi\nu}\frac{1}{\rho}\int_x^1 \frac{u^2}{r^3}dx - \frac{M}{4\pi\nu}\frac{1}{\rho}\int_x^1 \frac{dx}{r^4}.$$

The equilibrium satisfies

$$\text{(26)} \qquad -\frac{1}{\nu}\frac{\overline{p}}{\overline{\rho}} = -\frac{M}{4\pi\nu}\frac{1}{\overline{\rho}}\int_x^1 \frac{dx}{\overline{r}^4}.$$

Taking $\{(25)-(26)\} \times \left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\rho_0$, we get

(27)

$$\frac{\partial}{\partial t}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2-\frac{1}{\nu}\left(\frac{p}{\rho}-\frac{\overline{p}}{\overline{\rho}}\right)\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\rho_0$$
$$=-\frac{1}{4\pi\nu}\frac{\partial}{\partial t}\left(\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\int_x^1\frac{u}{r^2}dx\right)+\frac{1}{\nu}(r^2u)_x\left(\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)+\frac{\rho_0}{\rho}\right)\int_x^1\frac{u}{r^2}dx$$
$$-\frac{1}{2\pi\nu}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\int_x^1\frac{u^2}{r^3}dx-\frac{M}{4\pi\nu}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\int_x^1\frac{dx}{\overline{r}^4}$$
$$-\frac{M}{4\pi\nu}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\int_x^1\left(\frac{1}{r^4}-\frac{1}{\overline{r}^4}\right)dx.$$

Now we see

$$-\frac{1}{\nu}\left(\frac{p}{\rho}-\frac{\overline{p}}{\overline{\rho}}\right)\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\rho_0=\frac{a(\gamma-1)}{\nu}\widetilde{\rho}^{\gamma}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\rho_0\geq 0, \tag{28}$$

where $\widetilde{\rho}=\left(\frac{1}{\overline{\rho}}+\theta\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\right)^{-1}$ with $0<\theta<1$.
We assume temporarily

$$\rho_0(x)\leq C\overline{\rho}(x)\quad\text{for}\quad 0\leq x\leq 1. \tag{a.2}$$

Then we get

$$\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\leq\frac{C}{\rho_0},\quad\frac{\rho_0}{\rho}\leq C,\quad\rho_0\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|\leq C,\quad\frac{\rho_0}{\rho}\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|\leq\frac{C}{\rho_0}\quad\text{and so on.}$$

Moreover we assume

$$\rho_0(x)\geq\frac{(1-x)^{1/\gamma}}{C}. \tag{a.3}$$

Now, for $0<\mu$ and $0\leq\delta\leq 1$, we put

$$Q_{\delta,\mu}=\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{\mu}}. \tag{29}$$

$Q_{\delta,\mu}$ is finite as least if $\delta>0$.

Let $\delta>0$. Then it follows from (28) that

$$\frac{d}{dt}Q_{\delta,\mu}\leq\frac{d}{dt}F_{\delta,\mu}+G_{\delta,\mu}-\frac{M}{4\pi\nu}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{1}{(1-x)^{\mu}}\int_x^1\frac{d\xi}{\overline{r}^4}dx$$
$$-\frac{M}{4\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{1}{(1-x)^{\mu}}\int_x^1\left(\frac{1}{r^4}-\frac{1}{\overline{r}^4}\right)d\xi dx,$$

where

$$F_{\delta,\mu} = -\frac{1}{4\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{1}{(1-x)^\mu}\int_x^1\frac{u}{r^2}d\xi dx,$$

$$G_{\delta,\mu} = \frac{1}{\nu}\int_0^{1-\delta}(r^2u)_x\left(\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)+\frac{\rho_0}{\rho}\right)\frac{1}{(1-x)^\mu}\int_x^1\frac{u}{r^2}d\xi dx$$
$$-\frac{1}{2\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{1}{(1-x)^\mu}\int_x^1\frac{u^2}{r^3}d\xi dx.$$

First we consider the case $\mu = \frac{5}{4}$. Then it is easy to see

$$\left|F_{\delta,5/4}\right| \le C\int_0^{1-\delta}\frac{(1-x)^{1/2}dx}{(1-x)^{1/\gamma+5/4}}\sqrt{\int_x^1 u^2 d\xi} \le C'\delta^{1/4-1/\gamma}\sqrt{\int_0^1 u^2dx}$$

by (a.3), and

$$\left|G_{\delta,5/4}\right| \le C\delta^{-1/4}Y \quad \text{(see (20))}$$

Here and hereafter C stands for various constants independent of δ. On the other hand

$$-\frac{M}{4\pi\nu}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{5/4}}\int_x^1\frac{d\xi}{\overline{r}^4}$$
$$\le -\frac{M}{4\pi\nu}\frac{1}{\overline{R}^4}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}},$$

where $\overline{R} = \overline{r}(1)$, and

$$-\frac{M}{4\pi\nu}\int_0^{1-\delta}\frac{\rho_0}{\rho}\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{dx}{(1-x)^{5/4}}\int_x^1\left(\frac{1}{r^4}-\frac{1}{\overline{r}^4}\right)d\xi$$
$$\le \frac{MB}{4\pi^2\nu}\int_0^{1-\delta}\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|\frac{dx}{(1-x)^{1/4}}\left(\int_0^{1-\delta}\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|dx+\int_{1-\delta}^1\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|dx\right)$$
$$\le \frac{MB}{4\pi^2\nu}\left[\int_0^1\frac{dx}{\rho_0(1-x)^{1/4}}\right]^{1/2}\left[\int_0^1\frac{(1-x)^{1/4}}{\rho_0}dx\right]^{1/2}\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)\frac{dx}{(1-x)^{1/4}}$$
$$+C\delta^{(\gamma-1)/\gamma}\left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}\right].$$

Here we use the estimate

$$\int_{1-\delta}^1\left|\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right|dx \le C\delta^{(\gamma-1)/\gamma},$$

by (a.3). Let us suppose

$$B\left[\int_0^1 \frac{dx}{\rho_0(1-x)^{1/4}}\right]^{1/2}\left[\int_0^1 \frac{(1-x)^{1/4}}{\rho_0}dx\right]^{1/2} \le \frac{\pi}{2}\frac{1}{\overline{R}^4}. \tag{a.4}$$

Then

$$\frac{d}{dt}Q_{\delta,5/4} \le \frac{d}{dt}F_{\delta,5/4} + G_{\delta,5/4} - \alpha\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}} + C\delta^{(\gamma-1)/\gamma}\left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}\right]^{1/2},$$

where

$$\alpha = \frac{M}{8\pi\nu\overline{R}^4}.$$

Suppose

$$\frac{2C\delta^{(\gamma-1)/\gamma}}{\alpha} \le \left[\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}\right]^{1/2} \quad \text{for} \quad t \ge T.$$

Then

$$\frac{d}{dt}Q_{\delta,5/4} \le \frac{d}{dt}F_{\delta,5/4} + G_{\delta,5/4} - \frac{\alpha\delta}{2}Q_{\delta,5/4} \quad \text{for} \quad t \ge T.$$

Since $F_{\delta,5/4} \to 0$ as $t \to +\infty$ by Proposition 2 and

$$\int_0^{+\infty}|G_{\delta,5/4}(t)|dt < +\infty,$$

it follows that $Q_{\delta,5/4}(t) \to 0$ as $t \to +\infty$ from the above differential inequality. This is a contradiction, since

$$Q_{\delta,5/4} \ge \int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}}.$$

Therefore there exists a sequence $t_n(\delta) \to +\infty$ $(n \to +\infty)$ such that

$$\int_0^{1-\delta}\rho_0\left(\frac{1}{\rho}-\frac{1}{\overline{\rho}}\right)^2\frac{dx}{(1-x)^{1/4}} \le C\delta^{2(\gamma-1)/\gamma} \quad \text{at} \quad t = t_n(\delta). \tag{30}$$

Next we take $\mu = \frac{3}{4}$. Then

$$|F_{\delta,3/4}| \le C\int_0^1 \frac{(1-x)^{1/2}dx}{(1-x)^{1/\gamma+3/4}}\sqrt{\int_x^1 u^2 d\xi} \le C'\sqrt{\int_0^1 u^2 dx},$$

since $\frac{1}{2} - \frac{1}{\gamma} - \frac{3}{4} + 1 = \frac{3}{4} - \frac{1}{\gamma} > 0$ by (A.4)

$$|G_{\delta,3/4}| \le CY,$$

since $1 - \frac{3}{4} = \frac{1}{4} > 0$. And by a similar argument we get

$$\frac{d}{dt}Q_{\delta,3/4} \le \frac{d}{dt}F_{\delta,3/4} + G_{\delta,3/4} - \alpha \int_0^{1-\delta} \rho_0 \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)^2 (1-x)^{1/4} dx + C\delta^{(\gamma-1)/\gamma} \left[\int_0^{1-\delta} \rho_0 \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)^2 (1-x)^{1/4} dx\right]^{1/2}$$

under the assumption (a.4). Then, in this case, we have

$$\begin{aligned} Q_{\delta,3/4}(t) \le\ & Q_{\delta,3/4}(t_0) + F_{\delta,3/4}(t) - F_{\delta,3/4}(t_0) + \int_{t_0}^t G_{\delta,3/4}(\tau)d\tau \\ & + C\delta^{(\gamma-1)/\gamma} \int_{t_0}^t \left[\int_0^{1-\delta} \rho_0 \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)^2 (1-x)^{1/4} dx\right]^{1/2} (\tau) d\tau \\ \le\ & Q_{\delta,3/4}(t_0) + F_{\delta,3/4}(t) - F_{\delta,3/4}(t_0) \\ & + \int_{t_0}^t G_{\delta,3/4}(\tau)d\tau + C'\delta^{(\gamma-1)/\gamma}(t - t_0), \qquad \text{for} \quad 0 \le t_0 \le t. \end{aligned}$$

Take $t_0 = 0$. Assume

$$\text{(a.5)} \qquad Q_{0,3/4}(0) = \int_0^1 \rho_0 \left(\frac{1}{\rho_0} - \frac{1}{\bar{\rho}}\right)^2 \frac{dx}{(1-x)^{3/4}} < +\infty.$$

Then, since $F_{\delta,3/4} \to F_{0,3/4}$ and $G_{\delta,3/4} \to G_{0,3/4}$, we see

$$Q_{0,3/4}(t) \le C \quad \text{for} \quad t \ge 0$$

and

$$\text{(31)} \qquad \begin{aligned} Q_{0,3/4}(t) \le\ & Q_{0,3/4}(t_0) + F_{0,3/4}(t) - F_{0,3/4}(t_0) \\ & + \int_{t_0}^t G_{0,3/4}(\tau)d\tau \quad \text{for} \quad 0 \le t_0 \le t. \end{aligned}$$

Here we note

$$\text{(32)} \qquad Q_{\delta,3/4}(t_n(\delta)) \le \frac{1}{\delta^{1/2}} \int_0^{1-\delta} \rho_0 \left(\frac{1}{\rho} - \frac{1}{\bar{\rho}}\right)^2 \frac{dx}{(1-x)^{1/4}} \le C\delta^{(3\gamma-4)/2\gamma},$$

from (30).

Finally take μ such that

$$\text{(33)} \qquad \frac{3}{4} < \mu < \frac{3}{2} - \frac{1}{\gamma}.$$

Since $\frac{3}{2} - \frac{1}{\gamma} - \frac{3}{4} = \frac{3}{4} - \frac{1}{\gamma} > 0$ by (A.4), this is possible, and we see

$$|F_{\delta,\mu}| \le C \int_0^1 \frac{(1-x)^{1/2}dx}{(1-x)^{1/\gamma+\mu}} \sqrt{\int_x^1 u^2 d\xi} \le C' \sqrt{\int_0^1 u^2 dx},$$

since $\frac{1}{2} - \mu - \frac{1}{\gamma} + 1 = \frac{3}{2} - \frac{1}{\gamma} - \mu > 0$, and

$$|G_{\delta,\mu}| \le CY,$$

since $1 - \mu > 1 - \frac{3}{2} + \frac{1}{\gamma} = \frac{2-\gamma}{2\gamma} \ge 0$. Then, assuming

$$\text{(a.6)} \qquad B\left[\int_0^1 \frac{(1-x)^{1-\mu}}{\rho_0} dx\right]^{1/2} \left[\int_0^1 \frac{dx}{\rho_0 (1-x)^{1-\mu}}\right]^{1/2} \le \frac{\pi}{2}\frac{1}{\overline{R}^4},$$

we get

$$Q_{0,\mu}(t) \le C$$

by a similar argument to the case of $\mu = \frac{3}{4}$. Here we assume

$$\text{(a.7)} \qquad Q_{0,\mu}(0) = \int_0^1 \rho_0 \left(\frac{1}{\rho_0} - \frac{1}{\overline{\rho}}\right)^2 \frac{dx}{(1-x)^\mu} < +\infty$$

and use $1 - \mu > 1 - \left(\frac{3}{2} - \frac{1}{\gamma}\right) = \frac{2-\gamma}{\gamma} \ge 0$.

Then we get

$$\begin{aligned} Q_{0,3/4}(t_n(\delta)) &= Q_{\delta,3/4}(t_n(\delta)) + \int_{1-\delta}^1 \rho_0 \left(\frac{1}{\rho} - \frac{1}{\overline{\rho}}\right)^2 \frac{dx}{(1-x)^{3/4}} \\ &\le Q_{\delta,3/4}(t_n(\delta)) + \delta^{\mu-3/4} Q_{0,\mu}(t_n(\delta)) \\ &\le C\left(\delta^{(3\gamma-4)/2\gamma} + \delta^{\mu-3/4}\right). \end{aligned}$$

It follows from (31) that

$$\limsup_{t\to+\infty} Q_{0,3/4}(t) \le C\left(\delta^{(3\gamma-4)/2\gamma} + \delta^{\mu-3/4}\right).$$

Since $\delta > 0$ is arbitrary, we get

$$Q_{0,3/4}(t) \longrightarrow 0 \quad \text{as} \quad t \to +\infty.$$

This is our final goal.

We are going to give a sufficient condition for (a.1) (a.2) (a.3) (a.4) (a.5) (a.6).

We put the assumption

$$\text{(A.5)} \qquad \int_0^1 \rho_0 \left(\frac{1}{\rho_0} - \frac{1}{\overline{\rho}}\right)^2 \frac{dx}{(1-x)^\mu} < +\infty \quad \text{for some} \quad \mu > \frac{3}{4}.$$

This assumption (A.5) guarantees (a.5) (a.7).

On the other hand we consider the condition

$$La(1-x) \le p_0 \le M(1-x), \tag{a.8}$$

where L is a suitable constant depending upon γ, ν, E^*, M^* and $\overline{R}$ specified as follows, where $\overline{R} = \overline{R}(\gamma, \frac{a}{M})$ is the solution of

$$\frac{1}{4\pi}\left(\frac{\gamma}{\gamma-1}\frac{a}{M}\right)^{1/(\gamma-1)} = \int_1^{\overline{R}} \left(\frac{1}{r} - \frac{1}{\overline{R}}\right)^{1/(\gamma-1)} r^2 dr.$$

Under the condition (a.8), we have

$$B \le B(\gamma, \nu, E^*, M^*, R^*)$$

provided that $E_0 \le E^*$ and $M \le M^*$. We see $R_0 \le 2^{1/3}$ provided $\frac{3}{4\pi}\frac{\gamma}{\gamma-1}\left(\frac{1}{L}\right)^{1/\gamma} \le 1$, since

$$\frac{1}{\rho_0} \le \left(\frac{1}{L}\right)^{1/\gamma} \frac{1}{(1-x)^{1/\gamma}}. \tag{34}$$

It is easy to see (a.4) and (a.6) hold if we take L sufficiently large so that

$$B\left(\frac{1}{L}\right)^{1/\gamma}\frac{1}{\mu} \le \frac{\pi}{2}\frac{1}{\overline{R}^4},$$

using (34). The conditions (a.1) (a.2) (a.3) are direct consequences of the condition (a.8).

We note that, for the equilibrium,

$$\frac{M}{4\pi}\overline{R}^{(-2\gamma+1)/(\gamma-1)}(1-x) \le \overline{p} \le \frac{M}{4\pi}\overline{R}^{(-3\gamma+4)/(\gamma-1)}(1-x) \quad \text{if} \quad \gamma \le \frac{3}{2}$$

or

$$\frac{M}{4\pi}\overline{R}^{-4}(1-x) \le \overline{p} \le \frac{M}{4\pi}\overline{R}^{-1}(1-x) \quad \text{if} \quad \gamma \ge \frac{3}{2}.$$

Note that $\overline{R} \to 1$ as $\frac{a}{M} \to 0$. Therefore p_0 near $\overline{p}$ satisfies (a.8) for sufficiently small a. Thus we have proved the following:

THEOREM. *Under the assumptions* (A.0) (A.2) (A.3) (A.4) (A.5), *suppose that a is so small that*

$$La < M, \quad \textit{where} \quad L = L(\gamma, \nu, E^*, M^*, \overline{R}),$$

provided that $E_0 \le E^*$ and $M \le M^*$, and that the initial pressure p_0 satisfies

$$La(1-x) \le p_0 \le M(1-x).$$

Then the global solution (ρ, u) *satisfies*

$$\int_0^1 u(x,t)^2 dx \longrightarrow 0,$$
$$\int_0^1 \rho_0(x)\left(\frac{1}{\rho(x,t)} - \frac{1}{\overline{\rho}(x)}\right)^2 \frac{dx}{(1-x)^{3/4}} \longrightarrow 0 \quad \textit{as} \quad t \to +\infty.$$

As a corollary, $\rho(x,t) \to \overline{\rho}(x)$ a.e. x as $t \to +\infty$, and moreover, since $\rho \leq C$, we know

$$\int_0^1 |\rho(x,t) - \overline{\rho}(x)|^q dx \longrightarrow 0, \quad (1 \leq q < +\infty)$$

as $t \to +\infty$, for example. On the other hand, since $\frac{1}{\rho} \leq \frac{B}{\overline{\rho}}$, we know

$$\int_0^1 \left|\frac{1}{\rho(x,t)} - \frac{1}{\overline{\rho}(x)}\right| dx \longrightarrow 0$$

and

$$R(t) = r(1,t) \longrightarrow \overline{r}$$

as $t \to +\infty$.

REMARK 1. We conjecture that the assumption $\gamma > \frac{4}{3}$ can be removed in the present case in which the effect of the self-gravitation is neglected. However if we take into account the effect of self-gravitation, namely if we replace M by

$$M + 4\pi \int_1^r \rho r^2 dr = M + x,$$

then the number of the equilibria is one when $\gamma \geq \frac{4}{3}$ and more than two when $\gamma < \frac{4}{3}$ (see W.-Ch. Kuan and S.-S. Lin, [1]). Therefore the assumption that $\gamma > \frac{4}{3}$ may be essential for the case with self-gravitation. We note that our proof can work with a slight modification for this case with self-gravitation.

REMARK 2. We have not yet been able to remove the assumption that a is sufficiently small, although this is a serious restriction of our result.

REMARK 3. It is difficult to describe our conclusion in terms of the Eulerian coordinates, since in the Eulerian coordinates the support of $\rho(\cdot,t)$, $[1, R(t)]$, which corresponds to the fixed interval $[0,1]$ in the Lagrangean coordinates, varies with time t.

References

[1] W.-C. Kuan and S.-S. Lin, Numbers of equilibria of self-gravitating isentropic gas surrounding a solid ball, preprint.

[2] M. Okada and T. Makino, Free boundary problem for the equation of spherically symmetric motion of viscous gas. Japan J. Indust. Appl. Math., **10** (1993), 219–235.

[3] Š. Matušů-Nečasová, M. Okada and T. Makino, Free boundary problem for the equation of spherically symmetric motion of viscous gas (II). Japan J. Indust. Appl. Math., **12** (1995), 195–203.

[4] I. Straškraba, Asymptotic development of vacuums for 1-D Navier-Stokes equations of compressible flow. Preprint (Matematicky ustav, **90** (1994), Akademie red ceske republiky).

Nonlinear Analysis 72 (2010) 3258–3274

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Global existence of solutions for the one-dimensional motions of a compressible viscous gas with radiation: An "infrarelativistic model"

Bernard Ducomet[a], Šárka Nečasová[b,*]

[a] CEA, DAM, DIF, F-91297 Arpajon, France

[b] Mathematical Institute AS ČR Žitna 25, 115 67 Praha 1, Czech Republic

ARTICLE INFO

Article history:
Received 23 October 2009
Accepted 2 December 2009

MSC:
35Q30
76N10

Keywords:
Compressible
Viscous
Heat-conducting fluids
One-dimensional symmetry
Radiative transfer

ABSTRACT

We consider an initial–boundary value problem for the equations of 1D motions of a compressible viscous heat-conducting gas coupled with radiation through a radiative transfer equation. Assuming suitable hypotheses on the transport coefficients, we prove that the problem admits a unique weak solution.

1. Introduction

The aim of radiation hydrodynamics is to include the effects of radiation into the hydrodynamical framework. When equilibrium holds between the matter and the radiation, a simple way to do that is to include local radiative terms into the state functions and the transport coefficients. One knows from quantum mechanics that radiation is described by its quanta, the photons, which are massless particles traveling at the speed c of light, characterized by their frequency ν, their energy $E = h\nu$ (where h is Planck's constant), and their momentum $\vec{p} = \frac{\hbar\nu}{c}\vec{\Omega}$, where $\vec{\Omega}$ is a unit vector. Statistical mechanics allows us to describe macroscopically an assembly of massless photons of energy E and momentum $\vec{p}$ by using a distribution function: the radiative intensity $I(r, t, \vec{\Omega}, \nu)$. Using this fundamental quantity, one can derive global quantities by integrating with respect to the angular and frequency variables: the spectral radiative energy density $E_R(r, t)$ per unit volume is then $E_R(r, t) := \frac{1}{c}\iint I(r, t, \vec{\Omega}, \nu)\, d\Omega\, d\nu$, and the spectral radiative flux $\vec{F}_R(r, t) = \iint \vec{\Omega}\, I(r, t, \vec{\Omega}, \nu)\, d\Omega\, d\nu$. If matter is in thermodynamic equilibrium at constant temperature T and if radiation is also in thermodynamic equilibrium with matter, its temperature is also T and statistical mechanics tells us that the distribution function for photons is given by the Bose–Einstein statistics with zero chemical potential.

In the absence of radiation, one knows that the complete hydrodynamical system is derived from the standard conservation laws of mass, momentum and energy by using Boltzmann's equation satisfied by the $f_m(r, \vec{v}, t)$ and the

* Corresponding author. Tel.: +420 266611264.
E-mail addresses: bernard.ducomet@cea.fr (B. Ducomet), matus@math.cas.cz (Š. Nečasová).

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.12.005

Chapman–Enskog expansion [1]. One gets then formally the compressible Navier–Stokes system

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\vec{u})=0,\\ \partial_t(\rho\vec{u})+\nabla\cdot(\rho\vec{u}\otimes\vec{u})=-\nabla\cdot\overset{\Rightarrow}{\Pi}+\vec{f},\\ \partial_t(\rho\varepsilon)+\nabla\cdot(\rho\varepsilon\vec{u})=-\nabla\vec{q}-\overset{\Rightarrow}{D}:\overset{\Rightarrow}{\Pi}+g,\end{cases}\tag{1}$$

where $\overset{\Rightarrow}{\Pi}=-p(\rho,T)\ \overset{\Rightarrow}{I}+\overset{\Rightarrow}{\pi}$ is the material stress tensor for a newtonian fluid with the viscous contribution $\overset{\Rightarrow}{\pi}=2\mu\ \overset{\Rightarrow}{D}+\lambda\nabla\cdot\vec{u}\ \overset{\Rightarrow}{I}$ with $3\lambda+2\mu\geqslant 0$ and $\mu>0$, and the strain tensor $\overset{\Rightarrow}{D}$ such that $\overset{\Rightarrow}{D}_{ij}(\vec{u})=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$. $\vec{q}$ is the thermal heat flux and $\vec{F}$ and g are external force and energy source terms.

When radiation is present, the terms $\vec{f}$ and g include the terms for the coupling between the matter and the radiation (neglecting any other external field), depending on I, and I is driven by a transport equation: the so called radiative transfer integro-differential equation discussed by Chandrasekhar in [2].

Supposing that the matter is at LTE, the coupled system reads

$$\begin{cases}\partial_t\rho+\nabla\cdot(\rho\vec{u})=0,\\ \partial_t(\rho\vec{u})+\nabla\cdot(\rho\vec{u}\otimes\vec{u})=-\nabla\cdot\overset{\Rightarrow}{\Pi}-\vec{S}_F,\\ \partial_t(\rho\varepsilon)+\nabla\cdot(\rho\varepsilon\vec{u})=-\nabla\vec{q}-\overset{\Rightarrow}{D}:\overset{\Rightarrow}{\Pi}-S_E,\\ \dfrac{1}{c}\dfrac{\partial}{\partial t}I\left(r,t,\vec{\Omega},\nu\right)+\vec{\Omega}\cdot\nabla I\left(r,t,\vec{\Omega},\nu\right)=S_t\left(r,t,\vec{\Omega},\nu\right),\end{cases}\tag{2}$$

where the coupling terms are

$$S_t(r,t,\vec{\Omega},\nu)=\sigma_a\left(\nu,\vec{\Omega},\rho,T,\frac{\vec{\Omega}\cdot\vec{u}}{c}\right)\left[B(\nu,T)-I\left(r,t,\vec{\Omega},\nu\right)\right]+\iint\sigma_s\left(r,t,\rho,\vec{\Omega}'\cdot\vec{\Omega},\nu'\to\nu\right)$$
$$\times\left\{\frac{\nu}{\nu'}I\left(r,t,\vec{\Omega}',\nu'\right)I\left(r,t,\vec{\Omega},\nu\right)-\sigma_s\left(r,t,\rho,\vec{\Omega}\cdot\vec{\Omega}',\nu\to\nu'\right)I\left(r,t,\vec{\Omega},\nu\right)I\left(r,t,\vec{\Omega}',\nu'\right)\right\}\mathrm{d}\Omega'\,\mathrm{d}\nu',$$

the radiative energy source

$$S_E(r,t):=\iint S_t(r,t,\vec{\Omega},\nu)\,\mathrm{d}\Omega\,\mathrm{d}\nu,$$

the radiative flux

$$\vec{S}_F(r,t):=\frac{1}{c}\iint\vec{\Omega}\,S_t(r,t,\vec{\Omega},\nu)\,\mathrm{d}\Omega\,\mathrm{d}\nu.$$

In the radiative transfer equation (the last Eq. (2)) the functions σ_a and σ_s describe in a phenomenological way the absorption–emission and scattering properties of the photon–matter interaction, and Planck's function $B(\nu,\theta)$ describes the frequency–temperature black body distribution.

Let us note that the foundations for the previous system were described by Pomraning [3] and Mihalas and Weibel-Mihalas [4] in the full framework of special relativity (oversimplified in the previous considerations). The coupled system (2) has been investigated recently (in the inviscid case) by Lowrie, Morel and Hittinger [5], Buet and Després [6] with special attention paid to asymptotic regimes, and by Dubroca and Feugeas [7], Lin [8] and Lin, Coulombel and Goudon [9] as regards numerical aspects. As regards the existence of solutions, a proof of local-in-time existence and blow-up of solutions (in the inviscid case) has been proposed by Zhong and Jiang [10] (see also the recent papers by Jiang and Wang [11,12] for a 1D related "Euler–Boltzmann" model). Moreover, a simplified version of the system has been investigated by Golse and Perthame [13].

As the multidimensional viscous situation is far from been understood even at the formal level (however see [14] for a macroscopic treatment of radiation in the astrophysical context, and [15] for the associated mathematical treatment), we restrict the following to the monodimensional case.

In 1D the previous system reads

$$\begin{cases}\rho_\tau+(\rho v)_y=0,\\ (\rho v)_\tau+(\rho v^2)_y+p_y=\mu v_{yy}-(S_F)_R,\\ \left[\rho\left(e+\dfrac{1}{2}v^2\right)\right]_\tau+\left[\rho v\left(e+\dfrac{1}{2}v^2\right)+pv-\kappa\theta_y-\mu v v_y\right]_y=-(S_E)_R,\\ \dfrac{1}{c}I_t+\omega I_y=S.\end{cases}\tag{3}$$

In this preliminary study we only consider an "infrarelativistic" model of a compressible Navier–Stokes system for a 1D flow coupled to a the radiative transfer equation. As in the model studied by Amosov [16], we suppose that the fluid motion is so small with respect to the velocity of light c that one can drop all the $\frac{1}{c}$ factors in the previous formulation. We get then

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + (\rho v^2)_y + p_y = \mu v_{yy}, \\ \left[\rho\left(e + \frac{1}{2}\, v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}\, v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \\ \omega I_y = S, \end{cases} \tag{4}$$

in the domain $\mathcal{O} \times \mathbb{R}_+$ with $\mathcal{O} := (0, L)$, where the density ρ, the velocity v, the temperature θ depend on the coordinates (y, τ). The radiative intensity $I = I(y, \tau, \nu, \omega)$, depends also on two extra variables: the radiation frequency $\nu \in \mathbb{R}_+$ and the angular variable $\omega \in S^1 := [-1, 1]$. The state functions are the pressure p, the internal energy e, the heat conductivity κ and the viscosity coefficient μ.

In the standard radiative transfer equation, the source term is

$$S(y, \tau, \nu, \omega) := S_{a,e}(y, \tau, \nu, \omega) + S_s(y, \tau, \nu, \omega), \tag{5}$$

where the absorption–emission term is

$$S_{a,e}(y, \tau, \nu, \omega) = \sigma_a(\nu, \omega; \rho, \theta)\left[B(\nu; \theta) - I(y, \tau, \nu, \omega)\right], \tag{6}$$

and the scattering term is

$$S_s(y, \tau, \nu, \omega) = \sigma_s(\nu; \rho, \theta)\left[\tilde{I}(y, \tau, \nu, \theta) - I(y, \tau, \nu, \omega)\right], \tag{7}$$

where $\tilde{I}(y, \tau, \nu) := \frac{1}{2}\int_{-1}^{1} I(y, \tau, \nu, \omega)\, d\omega$ and B is a function of temperature and frequency describing the equilibrium state. We suppose that $\sigma_a(\nu, \omega; \rho, \theta)$ and $\sigma_s(\nu; \rho, \theta)$ are positive functions. We also define the radiative energy

$$E_R := \int_{-1}^{1}\int_0^\infty I(y, \tau, \nu, \omega)\, d\nu\, d\omega, \tag{8}$$

the radiative flux

$$F_R := \int_{-1}^{1}\int_0^\infty \omega I(y, \tau, \nu, \omega)\, d\nu\, d\omega, \tag{9}$$

and the radiative energy source

$$(S_E)_R := \int_{-1}^{1}\int_0^\infty S(y, \tau, \nu, \omega)\, d\nu\, d\omega. \tag{10}$$

It is convenient to switch now to Lagrange (mass) coordinates relative to matter flow: $(y, \tau) \to (x, t)$. With the transformation rules [17]: $\partial_y \to \rho\partial_x$ and $\partial_\tau + v\partial_y \to \partial_x$, the previous system reads now

$$\begin{cases} \eta_t = v_x, \\ v_t = \sigma_x, \\ \left(e + \frac{1}{2}\, v^2\right)_t = (\sigma v - q)_x - \eta(S_E)_R, \\ \omega I_x = \eta S, \end{cases} \tag{11}$$

in the transformed domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0, M)$ (M is the total mass of matter), where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v, the temperature θ and the radiative intensity I depend now on the lagrangian mass coordinates (x, t). We also denote by $\sigma := -p + \mu\frac{v_x}{\eta}$ the stress and by $q := -\kappa\frac{\theta_x}{\eta}$ the heat flux, and the source term in the last equation is

$$S(x, t, \nu, \omega) = \sigma_a(\nu, \omega; \eta, \theta)\left[B(\nu; \theta) - I(x, t; \nu, \omega)\right] + \sigma_s(\nu; \eta, \theta)\left[\tilde{I}(x, t, \nu) - I(x, t, \nu, \omega)\right]. \tag{12}$$

We consider Dirichlet–Neumann boundary conditions for the fluid unknowns

$$\begin{cases} v|_{x=0} = v|_{x=M} = 0, \\ q|_{x=0} = q|_{x=M} = 0, \end{cases} \tag{13}$$

and transparent boundary conditions for the radiative intensity

$$\begin{cases} I|_{x=0} = 0 \quad \text{for } \omega \in (0, 1) \\ I|_{x=M} = 0 \quad \text{for } \omega \in (-1, 0), \end{cases} \tag{14}$$

for $t > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \qquad v|_{t=0} = v^0(x), \qquad \theta|_{t=0} = \theta^0(x), \quad \text{on } \Omega \tag{15}$$

and

$$I|_{t=0} = I(x, \nu, \omega) \quad \text{on } \Omega \times \mathbb{R}_+ \times S^1. \tag{16}$$

Pressure and energy are related by the thermodynamical relation

$$e_\eta(\eta, \theta) = -p(\eta, \theta) + \theta p_\theta(\eta, \theta). \tag{17}$$

Finally we assume that state functions e, p and κ (resp. $\sigma_{a,e}$ and σ_s) are C^2 (resp. C^0) functions of their arguments for $0 < \eta < \infty$ and $0 \leqslant \theta < \infty$, and we suppose the following growth conditions:

$$\begin{cases} e(\eta, 0) \geqslant 0, \qquad c_1(1+\theta^r) \leqslant e_\theta(\eta, \theta) \leqslant C_1(1+\theta^r), \\ -c_2\eta^{-2}(1+\theta^{1+r}) \leqslant p_\eta(\eta, \theta) \leqslant -C_2\eta^{-2}(1+\theta^{1+r}), \\ |p_\theta(\eta, \theta)| \leqslant C_3\eta^{-1}(1+\theta^r), \\ c_4(1+\theta^{1+r}) \leqslant \eta p(\eta, \theta) \leqslant C_4(1+\theta^{1+r}), \qquad p_\eta(\eta, \theta_0) \leqslant 0, \\ 0 \leqslant p(\eta, \theta) \leqslant C_5(1+\theta^{1+r}), \\ c_6(1+\theta^q) \leqslant \kappa(\eta, \theta) \leqslant C_6(1+\theta^q), \\ |\kappa_\eta(\eta, \theta)| + |\kappa_{\eta\eta}(\eta, \theta)| \leqslant C_7(1+\theta^q), \\ \eta\sigma_a(\nu, \omega; \eta, \theta)B^m(\nu; \theta) \leqslant C_8|\omega|\theta^{\alpha+1}f(\nu, \omega) \quad \text{for } m = 1, 2, \\ \sigma_a(\nu, \omega; \eta, \theta) \leqslant C_9 g(\nu, \omega), \\ \left[\left|(\sigma_a)_\eta\right| + |(\sigma_a)_\theta|\right](\nu, \omega; \eta, \theta)\,[1 + B(\nu; \theta) + |B_\theta(\nu; \theta)|] \leqslant C_{10}h(\nu, \omega), \\ \sigma_s(\nu; \eta, \theta) \leqslant C_{11}k(\nu, \omega), \end{cases} \tag{18}$$

where $r \in [0, 1]$, $q \geqslant 2r + 1$, $0 \leqslant \alpha < r$, the numbers c_j, C_j, $j = 1, \ldots, 10$, are positive constants and the functions f, g, h, k are such that

$$f, g, h \in L^1(\mathbb{R}_+ \times S^1) \cap L^\infty(\mathbb{R}_+ \times S^1),$$

and

$$k \in L^{1+\gamma}(\mathbb{R}_+ \times S^1) \cap L^\infty(\mathbb{R}_+ \times S^1),$$

for an arbitrary small $\gamma > 0$.

We suppose also that the viscosity coefficient is a positive constant.

We study weak solutions for the above problem with properties

$$\begin{cases} \eta \in L^\infty(Q_T), \qquad \eta_t \in L^\infty([0, T], L^2(\Omega)), \\ v \in L^\infty([0, T], L^4(\Omega)), \qquad v_t \in L^\infty([0, T], L^2(\Omega)), \qquad v_x \in L^\infty([0, T], L^2(\Omega)), \\ \sigma_x \in L^\infty([0, T], L^2(\Omega)), \\ \theta \in L^\infty([0, T], L^2(\Omega)), \qquad \theta_x \in L^\infty([0, T], L^2(\Omega)), \\ I \in L^1(\Omega \times \mathbb{R}_+ \times S^1) \end{cases} \tag{19}$$

where $Q_T := \Omega \times (0, T)$.

We also assume the following conditions on the data:

$$\begin{cases} \eta^0 > 0 \quad \text{on } \Omega, \quad \eta^0 \in L^1(\Omega), \\ v_0 \in L^2(\Omega), \qquad v^0_x \in L^2(\Omega), \\ \theta^0 \in L^2(\Omega), \qquad \inf_\Omega \theta^0 \geqslant 0. \end{cases} \tag{20}$$

Then our definition of a weak solution for the previous problem is the following:

Definition 1.1. We call (η, v, θ, I) a weak solution of (11) if it satisfies

$$\eta(x, t) = \eta^0(x) + \int_0^t v_x \, ds, \tag{21}$$

for a.e. $x \in \Omega$ and any $t > 0$, and if, for any test function $\phi \in L^2([0, T], H^1(\Omega))$ with $\phi_t \in L^1([0, T], L^2(\Omega))$ such that $\phi(\cdot, T) = 0$, one has

$$\int_Q \left[\phi_t v + \phi_x p - \frac{\mu\phi_x}{\eta} v_x\right] dx\, dt = \int_\Omega \phi(0, x)\, v^0(x)\, dx, \tag{22}$$

$$\int_Q \left[\phi_t \left(e + \frac{1}{2}\, v^2\right) + \phi_x\,(\sigma v - q) + \phi\eta(S_E)_R\right] dx\, dt = \int_\Omega \phi(0, x) \left(e^0(x) + \frac{1}{2}\, v^0(x)^2\right) dx, \tag{23}$$

and if, for any test function $\psi \in H^1(\Omega) \times L^1(\mathbb{R}_+ \times S^1)$, one has

$$\int_{\mathbb{R}_+ \times S^1} [\psi_x \omega I + \psi \eta S] \, d\nu \, d\omega \, dx = 0. \tag{24}$$

In the following we use the following notation for the integrated radiative intensity:

$$\mathcal{I}(x,t) := \int_0^\infty \int_{S^1} I(x,t;\omega,\nu) \, d\omega \, d\nu.$$

Then our main result is the following:

Theorem 1. *Suppose that the initial data satisfy (20) and that T is an arbitrary positive number.*
Then the problem (11), (13) and (15) possesses a global weak solution satisfying (19) together with properties (21), (22) and (23).

Moreover one has the uniqueness result.

Theorem 2. *Suppose that the initial data satisfy (20) and that T is an arbitrary positive number.*
Then the problem (11), (13) and (15) possesses a global unique weak solution satisfying (19) together with properties (21), (22) and (23).

For that purpose, we first prove a classical existence result in the Hölder category. We denote by $C^\alpha(\Omega)$ the Banach space of real-valued functions on Ω which are uniformly Hölder continuous with exponent $\alpha \in \Omega$, and by $C^{\alpha,\alpha/2}(Q_T)$ the Banach space of real-valued functions on $Q_T := \Omega \times (0,T)$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t. The norms of $C^\alpha(\Omega)$ (resp. $C^{\alpha,\alpha/2}(Q_T)$) will be denoted by $\|\cdot\|_\alpha$ (resp. $|||\cdot|||_\alpha$).

Theorem 3. *Suppose that the initial data satisfy*

$$\left(\eta^0, \eta^0_x, v^0, v^0_x, v^0_{xx}, \theta^0, \theta^0_x, \theta^0_{xx}\right) \in (C^\alpha(\Omega))^8,$$

for some $\alpha \in \Omega$. Suppose also that $\eta^0(x) > 0$, $\theta^0(x) > 0$ and $I^0(x) > 0$ on Ω, and that the compatibility conditions

$$\theta^0_x(0) = \theta^0_x(M) = 0,$$

hold. Then, there exists a unique solution $(\eta(x,t), v(x,t), \theta(x,t), \mathcal{I}(x,t))$ with $\eta(x,t) > 0$, $\mathcal{I}(x,t) > 0$ and $\theta(x,t) > 0$ to the initial–boundary value problem (11), (13)–(16) on $Q = \Omega \times \mathbb{R}_+$ such that for any $T > 0$

$$(\eta, \eta_x, \eta_t, \eta_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}, \mathcal{I}, \mathcal{I}_x) \in (C^\alpha(Q_T))^{14},$$

and

$$(\eta_{tt}, v_{xt}, \theta_{xt}) \in \left(L^2(Q_T)\right)^3.$$

Then Theorem 1 will be obtained from Theorem 3 through a regularization process.
Let us recall that the investigation of 1D viscous flows for compressible media goes back to the pioneer work of Antontsev, Kazhikhov and Monakhov [17] (see also [18–20] for more recent presentations). The strategy that we use to prove these results consists in an adaptation to the radiative case of the ideas of Dafermos and Hsiao [21], Kawohl [22] and Jiang [23].

2. *A priori* estimates

We first suppose that the solution is classical in the following sense:

$$\begin{cases} \eta \in C^1(Q_T), \quad \rho > 0, \\ v, \theta \in C^1([0,T], C^0(\Omega)) \cap C^0([0,T], C^2(\Omega)), \\ I \in C^1(Q_T, C^0(\mathbb{R}_+ \times S^1)). \end{cases} \tag{25}$$

We first prove the following regularity result:

Theorem 4. *Suppose that the initial–boundary value problem (11), (13) and (15) has a classical solution described by Theorem 3. Then the solution $(\eta, v, v_x, \theta, \theta_x, I)$ is bounded in the Hölder space $C^{1/3,1/6}(Q_T)$ such that*

$$|||\eta|||_{1/3} + |||v|||_{1/3} + |||v_x|||_{1/3} + |||\theta|||_{1/3} + |||\theta_x|||_{1/3} \leqslant C,$$

and

$$|||\mathcal{I}|||_{1/3} \leqslant C,$$

where C depends on T, the physical data of the problem and the initial data. Moreover

$$0 < \underline{\eta} \leqslant \eta \leqslant \overline{\eta}, \qquad 0 < \underline{\theta} \leqslant \theta \leqslant \overline{\theta}.$$

Let T be an arbitrary positive number, and let us denote by C various positive constants which may possibly depend on T and the physical constants of the problem.

Lemma 1. *Under the following condition on the data:*

$$\|v^0\|_{L^2(\Omega)} + \|\eta^0\|_{L^1(\Omega)} + \|\theta^0\|_{L^1(\Omega)} \leqslant N, \tag{26}$$

there exists a constant $K(N)$ such that:

1. *the mass conservation*

$$\int_\Omega \eta \,\mathrm{d}x = \int_\Omega \eta^0 \,\mathrm{d}x, \tag{27}$$

2. *the energy equality*

$$\int_\Omega \left[e + \frac{1}{2} v^2\right] \mathrm{d}x = \int_\Omega \left[\mathrm{e}^0 + \frac{1}{2}(v^0)^2\right] \mathrm{d}x, \tag{28}$$

3. *the entropy inequality*

$$\frac{1}{4} \int_\Omega (\theta + \theta^{1+r}) \,\mathrm{d}x + \int_{Q_T} \left(\frac{\kappa(\eta,\theta)}{\eta\theta^2}\, \theta_x^2 + \frac{\mu}{\eta\theta}\, v_x^2\right) \mathrm{d}x\,\mathrm{d}s \leqslant C, \tag{29}$$

4. *the estimate*

$$\|\eta\|_{L^\infty(0,T;L^1(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C \tag{30}$$

hold.

Proof. 1. Integrating the first Eq. (11) and using boundary conditions gives (27).

2. From the radiative transfer equation, integrating over frequencies and angular momentum,

$$(F_R)_x = \eta(S_E)_R.$$

Plugging in the third Eq. (11), we get

$$\left(e + \frac{1}{2}\, v^2\right)_t = (\sigma v - q - F_R)_x\,, \tag{31}$$

which gives (28) after integrating over Ω and using boundary conditions.

3. Entropy $s = s_m + s_R$ is the sum of the entropy of matter s_m and entropy of radiation s_R. As in our simplified model $s_R = 0$, from the second principle of thermodynamics, one has that $\theta s_t = e_t + p\eta_t$, with $s_\eta = \frac{1}{\theta}\left(e_\eta - p\right)$ and $s_\theta = \frac{e_\theta}{\theta}$. From the third Eq. (11), we get

$$e_t = \sigma v_x - q_x - (F_R)_x\,. \tag{32}$$

Using (11), one finds

$$(s)_t + \left(\frac{\kappa\theta_x}{\eta\theta}\right)_x = \frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2} - \frac{\eta}{\theta}\,(S_E)_R. \tag{33}$$

We use the technique of [23] and define the free energy $\psi := e - \theta s$ of the fluid, with $\psi_\theta = -s$ and $\psi_\eta = -p$. Let us consider the auxiliary function

$$\mathcal{E}(\eta,\theta) := \psi(\eta,\theta) - \psi(1,\theta_0) - (\eta-1)\psi_\eta(1,\theta_0) - (\theta-\theta_0)\psi_\theta(\eta,\theta).$$

A direct computation gives

$$\left(\mathcal{E} + \frac{1}{2}\, v^2\right)_t + \theta_0\left(\frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}\right) = (\sigma v)_x + p(1,\theta_0)v_x + \left[\left(1 - \frac{\theta_0}{\theta}\right)\frac{\kappa\theta_x}{\eta\theta}\right]_x - \left(1 - \frac{\theta_0}{\theta}\right)\eta(S_E)_R. \tag{34}$$

Using the last Eq. (11), the last factor in (34) can be rewritten as

$$\begin{aligned} -\left(1 - \frac{\theta_0}{\theta}\right)\eta(S_E)_R \;=\; & -\eta(S_E)_R + \theta_0\frac{\eta}{\theta}\int_0^\infty\int_{S^1}\sigma_a(B - I)\,\mathrm{d}\omega\,\mathrm{d}\nu - \left[\int_0^\infty\int_{S^1}\omega I\,\mathrm{d}\omega\,\mathrm{d}\nu\right]_x \\ & + \theta_0\frac{\eta}{\theta}\int_0^\infty\int_{S^1}\sigma_a B\,\mathrm{d}\omega\,\mathrm{d}\nu - \theta_0\frac{\eta}{\theta}\,\eta\int_0^\infty\int_{S^1}\sigma_a I\,\mathrm{d}\omega\,\mathrm{d}\nu. \end{aligned}$$

The second term can be controlled by using (18):

$$\eta\frac{\theta_0}{\theta}\int_0^\infty\int_{S^1}\sigma_a B_0\,\mathrm{d}\omega\,\mathrm{d}\nu \leqslant C(1+\theta^r)\leqslant Ce(\eta,\theta).$$

Plugging all of these estimates into (34), we get finally

$$\left(\mathcal{E}+\frac{1}{2}v^2\right)_t+\theta_0\left(\frac{\mu v_x^2}{\eta\theta}+\frac{\kappa\theta_x^2}{\eta\theta^2}\right)+\frac{\eta}{\theta}\theta_0\int_0^\infty\int_{S^1}\sigma_a I\,\mathrm{d}\omega\,\mathrm{d}\nu$$
$$\leqslant\left[\sigma v+p(1,\theta_0)v+\left(1-\frac{\theta_0}{\theta}\right)\frac{\kappa\theta_x}{\eta\theta}\right]_x-\left[\int_0^\infty\int_{S^1}\omega I\,\mathrm{d}\omega\,\mathrm{d}\nu\right]_x+Ce(\eta,\theta).$$

Integrating over Q_t and using (28) and (13), the contribution of the first boundary term is zero.

Using (14), the contribution of the radiative term reads

$$\int_\Omega\left[\int_0^\infty\int_{S^1}\omega I\,\mathrm{d}\omega\,\mathrm{d}\nu\right]_x\mathrm{d}x=\int_0^\infty\int_0^1\omega I(M,t;\nu,\omega)\,\mathrm{d}\omega\,\mathrm{d}\nu-\int_0^\infty\int_{-1}^0\omega I(0,t;\nu,\omega)\,\mathrm{d}\omega\,\mathrm{d}\nu\geqslant 0,$$

and finally, we obtain

$$\int_\Omega\mathcal{E}\,\mathrm{d}x+\theta_0\int_{Q_t}\left(\frac{\mu v_x^2}{\eta\theta}+\frac{\kappa\theta_x^2}{\eta\theta^2}\right)\mathrm{d}x\,\mathrm{d}s+\theta_0\int_{Q_t}\frac{\eta}{\theta}\int_0^\infty\int_{S^1}\sigma_a I\,\mathrm{d}\omega\,\mathrm{d}\nu\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^t\int_0^\infty\int_0^1\omega I(M,t;\nu,\omega)\,\mathrm{d}\omega\,\mathrm{d}\nu\,\mathrm{d}s-\int_0^t\int_0^\infty\int_{-1}^0\omega I(0,t;\nu,\omega)\,\mathrm{d}\omega\,\mathrm{d}\nu\,\mathrm{d}s\leqslant C. \tag{35}$$

Now we argue in the same way as [23] noting that, by using Taylor's formula, for any $\eta>0$

$$\mathcal{E}(\eta,\theta)-\psi(\eta,\theta)+\psi(\eta,\theta_0)+(\theta-\theta_0)\psi_\theta(\eta,\theta)=\psi(\eta,\theta_0)-\psi(1,\theta_0)-(\eta-1)\psi_\eta(1,\theta_0)\geqslant 0,$$

and

$$\psi(\eta,\theta)-\psi(\eta,\theta_0)-(\theta-\theta_0)\psi_\theta(\eta,\theta)=-(\theta-\theta_0)^2\int_0^1(1-\alpha)\psi_{\theta\theta}(\eta,\theta+\alpha(\theta_0-\theta))\,\mathrm{d}\alpha.$$

Using $\psi_{\theta\theta}=\theta^{-1}e_\theta$ and estimates (18), we find

$$\psi(\eta,\theta)-\psi(\eta,\theta_0)-(\theta-\theta_0)\psi_\theta(\eta,\theta)\geqslant\frac{1}{4}C_1\left(\theta+\theta^{1+r}\right)-C.$$

So we deduce that

$$\mathcal{E}(\eta,\theta)\geqslant\frac{1}{4}C_1\left(\theta+\theta^{1+r}\right)-C,$$

and by plugging this into (35), we conclude that (29) holds.

3. The estimate (30) follows from (28). □

Using Lemma 1, we can quote verbatim all the results of [23] which only involve the first and second Eq. (11) and get first bounds for density.

Lemma 2. *Under the previous condition on the data (26), there exists positive constants $\underline{\eta}$ and $\overline{\eta}$ depending on T and N such that*

$$\underline{\eta}\leqslant\eta(x,t)\leqslant\overline{\eta}\quad\text{for }(t,x)\in Q_T. \tag{36}$$

As we will need bounds for the radiative intensity, we give the simple result:

Lemma 3. 1. *The solution of the integro-differential equation*

$$\begin{cases}\omega\dfrac{\partial}{\partial x}I(x;\nu,\omega)=\eta\sigma_a(\nu,\omega,\eta,\theta)\,[B(\nu,\theta)-I(x;\nu,\omega)]+\eta\sigma_s(\nu,\eta,\theta)\left[\tilde{I}(x;\nu)-I(x;\nu,\omega)\right]\\ \text{on }\Omega\times\mathbb{R}_+\times S^1,\\ I(0;\nu,\omega)=0\quad\text{for }\omega\in(0,1),\\ I(M;\nu,\omega)=0\quad\text{for }\omega\in(-1,0),\end{cases} \tag{37}$$

is given by the (implicit) formula

$$I(x;\nu,\omega)=\begin{vmatrix} \int_0^x e^{-\int_x^y \frac{\eta(\sigma_a+\sigma_s)}{\omega}\,dz}\,\frac{\eta}{\omega}\left(\sigma_a B+\sigma_s\tilde I\right)dy \quad \text{for } \omega\in(0,1),\\ -\int_x^M e^{\int_x^y \frac{\eta(\sigma_a+\sigma_s)}{\omega}\,dz}\,\frac{\eta}{\omega}\left(\sigma_a B+\sigma_s\tilde I\right)dy \quad \text{for } \omega\in(-1,0),\end{vmatrix} \tag{38}$$

for any $x\in\Omega$ and any $\nu\in\mathbb{R}_+$.

2. *The following inequalities hold:*

$$\begin{cases}\int_\Omega\int_0^\infty\int_{S^1}(\sigma_a+\sigma_s)I^2\,dx\,d\omega\leqslant C\left(1+\max_{Q_T}\theta^{\alpha+1}\right),\\ \int_\Omega\int_0^\infty\int_{S^1}\sigma_s\left(\tilde I-I\right)^2\,dx\,d\omega\,d\nu\leqslant C\left(1+\max_{Q_T}\theta^{\alpha+1}\right).\end{cases} \tag{39}$$

3. *The following bounds hold:*

$$\int_0^\infty\int_{S^1}I(x;\nu,\omega)\,d\omega\,d\nu\leqslant C\left(1+\max_{Q_T}\theta^{\alpha+1}\right). \tag{40}$$

Proof. 1. Identity (38) is straightforward after solving explicitly the ordinary differential equation and using boundary conditions. □

2. Multiplying (37) by I, integrating over $\Omega\times S^1$ and using boundary conditions, we get

$$\frac12\int_{S^1}\omega I^2(M;\nu,\omega)\,d\omega-\frac12\int_{S^1}\omega I^2(0;\nu,\omega)\,d\omega+\int_\Omega\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,dx\,d\omega$$
$$+\int_\Omega\int_{S^1}\eta\sigma_s\left(\tilde I-I\right)^2\,dx\,d\omega=\int_\Omega\int_{S^1}\eta\sigma_a BI\,dx\,d\omega.$$

Integrating over frequency, using (18) and estimating the right-hand side by the Cauchy–Schwarz inequality, we have

$$\frac12\int_0^\infty\int_{S^1}\omega I^2(M;\nu,\omega)\,d\omega\,d\nu-\frac12\int_0^\infty\int_{S^1}\omega I^2(0;\nu,\omega)\,d\omega\,d\nu$$
$$+\int_0^\infty\int_\Omega\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,dx\,d\omega+\int_0^\infty\int_\Omega\int_{S^1}\eta\sigma_s\left(\tilde I-I\right)^2\,dx\,d\omega\,d\nu$$
$$\leqslant\frac12\int_0^\infty\int_\Omega\int_{S^1}\eta(\sigma_a+\sigma_s)I^2\,dx\,d\omega+\frac12\int_0^\infty\int_\Omega\int_{S^1}\eta\frac{\sigma_a^2}{\sigma_a+\sigma_s}B^2\,dx\,d\omega,$$

and as, using (18), the last integral is bounded by

$$C_8\int_\Omega\theta^{\alpha+1}\int_0^\infty\int_{S^1}g\,dx\,d\omega\,d\nu,$$

we get (39).

3. From (38) we see that

$$\int_0^\infty\int_{S^1}I(x;\nu,\omega)\,d\omega\,d\nu\leqslant 2\int_\Omega\int_0^\infty\int_0^1\frac{\eta}{\omega}\sigma_a B\,dy\,d\omega\,d\nu+2\int_\Omega\eta\int_0^\infty\sigma_s\tilde I\left(\int_0^1\frac{e^{-\frac{\sigma_s}{\omega}\int_y^x\eta(z)\,dz}}{\omega}\,d\omega\right)dy\,d\nu.$$

After (18), the first integral in the right-hand side is bounded by $C(1+\theta^{\alpha+1})$. In the second, we introduce the function $E_1(s):=\int_c^\infty\frac{e^{-\tau}}{\tau}\,d\tau$, for $s>0$ and we get

$$\int_\Omega\eta\int_0^\infty\sigma_s\tilde I\left(\int_0^1\frac{e^{-\frac{\sigma_s}{\omega}\int_y^x\eta(z)\,dz}}{\omega}\,d\omega\right)dy\,d\nu=\int_\Omega\eta\int_0^\infty\sigma_s E_1\left(\sigma_s\left|\int_y^x\eta\,dz\right|\right)\tilde I(y,\nu)\,dy\,d\nu.$$

As $s\to E_1(s)$ is decreasing, we observe that

$$E_1\left(\sigma_s\left|\int_y^x\eta\,dz\right|\right)\leqslant E_1\left(\sigma_s(\nu)\underline{\eta}\,|x-y|\right).$$

Then using the Cauchy–Schwarz inequality,

$$\int_\Omega\eta\int_0^\infty\sigma_s\tilde I\left(\int_0^1\frac{e^{-\frac{\sigma_s}{\omega}\int_y^x\eta(z)\,dz}}{\omega}\,d\omega\right)dy\,d\nu\leqslant C\int_\Omega\int_0^\infty\sigma_s\tilde I^2\,dy\,d\nu+C\int_\Omega\int_0^\infty\sigma_s E_1^2\left(\sigma_s(\nu)\underline{\eta}\,|x-y|\right)dy\,d\nu.$$

As one knows (see [24, p. 229]) that $E_1(z) < e^{-x}\log(1+\frac{1}{x})$, one checks

$$E_1^2\left(\left(\sigma_s(\nu)\underline{\eta}\,|x-y|\right)\right) < \log^2\left(1+\sigma_s(\nu)\underline{\eta}\,|x-y|\right)+\log^2\left(\sigma_s(\nu)\underline{\eta}\,|x-y|\right),$$

and then we obtain

$$\int_\Omega\int_0^\infty \sigma_s(\nu)E_1^2\left(\sigma_s(\nu)\underline{\eta}\,|x-y|\right)\,dy\,d\nu \leqslant \int_\Omega\int_0^\infty \sigma_s\left[\log^2\left(1+\sigma_s(\nu)\underline{\eta}\,|x-y|\right)+\log^2\left(\sigma_s(\nu)\underline{\eta}\,|x-y|\right)\right]dy\,d\nu,$$
$$\leqslant \int_\Omega\int_0^\infty \sigma_s\left[\log^2(1+au)+\log^2(au)\right]du\,d\nu,$$

where $a=\sigma_s(\nu)\underline{\eta}$. An elementary computation shows that this integral is bounded by $C\int_0^\infty {\sigma_s}^{1+\gamma}\,d\nu$ for any positive γ, which is bounded after (18) and finally we get

$$\int_0^\infty\int_{S^1} I(x;\nu,\omega)\,d\omega\,d\nu \leqslant C(1+\theta^{\alpha+1}). \quad \square$$

We have now the following estimates (see [23] for the proof).

Lemma 4. 1.

$$\int_{Q_T} v_x^2\,dx\,dt \leqslant C. \tag{41}$$

2. *For any $\epsilon\in(0,1)$ if $r\in[0,1]$, and for $\epsilon=0$ if $r\in(0,1]$*

$$\int_{Q_T}\theta^{-r}v_x^2\,dx\,dt+\int_{Q_T}(1+\theta^q)\theta^{1-r-\epsilon}\theta_x^2\,dx\,dt \leqslant C. \tag{42}$$

3. *For any $\epsilon\in(0,1)$ if $r\in[0,1]$, and for $\epsilon=0$ if $r\in(0,1]$*

$$\int_{Q_T}\theta^{q+3+r-\epsilon}\,dx\,dt+\int_0^T\max_\Omega\theta^{q+2-\epsilon}\,dt \leqslant C. \tag{43}$$

4.

$$\max_{[0,T]}\int_\Omega \eta_x^2\,dx\,dt \leqslant C. \tag{44}$$

5.

$$\int_{Q_T} v_x^4\,dx\,dt \leqslant C\left(1+\max_\Omega\theta^{2(1+r+\epsilon)}\right). \tag{45}$$

Now we consider the two quantities

$$Y := \max_{[0,T]}\int_\Omega\left(1+\theta^{2q}\right)\theta_x^2\,dx,\qquad Z := \max_{[0,T]}\int_\Omega v_{xx}^2\,dx.$$

It is routine to show (see [21]) that

$$\max_{Q_T}\theta \leqslant C\left(1+Y^{\frac{1}{2q+3+r}}\right), \tag{46}$$

$$\max_{Q_T}\int_\Omega v_x^2\,dx\,dt \leqslant C\left(1+Z^{1/2}\right), \tag{47}$$

and

$$\max_{Q_T}|v_x| \leqslant C\left(1+Z^{3/8}\right). \tag{48}$$

Lemma 5. *Under the previous condition on the data (26), there exists a positive constant K depending on T and N such that*

$$Y \leqslant K\left(1+Z^{\frac{2q+1}{2q+2}}+Y^{\frac{2q+2\alpha-r+3}{2q+r+3}}\right), \tag{49}$$

and

$$X := \int_{Q_T}(1+\theta^{q+r})\theta_t^2\,dx\,dt \leqslant K\left(1+Z^{\frac{2q+1}{2q+2}}+Y^{\frac{2q+2\alpha-r+3}{2q+r+3}}\right). \tag{50}$$

Proof. From (11), the equation for the internal energy reads

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\, v_x^2 = \left(\frac{\kappa\theta_x}{\eta}\right)_x - \eta(S_E)_R.$$

Defining the auxiliary function $K(\eta,\theta) := \int_0^\theta \frac{\kappa(\eta,u)}{u}\,\mathrm{d}u$, multiplying the previous equation by K_t and integrating by parts, we get

$$\int_{Q_T}\left(e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\, v_x^2\right)K_t\,\mathrm{d}x\,\mathrm{d}s + \int_{Q_T}\left(\frac{\kappa\theta_x}{\eta}\right)K_{tx}\,\mathrm{d}x\,\mathrm{d}s - \int_{Q_T}\eta(S_E)_R K_t\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{51}$$

Observing that $K_t = K_\eta v_x + \frac{\kappa}{\eta}\,\theta_t$, $K_{xt} = \left(\frac{\kappa\theta_x}{\eta}\right)_t + K_{\eta\eta}v_x\eta_x + \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t$ and that after (18) $|K_\eta| + |K_{\eta\eta}| \leqslant C(1+\theta^{q+1})$, we can estimate all the contributions in (51).

After Lemma 4, the first two integrals lead to the same estimates as in [23] (see Lemma 2.8), and we have

$$\int_{Q_T} e_\theta\theta_t K_t\,\mathrm{d}x\,\mathrm{d}s \geqslant \frac{c_6c_1}{2\overline{\eta}}\,X - CZ^{3/4},$$

$$\left|\int_{Q_T}\left(\theta p_\theta v_x - \frac{\mu}{\eta}\, v_x^2\right)K_\eta v_x\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant a_1Y + C(1+Z^{3/4}),$$

$$\left|\int_{Q_T}\left(\theta p_\theta v_x - \frac{\mu}{\eta}\, v_x^2\right)\frac{\kappa}{\eta}\,\theta_t\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant b_1X + a_2Y + CZ^{3/4} + C,$$

$$\left|\int_{Q_T}\left(\frac{\kappa\theta_x}{\eta}\right)\left(\frac{\kappa\theta_x}{\eta}\right)_t\,\mathrm{d}x\,\mathrm{d}s\right| \geqslant \frac{c_6^2}{2\overline{\eta}^2}\,Y - C,$$

$$\left|\int_{Q_T}\left(\frac{\kappa\theta_x}{\eta}\right)K_\eta v_{xx}\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant a_3Y + CZ^{\frac{2q+1}{2q+2}} + C,$$

$$\left|\int_{Q_T}\left(\frac{\kappa\theta_x}{\eta}\right)K_{\eta\eta}v_x\eta_x\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant a_4Y + CZ^{\frac{3}{4}} + C,$$

$$\left|\int_{Q_T}\left(\frac{\kappa\theta_x}{\eta}\right)\left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant b_2X + a_5Y + CZ^{\frac{3}{4}} + C,$$

where a_j and b_j are positive numbers.

Let us estimate the last term in (51).

$$\left|\int_{Q_T}\eta(S_E)_R K_t\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant \int_{Q_T}\left(\int_0^\infty\int_{S^1}\eta\sigma_a(B+I)\,\mathrm{d}\nu\,\mathrm{d}\omega\right)|K_t|\,\mathrm{d}x\,\mathrm{d}s + \int_{Q_T}\left(\int_0^\infty\int_{S^1}\eta\sigma_s|\tilde{I}-I|\,\mathrm{d}\nu\,\mathrm{d}\omega\right)|K_t|\,\mathrm{d}x\,\mathrm{d}s =: P + Q.$$

After (18) and Lemma 3,

$$P \leqslant C\int_{Q_T}|K_t|(1+\theta^{\alpha+1})\,\mathrm{d}x\,\mathrm{d}s \leqslant C\int_{Q_T}\left(1+\theta^{q+\alpha+2}\right)|v_x|\,\mathrm{d}x\,\mathrm{d}s + C\int_{Q_T}\left(1+\theta^{q+\alpha+1}\right)|\theta_t|\,\mathrm{d}x\,\mathrm{d}s =: A + B.$$

Using Lemma 4 and the Cauchy–Schwarz inequality we have

$$A \leqslant C + CY^{\frac{2q+2\alpha-r+3}{2q+r+3}} + CZ^{3/4},$$

and

$$B \leqslant b_3X + CY^{\frac{2q+2\alpha-r+3}{2q+r+3}}.$$

Using (18), Lemmas 3, 4 and the Cauchy–Schwarz inequality we have in the same stroke

$$Q \leqslant C\int_{Q_T}\int_0^\infty\int_{S^1}\eta\sigma_s|\tilde{I}-I|^2\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s + \int_{Q_T}\int_0^\infty\int_{S^1}\eta\sigma_s K_t^2\,\mathrm{d}\nu\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s \leqslant C\left(1+\max_{Q_T}\theta^{\alpha+1}\right) + C\int_{Q_T}K_t^2\,\mathrm{d}x\,\mathrm{d}s,$$

and the last integral is bounded by

$$\leqslant \int_{Q_T} \left(1+\theta^{2(q+1)}\right) v_x^2 \,dx\,ds + C\int_{Q_T} \left(1+\theta^{2q}\right)\theta_t^2 \,dx\,ds =: C + D.$$

Exactly as for P, we get

$$C \leqslant C + CY^{\frac{2q-r+1}{2q+r+3}} + CZ^{3/4},$$

and

$$D \leqslant b_3 X + CY^{\frac{2q-r+1}{2q+r+3}},$$

so finally

$$\left|\int_{Q_T} \eta (S_E)_R K_t \,dx\,ds\right| \leqslant A + B + cY^{\frac{\alpha+1}{2q+r+3}} + C + D$$
$$\leqslant b_3 X + CY^{\frac{2q+2\alpha-r+3}{2q+r+3}} + CZ^{3/4},$$

where c is another positive constant.

Combining all the previous inequalities, choosing the numbers a_j such that $\sum_{j=1}^{5} a_j \leqslant \frac{c_6^2}{4\overline{\eta}^2}$ and the b_j such that $\sum_{j=1}^{3} b_j \leqslant \frac{c_6 c_1}{4\overline{\eta}}$, and observing that $\frac{2q+1}{2q+2} \geqslant \frac{3}{4}$, we get

$$\frac{c_6 c_1}{4\overline{\eta}} X + \frac{c_6^2}{4\overline{\eta}^2} Y \leqslant C + CZ^{\frac{2q+1}{2q+2}} + CY^{\frac{2q+2\alpha-r+3}{2q+r+3}}, \tag{52}$$

which ends the proof. □

Exactly as in [23] (see Lemma 2.9) one can prove now:

Lemma 6.

$$\max_{[0,T]} \int_\Omega v_t^2 \,dx + \int_{Q_T} v_{xt}^2 \,dx\,dt \leqslant C\left(1 + Z^{\frac{2q+1}{2q+2}}\right). \tag{53}$$

Proof. Differentiating the second Eq. (11) with respect to t, multiplying by v_t, integrating by parts, using boundary conditions together with Lemma 4, one gets (53) (see [23] for the details). □

Lemma 7. *All the quantities*

$$X,\ Y,\ \max_{Q_T} \theta, \tag{54}$$

are bounded.

Proof. From Lemma 5, it follows in particular that

$$\frac{Y}{1+Z^{\frac{2q+1}{2q+2}}} \leqslant C + C\left(\frac{Y}{1+Z^{\frac{2q+1}{2q+2}}}\right)^{\frac{2q+2\alpha-r+3}{2q+r+3}} \frac{1}{\left(1+Z^{\frac{2q+1}{2q+2}}\right)^{\frac{2(r-\alpha)}{2q+r+3}}},$$

and, as $r > \alpha$, we conclude that

$$Y \leqslant C + CZ^{\frac{2q+1}{2q+2}}. \tag{55}$$

Rewriting now the momentum equation as

$$v_{xx} = \frac{\eta}{\mu}\left[v_t + p_x - \left(\frac{\mu}{\eta}\right)_\eta \eta_x v_x\right],$$

we get, using Lemma 4,

$$
\begin{aligned}
Z &= \int_\Omega v_{xx}^2 \,\mathrm{d}x \\
&\leqslant C\int_\Omega v_t^2 \,\mathrm{d}x + C\int_\Omega (1+\theta^{2r+2})\eta_x^2 \,\mathrm{d}x + C\int_\Omega (1+\theta^{2r})\theta_x^2 \,\mathrm{d}x + C\int_\Omega \eta_x^2 v_x^2 \,\mathrm{d}x \\
&\leqslant C + CZ^{\frac{2q+1}{2q+2}} + \max_{Q_T}\theta^{2r+2} + CY + CZ^{3/4},
\end{aligned}
$$

and taking (55), we find

$$Z \leqslant C + CZ^{\frac{2q+1}{2q+2}}.$$

As $\frac{2q+1}{2q+2} < 1$, this implies that $Z \leqslant C$, and the bounds $X, Y, \max_{Q_T}\theta < C$ follow. □

Lemma 8. *All the quantities*

$$\max_{[0,T]}\int_\Omega v_{xx}^2 \,\mathrm{d}x,\ \max_{Q_T}|v_x|,\ \max_{[0,T]}\int_\Omega v_x^2 \,\mathrm{d}x,\ \int_{Q_T} v_x^4 \,\mathrm{d}x,\ \max_{[0,T]}\int_\Omega v_t^2 \,\mathrm{d}x,\ \int_{Q_T} v_{xt}^2 \,\mathrm{d}x\,\mathrm{d}t,$$

are bounded.

Proof. The first quantity is bounded after Lemma 7, the second one is bounded after (48), the third is bounded after (45) and the boundedness of the two last quantities follows after Lemma 6. □

Lemma 9. *Under the previous condition on the data there exist positive constants* $\overline{\theta}$ *and* $\underline{\theta}$ *depending on* T *and* N *such that*

$$0 < \underline{\theta} \leqslant \theta(x,t) \leqslant \overline{\theta} \quad \text{for } (t,x) \in Q_T. \tag{56}$$

Proof. Applying the maximum principle to the parabolic equation of the internal energy

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\, v_x^2 = \left(\frac{\kappa\theta_x}{\eta}\right)_x - \eta(S_E)_R,$$

and observing that the terms $\theta p_\theta v_x$ and $\eta(S_E)_R$ are bounded, we get (56). □

Lemma 10.

$$\left\| \int_0^\infty \int_{S^1} I \,\mathrm{d}\omega\,\mathrm{d}\nu \right\|_{L^\infty(Q_T)} \leqslant K(N), \tag{57}$$

$$\left\| \int_0^\infty \int_{S^1} |I_x| \,\mathrm{d}\omega\,\mathrm{d}\nu \right\|_{L^\infty(Q_T)} \leqslant K(N). \tag{58}$$

Proof. Estimate (57) follows after Lemmas 3 and 9.

After the last Eq. (11),

$$\int_0^\infty \int_{S^1} |I_x| \,\mathrm{d}\omega\,\mathrm{d}\nu \leqslant \int_0^\infty \int_{S^1} \left[\frac{\eta\sigma_a}{\omega} B + \frac{\eta(\sigma_a+\sigma_s)}{\omega} I + \frac{\eta\sigma_s}{\omega}\tilde{I} \right] \mathrm{d}\omega\,\mathrm{d}\nu.$$

It is now routine, revisiting part 3 of the proof of Lemma 3, that (58) follows after (57) and (18). □

Lemma 11. *All the quantities*

$$\int_{Q_T} \theta_t^2 \,\mathrm{d}x\,\mathrm{d}t,\ \max_{[0,T]}\int_\Omega \theta_x^2 \,\mathrm{d}x,\ \max_{[0,T]}\int_\Omega \theta_t^2 \,\mathrm{d}x,\ \max_{[0,T]}\int_\Omega \theta_{xx}^2 \,\mathrm{d}x,\ \int_{Q_T} \theta_{xt}^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{59}$$

are bounded.

Proof. 1. The first two quantities are bounded after Lemma 7.

2. In order to estimate the two remaining terms, by differentiating formally the internal energy equation with respect to t, multiplying by $e_\theta\theta_t$ and using integration by parts on Q_T, we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (e_\theta\theta_t)^2\,(x,t)\,dx - \frac{1}{2}\int_\Omega (e_\theta\theta_t)^2\,(x,0)\,dx + \int_{Q_T} p_\theta v_x e_\theta \theta_t^2\,dx\,dt \\
&\quad + \int_{Q_T} \theta p_{\theta\theta} v_x e_\theta\theta_t^2\,dx\,dt + \int_{Q_T} \theta p_{\theta\eta} v_x^2 e_\theta\theta_t\,dx\,dt + \int_{Q_T} \theta p_\theta v_{xt} e_\theta\theta_t\,dx\,dt - \int_{Q_T}\left[\left(\frac{\mu}{\eta}\right)_\eta v_x^3 + 2\frac{\mu}{\eta}\,v_x v_{xt}\right] e_\theta\theta_t\,dx\,dt \\
&= -\int_{Q_T}\frac{\kappa}{\eta} e_\theta\theta_{tx}^2\,dx\,dt - \int_{Q_T}\left[\left(\frac{\kappa}{\eta}\right)_\eta v_x\theta_x + \frac{\kappa_\theta}{\eta}\,\theta_t\theta_x\right](e_\theta\theta_t)_x\,dx\,dt \\
&\quad - \int_{Q_T}\theta_x\left(e_{\theta\eta}\eta_x + e_{\theta\theta}\theta_x\right)dx\,dt - \int_{Q_T}\eta\,[(S_E)_R]_t\; e_\theta\theta_t\,dx\,dt - \int_{Q_T} v_x (S_E)_R\, e_\theta\theta_t\,dx\,dt.
\end{aligned}$$

After [21] (see the proof of Lemma 3.6), we get

$$\frac{1}{2}\int_\Omega (e_\theta\theta_t)^2\,(x,t)\,dx + \int_{Q_T}\frac{\kappa}{\eta} e_\theta\theta_{tx}^2\,dx\,dt \leqslant C - \int_{Q_T}[(S_E)_R]_t\; e_\theta\theta_t\,dx\,dt - \int_{Q_T} v_x(S_E)_R\,e_\theta\theta_t\,dx\,dt. \tag{60}$$

The first integral in the right-hand side can be decomposed as follows:

$$\begin{aligned}
\int_{Q_T}[(S_E)_R]_t\; e_\theta\theta_t\,dx\,dt &= \int_{Q_T} e_\theta\theta_t v_x \int_0^\infty\int_{S^1}\sigma_a B\,d\nu\,d\omega\,dx\,dt + \int_{Q_T} e_\theta\theta_t\eta\int_0^\infty\int_{S^1}(\sigma_a)_\eta\, B\,d\nu\,d\omega\,dx\,dt \\
&\quad + \int_{Q_T} e_\theta\theta_t^2\int_0^\infty\int_{S^1}(\sigma_a)_\theta\, B\,d\nu\,d\omega\,dx\,dt + \int_{Q_T} e_\theta\theta_t^2\int_0^\infty\int_{S^1}\sigma_a\, B_\theta\,d\nu\,d\omega\,dx\,dt \\
&\quad - \int_{Q_T} e_\theta\theta_t v_x\int_0^\infty\int_{S^1}\sigma_a I\,d\nu\,d\omega\,dx\,dt + \int_{Q_T} e_\theta\theta_t\eta\int_0^\infty\int_{S^1}(\sigma_a)_\eta\, I\,d\nu\,d\omega\,dx\,dt \\
&\quad + \int_{Q_T} e_\theta\theta_t^2\int_0^\infty\int_{S^1}(\sigma_a)_\theta\, I\,d\nu\,d\omega\,dx\,dt + \int_{Q_T} e_\theta\theta_t\int_0^\infty\int_{S^1}\sigma_a I_t\,d\nu\,d\omega\,dx\,dt \\
&=: \sum_{j=1}^{8} A_j.
\end{aligned}$$

Using (18), the Cauchy–Schwarz inequality, Lemmas 8, 9 and the first part of the present lemma, one gets

$$\left|\sum_{j=1}^{4} A_j\right| \leqslant C + C\int_{Q_T}\theta_t^2\,dx\,dt \leqslant C.$$

After the formula (38) and the bounds (18), one has

$$\max_{Q_T}\int_0^\infty\int_{S^1} I\,d\omega\,d\nu \leqslant C_8\max_{Q_T}\theta^\alpha\,\|f\|_{L^1(\mathbb{R}_+\times S^1)} \leqslant C.$$

In the same stroke, after computing the time derivative, we have also

$$\max_{Q_T}\int_0^\infty\int_{S^1}|I_t|\,d\omega\,d\nu \leqslant C,$$

which gives finally that $|\sum_{j=5}^{8} A_j| \leqslant C$, and then

$$\left|\int_{Q_T}\eta\,[(S_E)_R]_t\; e_\theta\theta_t\,dx\,dt\right| \leqslant C.$$

It is readily checked that the same kind of estimate holds for the second integral in (60) (we omit the details):

$$\left|\int_{Q_T} v_x(S_E)_R\, e_\theta\theta_t\,dx\,dt\right| \leqslant C.$$

Plugging this estimate into (60), we obtain the first two estimates (59).

3. From the internal energy equation

$$\frac{\kappa}{\eta}\,\theta_{xx} = \left(\frac{\kappa-\eta\kappa_\eta}{\eta^2}\right)\eta_x\theta_x - \frac{\kappa_\theta}{\eta}\,\theta_x^2 + e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\,v_x^2 + \eta(S_E)_R,$$

then

$$|\theta_{xx}| \leqslant C\left(|\eta_x\theta_x| + \theta_x^2 + |\theta_t| + |v_x| + v_x^2 + |(S_E)_R|\right),$$

and one checks that all of the terms in the right-hand side are in $L^2(\Omega)$, which proves the last bound (59). □

Proof of Theorem 4. 1. As $\max_{Q_T} |v_x|$ is bounded we have

$$\left|\eta(x,t) - \eta(x,t')\right| \leqslant |t-t'|^{1/2}\left(\int_0^T v_x^2\,dt\right)^{1/2} \leqslant C|t-t'|^{1/2}.$$

After Lemma 4 we have also

$$\left|\eta(x,t) - \eta(x',t)\right| \leqslant C|x-x'|^{1/2}\left(1 + \int_\Omega \eta_x^2\,dx\right) \leqslant C|x-x'|^{1/2},$$

so we find that $\eta \in C^{1/2,1/4}(Q_T)$.

2. After Lemma 11 we have

$$\left|\theta(x,t) - \theta(x,t')\right| \leqslant |t-t'|^{1/2}\left(\int_0^T \theta_t^2\,dt\right)^{1/2} \leqslant C|t-t'|^{1/2}\left(\int_0^T\int_\Omega 2|\theta_t\theta_{xt}|\,dx\,dt\right)^{1/2} \leqslant C|t-t'|^{1/2}.$$

We see also that

$$\begin{aligned}\left|\theta(x,t) - \theta(x',t)\right| &\leqslant C|x-x'|^{1/2}\left(T\cdot\max_{[0,T]}\int_\Omega \theta_t^2\,dx + \int_0^T\int_\Omega \theta_{xt}^2\,dx\right)\\ &\leqslant C|x-x'|^{1/2},\end{aligned}$$

so we find that $\theta \in C^{1/2,1/4}(Q_T)$. We have also

$$\left|\theta_x(x,t) - \theta_x(x',t)\right| \leqslant |x-x'|^{1/2}\left(\int_\Omega \theta_{xx}^2\,dt\right)^{1/2} \leqslant |x-x'|^{1/2},$$

and we conclude, by using an interpolation argument of [25], that $\theta_x \in C^{1/3,1/6}(Q_T)$.

3. The same arguments holding verbatim for v and v_x, we have that $v, v_x \in C^{1/3,1/6}(Q_T)$.

4. Let us note $\mathcal{I}(x,t) := \int_0^\infty \int_{S^1} I(x,t;\omega,\nu)\,d\omega\,d\nu$.

From the explicit formula giving I and using (18), we have

$$\begin{aligned}|\mathcal{I}(x,t) - \mathcal{I}(x',t)| \leqslant \int_{x'}^x |I_y|\,dy &\leqslant \int_{x'}^x \left[\frac{\eta\sigma_a}{|\omega|}\,|B-I| + \frac{\eta\sigma_s}{\omega}\,(\tilde{I}-I)\right] dy\\ &\leqslant C|x-x'|^{1/2}.\end{aligned}$$

One also checks after an elementary computation from the explicit formula giving I, that $\max_{[0,T]} \|\mathcal{I}_t\|_{L^2(\Omega)} \leqslant C$. It follows that

$$|\mathcal{I}(x,t) - \mathcal{I}(x,t')| \leqslant C|t-t'|^{1/2},$$

and then $\mathcal{I} \in C^{1/3,1/6}(Q_T)$.

From the last Eq. (11), we get also the formula

$$\mathcal{I}_x(x,t) = \int_0^\infty \int_{S^1}\left[\frac{\eta\sigma_a}{\omega}\,(B-I) + \frac{\eta\sigma_s}{|\omega|}\,|\tilde{I}-I|\right] d\omega\,d\nu.$$

From the Hölder properties of the right-hand side, we get finally

$$|\mathcal{I}_x(x,t) - \mathcal{I}_x(x',t)| \leqslant C|x-x'|^{1/2},$$

and

$$|\mathcal{I}_x(x,t) - \mathcal{I}_x(x,t')| \leqslant C|t-t'|^{1/2},$$

and then $\mathcal{I}_x \in C^{1/3,1/6}(Q_T)$, which ends the proof of Theorem 4. □

3. Existence and uniqueness of solutions

In this section we prove the existence of a classical solution by means of the classical Leray–Schauder fixed point theorem in the same spirit as in [22,21]; then using a limiting process as in [26] we will get the existence of weak solutions.

Let us recall the Leray–Schauder fixed point theorem:

Theorem 5. *Let $\mathcal{B}$ be a Banach space and suppose that $P : [0, 1] \times \mathcal{B} \to \mathcal{B}$ has the following properties:*

(i) *For any fixed $\lambda \in [0, 1]$ the map $P(\lambda, .) : \mathcal{B} \to \mathcal{B}$ is completely continuous.*
(ii) *For every bounded subset $\mathcal{S} \subset \mathcal{B}$ the family of maps $P(., \chi) : [0, 1] \to \mathcal{B}$, $\chi \in \mathcal{S}$ is uniformly equicontinuous.*
(iii) *There is a bounded subset $\mathcal{S}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $P(\lambda, .)$, $\lambda \in [0, 1]$ is contained in $\mathcal{S}$.*
(iv) *$P(0, .)$ has precisely one fixed point in $\mathcal{B}$.*

Then, $P(1, .)$ has at least one fixed point in $\mathcal{B}$.

In our case $\mathcal{B}$ will be a Banach space of functions $\eta, v, \theta \in \mathcal{B}$ on Q_T with $\eta, v, v_x, \theta, \theta_x \in C^{1/3,1/6}(Q_T)$ with the norm

$$|\|(\eta, v, \theta)\||_{\mathcal{B}} := \|\eta\|_{1/3} + \|v\|_{1/3} + \|v_x\|_{1/3} + \|\theta\|_{1/3} + \|\theta_x\|_{1/3}.$$

For $\lambda \in [0, 1]$ we define $P(\lambda, .)$ as the map which carries $\{\tilde{\eta}, \tilde{v}, \tilde{\theta}\} \in \mathcal{B}$ into $\{\eta, v, \theta\} \in \mathcal{B}$ by solving the system

$$\begin{cases} \eta_t = v_x, \\ v_t - \dfrac{\mu}{\tilde{\eta}} v_{xx} = -\dfrac{\mu}{\tilde{\eta}^2}\tilde{\eta}_x\tilde{v}_x - \tilde{p}_\eta(\tilde{\eta}, \tilde{\theta})\eta_x - \tilde{p}_\theta(\tilde{\eta}, \tilde{\theta})\theta_x, \\ \tilde{e}_\theta(\tilde{u}, \tilde{\theta})\theta_t - \dfrac{\tilde{\kappa}(\tilde{\eta}, \tilde{\theta})}{\tilde{\eta}}\,\theta_{xx} = \left(\dfrac{\tilde{\kappa}(\tilde{\eta}, \tilde{\theta})}{\tilde{\eta}}\right)_\eta \tilde{\theta}_x\eta_x + \dfrac{\tilde{\kappa}_\theta(\tilde{\eta}, \tilde{\theta})}{\tilde{\eta}}\tilde{\theta}_x^2 + \dfrac{\mu}{\tilde{\eta}}\tilde{v}_x^2 - \tilde{\theta}\tilde{p}_\theta(\tilde{\eta}, \tilde{\theta})\tilde{v}_x - \tilde{\eta}\left(\tilde{S}_E\right)_R, \end{cases} \tag{61}$$

together with the boundary conditions

$$\begin{cases} v|_{x=0,M} = 0, \\ \theta_x|_{x=0} = 0, \qquad \theta|_{x=M} = 0, \end{cases} \tag{62}$$

for $t > 0$, and initial conditions

$$\begin{cases} \eta(x, 0) = (1 - \lambda) + \lambda\eta_0(x), \\ v(x, 0) = \lambda v_0(x), \\ \theta(x, 0) = (1 - \lambda) + \lambda\theta_0(x). \end{cases} \tag{63}$$

We can consider the second and the third equations of (64) as parabolic type and apply the classical Schauder–Friedmann estimates

$$\|v\|_{1/3} + \|v_x\|_{1/3} \leqslant c\{\|\eta\|_{1/3} + \|\tilde{v}\|_{1/3} + \|\tilde{\theta}_x\|_{1/3}\}$$

$$\|\theta_x\|_{1/3} + \|\theta\|_{1/3} \leqslant c\{\|\tilde{\theta}_x\|_{1/3} + \|\tilde{v}_x\|_{1/3}\}.$$

Moreover from the first Eq. (61), we get

$$\|\eta\|_{1/3} \leqslant \|v_x\|_{1/3}.$$

This implies that $P(\lambda, .) : \mathcal{B} \to \mathcal{B}$ is well defined and continuous.

Using *a priori* estimates from Section 2 it follows that for any $\{\tilde{\eta}, \tilde{v}, \tilde{\theta}\}$ from any fixed bounded subset the family $P(., \{\tilde{\eta}, \tilde{v}, \tilde{\theta}\}) : [0, 1] \to \mathcal{B}$ of mappings is uniformly equicontinuous.

Now, in order to verify (iii), we observe that any fixed point of P will initially satisfy the original problem; therefore η and θ cannot escape from $[\underline{\eta}, \overline{\eta}]$, $[\underline{\theta}, \overline{\theta}]$ up to time T. This fact follows clearly from Theorem 4. To check (iv) we see by inspection that the unique fixed point of $P(0, .)$ is given by $\eta(x, t) = 1$, $v(x, t) = 0$, $\theta(x, t) = 1$.

All the previous facts allow us to apply Theorem 5, which implies the existence of classical solutions of (11)–(15) in $\Omega \times (0, t^*)$.

This ends the proof of Theorem 3.

Let us now consider the existence of a weak solution. From previous results it follows that:

- $v_k \to v$ in $L^p(0, t^*; C^0(\Omega))$ strongly and in $L^p(0, t^*; H^1(\Omega))$ weakly for $1 < p < \infty$,
- $v_k \to v$ a.e. in $\Omega \times [0, t^*]$ and in $L^\infty(0, t^*; L^4(\Omega))$ weakly *,
- $(v_k)_t \to v_t$ in $L^2(0, t^*, L^2(\Omega))$ weakly,
- $\theta_k \to \theta$ in $L^2(0, t^*, C^0(\Omega))$ strongly and in $L^2(0, t^*, H^1(\Omega))$ weakly,
- $\theta_k \to \theta$ a.e. in $\Omega \times [0, t^*]$ and in $L^\infty(0, t^*; L^2(\Omega))$ weakly,
- $\sigma_k \to A_1$ in $L^2(0, t^*; H^1(\Omega))$ weakly.

This implies that

$$\eta_k \to \eta \quad \text{a.e. in } \Omega \times [0, t^*] \quad \text{and} \quad L^s(\Omega \times [0, t^*]) \text{ strongly for all } s \in [1, \infty[.$$

All this implies that

- $\frac{\kappa_k(\theta_k)_x}{\eta_k} \to A_2$ weakly in $L^2(0, t^*, H^1(\Omega))$,
- $\frac{\mu}{\eta_k}(v_k)_x \to A_3$ in $L^\infty(0, t^*, L^2(\Omega))$ weakly *,
- $\eta_k\{(S_E)_R\}_k \to A_4$ in $L^2(0, t^*; H^1(\Omega))$ weakly.

Then applying a technique similar to that of [26] it follows that

- $A_1 = \sigma$ in $L^2(0, t^*, H^1(\Omega))$,
- $A_2 = \frac{\kappa\theta_x}{\eta}$ in $L^2(0, t^*, L^2(\Omega))$,
- $A_3 = \frac{\mu}{\eta} v_x$ in $L^2(0, t^*, H^1(\Omega))$,
- $A_4 = \eta(S_E)_R$ in $L^2(0, t^*, H^1(\Omega))$, which ends the proof of the existence of a weak solution.

Finally we prove uniqueness of the solution.

Let $\eta_i, v_i, \theta_i, i = 1, 2$, be two solutions of (5), and let us consider the differences $E = \eta_1 - \eta_2, T = \theta_1 - \theta_2$ and $V = v_1 - v_2$.

From the first Eq. (11) written for η_1, w_1 and η_2, w_2, subtracting, multiplying by a test function χ, integrating by parts and putting $\chi = E$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E^2\,\mathrm{d}x = \int_\Omega EV_x\,\mathrm{d}x \leqslant \|E\|_2\|V_x\|_2.$$

Using the Cauchy–Schwarz inequality for $\varepsilon > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E^2\,\mathrm{d}x \leqslant \varepsilon\|V_x\|_2^2 + C_\varepsilon\|E\|_2^2. \tag{64}$$

Rewriting the second Eq. (11) for v_2 and v_1, subtracting, multiplying by a test function ϕ, integrating by parts and putting $\phi = V$ we obtain the following:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega V^2\,\mathrm{d}x + \int_\Omega \mu\frac{\mu_2}{\eta_2}V_x^2\,\mathrm{d}x = -\sum_{i=1}^{2}\mathcal{A}_i,$$

with

$$|\mathcal{A}_1| = \left|\int_\Omega (p_2 - p_1)\,V_x\,\mathrm{d}x\right| \leqslant C\|V_x\|_2\left(\|T\|_2 + \|E\|_2\right) \leqslant \varepsilon\|V_x\|_2^2 + C_\varepsilon\left(\|T\|_2^2 + \|E\|_2^2\right),$$

where we used the Cauchy–Schwarz inequality for $\varepsilon > 0$.

In the same stroke,

$$|\mathcal{A}_2| = \left|\int_\Omega \frac{\eta_2 - \eta_1}{\eta_2\eta_1}\,\mu(v_2)_xV_x\,\mathrm{d}x\right| \leqslant C\|E\|_2\|V_x\|_2 \leqslant \varepsilon\|V_x\|_2^2 + C_\varepsilon\|E\|_2^2.$$

So we get finally, taking ε small enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega V^2\,\mathrm{d}x + \int_\Omega V_x^2\,\mathrm{d}x \leqslant C\left(\|T\|_2^2 + \|E\|_2^2\right). \tag{65}$$

Now, dividing the energy equation by e_θ, we have

$$\theta_t = -\frac{\theta p_\theta}{e_\theta}\,w_x + \frac{q_x}{e_\theta} + \frac{\mu}{\eta e_\theta}\,v_x^2 - \frac{\eta}{e_\theta}\,(S_E)_R.$$

Subtracting this equation written for η_1, v_1, θ_1 from the same for η_2, v_2, θ_2, multiplying by a test function ψ, integrating by parts and putting $\psi = T$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^2\,\mathrm{d}x = -\int_\Omega\left[\frac{\theta_1p_\theta(\eta_1,\theta_1)}{e_\theta(\eta_1,\theta_1)}\,w_{1x} - \frac{\theta_2p_\theta(\eta_2,\theta_2)}{e_\theta(\eta_2,\theta_2)}\,w_{2x}\right]T\,\mathrm{d}x + \int_\Omega\left[\frac{\kappa(\eta_1,\theta_1)}{\eta_1e_\theta(\eta_1,\theta_1)} - \frac{\kappa(\eta_2,\theta_2)}{\eta_2e_\theta(\eta_2,\theta_2)}\right]T\,\mathrm{d}x$$
$$+ \int_\Omega\left[\frac{\mu {v_1}_x^2}{\eta_1e_\theta(\eta_1,\theta_1)} - \frac{\mu {v_2}_x^2}{\eta_2e_\theta(\eta_2,\theta_2)}\right]T\,\mathrm{d}x\int_\Omega\{\eta_1[(S_E)_R]_1 - \eta_2[(S_E)_R]_2\}\,T\,\mathrm{d}x := -\sum_{i=1}^{4}\mathcal{B}_i.$$

Bounding the $\mathcal{B}_i$, using as previously the Cauchy–Schwarz inequality for $\varepsilon > 0$, we get

$$|\mathcal{B}_1| \leqslant \varepsilon \left(\|V_x\|_2^2 + \|T_x\|_2^2\right) + C_\varepsilon \left(\|E\|_2^2 + \|T\|_2^2\right),$$

$$|\mathcal{B}_2| \leqslant -\int_\Omega \frac{\kappa(\eta_2, \theta_2) r_2^4}{\eta_2 e_\theta(\eta_2, \theta_2)} T_x^2 \, dx + \varepsilon \int_\Omega T_x^2 \, dx + C_\varepsilon \left(\|E\|_2^2 + \|T\|_2^2\right),$$

$$|\mathcal{B}_3| \leqslant \varepsilon \|V_x\|_2^2 + C_\varepsilon \|E\|_2^2,$$

and

$$|\mathcal{B}_4| \leqslant C\|E\|_2^2 + C\|T\|_2^2.$$

We obtain finally

$$\frac{d}{dt} \int_\Omega T^2 \, dx + \int_\Omega T_x^2 \, dx \leqslant \varepsilon \int_\Omega V_x^2 \, dx + C \left(\|E\|_2^2 + \|T\|_2^2\right). \tag{66}$$

Then adding inequalities (64)–(66) and choosing ε small enough, we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \left(E^2 + V^2 + T^2\right) \, dx \leqslant C \left(\|E\|_2^2 + \|V\|_2^2 + \|T\|_2^2\right),$$

which clearly implies uniqueness.

Acknowledgements

Š.N. was supported by the Grant Agency of the Czech Republic No. 201/08/0012 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

References

[1] G. Gallavotti, Foundations of Fluid Mechanics, Springer-Verlag, Berlin, Heidelberg, New York, 2002.
[2] S. Chandrasekhar, Radiative Transfer, Dover Publications, Inc., New York, 1960.
[3] G.C. Pomraning, Radiation Hydrodynamics, Dover Publications, Inc., Mineola, New York, 2005.
[4] D. Mihalas, B. Weibel-Mihalas, Foundations of Radiation Hydrodynamics, Oxford University Press, New York, 1984.
[5] R.B. Lowrie, J.E. Morel, J.A. Hittinger, The coupling of radiation and hydrodynamics, Astrophysical Journal 521 (1999) 432–450.
[6] C. Buet, B. Després, Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics, Journal of Quantitative Spectroscopy and Radiative Transfer 85 (2004) 385–418.
[7] B. Dubroca, J.-L. Feugeas, Etude théorique et numérique d'une hiérarchie de modèles aux moments pour le transfert radiatif, Comptes Rendus de l'Academie des Sciences, Série I 329 (1999) 915–920.
[8] C. Lin, Mathematical analysis of radiative transfer models, Ph.D. Thesis, 2007.
[9] C. Lin, J.-F. Coulombel, T. Goudon, Shock profiles for non-equilibrium radiating gases, Physica D 218 (2006) 83–94.
[10] X. Zhong, S. Jiang, Local existence and finite-time blow-up in multidimensional radiation hydrodynamics, Journal of Mathematical Fluid Mechanics 9 (2007) 543–564.
[11] P. Jiang, D. Wang, Formation of singularities of solutions to the three-dimensional Euler–Boltzmann equations in radiation hydrodynamics. Preprint, March 11, 2009.
[12] P. Jiang, D. Wang, Global weak solutions to the Euler–Boltzmann equations in radiation hydrodynamics. Preprint, June 27, 2009.
[13] F. Golse, B. Perthame, Generalized solutions of the radiative transfer equations in a singular case, Communications in Mathematical Physics 106 (1986) 211–239.
[14] R. Kippenhahn, A. Weigert, Stellar Structure and Evolution, Springer-Verlag, Berlin, Heidelberg, 1994.
[15] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, Oxford, 2003.
[16] A.A. Amosov, Well-posedness "in the large" initial and boundary-value problems for the system of dynamical equations of a viscous radiating gas, Soviet Physics - Doklady 30 (1985) 129–131.
[17] S.N. Antontsev, A.V. Kazhikhov, V.N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1990.
[18] L. Hsiao, Quasilinear Hyperbolic Systems and Dissipative Mechanisms, World Scientific, Singapore, New Jersey, London, Hong Kong, 1997.
[19] L. Hsiao, S. Jiang, Nonlinear parabolic equations and systems, in: C.M. Dafermos, E. Feireisl (Eds.), Handbook of Differential Equations. Vol. 1, Elsevier North Holland, Amsterdam, Boston, Heidelberg, 2004.
[20] Y. Qin, Nonlinear Parabolic–Hyperbolic Coupled Systems and their Attractors, Birkhäuser, Basel, Boston, Berlin, 2008.
[21] C.M. Dafermos, L. Hsiao, Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity, Nonlinear Analysis. Theory, Methods & Applications 6 (1982) 435–454.
[22] B. Kawohl, Global existence of large solutions to initial boundary value problems for a viscous heat-conducting, one-dimensional real gas, Journal of Differential Equations 58 (1985) 76–103.
[23] S. Jiang, On initial boundary value problems for a viscous heat conducting one-dimensional gas, Journal of Differential Equations 110 (1994) 157–181.
[24] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, 1972.
[25] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, AMS Providence, Rhode Island, 1968.
[26] H. Fujita-Yashima, R. Benabidallah, Equation à symétrie sphérique d'un gaz visqueux et calorifère avec la surface libre, Annali di Matematica Pura ed Applicata 168 (1995) 75–117.

126 (2001) MATHEMATICA BOHEMICA No. 2, 265–274

LAPLACE EQUATION IN THE HALF-SPACE WITH A NONHOMOGENEOUS DIRICHLET BOUNDARY CONDITION

Cherif Amrouche, Pau, Šárka Nečasová, Praha

Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We deal with the Laplace equation in the half space. The use of a special family of weigted Sobolev spaces as a framework is at the heart of our approach. A complete class of existence, uniqueness and regularity results is obtained for inhomogeneous Dirichlet problem.

Keywords: the Laplace equation, weighted Sobolev spaces, the half space, existence, uniqueness, regularity

MSC 2000: 35J05, 58J10

1. Introduction

The purpose of this paper is to solve the problem

$$\text{(P)} \qquad \begin{cases} -\Delta u = f & \text{in } \mathbb{R}^N_+, \\ u = g & \text{on } \Gamma = \mathbb{R}^{N-1}, \end{cases}$$

with the Dirichlet boundary condition on Γ. The approach is based on the use of a special class of weighted Sobolev spaces for describing the behavior at infinity. Many authors have studied the Laplace equation in the whole space $\mathbb{R}^N$ or in an exterior domain. The main difference is due to the nature of the boundary and one of difficulties is to obtain the appropriate spaces of traces. However, the half-space has a useful symmetric property.

Second author would like to thank the Grant Agency of the Czech Republic No. 201/99/0267 and both authors to the Barrande project between the Czech Republic and France.

Problem (P) has been investigated in weighted Sobolev spaces by several authors, but only in the Hilbert cases ($p = 2$) and without the critical cases corresponding to the logarithmic factor (cf. [2], [4]). We can also mention the book by Simader, Sohr [5] where the Dirichlet problem for the Laplacian is investigated.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geqslant 2$. Let $x = (x_1, \dots, x_N)$ be a typical point of $\mathbb{R}^N$ and $r = |x| = (x_1^2 + \dots + x_N^2)^{1/2}$. We use two basic weights:

$$\varrho = (1 + r^2)^{1/2} \quad \text{and} \; \lg \varrho = \ln(2 + r^2).$$

As usual, $\mathcal{D}(\mathbb{R}^N)$ denotes the spaces of indefinitely differentiable functions with a compact support and $\mathcal{D}'(\mathbb{R}^N)$ denotes its dual space, called the space of distributions. For any nonnegative integers n and m, real numbers $p > 1$, α and β, setting

$$k = k(m, N, p, \alpha) = \begin{cases} -1 & \text{if} \quad \frac{N}{p} + \alpha \notin \{1, \dots, m\}, \\ m - \frac{N}{p} - \alpha & \text{if} \quad \frac{N}{p} + \alpha \in \{1, \dots, m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \{u \in \mathcal{D}'(\Omega);\; & 0 \leqslant |\lambda| \leqslant k,\; \varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta - 1} D^\lambda u \in L^p(\Omega); \\ & k + 1 \leqslant |\lambda| \leqslant m,\; \varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta} D^\lambda u \in L^p(\Omega)\}. \end{aligned} \tag{1.1}$$

In case $\beta = 0$, we simply denote the space by $W^{m,p}_{\alpha}(\Omega)$. Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big[& \sum_{0 \leqslant |\lambda| \leqslant k} \|\varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta - 1} D^\lambda u\|^p_{L^p(\Omega)} \\ & + \sum_{k+1 \leqslant |\lambda| \leqslant m} \|\varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta} D^\lambda u\|^p_{L^p(\Omega)} \Big]^{1/p}. \end{aligned}$$

We also define the semi-norm

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda| = m} \|\varrho^{\alpha}(\lg \varrho)^{\beta} D^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p},$$

and for any integer q, we denote by P_q the space of polynomials in N variables of a degree smaller than or equal to q, with the convention that P_q is reduced to $\{0\}$ when q is negative. The weights defined in (1.1) are chosen so that the corresponding space satisfies two properties:

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \quad \text{is dense in} \quad W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \tag{1.2}$$

and the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$.

Theorem 1.1. *Let α and β be two real numbers and $m \geqslant 1$ an integer not satisfying simultaneously*

$$\frac{N}{p} + \alpha \in \{1, \ldots, m\} \quad \text{and} \quad (\beta - 1)p = -1. \tag{1.3}$$

Then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+),\ \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}} \leqslant c|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)} \tag{1.4}$$

with $q' = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$,

P r o o f. First, we construct a linear continuous extension operator such that

$$P\colon W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \to W^{m,p}_{\alpha,\beta}(\mathbb{R}^N) \tag{1.5}$$

satisfying

$$\|Pu\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)} \leqslant \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}. \tag{1.6}$$

Since

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N),\ \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)/P_{q'}} \leqslant c|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)} \tag{1.6}$$

holds [cf. 1], it automatically implies the statement of our theorem. □

Now, we define the space

$$\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}};$$

the dual space of $\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is denoted by $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$, where p' is the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ such that it is equivalent to the full norm $\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$.*

We recall now some properties of weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \subset W^{m-1,p}_{\alpha-1,\beta}(\mathbb{R}^N_+) \subset \ldots \subset W^{0,p}_{\alpha-m,\beta}(\mathbb{R}^N_+)$$

if $\frac{N}{p} + \alpha \notin \{1, \ldots, m\}$. When $\frac{N}{p} + \alpha = j \in \{1, \ldots, m\}$, then we have:

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \subset \ldots \subset W^{m-j+1,p}_{\alpha-j+1,\beta}(\mathbb{R}^N_+) \subset W^{m-j,p}_{\alpha-j,\beta-1}(\mathbb{R}^N_+) \subset \ldots \subset W^{0,p}_{\alpha-m,\beta-1}(\mathbb{R}^N_+).$$

Note that in the first case, the mapping $u \to \varrho^\gamma u$ is an isomorphism from $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ onto $W^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$ for any integer m. Moreover, in both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \to D^\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$$

is continuous.

Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m - (\frac{N}{p} + \alpha) - 1 & \text{if} \begin{cases} \frac{N}{p} + \alpha \in \{1, \ldots, m\} \text{ and } (\beta - 1)p \geqslant -1 \\ \frac{N}{p} + \alpha \in \{j \in Z; j \leqslant 0\} \text{ and } \beta p \geqslant -1 \end{cases} \\ [m - (\frac{N}{p} + \alpha)] & \text{otherwise,} \end{cases}$$

where $[s]$ denotes the integer part of s.

In the sequel, for any integer $q \geqslant 0$, we will use the following polynomial spaces:

— P_q (P^Δ_q) is the space of polynomials (respectively, harmonic polynomials) of degree $\leqslant q$,

— P'_q is the subspace of polynomials in P_q depending only on the $N-1$ first variables, $x' = (x_1, \ldots, x_{N-1})$,

— A^Δ_q (N^Δ_q) is the subspace of polynomials P^Δ_q satisfying the condition $p(x', 0) = 0$ (respectively, $\frac{\partial p}{\partial x_N}(x', 0) = 0$) or equivalently odd with respect to x_N (even with respect to x_N), with the convention that $P_q, P^\Delta_q, P'_q, \ldots$ are reduced to $\{0\}$ when q is negative.

2. The spaces of traces

In order to define the traces of functions of $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$, we introduce for any $\sigma \in]0, 1[$ the space

$$(2.1) \qquad W^{\sigma,p}_0(\mathbb{R}^N) = \Big\{u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma} u \in L^p(\mathbb{R}^N), \int_0^{+\infty} t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x + te_i) - u(x)|^p\,\mathrm{d}x < \infty\Big\},$$

where

$$w = \begin{cases} \varrho & \text{if } \frac{N}{p} \neq \sigma, \\ \varrho(\lg \varrho)^{1/\sigma} & \text{if } \frac{N}{p} = \sigma, \end{cases}$$

and $e_1, \dots, e_N$ is a canonical basis of $\mathbb{R}^N$. It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W_0^{\sigma,p}(\mathbb{R}^N)} = \Big(\Big\|\frac{u}{w^\sigma}\Big\|^p_{L^p(\mathbb{R}^N)} + \sum_{i=1}^N \int_0^\infty t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x+te_i) - u(x)|^p\,\mathrm{d}x\Big)^{1/p}$$

which is equivalent to the norm

$$\Big(\Big\|\frac{u}{w^\sigma}\Big\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}.$$

For any $s \in \mathbb{R}^+$, we set

$$\text{(2.2)} \qquad W_0^{s,p}(\mathbb{R}^N) = \Big\{u \in W^{[s],p}_{[s]-s}(\mathbb{R}^N);\ \forall|\lambda| = [s],\ D^\lambda u \in W_0^{s-[s],p}(\mathbb{R}^N)\Big\}.$$

It is a reflexive Banach space equipped with the norm

$$\|u\|_{W_0^{s,p}(\mathbb{R}^N)} = \|u\|_{W^{[s],p}_{[s]-s}(\mathbb{R}^N)} + \sum_{|\lambda|=s} \|D^\lambda u\|_{W_0^{s-[s],p}(\mathbb{R}^N)}.$$

We notice that this definition and the next one coincide with the definition in the first section when $s = m$ is a nonnegative integer. For any $s \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, we then set

$$\text{(2.3)} \qquad W_\alpha^{s,p}(\mathbb{R}^N) = \Big\{u \in W^{[s],p}_{[s]+\alpha-s}(\mathbb{R}^N),\ \forall|\lambda| = [s],\ \varrho^\alpha D^\lambda u \in W_0^{s-[s],p}(\mathbb{R}^N)\Big\}.$$

Finally, for any integer $m \geqslant 1$, we define the space

$$\text{(2.4)} \qquad \begin{aligned} X_0^{m,p}(\mathbb{R}^N_+) = \Big\{u \in \mathcal{D}'(\mathbb{R}^N_+);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho'^{|\lambda|-m}(\lg\varrho')^{-1} D^\lambda u \in L^p(\mathbb{R}^N_+),\\ k+1 \leqslant |\lambda| \leqslant m,\ \varrho'^{|\lambda|-m} D^\lambda u \in L^p(\mathbb{R}^N_+)\Big\}\end{aligned}$$

with $\varrho' = (1+|x'|^2)^{1/2}$ and $\lg\varrho' = \ln(2+|x'|^2)$. It is a reflexive Banach space. We can prove that

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \quad \text{is dense in} \quad X_0^{m,p}(\mathbb{R}^N_+).$$

We observe that the functions from $X_0^{m,p}(\mathbb{R}^N_+)$ and $W_0^{m,p}(\mathbb{R}^N_+)$ have the same traces on $\Gamma = \mathbb{R}^{N-1}$ (see below). If u is a function, we denote its traces on $\Gamma = \mathbb{R}^{N-1}$ by $x' \in \mathbb{R}^{N-1}$, $\gamma_0 u(x') = u(x',0), \dots, \gamma_j u(x') = \frac{\partial^j u}{\partial x_N^j}(x',0)$.

As in [3], we can prove the following trace lemma:

Lemma 2.1. *For any integer $m \geqslant 1$ and real number α, the mapping*

$$\gamma\colon\ \mathcal{D}(\overline{\mathbb{R}^N_+}) \to \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1})$$
$$u \mapsto (\gamma_0 u, \ldots, \gamma_{m-1} u)$$

can be extended by continuity to a linear and continuous mapping still denoted by γ from $W^{m,p}_\alpha(\mathbb{R}^N_+)$ to $\prod\limits_{j=0}^{m-1} W_\alpha^{m-j-\frac{1}{p},p}(\mathbb{R}^{N-1})$. Moreover, γ is onto and

$$\operatorname{Ker}\gamma = \mathring{W}^{m,p}_\alpha(\mathbb{R}^N_+).$$

3. The Laplace equation

The aim of this section is to study the problem (P):

$$\text{(P)}\qquad \begin{cases} -\Delta u = f & \text{in } \ \mathbb{R}^N_+, \\ u = g & \text{in } \ \Gamma = \mathbb{R}^{N-1}. \end{cases}$$

Theorem 3.1. *Let $\ell \geqslant 0$ be an integer and assume that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \tag{3.1}$$

with the convention that this set is empty if $\ell = 0$. For any f in $W^{-1,p}_\ell(\mathbb{R}^N_+)$ and g in $W_\ell^{\frac{1}{p'},p}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in A^\Delta{}_{[\ell+1-\frac{N}{p'}]},\ \langle f, \varphi\rangle_{W^{-1,p}_\ell \times W^{1,p'}_{-\ell}} = \Big\langle g, \frac{\partial \varphi}{\partial \gamma_N}\Big\rangle_\Gamma \tag{3.2}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality between $W_\ell^{\frac{1}{p'},p}(\Gamma)$ and $W_{-\ell}^{-\frac{1}{p'},p'}(\Gamma)$, problem (P) has a unique solution $u \in W^{1,p}_\ell(\mathbb{R}^N_+)$ and there exists a constant C independent of u, f and g such that

$$\|u\|_{W^{1,p}_\ell(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1,p}_\ell(\mathbb{R}^N_+)} + \|g\|_{W_\ell^{\frac{1}{p'},p}(\Gamma)}). \tag{3.3}$$

Proof. First, the kernel of the operator

$$(-\Delta, \gamma_0)\colon\ W^{1,p}_\ell(\mathbb{R}^N_+) \to W^{-1,p}_\ell(\mathbb{R}^N_+) \times W_\ell^{\frac{1}{p'},p}(\Gamma)$$

is precisely the space $A^{\Delta}_{[\ell+1-N/p']}$ for any integer ℓ and $A^{\Delta}_{[\ell+1-\frac{N}{p'}]}$ is reduced to $\{0\}$ when $\ell \geqslant 0$. Thanks to Lemma 2.1, let $u_g \in W^{1,p}_{\ell}(\mathbb{R}^N_+)$ be the lifting function of g such that

$$u_g = g \text{ on } \Gamma \text{ and } \|u_g\|_{W^{1,p}_{\ell}(\mathbb{R}^N_+)} \leqslant C_1 \|g\|_{W^{\frac{1}{p'},p}_{\ell}(\Gamma)}.$$

Then problem (P) is equivalent to

$$(3.4) \qquad \begin{cases} -\Delta v = f + \Delta u_g & \text{in} \quad \mathbb{R}^N_+, \\ v = 0 & \text{on} \quad \Gamma. \end{cases}$$

Set $h = f + \Delta u_g$. For any $\varphi \in W^{1,p'}_{-\ell}(\mathbb{R}^N)$ set

$$\sqcap\varphi(x', x_N) = \varphi(x', x_N) - \varphi(x', -x_N) \quad \text{if} \quad x_N > 0.$$

It is clear that $\sqcap\varphi \in \overset{\circ}{W}{}^{1,p'}_{-\ell}(\mathbb{R}^N_+)$. Then h can be extended to $h_\pi \in W^{-1,p}_{\ell}(\mathbb{R}^N)$ defined by

$$\varphi \in W^{1,p'}_{-\ell}(\mathbb{R}^N), \; h_\pi(\varphi) = \langle h, \sqcap\varphi \rangle_{W^{-1,p}_{\ell}(\mathbb{R}^N_+) \times W^{1,p'}_{-\ell}(\mathbb{R}^N_+)}.$$

Moreover,

$$\|h_\pi\|_{W^{-1,p}_{\ell}(\mathbb{R}^N)} = \|h\|_{W^{-1,p}_{\ell}(\mathbb{R}^N_+)}.$$

Let q be a polynomial in $P^{\Delta}_{[\ell+1-N/p']}$. We can write it in the form

$$q = r + s, \; r \in A^{\Delta}_{[\ell+1-N/p']} \text{ and } s \in N^{\Delta}_{[\ell+1-N/p]}.$$

Then,

$$\langle h_\pi, q \rangle = \langle f + \Delta u_g, r \rangle_{W^{-1,p}_{\ell}(\mathbb{R}^N_+) \times W^{1,p'}_{-\ell}(\mathbb{R}^N_+)}$$

and applying the Green formula we get

$$\begin{aligned} \langle \Delta u_g, r \rangle &= -\int_{\mathbb{R}^N_+} \nabla u_g \cdot \nabla r \, \mathrm{d}x \\ &= -\Big\langle g, \frac{\partial r}{\partial x_N} \Big\rangle_{W^{\frac{1}{p'},p}_{\ell}(\Gamma) \times W^{-\frac{1}{p'},p'}_{-\ell}(\Gamma)} \end{aligned}$$

(note that $\Delta r = 0$ in $\mathbb{R}^N_+$ and $r = 0$ on Γ). Thus, $h_\pi \in W^{-1,p}_{\ell}(\mathbb{R}^N)$ and it satisfies

$$\forall q \in P^{\Delta}_{[\ell+1-N/p']}, \; \langle h_\pi, q \rangle = 0.$$

Recall that (cf. [1]) since (3.1) holds, the operators

$$\begin{aligned} \Delta\colon \; W^{1,p}_{\ell}(\mathbb{R}^N) &\to W^{-1,p}_{\ell} \perp P^{\Delta}_{[\ell+1-\frac{N}{p'}]} \text{ if } \ell \geqslant 1, \\ \Delta\colon \; W^{1,p}_{0}(\mathbb{R}^N)/P_{[1-\frac{N}{p}]} &\to W^{-1,p}_{0}(\mathbb{R}^N) \perp P_{[1-\frac{N}{p'}]} \text{ if } \ell = 0 \end{aligned}$$

are isomorphisms. Hence, there exists $\tilde{v}$ in $W^{1,p}_{\ell}(\mathbb{R}^N)$ such that $-\Delta\tilde{v} = h_{\pi}$. Now we remark that the function $w = \frac{1}{2} \sqcap \tilde{v}$ belongs to $W^{1,p}_{\ell}(\mathbb{R}^N_+)$ and

$$-\Delta w = h \quad \text{in} \quad \mathbb{R}^N_+ \quad \text{and} \quad w = 0 \quad \text{on} \quad \Gamma,$$

i.e. w is a solution of (3.4). □

Remark. The kernel $A^{\Delta}_{[-\ell+1-N/p]}$ is reduced to $\{0\}$ if $\ell \geqslant 0$ and to $P_{[1-N/p]}$ if $\ell = 0$.

With similar arguments, we can prove the following theorem:

Theorem 3.2. *Let $\ell \geqslant 1$ be an integer and assume that*

$$\frac{N}{p} \notin \{1, \dots, -\ell\}. \tag{3.5}$$

Then for any f in $W^{-1,p}_{-\ell}(\mathbb{R}^N_+)$ and g in $W^{\frac{1}{p'},p}_{-\ell}(\Gamma)$, problem (P) has a unique solution $u \in W^{1,p}_{-\ell}(\mathbb{R}^N_+)/A^{\Delta}_{[\ell+1-N/p]}$ and there exists a constant C independent of u, f and g such that

$$\inf_{q \in A^{\Delta}_{[\ell+1-\frac{N}{p}]}} \|u + q\|_{W^{1,p}_{-\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1,p}_{-\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'},p}_{-\ell}(\Gamma)}).$$

Theorem 3.3. *Let m be a nonnegative integer, let g belong to $W^{\frac{1}{p'}+m,p}_{m}(\Gamma)$ and assume that*

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} \neq 1 \text{ or } m = 0, \tag{3.6}$$

or

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \cap W^{-1,p}_{0}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} = 1 \text{ and } m \neq 0. \tag{3.7}$$

Then problem (P) has a unique solution $u \in W^{1+m,p}_{m}(\mathbb{R}^N_+)$ and u satisfies

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\Gamma)}) \text{ if } \frac{N}{p'} \neq 1 \text{ or } m = 0$$

and

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{1,p}_{0}(\mathbb{R}^N_+)} + \|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\Gamma)}) \quad \text{if } \frac{N}{p'} = 1 \text{ and } m \neq 0.$$

P r o o f. First, we observe that for any integer $m \geqslant 0$ we have the inclusion

$$W_m^{-1+m,p}(\mathbb{R}_+^N) \subset W_0^{-1,p}(\mathbb{R}_+^N)$$

if $\frac{N}{p'} \neq 1$ or $m = 0$. Thus, under the assumptions (3.6) or (3.7) and thanks to Theorem 3.1, there exists a unique solution $u \in W_0^{1,p}(\mathbb{R}_+^N)$ of problem (P). Let us prove by induction that

$$(3.8) \qquad g \in W_m^{\frac{1}{p'}+m,p}(\Gamma) \text{ and } f \text{ satisfies (3.6) or (3.7)} \Longrightarrow u \in W_m^{m+1,p}(\mathbb{R}_+^N).$$

For $m = 0$, (3.8) is valid. Assume that (3.8) is valid for $0, 1, \ldots, m$ and suppose that $g \in W_{m+1}^{\frac{1}{p'}+m+1,p}(\Gamma)$ and $f \in W_{m+1}^{m,p}(\mathbb{R}_+^N)$ with $\frac{N}{p'} \neq 1$ (a similar argument can be used for f satisfying (3.7)). Let us prove that $u \in W_{m+1}^{m+2,p}(\mathbb{R}_+^N)$. We observe first that

$$W_{m+1}^{m,p}(\mathbb{R}_+^N) \subset W_m^{m-1,p}(\mathbb{R}_+^N) \text{ and } W_{m+1}^{\frac{1}{p'}+m+1,p}(\Gamma) \subset W_m^{\frac{1}{p'}+m,p}(\Gamma),$$

hence u belongs to $W_m^{m+1,p}(\mathbb{R}_+^N)$ thanks to the induction hypothesis. Now, for $i = 1, \ldots, N-1$,

$$\Delta(\varrho\partial_i u) = \varrho\partial_i f + \frac{2}{\varrho} r \cdot \nabla(\partial_i u) + \Big(\frac{2}{\varrho} + \frac{1}{\varrho^3}\Big)\partial_i u.$$

Thus, $\Delta(\varrho\partial_i u) \in W_m^{m-1,p}(\mathbb{R}_+^N)$ and $\gamma_0(\varrho\partial_i u) \in W_m^{m+1,p}(\mathbb{R}^{N-1})$. Applying the induction hypothesis, we can deduce that

$$\partial_i u \in W_{m+1}^{m+1,p}(\mathbb{R}_+^N) \text{ for } i = 1, \ldots, N-1.$$

It remains to prove that $v = \partial_N u \in W_{m+1}^{m+1,p}(\mathbb{R}_+^N)$. This is a consequence of the fact that v belongs to $W_m^{m,p}(\mathbb{R}_+^N)$ and

$$\partial_i\partial_N u = \partial_N\partial_i u \in W_{m+1}^{m,p}(\mathbb{R}_+^N), \ i = 1, \ldots, N-1,$$
$$\partial_N(\partial_N u) = \Delta u - \sum_{i=1}^{N-1} \partial_i^2 u \in W_{m+1}^{m,p}(\mathbb{R}_+^N).$$

We can conlude that $u \in W_{m+1}^{m+2,p}(\mathbb{R}_+^N)$. □

Corollary 3.4. *Let $\ell \geqslant 1$ and $m \geqslant 1$ be two integers.*
(i) *Under the assumption*

$$\frac{N}{p'} \notin \{1, \ldots, \ell+1\},$$

for any $f \in W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)$ *and* $g \in W^{\frac{1}{p'}+m,p}_{m+\ell}(\Gamma)$ *satisfying the compatibility condition* (3.2) *there exists a unique solution* $u \in W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)$ *of* (P) *and* u *satisfies*

$$\|u\|_{W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m+\ell}(\Gamma)})$$

where $C = C(m,p,\ell,N)$ *is a constant independent of* u, f *and* g.

(ii) *Under the assumption*

$$m \geqslant \ell \quad \text{or} \quad \frac{N}{p} \notin \{1,\ldots,\ell-m\},$$

for any $f \in W^{m-1,p}_{m-\ell}(\mathbb{R}^N_+)$ *and* $g \in W^{\frac{1}{p'}+m,p}_{m-\ell}(\Gamma)$ *there exists a unique solution* $u \in W^{m+1,p}_{m-\ell}(\mathbb{R}^N_+)/A^{\Delta}_{[1+\ell-N/p]}$ *of* (P) *and* u *satisfies*

$$\inf_{q\in A^{\Delta}_{[1+\ell-N/p]}} \|u+q\|_{W^{m+1,p}_{m-\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{m-1,p}_{m-\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m-\ell}(\Gamma)}).$$

References

[1] *C. Amrouche, V. Girault, J. Giroire*: Weighted Sobolev spaces for Laplace's equation in $\mathbb{R}^N$. J. Math. Pures Appl. *73* (1994), 579–606.

[2] *T. Z. Boulmezaoud*: Espaces de Sobolev avec poids pour l'équation de Laplace dans le demi-espace. C. R. Acad. Sci. Paris, Ser. I *328* (1999), 221–226.

[3] *B. Hanouzet*: Espaces de Sobolev avec poids. Application au problème de Dirichlet dans un demi-espace. Rend. Sem. Univ. Padova *46* (1971), 227–272.

[4] *V. G. Maz'ya, B. A. Plamenevskii, L. I. Stupyalis*: The three-dimensional problem of steady-state motion of a fluid with a free surface. Amer. Math. Soc. Transl. *123* (1984), 171–268.

[5] *C. C. Simader, H. Sohr*: The Dirichlet Problem for the Laplacian in Bounded and Unbounded Domains. Pitman Research Notes in Matehmatics Series 360, Addison Wesley Longman, Harlow, Great Britain, 1996.

Authors' addresses: *Cherif Amrouche*, Université de Pau et des Pays de L'Adour, Laboratoire de Mathématiques Appliquées, I.P.R.A, Av. de l'Université, 6400 Pau, France, e-mail: `cherif.amrouche@univ-pau.fr`; *Šárka Nečasová*, Mathematical Institute of the Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: `matus@math.cas.cz`.

Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 244 (2008) 887–915
Journal of Differential Equations
www.elsevier.com/locate/jde

Very weak, generalized and strong solutions to the Stokes system in the half-space

Chérif Amrouche [a,*], Šárka Nečasová [b,1], Yves Raudin [a]

[a] *Laboratoire de Mathématiques Appliquées, CNRS UMR 5142, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'Université, 64000 Pau, France*

[b] *Mathematical Institute of the Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic*

Received 5 April 2007; revised 19 July 2007

Available online 26 November 2007

Abstract

In this paper, we study the Stokes system in the half-space $\mathbb{R}^N_+$, with $N \geqslant 2$. We give existence and uniqueness results in weighted Sobolev spaces. After the central case of the generalized solutions, we are interested in strong solutions and symmetrically in very weak solutions by means of a duality argument.

MSC: 35J50; 35J55; 35Q30; 76D07; 76N10

Keywords: Stokes problem; Half-space; Weighted Sobolev spaces

1. Introduction

The purpose of this paper is the resolution of the Stokes system

$$(S^+) \quad \begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \mathbb{R}^N_+, \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \mathbb{R}^N_+, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma = \mathbb{R}^{N-1}, \end{cases}$$

* Corresponding author.

E-mail addresses: cherif.amrouche@univ-pau.fr (C. Amrouche), matus@math.cas.cz (Š. Nečasová), yves.raudin@etud.univ-pau.fr (Y. Raudin).

1 The work of the author was supported by the grant No. 201050005 from the Grant Agency of the Czech Republic and by Institutional Research Plan No. AV0Z10190503.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.10.007

with data and solutions which live in weighted Sobolev spaces, expressing at the same time their regularity and their behavior at infinity. We will naturally base on the previously established results on the harmonic and biharmonic operators (see [5–8]). We will also concentrate on the basic weights because they are the most usual and they avoid the question of the kernel for this operator and symmetrically the compatibility condition for the data. In a forthcoming work, we will complete these results for the other types of weights in this class of spaces.

Among the first works on the Stokes problem in the half-space, we can cite Cattabriga. In [11], he appeals to the potential theory to explicitly get the velocity and pressure fields. For the homogeneous problem ($\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$), for instance, he shows that if $\boldsymbol{g} \in \boldsymbol{L}^p(\Gamma)$ and the semi-norm $|\boldsymbol{g}|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)} < \infty$, then $\nabla \boldsymbol{u} \in \boldsymbol{L}^p(\mathbb{R}^N_+)$ and $\pi \in L^p(\mathbb{R}^N_+)$.

Similar results are given by Farwig and Sohr (see [12]) and Galdi (see [14]), who also have chosen the setting of homogeneous Sobolev spaces. On the other hand, Maz'ya, Plamenevskiĭ and Stupyalis (see [18]), work within the suitable setting of weighted Sobolev spaces and consider different sorts of boundary conditions. However, their results are limited to the dimension 3 and to the Hilbertian framework in which they give generalized and strong solutions. This is also the case of Boulmezaoud (see [10]), who only gives strong solutions. Otherwise, always in dimension 3, by Fourier analysis techniques, Tanaka considers the case of very regular data, corresponding to velocities which belong to $\boldsymbol{W}_2^{m+3,2}(\mathbb{R}^3_+)$, with $m \geqslant 0$ (see [19]).

Let us also quote, for the evolution Stokes or Navier–Stokes problems, Fujigaki and Miyakawa (see [13]), who are interested in the behavior in $t \to +\infty$; Bochers and Miyakawa (see [9]) and Kozono (see [17]), for the L^N-decay property; Ukai (see [20]), for the L^p–L^q estimates and Giga (see [15]), for the estimates in Hardy spaces.

This paper is organized as follows. Section 2 is devoted to the notations, functional setting and recalls about the Stokes system in the whole space. In Section 3, we give some results on homogeneous problems with singular boundary conditions and we complete them by Theorem 3.5 with a detailed proof, which is a model for analogous results. In Section 4, we start our study of the Stokes system in the half-space by the central case of generalized solutions which is the pivot of this work. In Section 5, we consider the strong solutions and give regularity results according to the data. In Section 6, we find very weak solutions to the homogeneous problem with singular boundary conditions. The main results of this paper are Theorem 4.2 for the generalized solutions, Theorems 5.2 and 5.6 for the strong solutions, Theorems 6.7 and 6.9 for the very weak solutions.

2. Notations, functional framework and known results

2.1. Notations

For any real number $p > 1$, we always take p' to be the Hölder conjugate of p, i.e.

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$. Writing a typical point $x \in \mathbb{R}^N$ as $x = (x', x_N)$, where $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$, we will especially look on the upper half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N;\ x_N > 0\}$. We let $\overline{\mathbb{R}^N_+}$ denote the closure of $\mathbb{R}^N_+$ in $\mathbb{R}^N$ and let $\Gamma = \{x \in \mathbb{R}^N;\ x_N = 0\} \equiv$

$\mathbb{R}^{N-1}$ denote its boundary. Let $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$ denote the Euclidean norm of x, we will use two basic weights

$$\varrho = \left(1 + |x|^2\right)^{1/2} \quad \text{and} \quad \lg \varrho = \ln\left(2 + |x|^2\right).$$

We denote by ∂_i the partial derivative $\frac{\partial}{\partial x_i}$, similarly $\partial_i^2 = \partial_i \circ \partial_i = \frac{\partial^2}{\partial x_i^2}$, $\partial_{ij}^2 = \partial_i \circ \partial_j = \frac{\partial^2}{\partial x_i \partial x_j}, \ldots$. More generally, if $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{N}^N$ is a multi-index, then

$$\partial^\lambda = \partial_1^{\lambda_1} \cdots \partial_N^{\lambda_N} = \frac{\partial^{|\lambda|}}{\partial x_1^{\lambda_1} \cdots \partial x_N^{\lambda_N}}, \quad \text{where } |\lambda| = \lambda_1 + \cdots + \lambda_N.$$

In the sequel, for any integer q, we will use the following polynomial spaces:

- $\mathcal{P}_q$ is the space of polynomials of degree smaller than or equal to q;
- $\mathcal{P}_q^\Delta$ is the subspace of harmonic polynomials of $\mathcal{P}_q$;
- $\mathcal{P}_q^{\Delta^2}$ is the subspace of biharmonic polynomials of $\mathcal{P}_q$;
- $\mathcal{A}_q^\Delta$ is the subspace of polynomials of $\mathcal{P}_q^\Delta$, odd with respect to x_N, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$;
- $\mathcal{N}_q^\Delta$ is the subspace of polynomials of $\mathcal{P}_q^\Delta$, even with respect to x_N, or equivalently, which satisfy the condition $\partial_N \varphi(x', 0) = 0$,

with the convention that these spaces are reduced to $\{0\}$ if $q < 0$. For any real number s, we denote by $[s]$ the integer part of s.

Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X, i.e.

$$B' \perp X = \left\{f \in B';\ \forall v \in X,\ \langle f, v\rangle = 0\right\} = (B/X)'.$$

Lastly, if $k \in \mathbb{Z}$, we will constantly use the notation $\{1, \ldots, k\}$ for the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

2.2. *Weighted Sobolev spaces*

For any nonnegative integer m, real numbers $p > 1$, α and β, we define the following space:

$$\begin{aligned} W_{\alpha,\beta}^{m,p}(\Omega) = \Big\{u \in \mathcal{D}'(\Omega);\ & 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-m+|\lambda|}(\lg \varrho)^{\beta-1}\partial^\lambda u \in L^p(\Omega); \\ & k+1 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha-m+|\lambda|}(\lg \varrho)^{\beta}\partial^\lambda u \in L^p(\Omega)\Big\}, \end{aligned} \tag{2.1}$$

where

$$k = \begin{cases} -1 & \text{if } \frac{N}{p} + \alpha \notin \{1, \ldots, m\}, \\ m - \frac{N}{p} - \alpha & \text{if } \frac{N}{p} + \alpha \in \{1, \ldots, m\}. \end{cases}$$

Note that $W_{\alpha,\beta}^{m,p}(\Omega)$ is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{0\leqslant|\lambda|\leqslant k} \big\|\varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta-1}\partial^\lambda u\big\|^p_{L^p(\Omega)} + \sum_{k+1\leqslant|\lambda|\leqslant m} \big\|\varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta}\partial^\lambda u\big\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

We also define the semi-norm

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda|=m} \big\|\varrho^{\alpha}(\lg\varrho)^{\beta}\partial^\lambda u\big\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

The weights in the definition (2.1) are chosen so that the corresponding space satisfies two fundamental properties. On the one hand, $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. On the other hand, the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ (see [5, Theorem 1.1]): if

$$\frac{N}{p} + \alpha \notin \{1,\dots,m\} \quad \text{or} \quad (\beta-1)p \neq -1, \tag{2.2}$$

then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_{\alpha,\beta}\big(\mathbb{R}^N_+\big), \quad \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathcal{P}_{q^*}} \leqslant C|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}, \tag{2.3}$$

with $q^* = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. Now, we define the space

$$\mathring{W}^{m,p}_{\alpha,\beta}\big(\mathbb{R}^N_+\big) = \overline{\mathcal{D}\big(\mathbb{R}^N_+\big)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}};$$

which will be characterized in Lemma 2.2 as the subspace of functions with null traces in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. From that, we can introduce the space $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$ as the dual space of $\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. In addition, under the assumption (2.2), $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ which is equivalent to the full norm $\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$. We will now recall some properties of the weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings:

$$W^{m,p}_{\alpha,\beta}\big(\mathbb{R}^N_+\big) \hookrightarrow W^{m-1,p}_{\alpha-1,\beta}\big(\mathbb{R}^N_+\big) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta}\big(\mathbb{R}^N_+\big) \quad \text{if } \frac{N}{p} + \alpha \notin \{1,\dots,m\}.$$

When $\frac{N}{p} + \alpha = j \in \{1,\dots,m\}$, then we have

$$W^{m,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{m-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{m-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-m,\beta-1}.$$

Note that in the first case, for any $\gamma \in \mathbb{R}$ such that $\frac{N}{p} + \alpha - \gamma \notin \{1,\dots,m\}$ and $m \in \mathbb{N}$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}\big(\mathbb{R}^N_+\big) \longmapsto \varrho^\gamma u \in W^{m,p}_{\alpha-\gamma,\beta}\big(\mathbb{R}^N_+\big)$$

is an isomorphism. In both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \longmapsto \partial^\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$$

is continuous. Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m - (\frac{N}{p} + \alpha) - 1, & \text{if } \begin{cases} \frac{N}{p} + \alpha \in \{1, \ldots, m\} \text{ and } (\beta - 1)p \geqslant -1, & \text{or} \\ \frac{N}{p} + \alpha \in \{j \in \mathbb{Z};\ j \leqslant 0\} \text{ and } \beta p \geqslant -1, \end{cases} \\ [m - (\frac{N}{p} + \alpha)], & \text{otherwise.} \end{cases} \tag{2.4}$$

Remark 2.1. In the case $\beta = 0$, we simply denote the space $W^{m,p}_{\alpha,0}(\Omega)$ by $W^{m,p}_{\alpha}(\Omega)$. In [16], Hanouzet introduced a class of weighted Sobolev spaces without logarithmic factors, with the same notation. We recall his definition under the notation $H^{m,p}_{\alpha}(\Omega)$:

$$H^{m,p}_{\alpha}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ 0 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha - m + |\lambda|} \partial^\lambda u \in L^p(\Omega)\}.$$

It is clear that if $\frac{N}{p} + \alpha \notin \{1, \ldots, m\}$, we have $W^{m,p}_{\alpha}(\Omega) = H^{m,p}_{\alpha}(\Omega)$. The fundamental difference between these two families of spaces is that the assumption (2.2) and thus the Poincaré-type inequality (2.3), hold for any value of (N, p, α) in $W^{m,p}_{\alpha}(\Omega)$, but not in $H^{m,p}_{\alpha}(\Omega)$ if $\frac{N}{p} + \alpha \in \{1, \ldots, m\}$.

2.3. *The spaces of traces*

In order to define the traces of functions of $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$ (here we do not consider the case $\beta \neq 0$), for any $\sigma \in]0, 1[$, we introduce the space

$$W^{\sigma,p}_0(\mathbb{R}^N) = \left\{u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma} u \in L^p(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\sigma p}}\, dx\, dy < \infty\right\}, \tag{2.5}$$

where $w = \varrho$ if $N/p \neq \sigma$ and $w = \varrho(\lg \varrho)^{1/\sigma}$ if $N/p = \sigma$. It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W^{\sigma,p}_0(\mathbb{R}^N)} = \left(\left\|\frac{u}{w^\sigma}\right\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+\sigma p}}\, dx\, dy\right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space

$$W^{\sigma,p}_{\alpha}(\mathbb{R}^N) = \left\{u \in \mathcal{D}'(\mathbb{R}^N);\ w^{\alpha-\sigma} u \in L^p(\mathbb{R}^N),\right.$$
$$\left.\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varrho^\alpha(x) u(x) - \varrho^\alpha(y) u(y)|^p}{|x - y|^{N+\sigma p}}\, dx\, dy < \infty\right\},$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho(\lg \varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$W_\alpha^{s,p}(\mathbb{R}^N) = \big\{u \in \mathcal{D}'(\mathbb{R}^N);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-s+|\lambda|}(\lg \varrho)^{-1}\partial^\lambda u \in L^p(\mathbb{R}^N);\\ k+1 \leqslant |\lambda| \leqslant [s]-1,\ \varrho^{\alpha-s+|\lambda|}\partial^\lambda u \in L^p(\mathbb{R}^N);\ |\lambda| = [s],\ \partial^\lambda u \in W_\alpha^{\sigma,p}(\mathbb{R}^N)\big\},$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \dots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm

$$\|u\|_{W_\alpha^{s,p}(\mathbb{R}^N)} = \bigg(\sum_{0 \leqslant |\lambda| \leqslant k} \big\|\varrho^{\alpha-s+|\lambda|}(\lg \varrho)^{-1}\partial^\lambda u\big\|^p_{L^p(\mathbb{R}^N)}\\ + \sum_{k+1 \leqslant |\lambda| \leqslant [s]-1} \big\|\varrho^{\alpha-s+|\lambda|}\partial^\lambda u\big\|^p_{L^p(\mathbb{R}^N)}\bigg)^{1/p}\\ + \sum_{|\lambda|=[s]} \big\|\partial^\lambda u\big\|_{W_\alpha^{\sigma,p}(\mathbb{R}^N)}.$$

We can similarly define, for any real number β, the space

$$W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) = \big\{v \in \mathcal{D}'(\mathbb{R}^N);\ (\lg \varrho)^\beta v \in W_\alpha^{s,p}(\mathbb{R}^N)\big\}.$$

We can prove some properties of the weighted Sobolev spaces $W_{\alpha,\beta}^{s,p}(\mathbb{R}^N)$. We have the algebraic and topological imbeddings in the case where $N/p + \alpha \notin \{\sigma, \dots, \sigma + [s] - 1\}$:

$$W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha-1,\beta}^{s-1,p}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W_{\alpha-[s],\beta}^{\sigma,p}(\mathbb{R}^N),$$
$$W_{\alpha,\beta}^{s,p}(\mathbb{R}^N) \hookrightarrow W_{\alpha+[s]-s,\beta}^{[s],p}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W_{\alpha-s,\beta}^{0,p}(\mathbb{R}^N).$$

When $N/p + \alpha = j \in \{\sigma, \dots, \sigma + [s] - 1\}$, then we have

$$W_{\alpha,\beta}^{s,p} \hookrightarrow \cdots \hookrightarrow W_{\alpha-j+1,\beta}^{s-j+1,p} \hookrightarrow W_{\alpha-j,\beta-1}^{s-j,p} \hookrightarrow \cdots \hookrightarrow W_{\alpha-[s],\beta-1}^{\sigma,p},$$
$$W_{\alpha,\beta}^{s,p} \hookrightarrow W_{\alpha+[s]-s,\beta}^{[s],p} \hookrightarrow \cdots \hookrightarrow W_{\alpha-\sigma-j+1,\beta}^{[s]-j+1,p} \hookrightarrow W_{\alpha-\sigma-j,\beta-1}^{[s]-j,p} \hookrightarrow \cdots \hookrightarrow W_{\alpha-s,\beta-1}^{0,p}.$$

If u is a function on $\mathbb{R}^N_+$, we denote its trace of order j on the hyperplane Γ by

$$\forall j \in \mathbb{N}, \quad \gamma_j u : x' \in \mathbb{R}^{N-1} \longmapsto \partial_N^j u(x', 0).$$

Let us recall the following trace lemma due to Hanouzet (see [16]) and extended by Amrouche and Nečasová (see [5]) to this class of weighted Sobolev spaces:

Lemma 2.2. *For any integer $m \geqslant 1$ and real number α, the mapping*

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{m-1}) : \mathcal{D}(\overline{\mathbb{R}^N_+}) \longrightarrow \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1})$$

can be extended to a linear continuous mapping, still denoted by γ,

$$\gamma : W_\alpha^{m,p}(\mathbb{R}_+^N) \longrightarrow \prod_{j=0}^{m-1} W_\alpha^{m-j-1/p,p}(\mathbb{R}^{N-1}).$$

Moreover γ *is surjective and* $\operatorname{Ker}\gamma = \mathring{W}_\alpha^{m,p}(\mathbb{R}_+^N)$.

2.4. The Stokes system in the whole space

On the Stokes problem in $\mathbb{R}^N$

$$(S): \quad -\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in } \mathbb{R}^N,$$

let us recall the fundamental result on which we are based in the sequel. First, for any $k \in \mathbb{Z}$, we introduce the space

$$\mathcal{S}_k = \{(\boldsymbol{\lambda}, \mu) \in \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_{k-1}^\Delta;\ \operatorname{div}\boldsymbol{\lambda} = 0,\ -\Delta\boldsymbol{\lambda} + \nabla\mu = \boldsymbol{0}\}.$$

Theorem 2.3. *(See Alliot and Amrouche [1].) Let* $\ell \in \mathbb{Z}$ *and assume that*

$$N/p' \notin \{1, \dots, \ell\} \quad \textit{and} \quad N/p \notin \{1, \dots, -\ell\}.$$

For any $(\boldsymbol{f}, g) \in (\boldsymbol{W}_\ell^{-1,p}(\mathbb{R}^N) \times W_\ell^{0,p}(\mathbb{R}^N)) \perp \mathcal{S}_{[1+\ell-N/p']}$, *problem* (S) *admits a solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_\ell^{1,p}(\mathbb{R}^N) \times W_\ell^{0,p}(\mathbb{R}^N)$, *unique up to an element of* $\mathcal{S}_{[1-\ell-N/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{S}_{[1-\ell-N/p]}} \left(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}_\ell^{1,p}(\mathbb{R}^N)} + \|\pi + \mu\|_{W_\ell^{0,p}(\mathbb{R}^N)}\right) \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_\ell^{-1,p}(\mathbb{R}^N)} + \|g\|_{W_\ell^{0,p}(\mathbb{R}^N)}\right).$$

We also have the following result for more regular data:

Theorem 2.4. *(See Alliot and Amrouche [1].) Let* $\ell \in \mathbb{Z}$ *and* $m \geqslant 1$ *be two integers and assume that*

$$N/p' \notin \{1, \dots, \ell + 1\} \quad \textit{and} \quad N/p \notin \{1, \dots, -\ell - m\}.$$

For any $(\boldsymbol{f}, g) \in (\boldsymbol{W}_{m+\ell}^{m-1,p}(\mathbb{R}^N) \times W_{m+\ell}^{m,p}(\mathbb{R}^N)) \perp \mathcal{S}_{[1+\ell-N/p']}$, *problem* (S) *admits a solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbb{R}^N) \times W_{m+\ell}^{m,p}(\mathbb{R}^N)$, *unique up to an element of* $\mathcal{S}_{[1-\ell-N/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\lambda},\mu)\in\mathcal{S}_{[1-\ell-N/p]}} \left(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}_{m+\ell}^{m+1,p}(\mathbb{R}^N)} + \|\pi + \mu\|_{W_{m+\ell}^{m,p}(\mathbb{R}^N)}\right) \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}_{m+\ell}^{m-1,p}(\mathbb{R}^N)} + \|g\|_{W_{m+\ell}^{m,p}(\mathbb{R}^N)}\right).$$

Note that if we suppose $\ell = 0$, then $\mathcal{S}_{[1-N/p']} = \boldsymbol{\mathcal{P}}_{[1-N/p']} \times \{0\}$ and the orthogonality condition $(\boldsymbol{f}, g) \perp \mathcal{S}_{[1-N/p']}$ is equivalent to $\boldsymbol{f} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$.

3. Homogeneous problems with singular boundary conditions

The way we will take to solve the Stokes system is based on the existence of very weak solutions to homogeneous problems with singular boundary conditions. The first one is the biharmonic problem: find $u \in W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$ solution to the problem

$$(\mathcal{P}):\quad \Delta^2 u = 0 \quad \text{in } \mathbb{R}^N_+, \quad u = g_0 \quad \text{and} \quad \partial_N u = g_1 \quad \text{on } \Gamma,$$

where $g_0 \in W^{1-1/p,p}_{\ell-1}(\Gamma)$ and $g_1 \in W^{-1/p,p}_{\ell-1}(\Gamma)$ are given. We begin to define for any integer q, the polynomial space $\mathcal{B}_q$ as follows:

$$\mathcal{B}_q = \big\{u \in \mathcal{P}^{\Delta^2}_q;\ u = \partial_N u = 0 \text{ on } \Gamma\big\}.$$

Theorem 3.1. *(See Amrouche and Raudin [8].) Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \dots, \ell - 1\} \quad \textit{and} \quad N/p \notin \{1, \dots, -\ell + 1\}. \tag{3.1}$$

For any $g_0 \in W^{1-1/p,p}_{\ell-1}(\Gamma)$ and $g_1 \in W^{-1/p,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{B}_{[2+\ell-N/p']}, \quad \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi\rangle_\Gamma = 0, \tag{3.2}$$

problem $(\mathcal{P})$ admits a solution $u \in W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{B}_{[2-\ell-N/p]}$, with the estimate

$$\inf_{q \in \mathcal{B}_{[2-\ell-N/p]}} \|u + q\|_{W^{1,p}_{\ell-1}(\mathbb{R}^N_+)} \leqslant C\big(\|g_0\|_{W^{1-1/p,p}_{\ell-1}(\Gamma)} + \|g_1\|_{W^{-1/p,p}_{\ell-1}(\Gamma)}\big).$$

Remark 3.2. (i) In the case where $\ell = 1$, if $1 - N/p' < 0$, then $\mathcal{B}_{[3-N/p']} = \{0\}$ and if $1 - N/p' \geqslant 0$, then $\mathcal{B}_{[3-N/p']} = \mathcal{B}_2 = \mathbb{R}x_N^2$.

(ii) We also established a result for the lower case, with $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$, but we do not use it in this paper.

We will also need a result of this type about the Neumann problem for the Laplacian: find $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$ satisfying the problem

$$(\mathcal{Q}):\quad \Delta u = 0 \quad \text{in } \mathbb{R}^N_+ \quad \text{and} \quad \partial_N u = g \quad \text{on } \Gamma,$$

where $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$.

Theorem 3.3. *(See Amrouche [6].) Let $\ell \in \mathbb{Z}$ and assume that*

$$N/p' \notin \{1, \dots, \ell - 2\} \quad \textit{and} \quad N/p \notin \{1, \dots, -\ell + 2\}. \tag{3.3}$$

For any $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{N}^{\Delta}_{[\ell-N/p']}, \quad \langle g, \varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma) \times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = 0, \tag{3.4}$$

problem ($\mathcal{Q}$) *admits a solution* $u \in W^{0,p}_{\ell-2}(\mathbb{R}^N_+)$*, unique up to an element of* $\mathcal{N}^{\Delta}_{[2-\ell-N/p]}$*, with the estimate*

$$\inf_{q\in\mathcal{N}^{\Delta}_{[2-\ell-N/p]}} \|u+q\|_{W^{0,p}_{\ell-2}(\mathbb{R}^N_+)} \leqslant C\|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}.$$

With the same arguments as for Theorem 3.3, we can prove an intermediate result for this problem:

Theorem 3.4. *Let* $\ell \in \mathbb{Z}$*. Under hypothesis* (3.1)*, for any* $g \in W^{-1/p,p}_{\ell-1}(\Gamma)$ *satisfying the compatibility condition* (3.4)*, problem* ($\mathcal{Q}$) *admits a solution* $u \in W^{1,p}_{\ell-1}(\mathbb{R}^N_+)$*, unique up to an element of* $\mathcal{N}^{\Delta}_{[2-\ell-N/p]}$*, with the estimate*

$$\inf_{q\in\mathcal{N}^{\Delta}_{[2-\ell-N/p]}} \|u+q\|_{W^{1,p}_{\ell-1}(\mathbb{R}^N_+)} \leqslant C\|g\|_{W^{-1/p,p}_{\ell-1}(\Gamma)}.$$

Now, we will establish a similar result about the Dirichlet problem for the Laplacian with very singular boundary conditions: find $u \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)$ satisfying the problem

$$(\mathcal{R}):\quad \Delta u = 0 \quad \text{in } \mathbb{R}^N_+ \quad \text{and} \quad u = g \quad \text{on } \Gamma,$$

where $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$.

Theorem 3.5. *Let* $\ell \in \mathbb{Z}$*. Under hypothesis* (3.3)*, for any* $g \in W^{-1-1/p,p}_{\ell-2}(\Gamma)$ *satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{A}^{\Delta}_{[1+\ell-N/p']},\quad \langle g, \partial_N\varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = 0, \tag{3.5}$$

problem ($\mathcal{R}$) *admits a solution* $u \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)$*, unique up to an element of* $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$*, with the estimate*

$$\inf_{q\in\mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u+q\|_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} \leqslant C\|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}.$$

Firstly, we must give a meaning to traces for a special class of distributions. We introduce the spaces

$$Y_\ell\big(\mathbb{R}^N_+\big) = \big\{v \in W^{-1,p}_{\ell-2}\big(\mathbb{R}^N_+\big);\ \Delta v \in W^{0,p}_{\ell+1}\big(\mathbb{R}^N_+\big)\big\},$$
$$Y_{\ell,1}\big(\mathbb{R}^N_+\big) = \big\{v \in W^{-1,p}_{\ell-2}\big(\mathbb{R}^N_+\big);\ \Delta v \in W^{0,p}_{\ell+1,1}\big(\mathbb{R}^N_+\big)\big\}.$$

They are reflexive Banach spaces equipped with their natural norms:

$$\|v\|_{Y_\ell(\mathbb{R}^N_+)} = \|v\|_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta v\|_{W^{0,p}_{\ell+1}(\mathbb{R}^N_+)},$$
$$\|v\|_{Y_{\ell,1}(\mathbb{R}^N_+)} = \|v\|_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \|\Delta v\|_{W^{0,p}_{\ell+1,1}(\mathbb{R}^N_+)}.$$

Lemma 3.6. *Let $\ell \in \mathbb{Z}$. Under hypothesis* (3.3)*, the space $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $Y_\ell(\mathbb{R}^N_+)$ and in $Y_{\ell,1}(\mathbb{R}^N_+)$.*

Proof. For every continuous linear form $T \in (Y_\ell(\mathbb{R}^N_+))'$, there exists a unique pair $(f,g) \in \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \times W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)$, such that

$$\forall v \in Y_\ell\big(\mathbb{R}^N_+\big), \quad \langle T, v\rangle = \langle f, v\rangle_{\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \int\limits_{\mathbb{R}^N_+} g\,\Delta v\,\mathrm{d}x. \tag{3.6}$$

Thanks to the Hahn–Banach theorem, it suffices to show that any T which vanishes on $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is actually zero on $Y_\ell(\mathbb{R}^N_+)$. Let us suppose that $T = 0$ on $\mathcal{D}(\overline{\mathbb{R}^N_+})$, thus on $\mathcal{D}(\mathbb{R}^N_+)$. Then we can deduce from (3.6) that

$$f + \Delta g = 0 \quad \text{in } \mathbb{R}^N_+,$$

hence we have $\Delta g \in \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Let $\tilde{f} \in W^{1,p'}_{-\ell+2}(\mathbb{R}^N)$ and $\tilde{g} \in W^{0,p'}_{-\ell-1}(\mathbb{R}^N)$ be respectively the extensions by 0 of f and g to $\mathbb{R}^N$. Thanks to (3.6), it is clear that $\tilde{f} + \Delta\tilde{g} = 0$ in $\mathbb{R}^N$, and thus $\Delta\tilde{g} \in W^{1,p'}_{-\ell+2}(\mathbb{R}^N)$. Now, thanks to the isomorphism results for the Laplace operator in $\mathbb{R}^N$ (see [4]), we can deduce that $\tilde{g} \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N)$, under hypothesis (3.3). Since $\tilde{g}$ is an extension by 0, it follows that $g \in \mathring{W}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Then, by density of $\mathcal{D}(\mathbb{R}^N_+)$ in $\mathring{W}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^N_+)$ such that $\varphi_k \to g$ in $\mathring{W}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$. Thus, for any $v \in Y_\ell(\mathbb{R}^N_+)$, we have

$$\begin{aligned}
\langle T, v\rangle &= \langle -\Delta g, v\rangle_{\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \langle g, \Delta v\rangle_{\mathring{W}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)} \\
&= \lim_{k\to\infty}\big\{\langle -\Delta\varphi_k, v\rangle_{\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)} + \langle \varphi_k, \Delta v\rangle_{\mathring{W}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)\times W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)}\big\} \\
&= \lim_{k\to\infty}\Bigg\{-\int\limits_{\mathbb{R}^N_+} \varphi_k\,\Delta v\,\mathrm{d}x + \int\limits_{\mathbb{R}^N_+} \varphi_k\,\Delta v\,\mathrm{d}x\Bigg\} \\
&= 0,
\end{aligned}$$

i.e. T is identically zero.

For the density of $\mathcal{D}(\overline{\mathbb{R}^N_+})$ in $Y_{\ell,1}(\mathbb{R}^N_+)$, the only difference in the proof concerns the logarithmic factors in the weights, with $g \in W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)$. □

Thanks to this density lemma, we can prove the following result of traces:

Lemma 3.7. *Let $\ell \in \mathbb{Z}$. Under hypothesis* (3.3)*, the trace mapping $\gamma_0 : \mathcal{D}(\overline{\mathbb{R}^N_+}) \to \mathcal{D}(\mathbb{R}^{N-1})$, can be extended to a linear continuous mapping*

$$\gamma_0 : Y_\ell(\mathbb{R}^N_+) \longrightarrow W^{-1-1/p,p}_{\ell-2}(\Gamma) \quad \text{if } N/p' \notin \{\ell-1, \ell, \ell+1\},$$
$$\big(\text{respectively } \gamma_0 : Y_{\ell,1}(\mathbb{R}^N_+) \longrightarrow W^{-1-1/p,p}_{\ell-2}(\Gamma) \quad \text{if } N/p' \in \{\ell-1, \ell, \ell+1\}\big).$$

Moreover, we have the following Green formula

$$\forall v \in Y_\ell(\mathbb{R}^N_+),\ \forall \varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \quad \text{such that} \quad \varphi = \Delta\varphi = 0 \quad \text{on } \Gamma,$$
$$\langle \Delta v, \varphi\rangle_{W^{0,p}_{\ell+1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)} - \langle v, \Delta\varphi\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}$$
$$= \langle v, \partial_N\varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} \tag{3.7}$$

(*respectively the Green formula for* $v \in Y_{\ell,1}(\mathbb{R}^N_+)$, *where the first term of the left-hand side is replaced by* $\langle \Delta v, \varphi\rangle_{W^{0,p}_{\ell+1,1}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)}$).

Proof. Firstly, let us remark that for any $\varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$, the boundary condition $\varphi = \Delta\varphi = 0$ on Γ is equivalent to $\varphi = \partial^2_N\varphi = 0$ on Γ. Moreover, if $N/p' \notin \{\ell-1, \ell, \ell+1\}$, we have the imbedding $W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-1}(\mathbb{R}^N_+)$. So we can write the following Green formula:

$$\forall v \in \mathcal{D}(\overline{\mathbb{R}^N_+}),\ \forall \varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \quad \text{such that} \quad \varphi = \Delta\varphi = 0 \quad \text{on } \Gamma,$$
$$\int_{\mathbb{R}^N_+} \varphi\Delta v\,\mathrm{d}x - \int_{\mathbb{R}^N_+} v\Delta\varphi\,\mathrm{d}x = \int_\Gamma v\partial_N\varphi\,\mathrm{d}x'. \tag{3.8}$$

Since $\Delta\varphi = 0$ on Γ, we have the identity

$$\int_{\mathbb{R}^N_+} v\Delta\varphi\,\mathrm{d}x = \langle v, \Delta\varphi\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

This implies

$$\big|\langle v, \partial_N\varphi\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}\big| \leqslant \|v\|_{Y_\ell(\mathbb{R}^N_+)}\|\varphi\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)}.$$

By Lemma 2.2, for any $\mu \in W^{2-1/p',p'}_{-\ell+2}(\Gamma)$, there exists a lifting function $\varphi \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $\varphi = 0$, $\partial_N\varphi = \mu$ and $\partial^2_N\varphi = 0$ on Γ, satisfying

$$\|\varphi\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} \leqslant C\|\mu\|_{W^{2-1/p',p'}_{-\ell+2}(\Gamma)},$$

where C is a constant not depending on φ and μ. Then we can deduce that

$$\|\gamma_0 v\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)} \leqslant C\|v\|_{Y_\ell(\mathbb{R}^N_+)}.$$

Thus the linear mapping $\gamma_0 : v \mapsto v|_\Gamma$ defined on $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is continuous for the norm of $Y_\ell(\mathbb{R}^N_+)$. Since $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $Y_\ell(\mathbb{R}^N_+)$, γ_0 can be extended by continuity to a mapping still called

$\gamma_0 \in \mathcal{L}(Y_\ell(\mathbb{R}^N_+);\ W^{-1-1/p,p}_{\ell-2}(\Gamma))$. Moreover, we also can deduce the formula (3.7) from (3.8) by density of $\mathcal{D}(\overline{\mathbb{R}^N_+})$ in $Y_\ell(\mathbb{R}^N_+)$. To finish, note that if $N/p' \in \{\ell-1, \ell, \ell+1\}$, we only have the imbedding $W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell-1,-1}(\mathbb{R}^N_+)$, hence the necessity to introduce the space $Y_{\ell,1}(\mathbb{R}^N_+)$ and the corresponding Green formula with logarithmic factors for these three critical values. □

Proof of Theorem 3.5. We can observe that solve problem $(\mathcal{R})$ is equivalent to find $u \in Y_\ell(\mathbb{R}^N_+)$ if $N/p' \notin \{\ell-1, \ell, \ell+1\}$ (respectively $u \in Y_{\ell,1}(\mathbb{R}^N_+)$ if $N/p' \in \{\ell-1, \ell, \ell+1\}$), satisfying

$$\forall v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \quad \text{such that} \quad v = \Delta v = 0 \quad \text{on } \Gamma,$$
$$\langle u, \Delta v\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)} = -\langle g, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}. \tag{3.9}$$

Indeed the direct implication is straightforward. Conversely, if u satisfies (3.9) then we have for any $\varphi \in \mathcal{D}(\mathbb{R}^N_+)$,

$$\langle \Delta u, \varphi\rangle_{W^{-3,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} = \langle u, \Delta\varphi\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)} = 0,$$

thus $\Delta u = 0$ in $\mathbb{R}^N_+$. Moreover, by the Green formula (3.7), we have

$$\forall v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+) \quad \text{such that} \quad v = \Delta v = 0 \quad \text{on } \Gamma,$$
$$\langle g, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = \langle u, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}.$$

By Lemma 2.2, for any $\mu \in W^{2-1/p',p'}_{-\ell+2}(\Gamma)$, there exists $v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)$ such that $v = 0$, $\partial_N v = \mu$, $\partial^2_N v = 0$ on Γ. Consequently,

$$\langle u - g, \mu\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)} = 0,$$

i.e. $u - g = 0$ on Γ. Thus u satisfies $(\mathcal{R})$.

Furthermore, for any $f \in \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{A}^\Delta_{[1-\ell-N/p]}$, we know that (see [5]) there exists a unique $v \in W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^\Delta_{[1+\ell-N/p']}$ such that

$$\Delta v = f \quad \text{in } \mathbb{R}^N_+, \qquad v = 0 \quad \text{on } \Gamma,$$

with the estimate

$$\|v\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^\Delta_{[1+\ell-N/p']}} \leqslant C\|f\|_{W^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)},$$

where C denotes a generic constant not depending on v and f. Now, let us consider the linear form $T : f \mapsto -\langle g, \partial_N v\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}$ defined on $\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{A}^\Delta_{[1-\ell-N/p]}$. Thanks to (3.5), we have for any $q \in \mathcal{A}^\Delta_{[1+\ell-N/p']}$,

$$
\begin{aligned}
|Tf| &= \left|\langle g, \partial_N(v+q)\rangle_{W^{-1-1/p,p}_{\ell-2}(\Gamma)\times W^{2-1/p',p'}_{-\ell+2}(\Gamma)}\right| \\
&\leqslant C\|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\|v+q\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)} \\
&\leqslant C\|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\|v\|_{W^{3,p'}_{-\ell+2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[3-N/p']}} \\
&\leqslant C\|g\|_{W^{-1-1/p,p}_{\ell-2}(\Gamma)}\|f\|_{W^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}.
\end{aligned}
$$

Thus we have shown that T is continuous on $\mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+) \perp \mathcal{A}^{\Delta}_{[1-\ell-N/p]}$ and then, according to Riesz representation theorem, there exists a unique $u \in W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)/\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$ such that $Tf = \langle u, f\rangle_{W^{-1,p}_{\ell-2}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell+2}(\mathbb{R}^N_+)}$. So we have (3.9) and u is the unique solution to problem $(\mathcal{R})$. □

Similarly to the Neumann problem, we can give an intermediate result:

Theorem 3.8. *Let $\ell \in \mathbb{Z}$. Under hypothesis* (3.1)*, for any $g \in W^{-1/p,p}_{\ell-1}(\Gamma)$ satisfying the compatibility condition* (3.5)*, problem $(\mathcal{R})$ admits a solution $u \in W^{0,p}_{\ell-1}(\mathbb{R}^N_+)$, unique up to an element of $\mathcal{A}^{\Delta}_{[1-\ell-N/p]}$, with the estimate*

$$
\inf_{q\in\mathcal{A}^{\Delta}_{[1-\ell-N/p]}} \|u+q\|_{W^{0,p}_{\ell-1}(\mathbb{R}^N_+)} \leqslant C\|g\|_{W^{-1/p,p}_{\ell-1}(\Gamma)}.
$$

4. Generalized solutions to the Stokes system in $\mathbb{R}^N_+$

We will establish a first result about the generalized solutions to (S^+) in the homogeneous case. The following proposition is quite natural and we can find similar results in the literature although not expressed in weighted Sobolev spaces (see e.g. Farwig and Sohr [12], Galdi [14], Cattabriga [11]). Moreover, we take up some ideas in [12] and we considerably simplify the proof.

Proposition 4.1. *For any $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}_0(\Gamma)$, the Stokes problem*

$$-\Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+, \tag{4.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+, \tag{4.2}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma, \tag{4.3}$$

has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}_0(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, with the estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}_0(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}^{1-1/p,p}_0(\Gamma)}. \tag{4.4}$$

Proof. (1) Firstly, we will show that system (4.1)–(4.3) can be reduced to three problems on the fundamental operators Δ^2 and Δ.

Applying the operator div to the first equation (4.1), we obtain

$$\Delta\pi = 0 \quad \text{in } \mathbb{R}^N_+. \tag{4.5}$$

Now, applying the operator Δ to the same equation (4.1), we deduce

$$\Delta^2 \boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+. \tag{4.6}$$

From the boundary condition (4.3), we take out

$$u_N = g_N \quad \text{on } \Gamma, \tag{4.7}$$

and moreover $\operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ, where $\operatorname{div}' \boldsymbol{u}' = \sum_{i=1}^{N-1} \partial_i u_i$.

Since $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, we also have $\operatorname{div} \boldsymbol{u} = 0$ on Γ, then we can write $\partial_N u_N + \operatorname{div}' \boldsymbol{u}' = 0$ on Γ, hence

$$\partial_N u_N = -\operatorname{div}' \boldsymbol{g}' \quad \text{on } \Gamma. \tag{4.8}$$

Combining (4.6)–(4.8), we obtain the following biharmonic problem

$$(\mathcal{P}): \quad \Delta^2 u_N = 0 \quad \text{in } \mathbb{R}^N_+, \quad u_N = g_N \quad \text{and} \quad \partial_N u_N = -\operatorname{div}' \boldsymbol{g}' \quad \text{on } \Gamma.$$

Then, combining (4.5) with the trace on Γ of the Nth component in Eqs. (4.1), we obtain the following Neumann problem

$$(\mathcal{Q}): \quad \Delta\pi = 0 \quad \text{in } \mathbb{R}^N_+ \quad \text{and} \quad \partial_N \pi = \Delta u_N \quad \text{on } \Gamma.$$

Lastly, if we consider the $N-1$ first components of Eqs. (4.1) and (4.3), we can write the following Dirichlet problem

$$(\mathcal{R}): \quad \Delta \boldsymbol{u}' = \nabla' \pi \quad \text{in } \mathbb{R}^N_+ \quad \text{and} \quad \boldsymbol{u}' = \boldsymbol{g}' \quad \text{on } \Gamma.$$

(2) Now, we will solve these three problems.

Step 1: Problem $(\mathcal{P})$. Since $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma)$, we have $g_N \in W_0^{1-1/p,p}(\Gamma)$ and $\operatorname{div}' \boldsymbol{g}' \in W_0^{-1/p,p}(\Gamma)$. So $(\mathcal{P})$ is a homogeneous biharmonic problem with singular boundary conditions, and we can apply Theorem 3.1 provided the compatibility condition (3.2) is fulfilled. If $1 - N/p' < 0$, then $\mathcal{B}_{[3-N/p']} = \{0\}$ and the condition vanishes. If $1 - N/p' \geqslant 0$, then $\mathcal{B}_{[3-N/p']} = \mathbb{R}x_N^2$ and this condition is equivalent to

$$\langle \operatorname{div}' \boldsymbol{g}', 1 \rangle_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)} = 0. \tag{4.9}$$

Since $\mathcal{D}(\mathbb{R}^{N-1})$ is dense in $W_0^{1/p,p'}(\Gamma)$, we know that there exists a sequence $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^{N-1})$ such that $\varphi_k \to 1$ in $W_0^{1/p,p'}(\Gamma)$, hence we can deduce

$$\langle \operatorname{div}' \boldsymbol{g}', 1 \rangle_{W_0^{-1/p,p}(\Gamma) \times W_0^{1/p,p'}(\Gamma)} = -\lim_{k\to\infty} \int_{\mathbb{R}^{N-1}} \boldsymbol{g}' . \nabla\varphi_k \, \mathrm{d}x' = 0.$$

Thus the orthogonality condition is fulfilled and problem $(\mathcal{P})$ has a unique solution $u_N \in W_0^{1,p}(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned}\|u_N\|_{W_0^{1,p}(\mathbb{R}^N_+)} &\leqslant C\big(\|g_N\|_{W_0^{1-1/p,p}(\Gamma)} + \|\operatorname{div}'\boldsymbol{g}'\|_{W_0^{-1/p,p}(\Gamma)}\big)\\ &\leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}.\end{aligned}\tag{4.10}$$

Step 2: Problem $(\mathcal{Q})$. Since $\Delta^2 u_N = 0$ in $\mathbb{R}^N_+$, we have $\Delta u_N \in Y_2(\mathbb{R}^N_+)$ and also $\Delta u_N \in Y_{2,1}(\mathbb{R}^N_+)$, hence $\Delta u_N|_\Gamma \in W_0^{-1-1/p,p}(\Gamma)$ by Lemma 3.7. Then we can apply Theorem 3.3, provided the compatibility condition (3.4) is fulfilled, i.e.

$$\forall \varphi \in \mathcal{N}^{\Delta}_{[2-N/p']}, \quad \langle \Delta u_N, \varphi\rangle_{W_0^{-1-1/p,p}(\Gamma)\times W_0^{2-1/p',p'}(\Gamma)} = 0.$$

Knowing that $\mathcal{N}^{\Delta}_{[2-N/p']} \subset \mathcal{P}_1$, an argument similar to that of the condition (4.9) in step 1 gives us this relation. We can conclude that problem $(\mathcal{Q})$ has a unique solution $\pi \in L^p(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned}\|\pi\|_{L^p(\mathbb{R}^N_+)} &\leqslant C\|\Delta u_N\|_{W_0^{-1-1/p,p}(\Gamma)}\\ &\leqslant C\|\Delta u_N\|_{Y_2(\mathbb{R}^N_+)} = C\|\Delta u_N\|_{W_0^{-1,p}(\mathbb{R}^N_+)}\\ &\leqslant C\|u_N\|_{W_0^{1,p}(\mathbb{R}^N_+)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}.\end{aligned}\tag{4.11}$$

Step 3: Problem $(\mathcal{R})$. By step 2, we have $\nabla'\pi \in W_0^{-1,p}(\mathbb{R}^N_+)^{N-1}$ and moreover $\boldsymbol{g}' \in W_0^{1-1/p,p}(\Gamma)^{N-1}$. Since $\mathcal{A}^{\Delta}_{[1-N/p']} = \{0\}$, we know that problem $(\mathcal{R})$ has a unique solution $\boldsymbol{u}' \in W_0^{1,p}(\mathbb{R}^N_+)^{N-1}$ (see [5, Theorem 3.1]), satisfying

$$\begin{aligned}\|\boldsymbol{u}'\|_{W_0^{1,p}(\mathbb{R}^N_+)^{N-1}} &\leqslant C\big(\|\nabla'\pi\|_{W_0^{-1,p}(\mathbb{R}^N_+)^{N-1}} + \|\boldsymbol{g}'\|_{W_0^{1-1/p,p}(\Gamma)^{N-1}}\big)\\ &\leqslant C\big(\|\pi\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}'\|_{W_0^{1-1/p,p}(\Gamma)^{N-1}}\big)\\ &\leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}.\end{aligned}\tag{4.12}$$

(3) In order, we have found u_N, π and $\boldsymbol{u}'$, which satisfy (4.3) and partially satisfy (4.1), i.e.

$$-\Delta\boldsymbol{u}' + \nabla'\pi = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+.$$

It remains to show they satisfy (4.2) and the Nth component of (4.1), i.e.

$$-\Delta u_N + \partial_N\pi = 0 \quad \text{in } \mathbb{R}^N_+.$$

Thanks to (4.5) and (4.6), we obtain

$$\Delta(\Delta u_N - \partial_N\pi) = \Delta^2 u_N = 0 \quad \text{in } \mathbb{R}^N_+.$$

With the boundary condition of $(\mathcal{Q})$, we can deduce that the distribution $\Delta u_N - \partial_N\pi \in W_0^{-1,p}(\mathbb{R}^N_+)$ satisfies the following Dirichlet problem

$$\Delta(\Delta u_N - \partial_N\pi) = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \Delta u_N - \partial_N\pi = 0 \quad \text{on } \Gamma.$$

Thanks to Theorem 3.5, we necessarily have $\Delta u_N - \partial_N \pi = 0$. Thus (u, π) completely satisfies (4.1).

Now, applying the operator div to (4.1), we have $-\Delta \operatorname{div} \boldsymbol{u} + \Delta\pi = 0$ in $\mathbb{R}^N_+$, and by the main equation of $(\mathcal{Q})$, i.e. (4.5), we obtain $\Delta \operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$. Moreover, from the boundary condition in $(\mathcal{R})$, we get $\operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ. Then, with the boundary condition in $(\mathcal{P})$, we can write

$$\operatorname{div} \boldsymbol{u} = \operatorname{div}'\boldsymbol{u}' + \partial_N u_N = \operatorname{div}' \boldsymbol{g}' - \operatorname{div}' \boldsymbol{g}' = 0 \quad \text{on } \Gamma.$$

So, we have

$$\Delta \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \operatorname{div} \boldsymbol{u} = 0 \quad \text{on } \Gamma,$$

with $\operatorname{div} \boldsymbol{u} \in L^p(\mathbb{R}^N_+)$ and then by Theorem 3.8, we can deduce that $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^N_+$, i.e. (4.2) is satisfied.

(4) Finally, let us remark that the uniqueness of $(\boldsymbol{u}, \pi)$ is a consequence of the uniqueness of the solutions to problems $(\mathcal{P})$, $(\mathcal{Q})$ and $(\mathcal{R})$. Moreover, the estimate (4.4) is a consequence of the estimates (4.10)–(4.12). □

Now, we can solve the complete problem (S^+). For this, we will show that it can be reduced to a homogeneous problem, solved by Proposition 4.1.

Theorem 4.2. *For any $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)$, $h \in L^p(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma)$, problem (S^+) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, and there exists a constant C such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+)} + \|\pi\|_{L^p(\mathbb{R}^N_+)} \leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}\big). \tag{4.13}$$

Proof. Firstly, let us write $\boldsymbol{f} = \operatorname{div} \mathbb{F}$, where $\mathbb{F} = (\boldsymbol{F}_i)_{1\leqslant i\leqslant N} \in \boldsymbol{L}^p(\mathbb{R}^N_+)^N$, with the estimate

$$\|\mathbb{F}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)^N} \leqslant C\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)};$$

and let us respectively denote by $\tilde{\mathbb{F}} = (\tilde{\boldsymbol{F}}_i)_{1\leqslant i\leqslant N} \in \boldsymbol{L}^p(\mathbb{R}^N)^N$ and $\tilde{h} \in L^p(\mathbb{R}^N)$ the extensions by 0 of $\mathbb{F}$ and h to $\mathbb{R}^N$. By Theorem 2.3, we know that there exists $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ solution to the problem

$$(\tilde{S}): \quad -\Delta\tilde{\boldsymbol{u}} + \nabla\tilde{\pi} = \operatorname{div}\tilde{\mathbb{F}} \quad \text{and} \quad \operatorname{div}\tilde{\boldsymbol{u}} = \tilde{h} \quad \text{in } \mathbb{R}^N,$$

provided the condition $\operatorname{div}\tilde{\mathbb{F}} \perp \boldsymbol{\mathcal{P}}_{[1-N/p']}$ is fulfilled. If $1 - N/p' < 0$, we obviously have $\boldsymbol{\mathcal{P}}_{[1-N/p']} = \{\boldsymbol{0}\}$, thus the condition vanishes. If $1 - N/p' \geqslant 0$, then we have $\boldsymbol{\mathcal{P}}_{[1-N/p']} = \mathbb{R}^N$ and this condition is equivalent to

$$\forall i = 1, \dots, N, \quad \langle \operatorname{div} \tilde{\boldsymbol{F}}_i, 1\rangle_{W_0^{-1,p}(\mathbb{R}^N)\times W_0^{1,p'}(\mathbb{R}^N)} = 0.$$

This is exactly the same argument as for the condition (4.9) in the previous proof. Thus the orthogonality condition is fulfilled, hence the existence of $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ solution to problem $(\tilde{S})$, satisfying

$$\begin{aligned}\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N)} + \|\tilde{\pi}\|_{L^p(\mathbb{R}^N)} &\leqslant C\big(\|\operatorname{div}\tilde{\mathbb{F}}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N)} + \|\tilde{h}\|_{L^p(\mathbb{R}^N)}\big)\\ &\leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)}\big).\end{aligned} \tag{4.14}$$

Consequently, we can reduce the system (S^+) to the homogeneous problem

$$\big(S^\sharp\big): \quad -\Delta \boldsymbol{v} + \nabla\vartheta = \boldsymbol{0} \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \boldsymbol{v} = \boldsymbol{g}^\sharp \quad \text{on } \Gamma,$$

where we have set $\boldsymbol{g}^\sharp = \boldsymbol{g} - \tilde{\boldsymbol{u}}|_\Gamma \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma)$. Now, thanks to Proposition 4.1, we know that $(S^\sharp)$ admits a unique solution $(\boldsymbol{v}, \vartheta) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$, satisfying

$$\begin{aligned}\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+)} + \|\vartheta\|_{L^p(\mathbb{R}^N_+)} &\leqslant C\big\|\boldsymbol{g}^\sharp\big\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}\\ &\leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^N_+)} + \|h\|_{L^p(\mathbb{R}^N_+)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{1-1/p,p}(\Gamma)}\big).\end{aligned} \tag{4.15}$$

Then, $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \tilde{\boldsymbol{u}}|_{\mathbb{R}^N_+}, \vartheta + \tilde{\pi}|_{\mathbb{R}^N_+}) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ is solution to (S^+) and the estimate (4.13) is a consequence of the estimates (4.14) and (4.15). Finally, the uniqueness of the solution to (S^+) is a straightforward consequence of Proposition 4.1. □

Remark 4.3. In a forthcoming work, we will show that under hypotheses of Theorem 4.2 and if moreover $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,q}(\mathbb{R}^N_+)$, $h \in L^q(\mathbb{R}^N_+)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/q,q}(\Gamma)$, for any real number $q > 1$, then the solution $(\boldsymbol{u}, \pi)$ given by Theorem 4.2 verifies, besides, $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,q}(\mathbb{R}^N_+) \times L^q(\mathbb{R}^N_+)$.

5. Strong solutions and regularity for the Stokes system in $\mathbb{R}^N_+$

In this section, we are interested in the existence of strong solutions (and then to regular solutions, see Corollaries 5.5 and 5.7), i.e. of solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{\ell+1}^{2,p}(\mathbb{R}^N_+) \times W_{\ell+1}^{1,p}(\mathbb{R}^N_+)$. Here, we limit ourselves to the two cases $\ell = 0$ or $\ell = -1$. Note that in the case $\ell = 0$, we have $W_1^{2,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{1,p}(\mathbb{R}^N_+)$ and $W_1^{1,p}(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$. The proposition and theorem which follow show that the generalized solution of Theorem 4.2, with a stronger hypothesis on the data, is in fact a strong solution.

Proposition 5.1. *Assume that* $\frac{N}{p'} \neq 1$. *For any* $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/p,p}(\Gamma)$, *the Stokes problem* (4.1)–(4.3) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^N_+) \times W_1^{1,p}(\mathbb{R}^N_+)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}^N_+)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_1^{2-1/p,p}(\Gamma)}.$$

Proof. The arguments for the estimate are unchanged with respect to the proof of Proposition 4.1. For the surjectivity and the uniqueness, note that we always have the imbedding $W_1^{2-1/p,p}(\Gamma) \hookrightarrow W_0^{1-1/p,p}(\Gamma)$. By Proposition 4.1, we can deduce that problem (4.1)–(4.3) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$, satisfying the estimate (4.4). Then, it suffices to go back to the proof of Proposition 4.1 and to use the established results about problems $(\mathcal{P})$, $(\mathcal{Q})$ and $(\mathcal{R})$, to show that in fact $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$. In order, for problem $(\mathcal{P})$, we find $u_N \in W_1^{2,p}(\mathbb{R}_+^N)$ (see [7, Lemma 4.9]); for problem $(\mathcal{Q})$, thanks to Theorem 3.4, we find $\pi \in W_1^{1,p}(\mathbb{R}_+^N)$; for problem $(\mathcal{R})$, we find $\boldsymbol{u}' \in W_1^{2,p}(\mathbb{R}_+^N)^{N-1}$ (see [5, Theorem 3.3]). Note that for these three results, the condition $N/p' \neq 1$ is always necessary. □

Now, we can study the strong solutions for the complete problem (S^+). As for the generalized solutions, we will show that it is equivalent to a homogeneous problem, solved by Proposition 5.1. The following theorem was established in the case $N = 3$, $p = 2$, by Maz'ya, Plamenevskiĭ and Stupyalis (see [18]).

Theorem 5.2. *Assume that* $\frac{N}{p'} \neq 1$. *For any* $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)$, $h \in W_1^{1,p}(\mathbb{R}_+^N)$ *and* $\boldsymbol{g} \in \boldsymbol{W}_1^{2-1/p,p}(\Gamma)$, *problem* (S^+) *admits a unique solution* $(\boldsymbol{u}, \pi)$ *which belongs to* $\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_1^{1,p}(\mathbb{R}_+^N)} \leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)} + \|h\|_{W_1^{1,p}(\mathbb{R}_+^N)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_1^{2-1/p,p}(\Gamma)}\big).$$

Proof. Here again, the arguments for the estimate are unchanged with respect to the proof of Theorem 4.2. For the surjectivity and the uniqueness, note that the imbedding $W_1^{0,p}(\mathbb{R}_+^N) \hookrightarrow W_0^{-1,p}(\mathbb{R}_+^N)$ holds if $N/p' \neq 1$. Moreover, we have $W_1^{1,p}(\mathbb{R}_+^N) \hookrightarrow L^p(\mathbb{R}_+^N)$ and $W_1^{2-1/p,p}(\Gamma) \hookrightarrow W_0^{1-1/p,p}(\Gamma)$. Thus, thanks to Theorem 4.2, we know that problem (S^+) admits a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}_+^N) \times L^p(\mathbb{R}_+^N)$, satisfying the estimate (4.13). To show that $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N) \times W_1^{1,p}(\mathbb{R}_+^N)$, we want to find an extension $\tilde{\boldsymbol{f}}$ of $\boldsymbol{f}$ to $\mathbb{R}^N$, such that the orthogonality condition for the extended problem to the whole space $(\tilde{S})$ holds. To this end, we still can write $\boldsymbol{f} = \operatorname{div} \mathbb{F}$. Indeed, if $N/p' \neq 1$, for any $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}_+^N)$, the Dirichlet problem

$$\Delta \boldsymbol{w} = \boldsymbol{f} \quad \text{in } \mathbb{R}_+^N, \qquad \boldsymbol{w} = \boldsymbol{0} \quad \text{in } \Gamma,$$

admits a unique solution $\boldsymbol{w} \in \boldsymbol{W}_1^{2,p}(\mathbb{R}_+^N)$ (see [5, Theorem 3.3]). So, if we consider $\mathbb{F} = \nabla \boldsymbol{w} \in \boldsymbol{W}_1^{1,p}(\mathbb{R}_+^N)^N$, we have $\boldsymbol{f} = \operatorname{div} \mathbb{F}$. Now, it suffices to go back to the proof of Theorem 4.2. Here again, we know that there exists a continuous linear extension operator from $W_1^{1,p}(\mathbb{R}_+^N)$ to $W_1^{1,p}(\mathbb{R}^N)$, so we get $\tilde{\boldsymbol{f}} = \operatorname{div} \tilde{\mathbb{F}} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^N)$ and $\tilde{h} \in W_1^{1,p}(\mathbb{R}^N)$, hence the extended problem $(\tilde{S})$, which has, by Theorem 2.4, a solution $(\tilde{\boldsymbol{u}}, \tilde{\pi}) \in \boldsymbol{W}_1^{2,p}(\mathbb{R}^N) \times W_1^{1,p}(\mathbb{R}^N)$. Then, we obtain the equivalent problem $(S^\sharp)$ with $\boldsymbol{g}^\sharp \in \boldsymbol{W}_1^{2-1/p,p}(\Gamma)$ and this problem is solved by Proposition 5.1. □

Remark 5.3. To give a variant to this proof, we also can consider the extension $\tilde{\boldsymbol{f}} \in \boldsymbol{W}_1^{0,p}(\mathbb{R}^N)$ of $\boldsymbol{f}$ to $\mathbb{R}^N$ defined by

$$\tilde{\boldsymbol{f}}(x', x_N) = \begin{cases} \boldsymbol{f}(x', x_N) & \text{if } x_N > 0, \\ -\boldsymbol{f}(x', -x_N) & \text{if } x_N < 0, \end{cases}$$

and $\tilde h \in W_1^{1,p}(\mathbb{R}^N)$ an extension of h to $\mathbb{R}^N$. Then by Theorem 2.4, there exists $(\tilde{\boldsymbol u}, \tilde\pi)$ solution to the problem

$$(\tilde S):\quad -\Delta\tilde{\boldsymbol u} + \nabla\tilde\pi = \tilde{\boldsymbol f} \quad\text{and}\quad \operatorname{div}\tilde{\boldsymbol u} = \tilde h \quad \text{in } \mathbb{R}^N,$$

provided the orthogonality condition $\tilde{\boldsymbol f} \perp \mathcal{P}_{[1-N/p']}$ is fulfilled. Here again, if $1 - N/p' < 0$ this condition vanishes and if $1 - N/p' > 0$, we have

$$\forall i = 1,\dots,N,\quad \int_{\mathbb{R}^N} \tilde{\boldsymbol f}_i(x', x_N)\,\mathrm{d}x = 0.$$

Thus the orthogonality condition holds. The rest of the proof is identical.

Remark 5.4. Similarly to Remark 4.3, we could show that under hypotheses of Theorem 5.2 and if moreover $\boldsymbol f \in \boldsymbol W_1^{0,q}(\mathbb{R}^N_+)$, $h \in W_1^{1,q}(\mathbb{R}^N_+)$ and $\boldsymbol g \in \boldsymbol W_1^{2-1/q,q}(\Gamma)$, with an arbitrary real number $q > 1$, then the solution $(\boldsymbol u, \pi)$ given by Theorem 4.2 verifies, besides, $(\boldsymbol u, \pi) \in \boldsymbol W_1^{2,q}(\mathbb{R}^N_+) \times W_1^{1,q}(\mathbb{R}^N_+)$.

We will now establish a global regularity result of solutions to the Stokes system (S^+), which includes the case of strong solutions and which rests on Theorem 4.2 and a regularity argument.

Corollary 5.5. *Let $m \in \mathbb{N}$ and assume that $\frac{N}{p'} \neq 1$ if $m \geqslant 1$. For any $\boldsymbol f \in \boldsymbol W_m^{m-1,p}(\mathbb{R}^N_+)$, $h \in W_m^{m,p}(\mathbb{R}^N_+)$ and $\boldsymbol g \in \boldsymbol W_m^{m+1-1/p,p}(\Gamma)$, problem (S^+) admits a unique solution $(\boldsymbol u, \pi) \in \boldsymbol W_m^{m+1,p}(\mathbb{R}^N_+) \times W_m^{m,p}(\mathbb{R}^N_+)$, with the estimate*

$$\begin{aligned}&\|\boldsymbol u\|_{\boldsymbol W_m^{m+1,p}(\mathbb{R}^N_+)} + \|\pi\|_{W_m^{m,p}(\mathbb{R}^N_+)}\\ &\quad\leqslant C\big(\|\boldsymbol f\|_{\boldsymbol W_m^{m-1,p}(\mathbb{R}^N_+)} + \|h\|_{W_m^{m,p}(\mathbb{R}^N_+)} + \|\boldsymbol g\|_{\boldsymbol W_m^{m+1-1/p,p}(\Gamma)}\big).\end{aligned}$$

Proof. Since we have $W_m^{m-1,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{-1,p}(\mathbb{R}^N_+)$, $W_m^{m,p}(\mathbb{R}^N_+) \hookrightarrow L^p(\mathbb{R}^N_+)$ and $W_m^{m+1-1/p,p}(\Gamma) \hookrightarrow W_0^{1-1/p,p}(\Gamma)$, thanks to Theorem 4.2, we know that problem (S^+) admits a unique solution $(\boldsymbol u, \pi) \in \boldsymbol W_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$. We will show by induction that

$$\begin{aligned}&(\boldsymbol f, h, \boldsymbol g) \in \boldsymbol W_m^{m-1,p}\big(\mathbb{R}^N_+\big) \times W_m^{m,p}\big(\mathbb{R}^N_+\big) \times \boldsymbol W_m^{m+1-1/p,p}(\Gamma)\\ &\quad\Rightarrow\quad (\boldsymbol u, \pi) \in \boldsymbol W_m^{m+1,p}\big(\mathbb{R}^N_+\big) \times W_m^{m,p}\big(\mathbb{R}^N_+\big).\end{aligned}\tag{5.1}$$

For $m = 0$, (5.1) is true. Assume that (5.1) is true for $0, 1, \dots, m$ and suppose that $(\boldsymbol f, h, \boldsymbol g) \in \boldsymbol W_{m+1}^{m,p}(\mathbb{R}^N_+) \times W_{m+1}^{m+1,p}(\mathbb{R}^N_+) \times \boldsymbol W_{m+1}^{m+2-1/p,p}(\Gamma)$. Let us prove that $(\boldsymbol u, \pi) \in \boldsymbol W_{m+1}^{m+2,p}(\mathbb{R}^N_+) \times W_{m+1}^{m+1,p}(\mathbb{R}^N_+)$. Since $W_{m+1}^{m,p}(\mathbb{R}^N_+) \hookrightarrow W_m^{m-1,p}(\mathbb{R}^N_+)$, $W_{m+1}^{m+1,p}(\mathbb{R}^N_+) \hookrightarrow W_m^{m,p}(\mathbb{R}^N_+)$ and $W_{m+1}^{m+2-1/p,p}(\Gamma) \hookrightarrow W_m^{m+1-1/p,p}(\Gamma)$, we know that $(\boldsymbol u, \pi) \in \boldsymbol W_m^{m+1,p}(\mathbb{R}^N_+) \times W_m^{m,p}(\mathbb{R}^N_+)$ thanks to the induction hypothesis. Now, for any $i \in \{1, \dots, N-1\}$, we have

$$
\begin{aligned}
&-\Delta(\varrho\partial_i \boldsymbol{u}) + \nabla(\varrho\partial_i\pi)\\
&\quad = \varrho\partial_i \boldsymbol{f} + \frac{2}{\varrho}x.\nabla\partial_i\boldsymbol{u} + \left(\frac{N-1}{\varrho} + \frac{1}{\varrho^3}\right)\partial_i\boldsymbol{u} + \frac{1}{\varrho}x\partial_i\pi.
\end{aligned}
$$

Thus, $-\Delta(\varrho\partial_i\boldsymbol{u}) + \nabla(\varrho\partial_i\pi) \in \boldsymbol{W}_m^{m-1,p}(\mathbb{R}_+^N)$. Moreover,

$$
\operatorname{div}(\varrho\partial_i\boldsymbol{u}) = \frac{1}{\varrho}x\partial_i\boldsymbol{u} + \varrho\partial_i h.
$$

Thus, $\operatorname{div}(\varrho\partial_i\boldsymbol{u}) \in W_m^{m,p}(\mathbb{R}_+^N)$. We also have $\gamma_0(\varrho\partial_i\boldsymbol{u}) = \varrho'\partial_i\gamma_0\boldsymbol{u} = \varrho'\partial_i\boldsymbol{g} \in \boldsymbol{W}_m^{m+1-1/p,p}(\Gamma)$. So, by induction hypothesis, we can deduce that

$$
\forall i \in \{1,\dots,N-1\},\quad (\partial_i\boldsymbol{u}, \partial_i\pi) \in \boldsymbol{W}_{m+1}^{m+1,p}(\mathbb{R}_+^N) \times W_{m+1}^{m,p}(\mathbb{R}_+^N).
$$

It remains to prove that $(\partial_N\boldsymbol{u}, \partial_N\pi) \in \boldsymbol{W}_{m+1}^{m+1,p}(\mathbb{R}_+^N) \times W_{m+1}^{m,p}(\mathbb{R}_+^N)$. For that, let us observe that for any $i \in \{1,\dots,N-1\}$, we have

$$
\begin{aligned}
\partial_i\partial_N\boldsymbol{u} &= \partial_N\partial_i\boldsymbol{u} \in \boldsymbol{W}_{m+1}^{m,p}(\mathbb{R}_+^N),\\
\partial_N^2 u_i &= -\Delta' u_i + \partial_i\pi - f_i \in W_{m+1}^{m,p}(\mathbb{R}_+^N),\\
\partial_N^2 u_N &= \partial_N h - \partial_N \operatorname{div}'\boldsymbol{u}' \in W_{m+1}^{m,p}(\mathbb{R}_+^N),\\
\partial_N\pi &= f_N + \Delta u_N \in W_{m+1}^{m,p}(\mathbb{R}_+^N).
\end{aligned}
$$

Hence, $\nabla(\partial_N\boldsymbol{u}) \in \boldsymbol{W}_{m+1}^{m,p}(\mathbb{R}_+^N)^N$ and knowing that $\partial_N\boldsymbol{u} \in \boldsymbol{W}_m^{m,p}(\mathbb{R}_+^N)$, we can deduce that $\partial_N\boldsymbol{u} \in \boldsymbol{W}_{m+1}^{m+1,p}(\mathbb{R}_+^N)$, according to definition (2.1). Consequently, we have $\nabla\boldsymbol{u} \in \boldsymbol{W}_{m+1}^{m+1,p}(\mathbb{R}_+^N)^N$. Likewise, we have $\nabla\pi \in \boldsymbol{W}_{m+1}^{m,p}(\mathbb{R}_+^N)$. Finally, we can conclude that $(\boldsymbol{u},\pi) \in \boldsymbol{W}_{m+1}^{m+2,p}(\mathbb{R}_+^N) \times W_{m+1}^{m+1,p}(\mathbb{R}_+^N)$. □

Now, we examine the basic case $\ell = -1$, corresponding to $f \in \boldsymbol{L}^p(\mathbb{R}_+^N)$. More precisely, we have the following result, corresponding to Theorem 5.2:

Theorem 5.6. *For any $\boldsymbol{f} \in \boldsymbol{L}^p(\mathbb{R}_+^N)$, $h \in W_0^{1,p}(\mathbb{R}_+^N)$ and $\boldsymbol{g} \in \boldsymbol{W}_0^{2-1/p,p}(\Gamma)$, problem (S^+) admits a solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}_0^{2,p}(\mathbb{R}_+^N) \times W_0^{1,p}(\mathbb{R}_+^N)$, unique if $N > p$, unique up to an element of $(\mathbb{R}x_N)^{N-1} \times \{0\} \times \mathbb{R}$ if $N \leqslant p$, with the following estimate if $N \leqslant p$ (eliminate $(\boldsymbol{\lambda},\mu)$ if $N > p$):*

$$
\begin{aligned}
&\inf_{(\boldsymbol{\lambda},\mu)\in(\mathbb{R}x_N)^{N-1}\times\{0\}\times\mathbb{R}} \left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}_0^{2,p}(\mathbb{R}_+^N)} + \|\pi+\mu\|_{W_0^{1,p}(\mathbb{R}_+^N)}\right)\\
&\quad \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\mathbb{R}_+^N)} + \|h\|_{W_0^{1,p}(\mathbb{R}_+^N)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_0^{2-1/p,p}(\Gamma)}\right).
\end{aligned}
$$

Proof. The idea is to go back to the proof of Theorem 4.2 and we will throw light on the modifications. In contrast to Theorem 5.2, the extension $\tilde{\boldsymbol{f}}$ of $\boldsymbol{f}$ is of no importance because there is no orthogonality condition for the extended problem $(\tilde{S})$ (see Theorem 2.4). Then, we get the reduced problem $(S^\sharp)$. Now, to solve $(S^\sharp)$, this is the proof of Proposition 4.1. Problem $(\mathcal{P})$

yields a unique $u_N \in W_0^{2,p}(\mathbb{R}_+^N)$, problem $(\mathcal{Q})$ gives $\pi \in W_0^{1,p}(\mathbb{R}_+^N)$ unique up to an element of $\mathcal{N}_{[1-N/p]}^{\Delta}$; and $(\mathcal{R})$ yields $\boldsymbol{u}' \in W_0^{2,p}(\mathbb{R}_+^N)^{N-1}$ unique up to an element of $(\mathcal{A}_{[2-N/p]}^{\Delta})^{N-1}$. The point (3) of the proof is identical for all N and p (the kernels of the two Dirichlet problems are always reduced to zero). The last point concerns the kernel of the operator associated to this problem. If $N > p$, it is clearly reduced to zero and if $N \leqslant p$, we have $\mathcal{A}_{[2-N/p]}^{\Delta} = \mathbb{R}x_N$ and $\mathcal{N}_{[1-N/p]}^{\Delta} = \mathcal{P}_{[1-N/p]} = \mathbb{R}$. □

Thanks to the corresponding imbeddings, we can give a regularity result with the same proof as Corollary 5.5.

Corollary 5.7. *Let $m \in \mathbb{N}$. For any $\boldsymbol{f} \in \boldsymbol{W}_m^{m,p}(\mathbb{R}_+^N)$, $h \in W_m^{m+1,p}(\mathbb{R}_+^N)$ and $\boldsymbol{g} \in \boldsymbol{W}_m^{m+2-1/p,p}(\Gamma)$, problem (S^+) admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_m^{m+2,p}(\mathbb{R}_+^N) \times W_m^{m+1,p}(\mathbb{R}_+^N)$, unique if $N > p$, unique up to an element of $(\mathbb{R}x_N)^{N-1} \times \{0\} \times \mathbb{R}$ if $N \leqslant p$, with the following estimate if $N \leqslant p$ (eliminate $(\boldsymbol{\lambda}, \mu)$ if $N > p$):*

$$\inf_{(\boldsymbol{\lambda},\mu)\in(\mathbb{R}x_N)^{N-1}\times\{0\}\times\mathbb{R}} \big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}_m^{m+2,p}(\mathbb{R}_+^N)} + \|\pi+\mu\|_{W_m^{m+1,p}(\mathbb{R}_+^N)}\big)$$
$$\leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{W}_m^{m,p}(\mathbb{R}_+^N)} + \|h\|_{W_m^{m+1,p}(\mathbb{R}_+^N)} + \|\boldsymbol{g}\|_{\boldsymbol{W}_m^{m+2-1/p,p}(\Gamma)}\big).$$

6. Very weak solutions for the Stokes system

The aim of this section is to study the Stokes problem with singular data on the boundary. For that, we will adapt a method employed in bounded domains (see [2, Section 4.2]). At first, we must give a meaning to singular data for the Stokes problem in the half-space. More precisely, we want to show that a boundary condition of the form $\boldsymbol{g} \in \boldsymbol{W}_{\ell-1}^{-1/p,p}(\Gamma)$ is meaningful. In mind of this paper, we limit ourselves to the two cases $\ell = 0$ or $\ell = 1$, i.e. to $\boldsymbol{g} \in \boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)$ corresponding to a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,p}(\mathbb{R}_+^N)$, or $\boldsymbol{g} \in \boldsymbol{W}_0^{-1/p,p}(\Gamma)$ corresponding to $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\mathbb{R}_+^N) \times W_0^{-1,p}(\mathbb{R}_+^N)$. In that way, for every $\ell \in \mathbb{Z}$, we introduce the space

$$\boldsymbol{M}_\ell(\mathbb{R}_+^N) = \big\{\boldsymbol{u} \in \boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbb{R}_+^N);\ \boldsymbol{u} = \boldsymbol{0} \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ on } \Gamma\big\}.$$

Lemma 6.1. *For any $\ell \in \mathbb{Z}$, we have the identity*

$$\boldsymbol{M}_\ell(\mathbb{R}_+^N) = \big\{\boldsymbol{u} \in \boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbb{R}_+^N);\ \boldsymbol{u} = \boldsymbol{0} \text{ and } \partial_N u_N = 0 \text{ on } \Gamma\big\} \tag{6.1}$$

and the range space of the normal derivative $\gamma_1 : \boldsymbol{M}_\ell(\mathbb{R}_+^N) \to \boldsymbol{W}_{-\ell+1}^{1/p,p'}(\Gamma)$ is

$$\boldsymbol{Z}_\ell(\Gamma) = \big\{\boldsymbol{w} \in \boldsymbol{W}_{-\ell+1}^{1/p,p'}(\Gamma);\ w_N = 0 \text{ on } \Gamma\big\}. \tag{6.2}$$

Proof. Let $\boldsymbol{u} \in \boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbb{R}_+^N)$ such that $\boldsymbol{u} = \boldsymbol{0}$ on Γ. Then $\operatorname{div} \boldsymbol{u} = \partial_N u_N$ on Γ and the identity (6.1) holds.

Moreover, it is clear that $\mathcal{I}m\,\gamma_1 \subset \boldsymbol{Z}_\ell(\Gamma)$. Conversely, given $\boldsymbol{w} \in \boldsymbol{Z}_\ell(\Gamma)$, by Lemma 2.2, there exists $\boldsymbol{u} \in \boldsymbol{W}_{-\ell+1}^{2,p'}(\mathbb{R}_+^N)$ such that $\boldsymbol{u} = \boldsymbol{0}$ and $\partial_N \boldsymbol{u} = \boldsymbol{w}$ on Γ. Since $w_N = 0$ on Γ, we have $\boldsymbol{u} \in \boldsymbol{M}_\ell(\mathbb{R}_+^N)$ and $\boldsymbol{w} \in \mathcal{I}m\,\gamma_1$. □

For any open subset Ω of $\mathbb{R}$, we also define the space

$$\boldsymbol{W}^{1,p'}_{-\ell}(\operatorname{div};\Omega)=\left\{\boldsymbol{v}\in\boldsymbol{W}^{1,p'}_{-\ell}(\Omega);\ \operatorname{div}\boldsymbol{v}\in W^{1,p'}_{-\ell+1}(\Omega)\right\},$$

which is a reflexive Banach space for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}_{-\ell}(\operatorname{div};\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p'}_{-\ell}(\Omega)}+\|\operatorname{div}\boldsymbol{v}\|_{W^{1,p'}_{-\ell+1}(\Omega)};$$

and the following subspace of $\boldsymbol{W}^{1,p'}_{-\ell}(\operatorname{div};\mathbb{R}^N_+)$

$$\boldsymbol{X}_\ell\big(\mathbb{R}^N_+\big)=\left\{\boldsymbol{v}\in\mathring{\boldsymbol{W}}^{1,p'}_{-\ell}\big(\mathbb{R}^N_+\big);\ \operatorname{div}\boldsymbol{v}\in\mathring{W}^{1,p'}_{-\ell+1}\big(\mathbb{R}^N_+\big)\right\}.$$

Lemma 6.2. *For any $\ell\in\mathbb{Z}$, the space $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ is dense in $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$.*

Proof. Let $\boldsymbol{v}\in\boldsymbol{X}_\ell(\mathbb{R}^N_+)$ and $\tilde{\boldsymbol{v}}$ be the extension by $\boldsymbol{0}$ of $\boldsymbol{v}$ to $\mathbb{R}^N$, then we have $\tilde{\boldsymbol{v}}\in\boldsymbol{W}^{1,p'}_{-\ell}(\operatorname{div};\mathbb{R}^N)$.

We begin to apply the cut off functions ϕ_k, defined on $\mathbb{R}^N$ for any $k\in\mathbb{N}$, by

$$\phi_k(x)=\begin{cases}\phi\big(\frac{k}{\ln|x|}\big), & \text{if } |x|>1,\\ 1, & \text{otherwise,}\end{cases}$$

where $\phi\in\mathcal{C}^\infty([0,\infty[)$ is such that

$$\phi(t)=0,\quad \text{if } t\in[0,1];\qquad 0\leqslant\phi(t)\leqslant1,\quad \text{if } t\in[1,2];\qquad \phi(t)=1,\quad \text{if } t\geqslant2.$$

Note that this truncation process is adapted to the logarithmic weights (see Lemma 7.1 in [3]). Then we have

$$\phi_k\tilde{\boldsymbol{v}}=\tilde{\boldsymbol{v}}_k\underset{k\to\infty}{\longrightarrow}\tilde{\boldsymbol{v}}\quad\text{in } \boldsymbol{W}^{1,p'}_{-\ell}\big(\mathbb{R}^N\big)$$

and

$$\operatorname{div}(\phi_k\tilde{\boldsymbol{v}})=\phi_k\operatorname{div}\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{v}}\cdot\nabla\phi_k\underset{k\to\infty}{\longrightarrow}\operatorname{div}\tilde{\boldsymbol{v}}\quad\text{in } W^{1,p'}_{-\ell+1}\big(\mathbb{R}^N\big).$$

Now, for any real number $\theta>0$ and $x\in\mathbb{R}^N$, we set $\tilde{\boldsymbol{v}}_{k,\theta}(x)=\tilde{\boldsymbol{v}}_k(x-\theta e_N)$. Then $\tilde{\boldsymbol{v}}_{k,\theta}\in\boldsymbol{W}^{1,p'}_{-\ell}(\operatorname{div};\mathbb{R}^N)$ and $\operatorname{supp}\tilde{\boldsymbol{v}}_{k,\theta}$ is compact in $\mathbb{R}^N_+$, moreover

$$\lim_{\theta\to0}\tilde{\boldsymbol{v}}_{k,\theta}=\tilde{\boldsymbol{v}}_k\quad\text{in } \boldsymbol{W}^{1,p'}_{-\ell}\big(\operatorname{div};\mathbb{R}^N\big).$$

Consequently, for any real number $\varepsilon>0$ small enough, $\rho_\varepsilon*\tilde{\boldsymbol{v}}_{k,\theta}\in\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ and

$$\lim_{\varepsilon\to0}\lim_{\theta\to0}\lim_{k\to\infty}\rho_\varepsilon*\tilde{\boldsymbol{v}}_{k,\theta}=\tilde{\boldsymbol{v}}\quad\text{in } \boldsymbol{W}^{1,p'}_{-\ell}\big(\operatorname{div};\mathbb{R}^N\big),$$

where ρ_ε is a mollifier. □

Let $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$ be the dual space of $\boldsymbol{X}_\ell(\mathbb{R}^N_+)$, we introduce the spaces

$$\boldsymbol{T}_\ell(\mathbb{R}^N_+) = \{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+);\ \Delta\boldsymbol{v} \in \boldsymbol{X}'_\ell(\mathbb{R}^N_+)\},$$
$$\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+) = \{\boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+);\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \mathbb{R}^N_+\},$$

which are reflexive Banach spaces for the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{T}_\ell(\mathbb{R}^N_+)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+)} + \|\Delta\boldsymbol{v}\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)},$$

where $\|\cdot\|_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)}$ denotes the dual norm of the space $\boldsymbol{X}'_\ell(\mathbb{R}^N_+)$.

Lemma 6.3. *Let $\ell \in \mathbb{Z}$. Under hypothesis* (3.1)*, the space $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ is dense in $\boldsymbol{T}_\ell(\mathbb{R}^N_+)$.*

Proof. For every continuous linear form $\boldsymbol{z} \in (\boldsymbol{T}_\ell(\mathbb{R}^N_+))'$, there exists a unique pair $(\boldsymbol{f}, \boldsymbol{g}) \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N_+) \times \boldsymbol{X}_\ell(\mathbb{R}^N_+)$, such that

$$\forall\boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+), \quad \langle\boldsymbol{z}, \boldsymbol{v}\rangle = \int_{\mathbb{R}^N_+} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \langle\Delta\boldsymbol{v}, \boldsymbol{g}\rangle_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)\times\boldsymbol{X}_\ell(\mathbb{R}^N_+)}. \tag{6.3}$$

Thanks to the Hahn–Banach theorem, it suffices to show that any $\boldsymbol{z}$ which vanishes on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ is actually zero on $\boldsymbol{T}_\ell(\mathbb{R}^N_+)$. Let us suppose that $\boldsymbol{z} = \boldsymbol{0}$ on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$, thus on $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$. Then we can deduce from (6.3) that

$$\boldsymbol{f} + \Delta\boldsymbol{g} = \boldsymbol{0} \quad \text{in } \mathbb{R}^N_+,$$

hence we have $\Delta\boldsymbol{g} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N_+)$, $\boldsymbol{g} \in \mathring{\boldsymbol{W}}^{1,p'}_{-\ell}(\mathbb{R}^N_+)$ and $\operatorname{div}\boldsymbol{g} \in \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Let $\tilde{\boldsymbol{f}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N)$ and $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{1,p'}_{-\ell}(\mathbb{R}^N)$ be respectively the extensions by $\boldsymbol{0}$ of $\boldsymbol{f}$ and $\boldsymbol{g}$ to $\mathbb{R}^N$. From (6.3), we get $\tilde{\boldsymbol{f}} + \Delta\tilde{\boldsymbol{g}} = \boldsymbol{0}$ in $\mathbb{R}^N$, and thus $\Delta\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^N)$. Now, according to the isomorphism results for Δ in $\mathbb{R}^N$ (see [4]), we can deduce that $\tilde{\boldsymbol{g}} \in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N)$, under hypothesis (3.1). Since $\tilde{\boldsymbol{g}}$ is an extension by $\boldsymbol{0}$, it follows that $\boldsymbol{g} = \mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Then, by density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ in $\mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$, there exists a sequence $(\boldsymbol{\varphi}_k)_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)$ such that $\boldsymbol{\varphi}_k \to \boldsymbol{g}$ in $\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^N_+)$. Thus, for any $\boldsymbol{v} \in \boldsymbol{T}_\ell(\mathbb{R}^N_+)$, we have

$$\begin{aligned}
\langle\boldsymbol{z}, \boldsymbol{v}\rangle &= -\int_{\mathbb{R}^N_+} \boldsymbol{v}\cdot\Delta\boldsymbol{g}\,\mathrm{d}x + \langle\Delta\boldsymbol{v}, \boldsymbol{g}\rangle_{\boldsymbol{X}'_\ell(\mathbb{R}^N_+)\times\boldsymbol{X}_\ell(\mathbb{R}^N_+)} \\
&= \lim_{k\to\infty}\left\{-\int_{\mathbb{R}^N_+} \boldsymbol{v}\cdot\Delta\boldsymbol{\varphi}_k\,\mathrm{d}x + \langle\Delta\boldsymbol{v}, \boldsymbol{\varphi}_k\rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^N_+)\times\boldsymbol{\mathcal{D}}(\mathbb{R}^N_+)}\right\} \\
&= 0,
\end{aligned}$$

i.e. $\boldsymbol{z}$ is identically zero. $\square$

We also can show that, under hypothesis (3.1), $\{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+});\ \operatorname{div} \boldsymbol{v} = 0\}$ is dense in $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$. To study the traces of functions which belong to $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$, we set

$$\boldsymbol{W}^{0,p}_{\ell}(\operatorname{div}; \mathbb{R}^N_+) = \{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+); \operatorname{div} \boldsymbol{v} \in W^{0,p}_{\ell}(\mathbb{R}^N_+)\}$$

and their normal traces are described in the following lemma:

Lemma 6.4. *Assume that $\ell \in \mathbb{Z}$ with $N/p' \neq \ell$. The linear mapping*

$$\gamma_{e_N} : \boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+}) \longrightarrow \mathcal{D}(\mathbb{R}^{N-1}),$$
$$\boldsymbol{v} \longmapsto v_N|_{\Gamma}$$

can be extended to a linear continuous mapping

$$\gamma_{e_N} : \boldsymbol{W}^{0,p}_{\ell}(\operatorname{div}; \mathbb{R}^N_+) \longrightarrow W^{-1/p,p}_{\ell-1}(\Gamma).$$

Moreover, we have the Green formula

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell}(\operatorname{div}; \mathbb{R}^N_+),\ \forall \varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+),$$
$$\int_{\mathbb{R}^N_+} \boldsymbol{v} \cdot \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N_+} \varphi \operatorname{div} \boldsymbol{v} \,\mathrm{d}x = -\langle v_N, \varphi\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1/p,p'}_{-\ell+1}(\Gamma)}. \tag{6.4}$$

Proof. Note that the assumption $N/p' \neq \ell$ is necessary for the imbedding $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell}(\mathbb{R}^N_+)$, which is underlying in the Green formula. We will show in remark how to do without.

Here again, we can show by truncation and regularization that $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ is dense in $\boldsymbol{W}^{0,p}_{\ell}(\operatorname{div}; \mathbb{R}^N_+)$ as in [3].

Let $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ and $\varphi \in \mathcal{D}(\overline{\mathbb{R}^N_+})$, then formula (6.4) obviously holds. Since $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$ and the mapping

$$\gamma_0 : W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \longrightarrow W^{1/p,p'}_{-\ell+1}(\Gamma),$$
$$\varphi \longmapsto \varphi|_{\Gamma}$$

is continuous, formula (6.4) holds for every $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ and $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. By Lemma 2.2, for every $\mu \in W^{1/p,p'}_{-\ell+1}(\Gamma)$, there exists $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$ such that $\varphi = \mu$ on Γ, with $\|\varphi\|_{W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)} \leqslant C\|\mu\|_{W^{1/p,p'}_{-\ell+1}(\Gamma)}$. Consequently,

$$\left|\langle v_N, \mu\rangle_{W^{-1/p,p}_{\ell-1}(\Gamma)\times W^{1/p,p'}_{-\ell+1}(\Gamma)}\right| \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell}(\operatorname{div};\mathbb{R}^N_+)}\|\mu\|_{W^{1/p,p'}_{-\ell+1}(\Gamma)}.$$

Thus

$$\|v_N\|_{W^{-1/p,p}_{\ell-1}(\Gamma)} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\mathbb{R}^N_+)}.$$

We can deduce that the linear mapping γ_{e_N} is continuous for the norm of $\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\mathbb{R}^N_+)$. Since $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ is dense in $\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\mathbb{R}^N_+)$, γ_{e_N} can be extended by continuity to $\gamma_{e_N} \in \mathcal{L}(\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\mathbb{R}^N_+); W^{-1/p,p}_{\ell-1}(\Gamma))$ and formula (6.4) holds for all $\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\mathbb{R}^N_+)$ and $\varphi \in W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+)$. $\square$

Remark 6.5. If $N/p' = \ell$, the imbedding $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell}(\mathbb{R}^N_+)$ fails, but in that case we have $W^{1,p'}_{-\ell+1}(\mathbb{R}^N_+) \hookrightarrow W^{0,p'}_{-\ell,-1}(\mathbb{R}^N_+)$. Thus, it suffices to introduce the space $\boldsymbol{W}^{0,p}_{\ell,1}(\mathrm{div};\mathbb{R}^N_+) = \{\boldsymbol{v} \in \boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^N_+); \mathrm{div}\,\boldsymbol{v} \in W^{0,p}_{\ell,1}(\mathbb{R}^N_+)\}$ instead of $\boldsymbol{W}^{0,p}_{\ell}(\mathrm{div};\mathbb{R}^N_+)$. Then, with the same proof, we can show that $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ is dense in the space $\boldsymbol{W}^{0,p}_{\ell,1}(\mathrm{div};\mathbb{R}^N_+)$ and that the mapping γ_{e_N} is continuous from $\boldsymbol{W}^{0,p}_{\ell,1}(\mathrm{div};\mathbb{R}^N_+)$ to $W^{-1/p,p}_{\ell-1}(\Gamma)$, with the corresponding Green formula.

It follows that the functions $\boldsymbol{v}$ from $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$ are such that their normal trace v_N belongs to $W^{-1/p,p}_{\ell-1}(\Gamma)$. Furthermore, for any $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ we have the following Green formula:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{M}_{\ell}(\mathbb{R}^N_+), \quad \int_{\mathbb{R}^N_+} \Delta \boldsymbol{v} \cdot \boldsymbol{\varphi}\,\mathrm{d}x = \int_{\mathbb{R}^N_+} \boldsymbol{v} \cdot \Delta\boldsymbol{\varphi}\,\mathrm{d}x + \int_{\Gamma} \boldsymbol{v} \cdot \partial_N \boldsymbol{\varphi}\,\mathrm{d}x'.$$

Let us now observe that the dual space $\boldsymbol{Z}'_{\ell}(\Gamma)$ of $\boldsymbol{Z}_{\ell}(\Gamma)$ can be identified with the space

$$\{\boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma);\ g_N = 0 \text{ on } \Gamma\},$$

and moreover that $\partial_N \boldsymbol{\varphi}$ sweeps $\boldsymbol{Z}_{\ell}(\Gamma)$ when $\boldsymbol{\varphi}$ sweeps $\boldsymbol{M}_{\ell}(\mathbb{R}^N_+)$. Thus, thanks to the density of $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^N_+})$ in $\boldsymbol{T}_{\ell}(\mathbb{R}^N_+)$, we can prove that the tangential trace of functions from $\boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+)$ belongs to $W^{-1/p,p}_{\ell-1}(\Gamma)$. So, their complete trace belongs to $\boldsymbol{W}^{-1/p,p}_{\ell-1}(\Gamma)$ and we have

$$\begin{gathered}\forall \boldsymbol{\varphi} \in \boldsymbol{M}_{\ell}(\mathbb{R}^N_+),\ \forall \boldsymbol{v} \in \boldsymbol{T}_{\ell,\sigma}(\mathbb{R}^N_+),\\ \langle \Delta \boldsymbol{v}, \boldsymbol{\varphi}\rangle_{\boldsymbol{X}'_{\ell} \times \boldsymbol{X}_{\ell}} = \langle \boldsymbol{v}, \Delta\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{0,p}_{\ell-1} \times \boldsymbol{W}^{0,p'}_{-\ell+1}} + \langle \boldsymbol{v}, \partial_N \boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1/p,p}_{\ell-1} \times \boldsymbol{W}^{1/p,p'}_{-\ell+1}}.\end{gathered} \tag{6.5}$$

Remark. Here again, if $N/p' = \ell$, we must add a logarithmic factor in the definition of $X_{\ell}(\mathbb{R}^N_+)$ to have the good imbedding.

We now can solve the homogeneous Stokes problem with singular boundary conditions. We will give separately the results for $\ell = 0$ and $\ell = 1$. The proofs are quite similar and we will just detail the first case. The following proposition and corollary yield the existence of very weak solutions when the data are singular, so extending Proposition 4.1. Note that $W^{1,p}_{0}(\mathbb{R}^N_+) \hookrightarrow W^{0,p}_{-1}(\mathbb{R}^N_+)$ and $W^{1-1/p,p}_{0}(\Gamma) \hookrightarrow W^{-1/p,p}_{-1}(\Gamma)$ if $N \neq p$.

Proposition 6.6. *Assume that* $\frac{N}{p} \neq 1$. *For any* $\boldsymbol{g} \in \boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)$ *such that* $g_N = 0$, *the Stokes problem* (4.1)–(4.3) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,p}(\mathbb{R}_+^N)$, *with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_{-1}^{-1,p}(\mathbb{R}_+^N)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)}.$$

Proof. (1) We will first show that if the pair $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,p}(\mathbb{R}_+^N)$ satisfies (4.1) and (4.2), then we have $\boldsymbol{u} \in \boldsymbol{T}_{0,\sigma}(\mathbb{R}_+^N)$ and thus the boundary condition (4.3) makes sense. With this aim, thanks to Lemma 6.2, observe that if $\pi \in W_{-1}^{-1,p}(\mathbb{R}_+^N)$, then we have $\nabla\pi \in \boldsymbol{X}_0'(\mathbb{R}_+^N)$ and

$$\|\nabla\pi\|_{\boldsymbol{X}_0'(\mathbb{R}_+^N)} \leqslant C\|\pi\|_{W_{-1}^{-1,p}(\mathbb{R}_+^N)}.$$

So, we have $\Delta\boldsymbol{u} \in \boldsymbol{X}_0'(\mathbb{R}_+^N)$ and the trace $\gamma_0\boldsymbol{u} \in \boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)$.

(2) Let us show that the problem (4.1)–(4.3) with $g_N = 0$ is equivalent to the variational formulation: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,p}(\mathbb{R}_+^N)$ such that

$$\begin{aligned} &\forall \boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}_+^N),\ \forall \vartheta \in W_1^{1,p'}(\mathbb{R}_+^N), \\ &\langle \boldsymbol{u}, -\Delta\boldsymbol{v} + \nabla\vartheta\rangle_{\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N)\times\boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N)} - \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{W_{-1}^{-1,p}(\mathbb{R}_+^N)\times\mathring{W}_1^{1,p'}(\mathbb{R}_+^N)} \\ &\quad = \langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)\times\boldsymbol{W}_1^{1/p,p'}(\Gamma)}. \end{aligned} \tag{6.6}$$

Indeed, let $(\boldsymbol{u}, \pi)$ be a solution to (4.1)–(4.3) with $g_N = 0$; then the Green formula (6.5) yields for all $\boldsymbol{v} \in \boldsymbol{M}_0(\mathbb{R}_+^N)$,

$$\begin{aligned} \langle -\Delta\boldsymbol{u} + \nabla\pi, \boldsymbol{v}\rangle_{\boldsymbol{X}_0'\times\boldsymbol{X}_0} &= -\langle \boldsymbol{u}, \Delta\boldsymbol{v}\rangle_{\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N)\times\boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N)} \\ &\quad - \langle \boldsymbol{g}, \partial_N\boldsymbol{v}\rangle_{\boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)\times\boldsymbol{W}_1^{1/p,p'}(\Gamma)} - \langle \pi, \operatorname{div}\boldsymbol{v}\rangle_{W_{-1}^{-1,p}(\mathbb{R}_+^N)\times\mathring{W}_1^{1,p'}(\mathbb{R}_+^N)} = 0. \end{aligned}$$

Moreover, using the density of the functions of $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}_+^N})$ with divergence zero in $\boldsymbol{T}_{0,\sigma}(\mathbb{R}_+^N)$, we obtain for all $\vartheta \in W_1^{1,p'}(\mathbb{R}_+^N)$,

$$\begin{aligned} \langle \boldsymbol{u}, \nabla\vartheta\rangle_{\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N)\times\boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N)} &= -\langle \operatorname{div}\boldsymbol{u}, \vartheta\rangle_{L^p(\mathbb{R}_+^N)\times L^{p'}(\mathbb{R}_+^N)} \\ &\quad - \langle u_N, \vartheta\rangle_{W_{-1}^{-1/p,p}(\Gamma)\times W_1^{1/p,p'}(\Gamma)} = 0. \end{aligned}$$

So we show that $(\boldsymbol{u}, \pi)$ satisfies the variational formulation (6.6). Conversely, we can readily prove that if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,p}(\mathbb{R}_+^N)$ satisfies the variational formulation (6.6), then $(\boldsymbol{u}, \pi)$ is a solution to problem (4.1)–(4.3).

(3) Let us solve problem (6.6). According to Theorem 5.2, we know that if $\frac{N}{p} \neq 1$, for all $\boldsymbol{f} \in \boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N)$ and $\varphi \in \mathring{W}_1^{1,p'}(\mathbb{R}_+^N)$, there exists a unique $(\boldsymbol{v}, \vartheta) \in \boldsymbol{M}_0(\mathbb{R}_+^N) \times W_1^{1,p'}(\mathbb{R}_+^N)$ solution to

$$-\Delta\boldsymbol{v} + \nabla\vartheta = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = \varphi \quad \text{in } \mathbb{R}_+^N, \qquad \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma,$$

with the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_1^{2,p'}(\mathbb{R}_+^N)} + \|\vartheta\|_{W_1^{1,p'}(\mathbb{R}_+^N)} \leqslant C\big(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N)} + \|\varphi\|_{W_1^{1,p'}(\mathbb{R}_+^N)}\big).$$

Then

$$\begin{aligned}\big|\langle \boldsymbol{g}, \partial_N \boldsymbol{v}\rangle_{\boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)\times \boldsymbol{W}_1^{1/p,p'}(\Gamma)}\big| &\leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)}\|\boldsymbol{v}\|_{W_1^{2,p'}(\mathbb{R}_+^N)} \\ &\leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_{-1}^{-1/p,p}}\big(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,p'}} + \|\varphi\|_{W_1^{1,p'}}\big).\end{aligned}$$

In other words, we can say that the linear mapping

$$T : (\boldsymbol{f}, \varphi) \longmapsto \langle \boldsymbol{g}, \partial_N \boldsymbol{v}\rangle$$

is continuous on $\boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N) \times \mathring{W}_1^{1,p'}(\mathbb{R}_+^N)$, and according to the Riesz representation theorem, there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,p}(\mathbb{R}_+^N)$ which is the dual space of $\boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N) \times \mathring{W}_1^{1,p'}(\mathbb{R}_+^N)$, such that

$$\begin{gathered}\forall (\boldsymbol{f}, \varphi) \in \boldsymbol{W}_1^{0,p'}\big(\mathbb{R}_+^N\big) \times \mathring{W}_1^{1,p'}\big(\mathbb{R}_+^N\big), \\ T(\boldsymbol{f}, \varphi) = \langle \boldsymbol{u}, \boldsymbol{f}\rangle_{\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N)\times \boldsymbol{W}_1^{0,p'}(\mathbb{R}_+^N)} + \langle \pi, -\varphi\rangle_{W_{-1}^{-1,p}(\mathbb{R}_+^N)\times \mathring{W}_1^{1,p'}(\mathbb{R}_+^N)},\end{gathered}$$

i.e. the pair $(\boldsymbol{u}, \pi)$ satisfies (6.6). □

We now can drop the hypothesis $g_N = 0$.

Theorem 6.7. *Assume that $\frac{N}{p} \neq 1$. For any $\boldsymbol{g} \in \boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)$, the Stokes problem* (4.1)–(4.3) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N) \times W_{-1}^{-1,p}(\mathbb{R}_+^N)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N)} + \|\pi\|_{W_{-1}^{-1,p}(\mathbb{R}_+^N)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)}.$$

Proof. According to Theorem 3.4, if $\frac{N}{p} \neq 1$, then there exists $\psi \in W_{-1}^{1,p}(\mathbb{R}_+^N)$ unique up to an element of $\mathcal{N}_{[2-N/p]}^{\Delta}$ solution to the following Neumann problem:

$$\Delta\psi = 0 \quad \text{in } \mathbb{R}_+^N, \qquad \partial_N\psi = g_N \quad \text{on } \Gamma.$$

Let us set $\boldsymbol{w} = \nabla\psi$ and $\boldsymbol{g}^* = \boldsymbol{g} - \gamma_0\boldsymbol{w}$. Then $\boldsymbol{w} \in \boldsymbol{T}_{0,\sigma}(\mathbb{R}_+^N)$ and

$$\|\boldsymbol{w}\|_{\boldsymbol{T}_0(\mathbb{R}_+^N)} = \|\boldsymbol{w}\|_{\boldsymbol{W}_{-1}^{0,p}(\mathbb{R}_+^N)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_{-1}^{-1/p,p}(\Gamma)}.$$

Furthermore, $\boldsymbol{g}^*$ satisfies the hypotheses of Proposition 6.6, hence the existence of a unique pair $(\boldsymbol{z}, \pi)$ which satisfies

$$-\Delta \boldsymbol{z} + \nabla\pi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in } \mathbb{R}_+^N, \qquad \boldsymbol{z} = \boldsymbol{g}^* \quad \text{on } \Gamma.$$

Then the pair $(z + \boldsymbol{w}, \pi)$ is the required solution. The uniqueness of this solution is a straightforward consequence of Proposition 6.6. □

Here is the corresponding results for the case $\ell = 1$.

Proposition 6.8. *For any $\boldsymbol{g} \in \boldsymbol{W}_0^{-1/p,p}(\Gamma)$ such that $g_N = 0$, and $\boldsymbol{g}' \perp \mathbb{R}^{N-1}$ if $N \leqslant p'$, the Stokes problem* (4.1)–(4.3) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\mathbb{R}^N_+) \times W_0^{-1,p}(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)} + \|\pi\|_{W_0^{-1,p}(\mathbb{R}^N_+)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{-1/p,p}(\Gamma)}.$$

Proof. The two differences from the weight $\ell = 0$ are the absence of critical value (the reason is that here the dual problem solved by Theorem 5.6 has no critical value), and the orthogonality condition in the case $N \leqslant p'$ (which corresponds by duality to the non-zero kernel in Theorem 5.6 if $N \leqslant p$). The rest of the proof is similar. □

Theorem 6.9. *For any $\boldsymbol{g} \in \boldsymbol{W}_0^{-1/p,p}(\Gamma)$ such that $\boldsymbol{g} \perp \mathbb{R}^N$ if $N \leqslant p'$, the Stokes problem* (4.1)–(4.3) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\mathbb{R}^N_+) \times W_0^{-1,p}(\mathbb{R}^N_+)$, with the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\mathbb{R}^N_+)} + \|\pi\|_{W_0^{-1,p}(\mathbb{R}^N_+)} \leqslant C\|\boldsymbol{g}\|_{\boldsymbol{W}_0^{-1/p,p}(\Gamma)}.$$

Remark 6.10. Let $p > 1$ be a real number. If $p < N$ and $r = Np/(N - p)$, then we have $W_0^{1-1/p,p}(\Gamma) \hookrightarrow W_0^{-1/r,r}(\Gamma)$. Indeed, for every $g \in W_0^{1-1/p,p}(\Gamma)$, there exists $u \in W_0^{2,p}(\mathbb{R}^N_+)$ such that

$$\Delta u = 0 \quad \text{in } \mathbb{R}^N_+, \qquad \partial_N u = g \quad \text{on } \Gamma$$

(see [6, Corollary 3.3]). Since we have the imbedding $W_0^{2,p}(\mathbb{R}^N_+) \hookrightarrow W_0^{1,r}(\mathbb{R}^N_+)$, we can deduce that $\boldsymbol{v} = \nabla u \in \boldsymbol{L}^r(\mathbb{R}^N_+)$ and $\operatorname{div} \boldsymbol{v} = 0 \in W_1^{0,r}(\mathbb{R}^N_+)$, i.e. $\boldsymbol{v} \in \boldsymbol{W}_1^{0,p}(\operatorname{div}; \mathbb{R}^N_+)$. Moreover, as $r' \neq N$, according to Lemma 6.4, we get $\gamma_{e_N} \boldsymbol{v} = \partial_N u|_\Gamma = g \in W_0^{-1/r,r}(\Gamma)$. Consequently, if $\boldsymbol{g} \in \boldsymbol{W}_0^{1-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{W}_0^{-1/r,r}(\Gamma)$, Proposition 4.1 and Theorem 6.9 respectively yield the unique solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^N_+) \times L^p(\mathbb{R}^N_+)$ and $(\boldsymbol{v}, \vartheta) \in \boldsymbol{L}^r(\mathbb{R}^N_+) \times W_0^{-1,r}(\mathbb{R}^N_+)$, which are identical thanks to the Sobolev imbeddings $W_0^{1,p}(\mathbb{R}^N_+) \hookrightarrow L^r(\mathbb{R}^N_+)$ and $L^p(\mathbb{R}^N_+) \hookrightarrow W_0^{-1,r}(\mathbb{R}^N_+)$.

References

[1] F. Alliot, C. Amrouche, The Stokes problem in $\mathbb{R}^N$: An approach in weighted Sobolev spaces, Math. Methods Appl. Sci. 9 (5) (1999) 723–754.

[2] C. Amrouche, V. Girault, Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension, Czechoslovak Math. J. 44 (119) (1994) 109–140.

[3] C. Amrouche, V. Girault, J. Giroire, Weighted Sobolev spaces for Laplace's equation in $\mathbb{R}^N$, J. Math. Pures Appl. 73 (6) (1994) 579–606.

[4] C. Amrouche, V. Girault, J. Giroire, Equation de Poisson et espaces de Sobolev avec poids critiques, Publications du LMA de l'Université de Pau, 06/25.

[5] C. Amrouche, S. Nečasová, Laplace equation in the half-space with a nonhomogeneous Dirichlet boundary condition, Math. Bohem. 126 (2) (2001) 265–274.

[6] C. Amrouche, The Neumann problem in the half-space, C. R. Acad. Sci. Paris Sér. I 335 (2002) 151–156.
[7] C. Amrouche, Y. Raudin, Nonhomogeneous biharmonic problem in the half-space, L^p theory and generalized solutions, J. Differential Equations 236 (2007) 57–81.
[8] C. Amrouche, Y. Raudin, Singular boundary conditions and regularity for the biharmonic problem in the half-space, Commun. Pure Appl. Anal. 6 (4) (2007) 957–982.
[9] W. Bochers, T. Miyakawa, L^2 decay for the Navier–Stokes flow in halfspaces, Math. Ann. 282 (1988) 139–155.
[10] T.Z. Boulmezaoud, On the Stokes system and the biharmonic equation in the half-space: An approach via weighted Sobolev spaces, Math. Methods Appl. Sci. 25 (2002) 373–398.
[11] L. Cattabriga, Su un problema al contorno relativo al sistema di equazioni di Stokes, Rend. Sem. Mat. Univ. Padova 31 (1961) 308–340.
[12] R. Farwig, H. Sohr, On the Stokes and Navier–Stokes system for domains with noncompact boundary in L^q-spaces, Math. Nachr. 170 (1994) 53–77.
[13] Y. Fujigaki, T. Miyakawa, Asymptotic profiles of nonstationary incompressible Navier–Stokes flows in the half-space, Methods Appl. Anal. 8 (1) (2001) 121–158.
[14] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, vols. I, II, Springer Tracts in Natural Philosophy, 1994.
[15] Y. Giga, S. Matsui, Y. Shimizu, On estimates in Hardy spaces for the Stokes flow in a half space, Math. Z. 231 (2) (1999) 383–396.
[16] B. Hanouzet, Espaces de Sobolev avec poids. Application au problème de Dirichlet dans un demi-espace, Rend. Sem. Univ. Padova 46 (1971) 227–272.
[17] H. Kozono, Global L^n-solution and its decay property for Navier–Stokes equations in half-space $\mathbb{R}^n_+$, J. Differential Equations 79 (1989) 79–88.
[18] V.G. Maz'ya, B.A. Plamenevskiĭ, L.I. Stupyalis, The three-dimensional problem of steady-state motion of a fluid with a free surface, Amer. Math. Soc. Transl. 123 (1984) 171–268.
[19] N. Tanaka, On the boundary value problem for the stationary Stokes system in the half-space, J. Differential Equations 115 (1995) 70–74.
[20] S. Ukai, A solution formula for the Stokes equation in $\mathbb{R}^n_+$, Comm. Pure Appl. Math. XL (1987) 611–621.

SIAM J. MATH. ANAL.
Vol. 41, No. 5, pp. 1792–1815

FROM STRONG TO VERY WEAK SOLUTIONS TO THE STOKES SYSTEM WITH NAVIER BOUNDARY CONDITIONS IN THE HALF-SPACE*

CHÉRIF AMROUCHE†, ŠÁRKA NEČASOVÁ‡, AND YVES RAUDIN†

Abstract. We consider the Stokes problem with slip-type boundary conditions in the half-space $\mathbb{R}^n_+$, with $n \geqslant 2$. The weighted Sobolev spaces yield the functional framework. We first study generalized and strong solutions and then the case with very low regularity of data on the boundary. We apply the method of decomposition introduced in our previous work [*J. Differential Equations*, 244 (2008), pp. 887–915] where it is necessary to solve particular problems for harmonic and biharmonic operators with very weak data. We also envisage a wide class of behaviors at infinity for data and solutions.

Key words. Stokes problem, half-space, weighted Sobolev spaces

AMS subject classifications. 35J50, 35J55, 35Q30, 76D07, 76N10

DOI. 10.1137/090749207

1. Introduction and preliminaries. The motion of a viscous incompressible fluid is described by the Navier–Stokes equations, which are nonlinear. The Stokes system is a linear approximation of this model, available for slow motions. For the stationary Stokes problem

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \Omega,$$

where Ω is a domain of $\mathbb{R}^n$, there are several possible boundary conditions. Under the hypothesis of impermeability of the boundary, the velocity field $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\boldsymbol{n}$ stands for the outer normal vector. According to the idea that the fluid cannot slip on the wall due to its viscosity, we get the no-slip condition

$$\boldsymbol{u}_\tau = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\, \boldsymbol{n}$ denotes, as usual, the tangential component of $\boldsymbol{u}$. The Dirichlet boundary value problem, which was suggested by Stokes, is the combination of (1.1) and (1.2). Concerning this problem, the literature is well known and extensive. Especially in the case of the half-space, we would like to mention the works of Cattabriga [11], Tanaka [25], Farwig and Sohr [14], and Galdi [15], where the solution of the problem is investigated in homogeneous Sobolev spaces, whereas in the works of Maz'ya, Plamenevskiĭ, and Stupyalis [21] and Boulmezaoud [10], we can find results

*Received by the editors February 10, 2009; accepted for publication (in revised form) August 17, 2009; published electronically November 5, 2009.
http://www.siam.org/journals/sima/41-5/74920.html
†Laboratoire de Mathématiques Appliquées, CNRS UMR 5142, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'Université, 64000 Pau, France (cherif.amrouche@univ-pau.fr, yves.raudin@univ-pau.fr).
‡Mathematical Institute of the Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic (matus@math.cas.cz). The work of this author was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503, and by the Grant Agency of the Academy of Sciences under grant IAA1001190804.

in weighted Sobolev spaces. This is also the functional framework of our previous work (see [7]).

The correctness of the no-slip hypothesis has been a subject of discussion for over two centuries among many distinguished scientists. Instead of (1.2), Navier had already proposed the following condition saying that the velocity on the boundary is proportional to the tangential component of the stress:

$$(\mathbb{T}\cdot\boldsymbol{n})_\tau + \beta\,\boldsymbol{u}_\tau = \mathbf{0} \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\mathbb{T}$ denotes the viscous stress tensor and β is a friction coefficient. For the incompressible isotropic fluids, the viscous stress tensor is given by

$$\mathbb{T} = -\pi\,\mathbb{I} + \nu\,(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T).$$

The case $\beta = 0$ is termed *complete slip*, while (1.3) reduces to (1.2) in the asymptotic limit $\beta \to \infty$.

Recent developments in micro- and nanofluidic technologies have renewed interest in the influence of surface roughness on the slip behavior of viscous fluids (see Priezjev and Troian [23]). Intuitively much closer to the observed reality, the Navier slip conditions have been often replaced by (1.2), as the slip length is likely to be too small to influence the motion on the macroscopic scale. However, numerous experiments and simulations, as well as theoretical studies, have shown that the classical no-slip assumption can fail when the walls are sufficiently smooth (see Einzel, Panzer, and Liu [12], Lauga, Brenner, and Stone [20], Priezjev, Darhuber, and Troian [22], Qian, Wang, and Sheng [24], and Zhu and Granick [27]). Strictly speaking, the slip length characterizing the contact between a fluid and a solid wall in relative motion is influenced by many different factors, among which the intrinsic affinity and commensurability between the liquid and solid molecular size, as well as the macroscopic surface roughness caused by imperfections and tiny asperities, play a significant role. Navier's boundary conditions have been considered by many authors. Let us quote Jäger and Mikelić [18] and Zajączkowski [26]. In the three-dimensional case, we can find other boundary conditions in the work of Ladyzhenskaya and Solonnikov [19] and intensively studied by Babin, Mahalov, and Nicolaenko [8, 9]. These conditions can be expressed by (1.1) combined with the equations

$$\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}. \tag{1.4}$$

In the half-space $\mathbb{R}^n_+$, where $\boldsymbol{n} = (0, \ldots, 0, -1)$, Navier's conditions (1.1) and (1.3) with $\beta = 0$ can be written as

$$u_n = 0, \quad \partial_n\boldsymbol{u}' = \mathbf{0} \quad \text{on } \Gamma = \partial\mathbb{R}^n_+.$$

Let us remark that in the case of $\mathbb{R}^3_+$, we would get the same boundary conditions from (1.1) and (1.4).

The aim of this paper is to investigate the Stokes problem in the half-space with the following type of slip boundary conditions:

$$(\mathrm{S}^\sharp)\ \begin{cases} -\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} & \text{and} \quad \operatorname{div}\boldsymbol{u} = h \quad \text{in } \mathbb{R}^n_+, \\ \qquad\qquad u_n = g_n & \text{and} \quad \partial_n\boldsymbol{u}' = \boldsymbol{g}' \quad \text{on } \Gamma. \end{cases}$$

This paper is organized as follows. The second part of this section is devoted to notation, functional setting, and useful results. In section 2, we establish the existence of generalized solutions in the central case of weight zero. In section 3, we extend this result to a wide class of weights, and we also deal with strong solutions. Last, in section 4, we are interested in the case of very low regularity at the boundary which yields very weak solutions.

For any real number $p > 1$, we always take p' to be the Hölder conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

For any integer $n \geqslant 2$, writing a typical point $x \in \mathbb{R}^n$ as $x = (x', x_n)$, we denote by $\mathbb{R}^n_+$ the upper half-space of $\mathbb{R}^n$ and by $\Gamma \equiv \mathbb{R}^{n-1}$ its boundary. We will use the two basic weights $\varrho = (1 + |x|^2)^{1/2}$ and $\lg \varrho = \ln(2 + |x|^2)$, where $|x|$ is the Euclidean norm of x.

For any integer q, $\mathcal{P}_q$ stands for the space of polynomials of degree smaller than or equal to q; $\mathcal{P}_q^{\Delta}$ (resp., $\mathcal{P}_q^{\Delta^2}$) is the subspace of harmonic (resp., biharmonic) polynomials of $\mathcal{P}_q$; $\mathcal{A}_q^{\Delta}$ (resp., $\mathcal{N}_q^{\Delta}$) is the subspace of polynomials of $\mathcal{P}_q^{\Delta}$, odd (resp., even) with respect to x_n, or equivalently, which satisfy the condition $\varphi(x', 0) = 0$ (resp., $\partial_n \varphi(x', 0) = 0$), with the convention that these spaces are reduced to $\{0\}$ if $q < 0$. For any real number s, we denote by $[s]$ the integer part of s.

Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B' \perp X$ the subspace of B' orthogonal to X. For any $k \in \mathbb{Z}$, we will denote by $\{1, \ldots, k\}$ the set of the first k positive integers, with the convention that this set is empty if k is nonpositive.

Throughout this paper, bold characters are used for the vector fields; depending on the context, $\boldsymbol{f} \in \boldsymbol{X}$ stands for $\boldsymbol{f} = (f_1, \ldots, f_n) \in \boldsymbol{X} = X^n$, and $\boldsymbol{g'} \in \boldsymbol{X}$ stands for $\boldsymbol{g'} = (g_1, \ldots, g_{n-1}) \in \boldsymbol{X} = X^{n-1}$.

For weighted Sobolev spaces, we refer the reader to Hanouzet's classic article [17] and especially to [2] for logarithmic weights. Let Ω be an open set of $\mathbb{R}^n$. For any $m \in \mathbb{N}$, $p \in \,]1, \infty[$, $(\alpha, \beta) \in \mathbb{R}^2$, we define the following space:

$$
\begin{aligned}
W^{m,p}_{\alpha,\beta}(\Omega) = \Big\{ u \in \mathcal{D}'(\Omega);\ & 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta - 1} \partial^\lambda u \in L^p(\Omega); \\
& k + 1 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta} \partial^\lambda u \in L^p(\Omega) \Big\},
\end{aligned} \tag{1.5}
$$

where $k = m - n/p - \alpha$ if $n/p + \alpha \in \{1, \ldots, m\}$, and $k = -1$ otherwise. In the case $\beta = 0$, we simply denote the space by $W^{m,p}_{\alpha}(\Omega)$. Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$
\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Bigg(\sum_{0 \leqslant |\lambda| \leqslant k} \|\varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta - 1} \partial^\lambda u\|^p_{L^p(\Omega)} + \sum_{k+1 \leqslant |\lambda| \leqslant m} \|\varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{\beta} \partial^\lambda u\|^p_{L^p(\Omega)} \Bigg)^{1/p}.
$$

We also define the seminorm:

$$
|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Bigg(\sum_{|\lambda| = m} \|\varrho^{\alpha} (\lg \varrho)^{\beta} \partial^\lambda u\|^p_{L^p(\Omega)} \Bigg)^{1/p}.
$$

The weights in definition (1.5) are chosen so that $\mathcal{D}\big(\overline{\mathbb{R}^n_+}\big)$ is dense in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^n_+)$ and so that the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^n_+)$ (see [3]): Let

$q^* = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)$. If $n/p + \alpha \notin \{1, \dots, m\}$ or $(\beta - 1)p \neq -1$, then

$$\forall u \in W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+), \quad \|u\|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)/\mathcal{P}_{q^*}} \leqslant C\,|u|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)},$$

and

$$\forall u \in \overset{\circ}{W}{}^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+) = \overline{\mathcal{D}(\mathbb{R}^n_+)}^{\|\cdot\|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)}}, \quad \|u\|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)} \leqslant C\,|u|_{W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)}.$$

We denote by $W^{-m,\,p'}_{-\alpha,\,-\beta}(\mathbb{R}^n_+)$ the dual space of $\overset{\circ}{W}{}^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+)$, and we notice that it is a space of distributions. If $n/p + \alpha \notin \{1, \dots, m\}$, we have the imbeddings

$$W^{m,\,p}_{\alpha,\,\beta}(\mathbb{R}^n_+) \hookrightarrow W^{m-1,\,p}_{\alpha-1,\,\beta}(\mathbb{R}^n_+) \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-m,\,\beta}(\mathbb{R}^n_+).$$

If $n/p + \alpha = j \in \{1, \dots, m\}$, then we have

$$W^{m,\,p}_{\alpha,\,\beta} \hookrightarrow \cdots \hookrightarrow W^{m-j+1,\,p}_{\alpha-j+1,\,\beta} \hookrightarrow W^{m-j,\,p}_{\alpha-j,\,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-m,\,\beta-1}.$$

In order to define the traces of functions of $W^{m,\,p}_{\alpha}(\mathbb{R}^n_+)$ (here we do not consider the case $\beta \neq 0$), for any $\sigma \in \,]0,1[$, we introduce the space

$$W^{\sigma,\,p}_{\alpha}(\mathbb{R}^n) = \Big\{ u \in \mathcal{D}'(\mathbb{R}^n);\ w^{\alpha-\sigma} u \in L^p(\mathbb{R}^n), \\ \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\varrho^{\alpha}(x)\,u(x) - \varrho^{\alpha}(y)\,u(y)|^p}{|x-y|^{n+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < \infty \Big\},$$

where $w = \varrho$ if $n/p + \alpha \neq \sigma$ and $w = \varrho\,(\lg \varrho)^{1/(\sigma-\alpha)}$ if $n/p + \alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$W^{s,\,p}_{\alpha}(\mathbb{R}^n) = \Big\{ u \in \mathcal{D}'(\mathbb{R}^n);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-s+|\lambda|}\,(\lg \varrho)^{-1}\,\partial^{\lambda} u \in L^p(\mathbb{R}^n); \\ k+1 \leqslant |\lambda| \leqslant [s]-1,\ \varrho^{\alpha-s+|\lambda|}\,\partial^{\lambda} u \in L^p(\mathbb{R}^n);\ \partial^{[s]} u \in W^{\sigma,\,p}_{\alpha}(\mathbb{R}^n) \Big\},$$

where $k = s - n/p - \alpha$ if $n/p + \alpha \in \{\sigma, \dots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. In the same way, we define, for any real number β, the space $W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^n) = \big\{ v \in \mathcal{D}'(\mathbb{R}^n);\ (\lg \varrho)^{\beta}\, v \in W^{s,\,p}_{\alpha}(\mathbb{R}^n) \big\}$. These two spaces are reflexive Banach spaces equipped with their natural norms. If $n/p + \alpha \notin \{\sigma, \dots, \sigma + [s] - 1\}$, we have the imbeddings

$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^n) \hookrightarrow W^{s-1,\,p}_{\alpha-1,\,\beta}(\mathbb{R}^n) \hookrightarrow \cdots \hookrightarrow W^{\sigma,\,p}_{\alpha-[s],\,\beta}(\mathbb{R}^n),$$
$$W^{s,\,p}_{\alpha,\,\beta}(\mathbb{R}^n) \hookrightarrow W^{[s],\,p}_{\alpha+[s]-s,\,\beta}(\mathbb{R}^n) \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-s,\,\beta}(\mathbb{R}^n).$$

If $n/p + \alpha = j \in \{\sigma, \dots, \sigma + [s] - 1\}$, then we have

$$W^{s,\,p}_{\alpha,\,\beta} \hookrightarrow \cdots \hookrightarrow W^{s-j+1,\,p}_{\alpha-j+1,\,\beta} \hookrightarrow W^{s-j,\,p}_{\alpha-j,\,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{\sigma,\,p}_{\alpha-[s],\,\beta-1},$$
$$W^{s,\,p}_{\alpha,\,\beta} \hookrightarrow W^{[s],\,p}_{\alpha+[s]-s,\,\beta} \hookrightarrow \cdots \hookrightarrow W^{[s]-j+1,\,p}_{\alpha-\sigma-j+1,\,\beta} \hookrightarrow W^{[s]-j,\,p}_{\alpha-\sigma-j,\,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,\,p}_{\alpha-s,\,\beta-1}.$$

If u is a function on $\mathbb{R}^n_+$, we denote its trace of order j on the hyperplane Γ by

$$\forall j \in \mathbb{N}, \quad \gamma_j u : x' \in \mathbb{R}^{n-1} \longmapsto \partial^j_n u(x', 0).$$

Let us recall the following trace lemma due to Hanouzet (see [17]) and extended by Amrouche and Nečasová (see [3]) to the critical values with logarithmic weights.

LEMMA 1.1 (the trace lemma). *For any integer $m \geqslant 1$ and real number α, we have the linear continuous mapping*

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{m-1}) : \; W^{m,\,p}_{\alpha}(\mathbb{R}^n_+) \longrightarrow \prod_{j=0}^{m-1} W^{m-j-1/p,\,p}_{\alpha}(\mathbb{R}^{n-1}).$$

Moreover, γ is surjective and $\mathrm{Ker}\gamma = \overset{\circ}{W}{}^{m,\,p}_{\alpha}(\mathbb{R}^n_+)$.

On the Stokes problem in $\mathbb{R}^n$,

$$\text{(S)} \qquad -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in } \mathbb{R}^n,$$

let us recall the fundamental results on which the text to follow is based. First, for any $k \in \mathbb{Z}$, we introduce the space

$$\mathcal{S}_k = \left\{ (\boldsymbol{\lambda},\, \mu) \in \boldsymbol{\mathcal{P}}_k \times \mathcal{P}^{\Delta}_{k-1};\; \operatorname{div} \boldsymbol{\lambda} = 0,\; -\Delta \boldsymbol{\lambda} + \nabla \mu = \boldsymbol{0} \right\}.$$

THEOREM 1.2 (see Alliot and Amrouche [1]). *Let $\ell \in \mathbb{Z}$, and assume that*

$$(1.6) \qquad n/p' \notin \{1, \dots, \ell\} \quad \textit{and} \quad n/p \notin \{1, \dots, -\ell\}.$$

For any $(\boldsymbol{f},\, h) \in \left(\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^n) \times W^{0,\,p}_{\ell}(\mathbb{R}^n) \right) \perp \mathcal{S}_{[1+\ell-n/p']}$, *problem* (S) *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n) \times W^{0,\,p}_{\ell}(\mathbb{R}^n)$, *unique up to an element of* $\mathcal{S}_{[1-\ell-n/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}_{[1-\ell-n/p]}} \left(\| \boldsymbol{u} + \boldsymbol{\lambda} \|_{\boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n)} + \| \pi + \mu \|_{W^{0,\,p}_{\ell}(\mathbb{R}^n)} \right) \leqslant C \left(\| \boldsymbol{f} \|_{\boldsymbol{W}^{-1,\,p}_{\ell}(\mathbb{R}^n)} + \| h \|_{W^{0,\,p}_{\ell}(\mathbb{R}^n)} \right).$$

THEOREM 1.3 (see Alliot and Amrouche [1]). *Let $\ell \in \mathbb{Z}$ and $m \geqslant 1$ be two integers, and assume that*

$$(1.7) \qquad n/p' \notin \{1, \dots, \ell + 1\} \quad \textit{and} \quad n/p \notin \{1, \dots, -\ell - m\}.$$

For any $(\boldsymbol{f},\, h) \in \left(\boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^n) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^n) \right) \perp \mathcal{S}_{[1+\ell-n/p']}$, *problem* (S) *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^n)$, *unique up to an element of* $\mathcal{S}_{[1-\ell-n/p]}$, *with the estimate*

$$\inf_{(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}_{[1-\ell-n/p]}} \left(\| \boldsymbol{u} + \boldsymbol{\lambda} \|_{\boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n)} + \| \pi + \mu \|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^n)} \right) \leqslant C \left(\| \boldsymbol{f} \|_{\boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^n)} + \| h \|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^n)} \right).$$

2. Generalized solutions for the weight zero. In this section, we will concentrate on the central case of weight zero—that is, solutions $(\boldsymbol{u},\, \pi)$ which belong to $\boldsymbol{W}^{1,\,p}_{0}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$. This restriction allows us to avoid the question of kernel and, above all, of compatibility conditions for the data. However, in the next section, we will rest on this construction to envisage a wide class of weights.

First, we will establish the result about the generalized solutions to ($\mathrm{S}^\sharp$) in the homogeneous case. The method is similar to the one employed for the Dirichlet conditions (see [7]), but the auxiliary problems and the arguments for their resolution are appreciably different.

2.1. The homogeneous case. Here, we assume that $\boldsymbol{f} = \boldsymbol{0}$ and $h = 0$.

PROPOSITION 2.1. *For any $g_n \in W_0^{1-1/p,\,p}(\Gamma)$ and $\boldsymbol{g}' \in \boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$ such that $\boldsymbol{g}' \perp \mathbb{R}^{n-1}$ if $n \leqslant p'$, the Stokes problem*

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{0} \quad \text{in } \mathbb{R}^n_+, \tag{2.1a}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^n_+, \tag{2.1b}$$

$$u_n = g_n \quad \text{on } \Gamma, \tag{2.1c}$$

$$\partial_n \boldsymbol{u}' = \boldsymbol{g}' \quad \text{on } \Gamma \tag{2.1d}$$

has a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$, unique if $n > p$, unique up to an element of $\mathbb{R}^{n-1} \times \{0\}^2$ if $n \leqslant p$, with the estimate

$$\inf_{\boldsymbol{\xi} \in \mathbb{R}^{n-1} \times \{0\}} \|\boldsymbol{u} + \boldsymbol{\xi}\|_{\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+)} + \|\pi\|_{L^p(\mathbb{R}^n_+)} \leqslant C \left(\|g_n\|_{W_0^{1-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}_0^{-1/p,\,p}(\Gamma)} \right)$$

if $n \leqslant p$, and the corresponding estimate without inf ($\boldsymbol{\xi} = 0$) *if $n > p$.*

Remark 2.2. Before giving the proof, let us notice that this problem is not standard. Indeed, we find the velocity field $\boldsymbol{u}$ in $\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+)$ with a boundary condition $\partial_n \boldsymbol{u}' = \boldsymbol{g}' \in \boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$ for its tangential components.

Such a velocity field is possible because $\Delta^2 \boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^n_+$, and then, we find an ad hoc space in which we can give a meaning (see [6, Lemma 4.8]) to the trace of $\partial_n \boldsymbol{u}'$ precisely in the space $\boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$.

Proof. (i) First, we reduce system (2.1) to three problems on the fundamental operators Δ^2 and Δ.

According to (2.1b) and applying the operators div and Δ to (2.1a), we get both $\Delta \pi = 0$ and $\Delta^2 \boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^n_+$.

From the boundary condition (2.1c), we take out

$$\forall i \in \{1, 2, \ldots, n-1\}, \qquad \partial_i^2 u_n = \partial_i^2 g_n \quad \text{on } \Gamma.$$

In addition, from (2.1d), we take out

$$\partial_n^2 u_n = \partial_n(\partial_n u_n) = \partial_n(-\operatorname{div}' \boldsymbol{u}') = -\operatorname{div}' \boldsymbol{g}' \quad \text{on } \Gamma,$$

and hence, the boundary condition

$$\Delta u_n = \Delta' g_n - \operatorname{div}' \boldsymbol{g}' \quad \text{on } \Gamma,$$

where $\Delta' = \sum_{j=1}^{n-1} \partial_j^2$. So, we get the biharmonic problem

$$\text{(B)} \qquad \Delta^2 u_n = 0 \ \text{ in } \mathbb{R}^n_+, \quad u_n = g_n, \ \text{ and } \ \Delta u_n = \Delta' g_n - \operatorname{div}' \boldsymbol{g}' \ \text{ on } \Gamma.$$

Moreover, we have two Neumann problems,

$$\text{(N1)} \qquad \Delta \pi = 0 \ \text{ in } \mathbb{R}^n_+, \quad \partial_n \pi = \Delta u_n \ \text{ on } \Gamma,$$

$$\text{(N2)} \qquad \Delta \boldsymbol{u}' = \nabla' \pi \ \text{ in } \mathbb{R}^n_+, \quad \partial_n \boldsymbol{u}' = \boldsymbol{g}' \ \text{ on } \Gamma.$$

(ii) Now, we will solve these three problems.

Step 1. We deal with problem (B). Denoting $z_n = \Delta u_n$, we can split our problem into the following two Dirichlet problems:

$$\Delta z_n = 0 \;\text{ in } \mathbb{R}^n_+, \quad z_n = \Delta' g_n - \operatorname{div}' \boldsymbol{g}' \;\text{ on } \Gamma, \tag{2.2}$$

$$\Delta u_n = z_n \;\text{ in } \mathbb{R}^n_+, \quad u_n = g_n \;\text{ on } \Gamma. \tag{2.3}$$

Concerning (2.2), we notice that $\Delta' g_n - \operatorname{div}' \boldsymbol{g}' \in W_0^{-1-1/p,\,p}(\Gamma)$, and then we can apply the result on the singular boundary conditions for the homogeneous Dirichlet problem (see [7, Theorem 3.5]), provided the following orthogonality condition is satisfied:

$$\forall \varphi \in \mathcal{A}^{\Delta}_{[3-n/p']}, \quad \left\langle \Delta' g_n - \operatorname{div}' \boldsymbol{g}', \partial_n \varphi \right\rangle_{W_0^{-1-1/p,\,p}(\Gamma) \times W_0^{2-1/p',\,p'}(\Gamma)} = 0.$$

According to the degree of polynomials in $\mathcal{A}^{\Delta}_{[3-n/p']}$, this condition reduces to $\boldsymbol{g}' \perp \boldsymbol{\mathcal{P}}_{[1-n/p']}$, which is precisely the assumption of Proposition 2.1. Thus problem (2.2) has a unique solution $z_n \in W_0^{-1,\,p}(\mathbb{R}^n_+)$.

Concerning (2.3), we can apply the result on the generalized solutions to the Dirichlet problem (see [3, Theorem 3.1]) without any condition since $\mathcal{A}^{\Delta}_{[1-n/p']} = \{0\}$. Thus problem (2.3) has a unique solution $u_n \in W_0^{1,\,p}(\mathbb{R}^n_+)$.

Step 2. Next, we study problem (N1). Since $\Delta u_n \in W_0^{-1,\,p}(\mathbb{R}^n_+)$, it is necessary to check that the trace of Δu_n has meaning. We have both $\Delta u_n \in W_0^{-1,\,p}(\mathbb{R}^n_+)$ and $\Delta^2 u_n = 0$, and then it follows that $\Delta u_n \in W_0^{-1-1/p,\,p}(\Gamma)$ (see [7, Lemma 3.7]). Next, the result on the singular boundary conditions for the homogeneous Neumann problem (see [4] or [7, Theorem 3.3]) holds, provided the following orthogonality condition is satisfied:

$$\forall \varphi \in \mathcal{N}^{\Delta}_{[2-n/p']}, \quad \langle \Delta u_n, \varphi \rangle_{W_0^{-1-1/p,\,p}(\Gamma) \times W_0^{2-1/p',\,p'}(\Gamma)} = 0.$$

But, according to the degree of polynomials in $\mathcal{N}^{\Delta}_{[2-n/p']}$, it is clear that this condition is always satisfied. It implies the existence of a unique solution $\pi \in L^p(\mathbb{R}^n_+)$ to problem (N1).

Step 3. Finally, we are dealing with problem (N2). We split it into two parts:

$$\Delta \boldsymbol{v}' = \nabla' \pi \;\text{ in } \mathbb{R}^n_+, \quad \partial_n \boldsymbol{v}' = \boldsymbol{0} \;\text{ on } \Gamma, \tag{2.4}$$

and

$$\Delta \boldsymbol{z}' = \boldsymbol{0} \;\text{ in } \mathbb{R}^n_+, \quad \partial_n \boldsymbol{z}' = \boldsymbol{g}' \;\text{ on } \Gamma. \tag{2.5}$$

To solve (2.4), we introduce the auxiliary problem

$$\Delta w = \pi \;\text{ in } \mathbb{R}^n_+, \quad \partial_n w = 0 \;\text{ on } \Gamma. \tag{2.6}$$

Since we have $\pi \in L^p(\mathbb{R}^n_+)$, problem (2.6) has a solution $w \in W_0^{2,\,p}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{N}^{\Delta}_{[2-n/p]}$ (see [4]). Next, it suffices to put $\boldsymbol{v}' = \nabla' w$ to obtain a (nonunique) solution $\boldsymbol{v}' \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+)$ to problem (2.4).

For problem (2.5), with $\boldsymbol{g}' \in \boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$, we must use an intermediate result for the Neumann problem (see [7, Theorem 3.4]). With this weight, the compatibility condition is $\boldsymbol{g}' \perp \boldsymbol{\mathcal{P}}_{[1-n/p']}$. Thus it is realized by the assumption of Proposition 2.1. So, this problem has a solution $\boldsymbol{z}' \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+)$, unique up to an element of $\boldsymbol{\mathcal{P}}_{[1-n/p]}$.

Then, it is clear that the function $\boldsymbol{u}' = \boldsymbol{v}' + \boldsymbol{z}' \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+)$ is the solution to problem (N2).

(iii) Conversely, it is necessary to show that from $u_n, \pi, \boldsymbol{u}'$, we get a solution $(\boldsymbol{u}, \pi)$ of the original problem (2.1).

From the previous steps it is clear that

$$\begin{aligned} -\Delta \boldsymbol{u}' + \nabla' \pi &= 0 && \text{in } \mathbb{R}^n_+, \\ u_n &= g_n && \text{on } \Gamma, \\ \partial_n \boldsymbol{u}' &= \boldsymbol{g}' && \text{on } \Gamma. \end{aligned}$$

It remains to prove that

$$-\Delta u_n + \partial_n \pi = 0 \quad \text{in } \mathbb{R}^n_+, \tag{2.7}$$

and finally, to prove relation (2.1b).

For (2.7), thanks to the first equations of (B) and (N1), we get

$$\Delta(\Delta u_n - \partial_n \pi) = \Delta^2 u_n = 0 \quad \text{in } \mathbb{R}^n_+.$$

With the boundary condition of (N1), it follows that $\Delta u_n - \partial_n \pi$ satisfies the problem

$$\Delta(\Delta u_n - \partial_n \pi) = 0 \quad \text{in } \mathbb{R}^n_+, \quad \Delta u_n - \partial_n \pi = 0 \quad \text{on } \Gamma. \tag{2.8}$$

As well, $\Delta u_n - \partial_n \pi \in W_0^{-1,\,p}(\mathbb{R}^n_+)$, and then by a uniqueness argument, we necessarily have $\Delta u_n - \partial_n \pi = 0$ in $\mathbb{R}^n_+$ (see [7, Theorem 3.5]).

For (2.1b), the boundary conditions of (N2) imply $\partial_n \operatorname{div}' \boldsymbol{u}' = \operatorname{div}' \boldsymbol{g}'$ on Γ. Besides, from the boundary conditions of (B), we get $\partial_n^2 u_n = -\operatorname{div}' \boldsymbol{g}'$ on Γ. Then we have

$$\partial_n \operatorname{div} \boldsymbol{u} = \partial_n \operatorname{div}' \boldsymbol{u}' + \partial_n^2 u_n = \operatorname{div}' \boldsymbol{g}' - \operatorname{div}' \boldsymbol{g}' = 0 \quad \text{on } \Gamma.$$

So, $\operatorname{div} \boldsymbol{u}$ satisfies the problem

$$\Delta \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^n_+, \quad \partial_n \operatorname{div} \boldsymbol{u} = 0 \quad \text{on } \Gamma. \tag{2.9}$$

As well, $\operatorname{div} \boldsymbol{u} \in L^p(\mathbb{R}^n_+)$, and hence $\operatorname{div} \boldsymbol{u} = 0$ in $\mathbb{R}^n_+$ (see [4] or [7, Theorem 3.3]).

(iv) Concerning the uniqueness question, we notice that u_n and π are unique. Let $\boldsymbol{u}' = (u_i)_{1 \leqslant i \leqslant n-1}$ and $\boldsymbol{u}'_\dagger = (u_i^\dagger)_{1 \leqslant i \leqslant n-1}$ be solutions to (N2); then

$$\begin{aligned} \Delta(u_i - u_i^\dagger) &= 0 && \text{in } \mathbb{R}^n_+, \\ \partial_n(u_i - u_i^\dagger) &= 0 && \text{on } \Gamma, \end{aligned}$$

where $u_i - u_i^\dagger \in W_0^{1,p}(\mathbb{R}^n_+)$. So, we can deduce that $u_i - u_i^\dagger \in \mathcal{N}^\Delta_{[1-n/p]}$ (see [4]). It remains to remark that $\mathcal{N}^\Delta_{[1-n/p]} = \mathbb{R}$ if $n \leqslant p$, and $\mathcal{N}^\Delta_{[1-n/p]} = \{0\}$ if $n > p$.

Finally, the estimate of Proposition 2.1 is a straightforward consequence of the Banach theorem. Let us notice that we can also get it from the estimates of the auxiliary problems as we showed in [7] for the no-slip boundary conditions. ☐

2.2. The nonhomogeneous case. Resting on the previous result, we now can deal with the complete problem.

THEOREM 2.3. *Assume that* $\frac{n}{p'} \neq 1$. *For any* $\boldsymbol{f} \in \boldsymbol{W}_1^{0,\,p}(\mathbb{R}^n_+)$, $h \in W_1^{1,\,p}(\mathbb{R}^n_+)$, $g_n \in W_0^{1-1/p,\,p}(\Gamma)$, *and* $\boldsymbol{g}' \in \boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$, *satisfying the following compatibility condition if* $n < p'$*:*

$$\forall i \in \{1, \dots, n-1\}, \quad \int_{\mathbb{R}^n_+} f_i \, \mathrm{d}x = \langle g_i,\, 1 \rangle_{W_0^{-1/p,\,p}(\Gamma) \times W_0^{1/p,\,p'}(\Gamma)}, \tag{2.10}$$

problem (S$^\sharp$) *admits a solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$*, unique if* $n > p$*, unique up to an element of* $\mathbb{R}^{n-1} \times \{0\}^2$ *if* $n \leqslant p$*, with the estimate*

$$\inf_{\boldsymbol{\xi} \in \mathbb{R}^{n-1}\times\{0\}} \|\boldsymbol{u} + \boldsymbol{\xi}\|_{\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+)} + \|\pi\|_{L^p(\mathbb{R}^n_+)} \leqslant C \left(\|\boldsymbol{f}\|_{\boldsymbol{W}_1^{0,\,p}(\mathbb{R}^n_+)} + \|h\|_{W_1^{1,\,p}(\mathbb{R}^n_+)} + \|g_n\|_{W_0^{1-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}_0^{-1/p,\,p}(\Gamma)} \right)$$

if $n \leqslant p$*, and the corresponding estimate without* inf $(\boldsymbol{\xi} = 0)$ *if* $n > p$*.*

Proof. We can give a proof quite similar to that of the nonhomogeneous case for the Stokes system with Dirichlet boundary conditions, by extension of the data $\boldsymbol{f}$ and h to the whole space (see [7]). But another way is to combine this result with the homogeneous case for the Stokes system with Navier boundary conditions. We will follow this one.

First, we introduce the auxiliary problem

$$\begin{aligned} -\Delta \boldsymbol{z} + \nabla \eta &= \boldsymbol{f} \quad \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \boldsymbol{z} &= h \quad \text{in } \mathbb{R}^n_+, \\ \boldsymbol{z} &= \boldsymbol{0} \quad \text{on } \Gamma. \end{aligned} \tag{2.11}$$

With the assumption $\frac{n}{p'} \neq 1$, we know that problem (2.11) admits a unique solution $(\boldsymbol{z},\, \eta) \in \boldsymbol{W}_1^{2,\,p}(\mathbb{R}^n_+) \times W_1^{1,\,p}(\mathbb{R}^n_+)$ (see [7, Theorem 5.2]). Thus we can deduce that $\partial_n \boldsymbol{z}'|_\Gamma \in \boldsymbol{W}_1^{1-1/p,\,p}(\Gamma)$. In addition, we can notice that we have the imbeddings $W_1^{2,\,p}(\mathbb{R}^n_+) \hookrightarrow W_0^{1,\,p}(\mathbb{R}^n_+)$ and $W_1^{1,\,p}(\mathbb{R}^n_+) \hookrightarrow L^p(\mathbb{R}^n_+)$ without condition, whereas we have $W_1^{1-1/p,\,p}(\Gamma) \hookrightarrow W_0^{-1/p,\,p}(\Gamma)$ only if $\frac{n}{p'} \neq 1$.

Indeed, we can break it down into

$$W_1^{1-1/p,\,p}(\Gamma) \hookrightarrow W_{1/p}^{0,\,p}(\Gamma) \quad \text{and} \quad W_{1/p}^{0,\,p}(\Gamma) \hookrightarrow W_0^{-1/p,\,p}(\Gamma).$$

The first imbedding holds without condition. By duality, the second is equivalent to $W_0^{1/p,\,p'}(\Gamma) \hookrightarrow W_{-1/p}^{0,\,p'}(\Gamma)$, which holds if $\frac{n-1}{p'} \neq \frac{1}{p}$, i.e., $\frac{n}{p'} \neq 1$.

So, $(\boldsymbol{z},\, \eta) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$, and above all $\gamma_1 \boldsymbol{z}' \in \boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$, which allows us to consider the second auxiliary problem

$$\begin{aligned} -\Delta \boldsymbol{v} + \nabla \vartheta = \boldsymbol{0} \quad &\text{and} \quad \operatorname{div} \boldsymbol{v} = 0 \quad &&\text{in } \mathbb{R}^n_+, \\ v_n = g_n \quad &\text{and} \quad \partial_n \boldsymbol{v}' = \boldsymbol{g}' - \partial_n \boldsymbol{z}' \quad &&\text{on } \Gamma, \end{aligned} \tag{2.12}$$

where $\boldsymbol{g}' - \partial_n \boldsymbol{z}'|_\Gamma = \boldsymbol{g}' - \gamma_1 \boldsymbol{z}' \in \boldsymbol{W}_0^{-1/p,\,p}(\Gamma)$. Then, Proposition 2.1 yields a pair $(\boldsymbol{v},\, \vartheta) \in \boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ which is a solution to (2.12), provided the orthogonality condition

$$\forall \boldsymbol{\varphi}' \in \mathbb{R}^{n-1}, \quad \langle \boldsymbol{g}' - \gamma_1 \boldsymbol{z}',\, \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}_0^{-1/p,\,p}(\Gamma) \times \boldsymbol{W}_0^{1/p,\,p'}(\Gamma)} = 0 \tag{2.13}$$

is satisfied if $n < p'$. Now, we must write this condition by only the means of data. It suffices to notice that we have for all $\boldsymbol{\varphi} \in \mathbb{R}^{n-1} \times \{0\}$,

$$\begin{aligned} \int_{\mathbb{R}^n_+} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x &= \int_{\mathbb{R}^n_+} (-\Delta \boldsymbol{z} + \nabla \eta) \cdot \boldsymbol{\varphi} \, \mathrm{d}x \\ &= \langle \gamma_1 \boldsymbol{z}',\, \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}_0^{-1/p,\,p}(\Gamma) \times \boldsymbol{W}_0^{1/p,\,p'}(\Gamma)}, \end{aligned}$$

to deduce that condition (2.13) is written

$$\forall \boldsymbol{\varphi}' \in \mathbb{R}^{n-1}, \quad \int_{\mathbb{R}^n_+} \boldsymbol{f}' \cdot \boldsymbol{\varphi}' \, \mathrm{d}x = \langle \boldsymbol{g}', \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}_0^{-1/p,\,p}(\Gamma) \times \boldsymbol{W}_0^{1/p,\,p'}(\Gamma)},$$

that is, more simply, the compatibility condition (2.10).

Then, the pair $(\boldsymbol{u}, \pi) = (\boldsymbol{v} + \boldsymbol{z}, \vartheta + \eta)$ which belongs to $\boldsymbol{W}_0^{1,\,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$ is a solution to ($\mathrm{S}^\sharp$).

Finally, the uniqueness of solutions to ($\mathrm{S}^\sharp$) is a straightforward consequence of Proposition 2.1. ∎

Remark 2.4. Unlike Dirichlet boundary conditions, with Navier conditions it is not reasonable to consider data $(\boldsymbol{f}, h)$ in $\boldsymbol{W}_0^{-1,\,p}(\mathbb{R}^n_+) \times L^p(\mathbb{R}^n_+)$. Indeed, with such data for problem (2.11) we should get the velocity field $\boldsymbol{z}$ in the space $W_0^{1,\,p}(\mathbb{R}^n_+)$, and we cannot give a meaning to the trace of $\partial_n \boldsymbol{z}'$ in that case without an ad hoc assumption. This limitation is not due to the method employed here; it is the same situation as in the Neumann problem for the Laplacian (see [4]).

3. A wide class of behavior at infinity. Naturally, this problem will be solved by the consideration of a scale of weights which extends the weight zero of the previous section. After the study of the kernel of the operator associated to this problem, we will show that the method established for the homogeneous system with the weight zero works in fact for any weight. The main difficulty is getting compatibility conditions into all the auxiliary problems from that of the original problem. Next, the treatment of the nonhomogeneous system will be noticeably different.

3.1. The kernel. In the half-space, the key to this question is the reflection principle. We can find an extensive study of this principle in the work of Farwig (see [13]). With these boundary conditions, the reflection principle is simpler than that for the Dirichlet conditions, and it can be deduced from the classical Schwarz reflection principle for the harmonic functions.

Let $\ell \in \mathbb{Z}$, and let $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+)$ be an element of the kernel of the Stokes operator with Navier boundary conditions—that is, a solution of (2.1) with homogeneous boundary conditions; then the unique extension $(\tilde{\boldsymbol{u}}, \tilde{\pi})$ of $(\boldsymbol{u}, \pi)$ to the whole space, satisfying

$$-\Delta \tilde{\boldsymbol{u}} + \nabla \tilde{\pi} = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \tilde{\boldsymbol{u}} = 0 \qquad \text{in } \mathbb{R}^n,$$

is given by the continuation formulae as follows: for all $x = (x', x_n) \in \mathbb{R}^n_-$,

$$\tilde{\boldsymbol{u}}'(x) = \boldsymbol{u}'(x^*), \ \tilde{u}_n(x) = -u_n(x^*), \ \tilde{\pi}(x) = \pi(x^*), \qquad \text{where } x^* = (x', -x_n).$$

Moreover, such $\tilde{\pi}$ and $\tilde{\boldsymbol{u}}$ are, respectively, harmonic and biharmonic tempered distributions in $\mathbb{R}^n$, and thus polynomials. For all $k \in \mathbb{Z}$, let us denote

$$\mathcal{S}^\sharp_k = \big\{ (\boldsymbol{\lambda}, \mu) \in \boldsymbol{\mathcal{P}}_k^{\Delta^2} \times \mathcal{P}_{k-1}^{\Delta}; \ -\Delta \boldsymbol{\lambda} + \nabla \mu = \boldsymbol{0} \text{ and } \operatorname{div} \boldsymbol{\lambda} = 0 \text{ in } \mathbb{R}^n_+, \\ \partial_n \boldsymbol{\lambda}' = \boldsymbol{0} \text{ and } \lambda_n = 0 \text{ on } \Gamma \big\}.$$

According to the maximum degree of polynomials in weighted Sobolev spaces (see [2]), we can characterize this kernel as follows.

Corollary 3.1. *Let $\ell \in \mathbb{Z}$ with hypothesis* (1.6)*; then the kernel of the Stokes operator with Navier boundary conditions in $\boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}^n_+) \times W_\ell^{0,\,p}(\mathbb{R}^n_+)$ is the space $\mathcal{S}^\sharp_{[1-\ell-n/p]}$.*

In fact, this kernel does not depend on the regularity according to the Sobolev imbbedings. More precisely, we have the following result.

COROLLARY 3.2. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}^*$ with (1.7); then the kernel of the Stokes operator with Navier boundary conditions in $\boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n_+) \times W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)$ is the space $\mathcal{S}^\sharp_{[1-\ell-n/p]}$.*

Using an idea due to Boulmezaoud (see [10]), we can also express this space from the polynomial spaces $\mathcal{A}^\Delta_k$ and $\mathcal{N}^\Delta_k$ which define the kernels of the Laplacian with Dirichlet and Neumann boundary conditions in the half-space. With this aim, we will use the operator Π_N—introduced in [5] for the biharmonic problem—defined as follows:

$$\forall s \in \mathcal{N}^\Delta_k, \quad \Pi_N s(x', x_n) = \frac{1}{2}\, x_n \int_0^{x_n} s(x', t)\,\mathrm{d}t$$

and satisfying for all $s \in \mathcal{N}^\Delta_k$, $\Delta\Pi_N s = s$ in $\mathbb{R}^n_+$ and $\Pi_N s = \partial_n \Pi_N s = 0$ on Γ.

PROPOSITION 3.3. *Let $\ell \in \mathbb{Z}$. The pair $(\boldsymbol{\lambda}, \mu) \in \mathcal{S}^\sharp_{[1-\ell-n/p]}$ if and only if there exists $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{N}}^\Delta_{[1-\ell-n/p]} \times \mathcal{A}^\Delta_{[1-\ell-n/p]}$ such that*

$$\boldsymbol{\lambda} = \boldsymbol{\varphi} - \nabla\Pi_N \operatorname{div} \boldsymbol{\varphi}, \qquad \mu = -\operatorname{div} \boldsymbol{\varphi}. \tag{3.1}$$

Proof. Given $(\boldsymbol{\lambda}, \mu) \in \mathcal{S}^\sharp_{[1-\ell-n/p]}$, we have $\Delta\mu = 0$ in $\mathbb{R}^n_+$ and $\partial_n \mu = 0$ on Γ, and hence $\mu \in \mathcal{N}^\Delta_{[1-\ell-n/p]}$. So we can write $\Delta(\boldsymbol{\lambda} - \nabla\Pi_N\mu) = \Delta\boldsymbol{\lambda} - \nabla\mu = \mathbf{0}$, which implies the existence of $\boldsymbol{\varphi} \in \mathcal{P}^\Delta_{[1-\ell-n/p]}$ such that

$$\boldsymbol{\varphi} = \boldsymbol{\lambda} - \nabla\Pi_N\mu. \tag{3.2}$$

In fact, we can see that $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{N}}^\Delta_{[1-\ell-n/p]} \times \mathcal{A}^\Delta_{[1-\ell-n/p]}$ by considerations on the parity of $\boldsymbol{\lambda}'$, λ_n, and $\nabla\Pi_N\mu$. In addition, applying the operator div to (3.2), we get $\operatorname{div}\boldsymbol{\varphi} = -\mu$, which yields (3.1) by substitution in (3.2).

Conversely, we can verify that such a pair $(\boldsymbol{\lambda}, \mu)$ belongs to $\mathcal{S}^\sharp_{[1-\ell-n/p]}$. ∎

Remark 3.4. It is clear that if $\ell > 0$, this kernel is reduced to $\{\mathbf{0}\}$, and if $\ell = 0$, we find $\mathbb{R}^{n-1} \times \{0\}^2$ as in Proposition 2.1 and Theorem 2.3. However, for $\ell > 0$, a compatibility condition symmetrically appears for the data, which extends that of the weight zero.

3.2. Generalized solutions. Here is the generalization of Theorem 2.3 for any weight $\ell \in \mathbb{Z}$. This result will be completely proved at the end of this section.

THEOREM 3.5. *Let $\ell \in \mathbb{Z}$ and assume that*

$$n/p' \notin \{1, \ldots, \ell+1\} \quad \text{and} \quad n/p \notin \{1, \ldots, -\ell\}. \tag{3.3}$$

For any $\boldsymbol{f} \in \boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)$, $h \in W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)$, $g_n \in W^{1-1/p,\,p}_{\ell}(\Gamma)$, and $\boldsymbol{g}' \in \boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)$ satisfying the compatibility condition

$$\begin{aligned}
&\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{N}}^\Delta_{[1+\ell-n/p']} \times \mathcal{A}^\Delta_{[1+\ell-n/p']},\\
&\int_{\mathbb{R}^n_+} (\boldsymbol{f} - \nabla h) \cdot \boldsymbol{\varphi}\,\mathrm{d}x \;+\; \langle \operatorname{div}\boldsymbol{f},\, \Pi_N \operatorname{div}\boldsymbol{\varphi}\rangle_{W^{-1,\,p}_{\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,\,p'}_{-\ell-1}(\mathbb{R}^n_+)}\\
&\quad + \int_\Gamma g_n\, \partial_n\varphi_n\,\mathrm{d}x' \;-\; \langle \boldsymbol{g}',\, \boldsymbol{\varphi}'\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{1-1/p',\,p'}_{-\ell}(\Gamma)} \;=\; 0,
\end{aligned} \tag{3.4}$$

problem $(S^\sharp)$ *admits a solution* $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+) \times W^{0,\,p}_{\ell}(\mathbb{R}^n_+)$*, unique up to an element of* $\mathcal{S}^\sharp_{[1-\ell-n/p]}$*, with the estimate*

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^\sharp_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+)} + \|\pi+\mu\|_{W^{0,\,p}_{\ell}(\mathbb{R}^n_+)}\Big)$$
$$\leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)} + \|h\|_{W^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)} + \|g_n\|_{W^{1-1/p,\,p}_{\ell}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)}\right).$$

Since the kernel has been characterized before, now it remains to show the necessity of condition (3.4) and, above all, the existence of a solution, that is, the surjectivity of this operator. As for the weight zero, we will start with the homogeneous problem, and then we will consider more regular data on the boundary to finish by this theorem.

3.3. The compatibility condition. If $(\boldsymbol{u},\,\pi)$ is a solution to $(S^\sharp)$, then we have the following Green's formula:

$$\forall(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^\sharp_{[1+\ell-n/p']},$$
$$\int_{\mathbb{R}^n_+}(-\Delta\boldsymbol{u}+\nabla\pi)\cdot\boldsymbol{\lambda}\,\mathrm{d}x \;-\; \int_{\mathbb{R}^n_+}(\operatorname{div}\boldsymbol{u})\,\mu\,\mathrm{d}x$$
$$= -\int_\Gamma u_n\,\partial_n\lambda_n\,\mathrm{d}x' \;+\; \big\langle \partial_n\boldsymbol{u}',\,\boldsymbol{\lambda}'\big\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)\times\boldsymbol{W}^{1-1/p',\,p'}_{-\ell}(\Gamma)} \;+\; \int_\Gamma u_n\,\mu\,\mathrm{d}x'.$$

Hence we have a first formulation of the compatibility condition for data $\boldsymbol{f},\ h,\ g_n,\ \boldsymbol{g}'$:

$$\begin{aligned}&\forall(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^\sharp_{[1+\ell-n/p']}, \qquad \int_{\mathbb{R}^n_+}\boldsymbol{f}\cdot\boldsymbol{\lambda}\,\mathrm{d}x \;-\; \int_{\mathbb{R}^n_+}h\,\mu\,\mathrm{d}x\\ &=\quad -\int_\Gamma g_n\,(\partial_n\lambda_n-\mu)\,\mathrm{d}x' \;+\; \big\langle \boldsymbol{g}',\,\boldsymbol{\lambda}'\big\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)\times\boldsymbol{W}^{1-1/p',\,p'}_{-\ell}(\Gamma)}.\end{aligned}\tag{3.5}$$

Now, in order to use Proposition 3.3, we can observe that

$$\forall\boldsymbol{\varphi}\in\ \boldsymbol{\mathcal{N}}^{\Delta}_{[1+\ell-n/p']}\times\mathcal{A}^{\Delta}_{[1+\ell-n/p']},$$
$$\int_{\mathbb{R}^n_+}\boldsymbol{f}\cdot(\nabla\Pi_N\operatorname{div}\boldsymbol{\varphi})\,\mathrm{d}x \;=\; \big\langle -\operatorname{div}\boldsymbol{f},\,\Pi_N\operatorname{div}\boldsymbol{\varphi}\big\rangle_{W^{-1,\,p}_{\ell+1}(\mathbb{R}^n_+)\times\mathring{W}^{1,\,p'}_{-\ell-1}(\mathbb{R}^n_+)}$$

and

$$\int_{\mathbb{R}^n_+}h\ \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x \;=\; -\int_{\mathbb{R}^n_+}\nabla h\cdot\boldsymbol{\varphi}\,\mathrm{d}x.$$

On the other hand, for the trace terms, we have

$$\boldsymbol{\lambda}'=\boldsymbol{\varphi}' \ \text{ and }\ \partial_n\lambda_n-\mu=\partial_n\varphi_n \ \text{ on }\ \Gamma.$$

According to Proposition 3.3 and introducing these identities in (3.5), we get (3.4) as a second formulation for the compatibility condition.

3.4. Weak and strong solutions in the homogeneous case. Here again, we start with the homogeneous Stokes system (2.1). In fact the method of subsection 2.1 works for any weight. The extra trouble comes from the compatibility conditions for the auxiliary problems. Following step by step the proof of Proposition 2.1, we throw light on this problem.

Proof of Proposition 2.1 *revisited.* Point (i) is unchanged.

(ii) The compatibility condition (3.4) adapted to problem (2.1) is written as

$$\begin{aligned}&\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{N}}^{\Delta}_{[1+\ell-n/p']} \times \mathcal{A}^{\Delta}_{[1+\ell-n/p']},\\ &\int_{\Gamma} g_n\, \partial_n \varphi_n \,\mathrm{d}x' \;-\; \langle \boldsymbol{g}',\, \boldsymbol{\varphi}' \rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{1-1/p',\,p'}_{-\ell}(\Gamma)} \;=\; 0.\end{aligned} \tag{3.6}$$

Step 1. *Problem* (B). For (2.2), the compatibility condition is

$$\forall \psi \in \mathcal{A}^{\Delta}_{[3+\ell-n/p']},\ \left\langle \Delta' g_n - \mathrm{div}' \boldsymbol{g}',\, \partial_n \psi \right\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{2-1/p',\,p'}_{-\ell}(\Gamma)} = 0 \tag{3.7}$$

(see [7, Theorem 3.5]). By means of Green's formulae, we can rewrite it as

$$\forall \psi \in \mathcal{A}^{\Delta}_{[3+\ell-n/p']}, \quad \int_{\Gamma} g_n\, \partial_n \Delta' \psi \,\mathrm{d}x' + \langle \boldsymbol{g}',\, \partial_n \nabla' \psi \rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{1-1/p',\,p'}_{-\ell}(\Gamma)} = 0.$$

Now, to see that it is a consequence of (3.6), it suffices to remark that

$$\forall \psi \in \mathcal{A}^{\Delta}_{[3+\ell-n/p']}, \quad \Delta' \psi \in \mathcal{A}^{\Delta}_{[1+\ell-n/p']}, \quad \text{and} \ \ \partial_n \nabla' \psi \in \boldsymbol{\mathcal{N}}^{\Delta}_{[1+\ell-n/p']}.$$

So, we get $z_n \in W^{-1,\,p}_{\ell}(\mathbb{R}^n_+)/\mathcal{A}^{\Delta}_{[-1-\ell-n/p]}$ as a solution to (2.2).

For (2.3), the compatibility condition is

$$\forall \psi \in \mathcal{A}^{\Delta}_{[1+\ell-n/p']}, \quad \langle z_n,\, \psi \rangle_{W^{-1,\,p}_{\ell}(\mathbb{R}^n_+)\times \mathring{W}^{1,\,p'}_{-\ell}(\mathbb{R}^n_+)} = \int_{\Gamma} g_n\, \partial_n \psi \,\mathrm{d}x' \tag{3.8}$$

(see [5, Theorem 2.5]). First, (3.6) implies that for any $\psi \in \mathcal{A}^{\Delta}_{[1+\ell-n/p']}$, we have $\int_{\Gamma} g_n\, \partial_n \psi \,\mathrm{d}x' = 0$. It remains to show that the left-hand term is also zero. For this, we need to express $\mathcal{A}^{\Delta}_{[1+\ell-n/p']}$ by means of the kernel of the biharmonic operator $\mathcal{B}_k$—that is, the space of polynomials ζ such that $\Delta^2 \zeta = 0$ in $\mathbb{R}^n_+$ and $\zeta = \partial_n \zeta = 0$ on Γ. We showed in [5, Lemma 4.4] that

$$\forall k \in \mathbb{Z}, \qquad \mathcal{B}_{k+2} = \Pi_D \mathcal{A}^{\Delta}_{k} \oplus \Pi_N \mathcal{N}^{\Delta}_{k}, \tag{3.9}$$

where Π_D—which is the equivalent for the odd harmonic polynomials with respect to x_n of the operator Π_N for the even harmonic polynomials with respect to x_n—is defined as follows:

$$\forall r \in \mathcal{A}^{\Delta}_{k}, \quad \Pi_D r(x', x_n) = \frac{1}{2} \int_0^{x_n} t\, r(x', t) \,\mathrm{d}t$$

and satisfies for all $r \in \mathcal{A}^{\Delta}_{k}$, $\Delta \Pi_D r = r$ in $\mathbb{R}^n_+$ and $\Pi_D r = \partial_n \Pi_D r = 0$ on Γ. From (3.9), we get for any $\psi \in \mathcal{A}^{\Delta}_{[1+\ell-n/p']}$, $\Pi_D \psi = \zeta \in \mathcal{B}_{[3+\ell-n/p']}$ and thus we have $\psi = \Delta \zeta$. So, by means of a Green's formula (see [7, Lemma 3.7] for the justification), we get

$$\begin{aligned}&\forall \psi \in \mathcal{A}^{\Delta}_{[1+\ell-n/p']},\ \exists \zeta \in \mathcal{B}_{[3+\ell-n/p']} \ \text{ such that}\\ &\qquad\qquad \langle z_n,\, \psi \rangle_{W^{-1,\,p}_{\ell}(\mathbb{R}^n_+)\times \mathring{W}^{1,\,p'}_{-\ell}(\mathbb{R}^n_+)} = \langle z_n,\, \Delta \zeta \rangle = \langle \Delta z_n,\, \zeta \rangle = 0.\end{aligned}$$

So (3.8) is proved and we get $u_n \in W^{1,\,p}_{\ell}(\mathbb{R}^n_+)/\mathcal{A}^{\Delta}_{[1-\ell-n/p]}$ as a solution to (2.3).

Step 2. *Problem* (N1). Here $\Delta u_n \in W^{-1-1/p,\,p}_{\ell}(\Gamma)$ (see [7, Lemma 3.7]), and for this problem, the compatibility condition is

$$\forall \psi \in \mathcal{N}^{\Delta}_{[2+\ell-n/p']}, \quad \langle \Delta u_n,\, \psi\rangle_{W^{-1-1/p,\,p}_{\ell}(\Gamma)\times W^{2-1/p',\,p'}_{-\ell}(\Gamma)} = 0 \tag{3.10}$$

(see [7, Theorem 3.3]). For any $\psi \in \mathcal{N}^{\Delta}_{[2+\ell-n/p']}$, if we put $\zeta = \int_0^{x_n} \psi(x', t)\,\mathrm{d}t$, this yields $\psi = \partial_n \zeta$ with $\zeta \in \mathcal{A}^{\Delta}_{[3+\ell-n/p']}$. Since $\Delta u_n = \Delta' g_n - \operatorname{div}' \boldsymbol{g}'$ on Γ, we see that (3.10) is exactly written as condition (3.7), which is satisfied.

So, we get $\pi \in W^{0,\,p}_{\ell}(\mathbb{R}^n_+)/\mathcal{N}^{\Delta}_{[-\ell-n/p]}$ as a solution to (N1).

Step 3. *Problem* (N2). For (2.6), the compatibility condition is

$$\forall \psi \in \mathcal{N}^{\Delta}_{[\ell-n/p']}, \qquad \int_{\mathbb{R}^n_+} \pi\, \psi \,\mathrm{d}x = 0 \tag{3.11}$$

(see [4, Theorem 3.1]). According to (3.9), we also have, for any $\psi \in \mathcal{N}^{\Delta}_{[\ell-n/p']}$, $\Pi_N \psi = \zeta \in \mathcal{B}_{[2+\ell-n/p']}$, and thus $\psi = \Delta\zeta$. So, we have

$$\forall \psi \in \mathcal{N}^{\Delta}_{[\ell-n/p']},\ \exists \zeta \in \mathcal{B}_{[2+\ell-n/p']} \ \text{ such that} \quad \int_{\mathbb{R}^n_+} \pi\, \psi \,\mathrm{d}x = \int_{\mathbb{R}^n_+} \pi\, \Delta\zeta \,\mathrm{d}x = \langle \Delta\pi,\, \zeta\rangle = 0.$$

Thus (3.11) is proved, and we get $w \in W^{2,\,p}_{\ell}(\mathbb{R}^n_+)/\mathcal{N}^{\Delta}_{[2-\ell-n/p]}$ as a solution to (2.6). Consequently, $\boldsymbol{v}' = \nabla' w \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+)/\boldsymbol{\mathcal{N}}^{\Delta}_{[1-\ell-n/p]}$ is a solution to problem (2.4).

Finally, for problem (2.5), the compatibility condition is

$$\forall \boldsymbol{\varphi}' \in \boldsymbol{\mathcal{N}}^{\Delta}_{[1+\ell-n/p']}, \qquad \langle \boldsymbol{g}',\, \boldsymbol{\varphi}'\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)\times \boldsymbol{W}^{1-1/p',\,p'}_{-\ell}(\Gamma)} \ = \ 0 \tag{3.12}$$

(see [7, Theorem 3.4]). It is clear that (3.12), is include in (3.6), and then we get $\boldsymbol{z}' \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+)/\boldsymbol{\mathcal{N}}^{\Delta}_{[1-\ell-n/p]}$ as a solution to (2.5).

So $\boldsymbol{u}' = \boldsymbol{v}' + \boldsymbol{z}' \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+)/\boldsymbol{\mathcal{N}}^{\Delta}_{[1-\ell-n/p]}$ is a solution to (N2).

Remark 3.6. The set of critical values for all these auxiliary problems is given by hypothesis (1.6). This is the good set of critical values for the homogeneous problem (2.1), and the supplementary value $n/p' = \ell + 1$ will appear only in the nonhomogeneous problem $(S^\sharp)$.

(iii) To recover the nth component of (2.1a)—that is, (2.7)—and (2.1b) from (B), (N1), and (N2), we will use the nonuniqueness of their respective solutions u_n, π, and $\boldsymbol{u}'$, constructed in (ii), to select a "good one."

Since $\Delta u_n - \partial_n \pi$ satisfies (2.8) and belongs to $W^{-1,\,p}_{\ell}(\mathbb{R}^n_+)$, we can deduce that $\Delta u_n - \partial_n \pi \in \mathcal{A}^{\Delta}_{[-1-\ell-n/p]}$. As π is defined up to an element of $\mathcal{N}^{\Delta}_{[-\ell-n/p]}$, $\partial_n \pi$ is defined up to an element of $\mathcal{A}^{\Delta}_{[-1-\ell-n/p]}$, and thus we can choose π such that $\Delta u_n - \partial_n \pi = 0$.

Since $\operatorname{div} \boldsymbol{u}$ satisfies (2.9) and belongs to $W^{0,\,p}_{\ell}(\mathbb{R}^n_+)$, we can deduce that $\operatorname{div} \boldsymbol{u} \in \mathcal{N}^{\Delta}_{[-\ell-n/p]}$. As $\boldsymbol{u}'$ is defined up to an element of $\boldsymbol{\mathcal{N}}^{\Delta}_{[1-\ell-n/p]}$, $\operatorname{div}' \boldsymbol{u}'$ is defined up to an element of $\mathcal{N}^{\Delta}_{[-\ell-n/p]}$, and thus we can choose $\boldsymbol{u}'$ such that $\operatorname{div} \boldsymbol{u} = 0$.

To finish this proof, let us notice that the characterization of the kernel gives an answer to point (iv). ∎

So we have established the existence of weak solutions to the homogeneous problem, and we can sum up in the following result.

PROPOSITION 3.7. *Let $\ell \in \mathbb{Z}$ with hypothesis* (1.6). *For any $g_n \in W_\ell^{1-1/p,\,p}(\Gamma)$ and $\boldsymbol{g}' \in \boldsymbol{W}_\ell^{-1/p,\,p}(\Gamma)$, satisfying the compatibility condition* (3.6), *problem* (2.1) *admits a solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}_+^n) \times W_\ell^{0,\,p}(\mathbb{R}_+^n)$, unique up to an element of $\mathcal{S}^\sharp_{[1-\ell-n/p]}$, with the estimate*

$$\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^\sharp_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}_+^n)} + \|\pi+\mu\|_{W_\ell^{0,\,p}(\mathbb{R}_+^n)}\Big) \leqslant C\Big(\|g_n\|_{W_\ell^{1-1/p,\,p}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}_\ell^{-1/p,\,p}(\Gamma)}\Big).$$

Now, always for the homogeneous problem, we will consider the case of more regular boundary conditions, which yields strong solutions.

PROPOSITION 3.8. *Let $\ell \in \mathbb{Z}$ and assume that*

$$n/p' \notin \{1,\dots,\ell+1\} \quad \text{and} \quad n/p \notin \{1,\dots,-\ell-1\}. \tag{3.13}$$

For any $g_n \in W_{\ell+1}^{2-1/p,\,p}(\Gamma)$ and $\boldsymbol{g}' \in \boldsymbol{W}_{\ell+1}^{1-1/p,\,p}(\Gamma)$, satisfying condition (3.6), *problem* (2.1) *admits a solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_{\ell+1}^{2,\,p}(\mathbb{R}_+^n) \times W_{\ell+1}^{1,\,p}(\mathbb{R}_+^n)$, unique up to an element of $\mathcal{S}^\sharp_{[1-\ell-n/p]}$, with the corresponding estimate.*

Proof. We simply resume the proof of Proposition 3.7, which we named "Proof of Proposition 2.1 revisited" at the beginning of subsection 3.4, using the regularity results for the harmonic and biharmonic operators.

First, for the biharmonic problem (B), split into the Dirichlet problems (2.2) and (2.3), we find $z_n \in W_{\ell+1}^{0,\,p}(\mathbb{R}_+^n)$ to be a solution to (2.2) (see [7, Theorem 3.8]); as well, we find $u_n \in W_{\ell+1}^{2,\,p}(\mathbb{R}_+^n)$ to be a solution to (2.3) (see [3, Corollary 3.4]). Second, for the first Neumann problem (N1), we find $\pi \in W_{\ell+1}^{1,\,p}(\mathbb{R}_+^n)$ (see [7, Theorem 3.4]). Last, concerning the second Neumann problem (N2), we find $\boldsymbol{u}' \in \boldsymbol{W}_{\ell+1}^{2,\,p}(\mathbb{R}_+^n)$ (see [4, Corollary 3.3]). Moreover, all these results hold under hypothesis (3.13), which yields the optimal set of critical values for such data. ∎

Remark 3.9. We can also get Proposition 3.8 as a regularity result from Proposition 3.7. Indeed, we have $W_{\ell+1}^{2-1/p,\,p}(\Gamma) \hookrightarrow W_\ell^{1-1/p,\,p}(\Gamma)$ if $\frac{n}{p} \neq -\ell$. On the other hand, the imbedding $W_{\ell+1}^{1-1/p,\,p}(\Gamma) \hookrightarrow W_\ell^{-1/p,\,p}(\Gamma)$ can be broken down into $W_{\ell+1}^{1-1/p,\,p}(\Gamma) \hookrightarrow W_{\ell+1/p}^{0,\,p}(\Gamma)$ and $W_{\ell+1/p}^{0,\,p}(\Gamma) \hookrightarrow W_\ell^{-1/p,\,p}(\Gamma)$. The first imbedding also holds if $\frac{n}{p} \neq -\ell$, and, by duality, we find $\frac{n}{p'} \neq \ell+1$ as a condition for the second imbedding. So, under hypothesis (3.13), if in addition $\frac{n}{p} \neq -\ell$, we can deduce from Proposition 3.7 that problem (2.1) admits a solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_\ell^{1,\,p}(\mathbb{R}_+^n) \times W_\ell^{0,\,p}(\mathbb{R}_+^n)$. Then, as in [7, Corollary 5.5], we can show by regularity arguments that in fact $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_{\ell+1}^{2,\,p}(\mathbb{R}_+^n) \times W_{\ell+1}^{1,\,p}(\mathbb{R}_+^n)$. The cost of this approach is thus the supplementary critical value $\frac{n}{p} = -\ell$.

3.5. The nonhomogeneous case. We start with enough regular data on the boundary—that is, the data of Proposition 3.8—to get strong solutions to the complete problem.

THEOREM 3.10. *Let $\ell \in \mathbb{Z}$ with hypothesis* (3.13). *For any $\boldsymbol{f} \in \boldsymbol{W}_{\ell+1}^{0,\,p}(\mathbb{R}_+^n)$, $h \in W_{\ell+1}^{1,\,p}(\mathbb{R}_+^n)$, $g_n \in W_{\ell+1}^{2-1/p,\,p}(\Gamma)$, $\boldsymbol{g}' \in \boldsymbol{W}_{\ell+1}^{1-1/p,\,p}(\Gamma)$, satisfying condition* (3.4), *problem $(S^\sharp)$ admits a solution $(\boldsymbol{u},\,\pi) \in \boldsymbol{W}_{\ell+1}^{2,\,p}(\mathbb{R}_+^n) \times W_{\ell+1}^{1,\,p}(\mathbb{R}_+^n)$, unique up to an* *element of* $\mathcal{S}^{\sharp}_{[1-\ell-n/p]}$*, with the estimate*

$$\inf_{(\boldsymbol{\lambda},\, \mu)\in \mathcal{S}^{\sharp}_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^n_+)} + \|\pi+\mu\|_{W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)} \Big)$$
$$\leqslant C \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)} + \|h\|_{W^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)} + \|g_n\|_{W^{2-1/p,\,p}_{\ell+1}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{1-1/p,\,p}_{\ell+1}(\Gamma)} \Big).$$

Proof. It suffices to show the existence. We will naturally use the result in the homogeneous case established above. First, we consider the lifted problem

$$(S_\flat)\ \begin{cases} -\Delta \boldsymbol{v} + \nabla \pi = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = H & \text{in } \mathbb{R}^n_+, \\ v_n = 0 \quad \text{and} \quad \partial_n \boldsymbol{v}' = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

Indeed, according to Lemma 1.1, there exists a lifting function $\boldsymbol{u}_{\boldsymbol{g}'} \in \boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^n_+)$ of $\boldsymbol{g}'$ such that $\partial_n \boldsymbol{u}_{\boldsymbol{g}'} = \boldsymbol{g}'$ on Γ, and there also exists $u_{g_n} \in W^{2,\,p}_{\ell+1}(\mathbb{R}^n_+)$ such that $u_{g_n} = g_n$ on Γ. Then, if we put $\boldsymbol{u}_{\boldsymbol{g}} = (\boldsymbol{u}_{\boldsymbol{g}'},\, u_{g_n})$, $\boldsymbol{F} = \boldsymbol{f} + \Delta \boldsymbol{u}_{\boldsymbol{g}}$, $H = h - \operatorname{div} \boldsymbol{u}_{\boldsymbol{g}}$, and $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{g}}$, the two problems $(S_\flat)$ and $(S^\sharp)$ are equivalent. In addition, by means of Green's formulae, we can easily verify that condition (3.5)—i.e., the alternative form of (3.4)—becomes, for $(S_\flat)$,

$$\forall (\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^{\sharp}_{[1+\ell-n/p']}, \quad \int_{\mathbb{R}^n_+} \boldsymbol{F}\cdot\boldsymbol{\lambda}\,\mathrm{d}x \ - \int_{\mathbb{R}^n_+} H\,\mu\,\mathrm{d}x \ = 0. \tag{3.14}$$

Next, we extend $\boldsymbol{F}$ and H to the whole space by $\tilde{\boldsymbol{F}} \in \boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n)$ and $\tilde{H} \in W^{1,\,p}_{\ell+1}(\mathbb{R}^n)$ as follows:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n),\ \forall \psi \in \mathcal{D}(\mathbb{R}^n),$$
$$\begin{aligned} \int_{\mathbb{R}^n} \tilde{\boldsymbol{F}}\cdot\boldsymbol{\varphi}\,\mathrm{d}x \ &= \int_{\mathbb{R}^n_+} \boldsymbol{F}\cdot(\boldsymbol{\varphi}' + \boldsymbol{\varphi}'^{*},\, \varphi_n - \varphi_n^{*})\,\mathrm{d}x, \\ \int_{\mathbb{R}^n} \tilde{H}\,\psi\,\mathrm{d}x \ &= \int_{\mathbb{R}^n_+} H\,(\psi + \psi^{*})\,\mathrm{d}x, \end{aligned} \tag{3.15}$$

where $\psi^*(x) = \psi(x^*)$ for any $x = (x',\, x_n) \in \mathbb{R}^n$ with $x^* = (x',\, -x_n)$. We can also give a functional writing for this extension:

$$(\tilde{\boldsymbol{F}},\, \tilde{H})(x',\, x_n) = \begin{cases} (\boldsymbol{F},\, H)(x',\, x_n) & \text{if } x_n > 0, \\ (\boldsymbol{F}',\, -F_n,\, H)(x',\, -x_n) & \text{if } x_n < 0. \end{cases}$$

Now, by Theorem 1.3, with hypothesis (3.13), we know that there exists $(\boldsymbol{w},\, \vartheta) \in \boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^n) \times W^{1,\,p}_{\ell+1}(\mathbb{R}^n)$, a solution to the problem

$$(\tilde{S}): \qquad -\Delta \boldsymbol{w} + \nabla \vartheta = \tilde{\boldsymbol{F}} \ \text{ and } \ \operatorname{div} \boldsymbol{w} = \tilde{H} \ \text{ in } \mathbb{R}^n,$$

provided the condition $(\tilde{\boldsymbol{F}},\, \tilde{H}) \perp \mathcal{S}_{[1+\ell-n/p']}$ is fulfilled, that is,

$$\forall (\boldsymbol{\Lambda},\, M) \in \mathcal{S}_{[1+\ell-n/p']}, \quad \int_{\mathbb{R}^n} \tilde{\boldsymbol{F}}\cdot\boldsymbol{\Lambda}\,\mathrm{d}x \ - \int_{\mathbb{R}^n} \tilde{H}\,M\,\mathrm{d}x \ = 0. \tag{3.16}$$

Thanks to (3.15), we can write (3.16) as

$$\int_{\mathbb{R}^n_+} \boldsymbol{F}\cdot(\boldsymbol{\Lambda}' + \boldsymbol{\Lambda}'^{*},\, \Lambda_n - \Lambda_n^{*})\,\mathrm{d}x \ - \int_{\mathbb{R}^n_+} H\,(M + M^{*})\,\mathrm{d}x \ = 0. \tag{3.17}$$

Since $(\boldsymbol{\Lambda}'+\boldsymbol{\Lambda}'^*,\, \Lambda_n-\Lambda_n^*,\, M+M^*) \in \mathcal{S}^\sharp_{[1+\ell-n/p']}$, condition (3.17)—and thus (3.16)—is a simple consequence of (3.14). Then, the pair of functions $(\boldsymbol{v},\, \pi)$ defined in $\mathbb{R}^n_+$ by

$$(\boldsymbol{v},\, \pi) = \frac{1}{2}\,(\boldsymbol{w}' + \boldsymbol{w}'^*,\, w_n - w_n^*,\, \vartheta + \vartheta^*)$$

belongs to $\boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)$ and, by a straightforward calculation, we can see that it satisfies $(S_\flat)$. $\square$

Now, we can establish the existence of generalized solutions announced in subsection 3.2.

Proof of the existence in Theorem 3.5. (i) Assume that $\frac{n}{p'} > \ell + 1$. According to Proposition 3.7, there exists $(\boldsymbol{v},\, \vartheta) \in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+) \times W^{0,\,p}_{\ell}(\mathbb{R}^n_+)$ satisfying

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla\vartheta = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^n_+, \\ \qquad v_n = g_n \quad \text{and} \quad \partial_n \boldsymbol{v}' = \boldsymbol{g}' \quad \text{on } \Gamma. \end{cases}$$

In addition, by Theorem 3.10, there exists $(\boldsymbol{w},\, \zeta) \in \boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)$ satisfying

$$\begin{cases} -\Delta \boldsymbol{w} + \nabla\zeta = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = h \quad \text{in } \mathbb{R}^n_+, \\ \qquad w_n = 0 \quad \text{and} \quad \partial_n \boldsymbol{w}' = \boldsymbol{0} \quad \text{on } \Gamma. \end{cases}$$

The pair $(\boldsymbol{u},\, \pi) = (\boldsymbol{v} + \boldsymbol{w},\, \vartheta + \zeta)$ gives an answer to the question.

(ii) Assume that $\frac{n}{p'} < \ell + 1$. We cannot directly construct a solution as above. Indeed, the compatibility conditions—which are now nontrivial—of the auxiliary and initial problems must coincide. Let N be the dimension of the subspace $\mathcal{S}^\sharp_{[1+\ell-n/p']}$ of $\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$, which is imbedded in $\boldsymbol{W}^{0,\,p'}_{-\ell-1}(\mathbb{R}^n_+) \times W^{-1,\,p'}_{-\ell-1}(\mathbb{R}^n_+)$, and let $\{\boldsymbol{e}_1,\, \dots,\, \boldsymbol{e}_N\}$ be a basis of $\mathcal{S}^\sharp_{[1+\ell-n/p']}$. According to the Hahn–Banach theorem, there exists a family $\{\boldsymbol{e}_1^*,\, \dots,\, \boldsymbol{e}_N^*\}$ of elements of $\boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)$, which extends the dual basis of the dual space $\big(\mathcal{S}^\sharp_{[1+\ell-n/p']}\big)'$. First, we can give a more compact writing of the compatibility condition (3.5)—which is equivalent to (3.4)—as

$$\begin{aligned} &\forall(\boldsymbol{\lambda},\, \mu) \in \mathcal{S}^\sharp_{[1+\ell-n/p']}, \\ &\qquad \langle(\boldsymbol{f},\, -h),\, (\boldsymbol{\lambda},\, \mu)\rangle_{\boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)\times W^{0,\,p}_{\ell}(\mathbb{R}^n_+),\, \boldsymbol{W}^{0,\,p'}_{-\ell-1}(\mathbb{R}^n_+)\times W^{0,\,p'}_{-\ell}(\mathbb{R}^n_+)} \\ &\qquad = \big\langle \boldsymbol{g},\, (\boldsymbol{\lambda}',\, \mu - \partial_n\lambda_n)\big\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell}(\Gamma)\times W^{1-1/p,\,p}_{\ell}(\Gamma),\, \boldsymbol{W}^{1-1/p',\,p'}_{-\ell}(\Gamma)\times W^{-1/p',\,p'}_{-\ell}(\Gamma)}, \end{aligned}$$

where $\boldsymbol{g} = (\boldsymbol{g}',\, g_n)$. We denote the corresponding trace mapping by

$$\begin{aligned} \kappa:\ \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \quad &\longrightarrow \quad \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma) \times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma), \\ (\boldsymbol{\lambda},\, \mu) \quad &\longmapsto \quad (\gamma_0\boldsymbol{\lambda}',\, \gamma_0\mu - \gamma_1\lambda_n), \end{aligned}$$

and $\boldsymbol{\varepsilon}_i = \kappa(\boldsymbol{e}_i)$. With a suitable numbering of the family, $\{\boldsymbol{\varepsilon}_1,\, \dots,\, \boldsymbol{\varepsilon}_d\}$ form a basis of the subspace $\kappa\big(\mathcal{S}^\sharp_{[1+\ell-n/p']}\big)$ of $\boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma) \times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma) \hookrightarrow \boldsymbol{W}^{-1/p',\,p'}_{-\ell-1}(\Gamma) \times W^{-1-1/p',\,p'}_{-\ell-1}(\Gamma)$ and $\boldsymbol{\varepsilon}_i = \boldsymbol{0}$ for $i \in \{d{+}1,\, \dots,\, N\}$. Here again, according to the Hahn–Banach theorem, there exists a family $\{\boldsymbol{\varepsilon}_1^*,\, \dots,\, \boldsymbol{\varepsilon}_d^*\}$ of elements of $\boldsymbol{W}^{1-1/p,\,p}_{\ell+1}(\Gamma) \times W^{2-1/p,\,p}_{\ell+1}(\Gamma)$ which extends the dual basis of $\{\boldsymbol{\varepsilon}_1,\, \dots,\, \boldsymbol{\varepsilon}_d\}$. Now, let us consider the functions defined by

$$(\boldsymbol{F},\, -H) = \sum_{i=1}^{N} \boldsymbol{e}_i^*\, \langle(\boldsymbol{f},\, -h),\, \boldsymbol{e}_i\rangle \quad \text{and} \quad \boldsymbol{G} = \sum_{i=1}^{d} \boldsymbol{\varepsilon}_i^*\, \langle \boldsymbol{g},\, \boldsymbol{\varepsilon}_i\rangle\,.$$

They satisfy

$$\begin{aligned}&\langle(\boldsymbol{F},\,-H),\,e_k\rangle = \langle(\boldsymbol{f},\,-h),\,e_k\rangle = \langle\boldsymbol{g},\,\boldsymbol{\varepsilon}_k\rangle \quad \text{for } k\in\{1,\,\dots,\,N\},\\ &\langle(\boldsymbol{F},\,-H),\,e_k\rangle = \langle\boldsymbol{G},\,\boldsymbol{\varepsilon}_k\rangle = \langle\boldsymbol{g},\,\boldsymbol{\varepsilon}_k\rangle \quad \text{for } k\in\{1,\,\dots,\,d\},\\ &\langle(\boldsymbol{F},\,-H),\,e_k\rangle = \langle\boldsymbol{G},\,\boldsymbol{\varepsilon}_k\rangle = 0 \quad \text{for } k\in\{d+1,\,\dots,\,N\}.\end{aligned}$$

By Theorem 3.10, there exists $(\boldsymbol{v},\,\vartheta)\in \boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)$ satisfying

$$\begin{cases} (-\Delta\boldsymbol{v}+\nabla\vartheta,\,\operatorname{div}\boldsymbol{v}) = (\boldsymbol{f}-\boldsymbol{F},\,h-H) & \text{in } \mathbb{R}^n_+,\\ (\partial_n\boldsymbol{v}',\,v_n) = \boldsymbol{0} & \text{on } \Gamma.\end{cases}$$

By Proposition 3.7, there exists $(\boldsymbol{w},\,\zeta)\in \boldsymbol{W}^{1,\,p}_{\ell}(\mathbb{R}^n_+)\times W^{0,\,p}_{\ell}(\mathbb{R}^n_+)$ satisfying

$$\begin{cases} (-\Delta\boldsymbol{w}+\nabla\zeta,\,\operatorname{div}\boldsymbol{w}) = \boldsymbol{0} & \text{in } \mathbb{R}^n_+,\\ (\partial_n\boldsymbol{w}',\,w_n) = \boldsymbol{g}-\boldsymbol{G} & \text{on } \Gamma.\end{cases}$$

By Theorem 3.10, there exists $(\boldsymbol{z},\,\eta)\in \boldsymbol{W}^{2,\,p}_{\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)$ satisfying

$$\begin{cases} (-\Delta\boldsymbol{z}+\nabla\eta,\,\operatorname{div}\boldsymbol{z}) = (\boldsymbol{F},\,H) & \text{in } \mathbb{R}^n_+,\\ (\partial_n\boldsymbol{z}',\,z_n) = \boldsymbol{G} & \text{on } \Gamma.\end{cases}$$

Finally, the pair $(\boldsymbol{u},\,\pi)=(\boldsymbol{v}+\boldsymbol{w}+\boldsymbol{z},\,\vartheta+\zeta+\eta)$ proves the existence. ∎

To end this section, we can give a global regularity result which extends the strong solutions of Theorem 3.10.

COROLLARY 3.11. *Let $\ell\in\mathbb{Z}$ and $m\geqslant 1$ be two integers, and assume* (1.7). *For all $\boldsymbol{f}\in \boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^n_+)$, $h\in W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)$, $g_n\in W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$, and $\boldsymbol{g}'\in \boldsymbol{W}^{m-1/p,\,p}_{m+\ell}(\Gamma)$, satisfying the compatibility condition* (3.4), *problem* $(S^\sharp)$ *admits a solution* $(\boldsymbol{u},\,\pi)\in \boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n_+)\times W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)$, *unique up to an element of $\mathcal{S}^\sharp_{[1-\ell-n/p]}$, with the estimate*

$$\begin{aligned}&\inf_{(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^\sharp_{[1-\ell-n/p]}}\left(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{m+1,\,p}_{m+\ell}(\mathbb{R}^n_+)}+\|\pi+\mu\|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)}\right)\\ &\leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{m-1,\,p}_{m+\ell}(\mathbb{R}^n_+)}+\|h\|_{W^{m,\,p}_{m+\ell}(\mathbb{R}^n_+)}+\|g_n\|_{W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)}+\|\boldsymbol{g}'\|_{\boldsymbol{W}^{m-1/p,\,p}_{m+\ell}(\Gamma)}\right).\end{aligned}$$

Proof. The case $m=1$ corresponds to Theorem 3.10. We suppose that $m\geqslant 2$.

(1) Assuming that $\ell\leqslant -2$, hypothesis (1.7)—which yields the set of critical values—is reduced to $n/p\notin\{1,\dots,-\ell-m\}$. We begin by establishing the result for the homogeneous problem, as in Proposition 3.8. The arguments are the same, using the regularity results for the Laplacian with Dirichlet and Neumann boundary conditions (see [3, 5]). Next, for the complete problem, we apply the method of the proof of Theorem 3.10 with an ad hoc extension for $\boldsymbol{F}$ and H—in this case, there is no compatibility condition.

(2) Assuming that $\ell\geqslant -1$ and $n/p'\notin\{1,\dots,\ell+1\}$, we can adapt the proof by induction of the regularity result for the Stokes system with Dirichlet boundary conditions (see [7, Corollary 5.5]). ∎

4. Very weak solutions. The aim of this section is to return to the homogeneous problem (2.1) in which we envisage now very singular data on the boundary; that is, more precisely,

$$\boldsymbol{g}'\in \boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma) \quad\text{and}\quad g_n\in W^{-1/p,\,p}_{\ell-1}(\Gamma).$$

First, we will establish two preliminary lemmas. The second one yields a Green's formula in order to solve this new problem by a duality argument.

Let us denote by

$$T: \ (\boldsymbol{u},\, \pi) \longmapsto (-\Delta \boldsymbol{u} + \nabla \pi,\, -\operatorname{div} \boldsymbol{u})$$

the Stokes operator. For any $\ell \in \mathbb{Z}$, we introduce the space

$$T^{p}_{\ell,1}(\mathbb{R}^n_+) = \Big\{(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+); \\ T(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell+1,\,1}(\mathbb{R}^n_+) \times W^{0,\,p}_{\ell,\,1}(\mathbb{R}^n_+)\Big\},$$

which is a reflexive Banach space equipped with the graph-norm. Then we have the following density result.

LEMMA 4.1. *Let $\ell \in \mathbb{Z}$ and assume that*

$$n/p' \notin \{1, \ldots, \ell - 1\} \quad \text{and} \quad n/p \notin \{1, \ldots, -\ell + 1\}; \tag{4.1}$$

then the space $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in $T^{p}_{\ell,1}(\mathbb{R}^n_+)$.

Proof. For every continuous linear form $\Lambda \in \big(T^{p}_{\ell,1}(\mathbb{R}^n_+)\big)'$, there exists a unique $(\boldsymbol{v},\, \zeta,\, \boldsymbol{w},\, \vartheta) \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \boldsymbol{W}^{0,\,p'}_{-\ell-1,\,-1}(\mathbb{R}^n_+) \times W^{0,\,p'}_{-\ell,\,-1}(\mathbb{R}^n_+)$ such that for all $(\boldsymbol{u},\, \pi) \in T^{p}_{\ell,1}(\mathbb{R}^n_+)$,

$$\langle \Lambda,\, (\boldsymbol{u},\, \pi) \rangle = \langle (\boldsymbol{v},\, \zeta),\, (\boldsymbol{u},\, \pi) \rangle + \langle (\boldsymbol{w},\, \vartheta),\, T(\boldsymbol{u},\, \pi) \rangle\,. \tag{4.2}$$

Thanks to the Hahn–Banach theorem, it suffices to show that any Λ which vanishes on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ is actually zero on $T^{p}_{\ell,1}(\mathbb{R}^n_+)$. Let us suppose that $\Lambda = 0$ on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ and thus on $\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)$. Then we can deduce from (4.2) that

$$(\boldsymbol{v},\, \zeta) + T(\boldsymbol{w},\, \vartheta) = 0 \quad \text{in } \mathbb{R}^n_+,$$

and hence $T(\boldsymbol{w},\, \vartheta) \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$. Let $\tilde{\boldsymbol{v}},\, \tilde{\zeta},\, \tilde{\boldsymbol{w}},\, \tilde{\vartheta}$ be, respectively, the zero extensions of $\boldsymbol{v},\, \zeta,\, \boldsymbol{w},\, \vartheta$ to $\mathbb{R}^n$. By (4.2), it is clear that we have

$$(\tilde{\boldsymbol{v}},\, \tilde{\zeta}) + T(\tilde{\boldsymbol{w}},\, \tilde{\vartheta}) = 0 \ \text{ in } \mathbb{R}^n,$$

and thus $T(\tilde{\boldsymbol{w}},\, \tilde{\vartheta}) \in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n)$. Besides, we have the following Green's formula: for any $(\boldsymbol{\varphi},\, \psi) \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)$,

$$\Big\langle T(\tilde{\boldsymbol{w}},\, \tilde{\vartheta}),\, (\boldsymbol{\varphi},\, \psi) \Big\rangle = \Big\langle (\tilde{\boldsymbol{w}},\, \tilde{\vartheta}),\, T(\boldsymbol{\varphi},\, \psi) \Big\rangle\,. \tag{4.3}$$

On the other hand, we have both $\mathcal{S}_{[1-\ell-n/p]} \subset \boldsymbol{W}^{2,\,p}_{\ell+1,\,1}(\mathbb{R}^n) \times W^{1,\,p}_{\ell+1,\,1}(\mathbb{R}^n)$ and the imbedding $\boldsymbol{W}^{2,\,p}_{\ell+1,\,1}(\mathbb{R}^n) \times W^{1,\,p}_{\ell+1,\,1}(\mathbb{R}^n) \hookrightarrow \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n)$ under hypothesis (4.1); then by the density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^n) \times \mathcal{D}(\mathbb{R}^n)$ in $\boldsymbol{W}^{2,\,p}_{\ell+1,\,1}(\mathbb{R}^n) \times W^{1,\,p}_{\ell+1,\,1}(\mathbb{R}^n)$, we can deduce that (4.3) holds for any $(\boldsymbol{\varphi},\, \psi) \in \mathcal{S}_{[1-\ell-n/p]}$, and thus $T(\tilde{\boldsymbol{w}},\, \tilde{\vartheta}) \perp \mathcal{S}_{[1-\ell-n/p]}$.

With this orthogonality condition, we can apply Theorem 1.3, and it follows that $(\tilde{\boldsymbol{w}},\, \tilde{\vartheta}) \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n)$. Since $\tilde{\boldsymbol{w}}$ and $\tilde{\vartheta}$ are the zero extensions of $\boldsymbol{w}$ and ϑ, it follows that $(\boldsymbol{w},\, \vartheta) \in \overset{\circ}{\boldsymbol{W}}{}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$. Then, by the density of $\boldsymbol{\mathcal{D}}(\mathbb{R}^n_+) \times$

$\mathcal{D}(\mathbb{R}^n_+)$ in $\mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)$, we can construct a sequence $(\boldsymbol{w}_k, \vartheta_k)_{k\in\mathbb{N}} \subset \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+)\times\mathcal{D}(\mathbb{R}^n_+)$ such that $(\boldsymbol{w}_k, \vartheta_k)\to(\boldsymbol{w}, \vartheta)$ in $\mathring{\boldsymbol{W}}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)$. Thus, for any $(\boldsymbol{u}, \pi)\in T^p_{\ell,1}(\mathbb{R}^n_+)$, we have

$$\begin{aligned}\langle\Lambda, (\boldsymbol{u}, \pi)\rangle &= -\langle T(\boldsymbol{w}, \vartheta), (\boldsymbol{u}, \pi)\rangle + \langle(\boldsymbol{w}, \vartheta), T(\boldsymbol{u}, \pi)\rangle\\ &= \lim_{k\to\infty}\{-\langle T(\boldsymbol{w}_k, \vartheta_k), (\boldsymbol{u}, \pi)\rangle + \langle(\boldsymbol{w}_k, \vartheta_k), T(\boldsymbol{u}, \pi)\rangle\}\\ &= 0;\end{aligned}$$

i.e., Λ is identically zero. ◻

Thanks to this density lemma, we can prove the following result.

LEMMA 4.2. *Let $\ell\in\mathbb{Z}$. Under hypothesis* (4.1)*, we can define the following linear continuous mapping—that is, the traces of order* 1 *for the tangential component and of order* 0 *for the normal component of the velocity field:*

$$\begin{aligned}\gamma^\sharp : T^p_{\ell,1}(\mathbb{R}^n_+) &\longrightarrow \boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)\times W^{-1/p,\,p}_{\ell-1}(\Gamma),\\ (\boldsymbol{u}, \pi) &\longmapsto (\partial_n\boldsymbol{u}', u_n)|_\Gamma = (\gamma_1 u_1, \ldots, \gamma_1 u_{n-1}, \gamma_0 u_n).\end{aligned}$$

Moreover, we have the following Green's formula:

$$\begin{aligned}&\forall(\boldsymbol{u}, \pi)\in T^p_{\ell,1}(\mathbb{R}^n_+),\ \forall(\boldsymbol{\varphi}, \psi)\in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)\\ &\qquad \textit{such that } \varphi_n = 0,\ \partial_n\boldsymbol{\varphi}' = \boldsymbol{0},\ \textit{and } \operatorname{div}\boldsymbol{\varphi} = 0 \textit{ on } \Gamma,\\ &\langle T(\boldsymbol{u}, \pi), (\boldsymbol{\varphi}, \psi)\rangle_{\boldsymbol{W}^{0,p}_{\ell+1,1}(\mathbb{R}^n_+)\times W^{0,p}_{\ell,1}(\mathbb{R}^n_+),\ \boldsymbol{W}^{0,p'}_{-\ell-1,-1}(\mathbb{R}^n_+)\times W^{0,p'}_{-\ell,-1}(\mathbb{R}^n_+)}\\ &= \langle(\boldsymbol{u}, \pi), T(\boldsymbol{\varphi}, \psi)\rangle_{\boldsymbol{W}^{0,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{-1,p}_{\ell-1}(\mathbb{R}^n_+),\ \boldsymbol{W}^{0,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\ &\qquad + \langle(\partial_n\boldsymbol{u}', u_n), (\boldsymbol{\varphi}', \psi - \partial_n\varphi_n)\rangle_\Gamma .\end{aligned} \tag{4.4}$$

Proof. Let us make two remarks to start. First, the left-hand term in (4.4) is nothing but the integral $\int_{\mathbb{R}^n_+} T(\boldsymbol{u}, \pi)\cdot(\boldsymbol{\varphi}, \psi)\,\mathrm{d}x$. Second, the reason for the logarithmic factor in the definition of $T^p_{\ell,1}(\mathbb{R}^n_+)$ is that the imbeddings $\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\hookrightarrow \boldsymbol{W}^{0,p'}_{-\ell-1,-1}(\mathbb{R}^n_+)$ and $W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)\hookrightarrow W^{0,p'}_{-\ell,-1}(\mathbb{R}^n_+)$ hold without supplementary critical value with respect to (4.1)—whereas the imbedding $\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\hookrightarrow \boldsymbol{W}^{0,p'}_{-\ell-1}(\mathbb{R}^n_+)$ fails if $n/p'\in\{\ell, \ell+1\}$.

So we can write the following Green's formula:

$$\begin{aligned}&\forall(\boldsymbol{u}, \pi)\in \boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+})\times\mathcal{D}(\overline{\mathbb{R}^n_+}),\ \forall(\boldsymbol{\varphi}, \psi)\in \boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)\\ &\qquad \text{such that } \varphi_n = 0,\ \partial_n\boldsymbol{\varphi}' = \boldsymbol{0} \text{ and } \operatorname{div}\boldsymbol{\varphi} = 0 \text{ on } \Gamma,\\ &\int_{\mathbb{R}^n_+} T(\boldsymbol{u}, \pi)\cdot(\boldsymbol{\varphi}, \psi)\,\mathrm{d}x = \int_{\mathbb{R}^n_+}(\boldsymbol{u}, \pi)\cdot T(\boldsymbol{\varphi}, \psi)\,\mathrm{d}x\\ &\qquad + \int_\Gamma(\partial_n\boldsymbol{u}', u_n)\cdot(\boldsymbol{\varphi}', \psi - \partial_n\varphi_n)\,\mathrm{d}x'.\end{aligned} \tag{4.5}$$

We can deduce the following estimate:

$$|\langle(\partial_n\boldsymbol{u}', u_n), (\boldsymbol{\varphi}', \psi - \partial_n\varphi_n)\rangle_\Gamma| \leqslant \|(\boldsymbol{u}, \pi)\|_{T^p_{\ell,1}(\mathbb{R}^n_+)}\ \|(\boldsymbol{\varphi}, \psi)\|_{\boldsymbol{W}^{2,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,p'}_{-\ell+1}(\mathbb{R}^n_+)}.$$

According to Lemma 1.1, for any $\boldsymbol{\mu} = (\boldsymbol{\mu}', \mu_n) \in \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma) \times W^{1-1/p',\,p}_{-\ell+1}(\Gamma)$, there exists a lifting function $(\boldsymbol{\varphi}, \psi) \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)$ such that

$$
\begin{aligned}
(\gamma_0, \gamma_1)\boldsymbol{\varphi}' &= (\boldsymbol{\mu}', \mathbf{0}) \in \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma) \times \boldsymbol{W}^{1-1/p',\,p}_{-\ell+1}(\Gamma),\\
(\gamma_0, \gamma_1)\varphi_n &= (0, -\operatorname{div}' \boldsymbol{\mu}') \in W^{2-1/p',\,p'}_{-\ell+1}(\Gamma) \times W^{1-1/p',\,p}_{-\ell+1}(\Gamma),\\
\gamma_0\psi &= \mu_n - \operatorname{div}' \boldsymbol{\mu}' \in W^{1-1/p',\,p}_{-\ell+1}(\Gamma),
\end{aligned}
$$

i.e., $(\boldsymbol{\varphi}', \psi - \partial_n\varphi_n) = \boldsymbol{\mu}$ with $\varphi_n = 0$, $\partial_n\boldsymbol{\varphi}' = \mathbf{0}$, and $\operatorname{div}\boldsymbol{\varphi} = 0$ on Γ, satisfying

$$
\|(\boldsymbol{\varphi}, \psi)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)} \leqslant C\, \|\boldsymbol{\mu}\|_{\boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma) \times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)},
$$

where C is a constant not depending on $(\boldsymbol{\varphi}, \psi)$ and $\boldsymbol{\mu}$. Then we can deduce that

$$
\|(\partial_n\boldsymbol{u}', u_n)\|_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma) \times W^{-1/p,\,p}_{\ell-1}(\Gamma)} \;\leqslant\; C\, \|(\boldsymbol{u}, \pi)\|_{T^p_{\ell,\,1}(\mathbb{R}^n_+)}.
$$

Thus the linear mapping $\gamma^\sharp : (\boldsymbol{u}, \pi) \longmapsto (\partial_n\boldsymbol{u}', u_n)|_\Gamma$ defined on $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ is continuous for the norm of $T^p_{\ell,\,1}(\mathbb{R}^n_+)$. In addition, since $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in $T^p_{\ell,\,1}(\mathbb{R}^n_+)$, the mapping $\gamma^\sharp$ can be extended by continuity to a mapping still called $\gamma^\sharp \in \mathcal{L}\big(T^p_{\ell,\,1}(\mathbb{R}^n_+);\ \boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma) \times W^{-1/p,\,p}_{\ell-1}(\Gamma)\big)$. Moreover, we can also deduce the formula (4.4) from (4.5) by density of $\boldsymbol{\mathcal{D}}(\overline{\mathbb{R}^n_+}) \times \mathcal{D}(\overline{\mathbb{R}^n_+})$ in $T^p_{\ell,\,1}(\mathbb{R}^n_+)$. ∎

Thanks to this lemma, we now can give the result for singular boundary conditions.

THEOREM 4.3. *Let $\ell \in \mathbb{Z}$ with hypothesis* (4.1). *For any $\boldsymbol{g} = (\boldsymbol{g}', g_n) \in \boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma) \times W^{-1/p,\,p}_{\ell-1}(\Gamma)$, satisfying the compatibility condition*

$$
\begin{aligned}
&\forall \boldsymbol{\varphi} = (\boldsymbol{\varphi}', \varphi_n) \in \boldsymbol{\mathcal{N}}^{\Delta}_{[1+\ell-n/p']} \times \mathcal{A}^{\Delta}_{[1+\ell-n/p']},\\
&\langle \boldsymbol{g}', \boldsymbol{\varphi}'\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma) \times \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma)} = \langle g_n, \partial_n\varphi_n\rangle_{W^{-1/p,\,p}_{\ell-1}(\Gamma) \times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)},
\end{aligned}
\tag{4.6}
$$

problem (2.1) *admits a solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{S}^\sharp_{[1-\ell-n/p]}$, and there exists a constant C such that*

$$
\inf_{(\boldsymbol{\lambda},\,\mu) \in \mathcal{S}^\sharp_{[1-\ell-n/p]}} \Big(\|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)} + \|\pi + \mu\|_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)} \Big) \;\leqslant\; C\, \|\boldsymbol{g}\|_\Gamma.
$$

Proof. To start with, let us observe that such a pair $(\boldsymbol{u}, \pi)$ belongs to $T^p_{\ell,\,1}(\mathbb{R}^n_+)$, and then Lemma 4.2 gives meaning to these boundary conditions. Next, we can observe that problem (2.1) is equivalent to the variational formulation: find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)$ satisfying

$$
\begin{aligned}
&\forall (\boldsymbol{v}, \vartheta) \in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\\
&\qquad \text{such that } v_n = 0,\ \partial_n\boldsymbol{v}' = \mathbf{0}, \text{ and } \operatorname{div}\boldsymbol{v} = 0 \text{ on } \Gamma,\\
&\langle (\boldsymbol{u}, \pi), T(\boldsymbol{v}, \vartheta)\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+),\ \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+) \times \mathring{W}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\\
&= -\,\langle \boldsymbol{g}, (\boldsymbol{v}', \vartheta - \partial_n v_n)\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma) \times W^{-1/p,\,p}_{\ell-1}(\Gamma),\ \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma) \times W^{1/p,\,p'}_{-\ell+1}(\Gamma)}.
\end{aligned}
\tag{4.7}
$$

Indeed, the direct implication is straightforward. Conversely, if the pair $(\boldsymbol{u}, \pi)$ satisfies (4.7), then we have for any $(\boldsymbol{\varphi}, \psi) \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)$,

$$
\langle T(\boldsymbol{u}, \pi), (\boldsymbol{\varphi}, \psi)\rangle_{\boldsymbol{\mathcal{D}}'(\mathbb{R}^n_+) \times \mathcal{D}'(\mathbb{R}^n_+),\, \boldsymbol{\mathcal{D}}(\mathbb{R}^n_+) \times \mathcal{D}(\mathbb{R}^n_+)} = \langle (\boldsymbol{u}, \pi), T(\boldsymbol{\varphi}, \psi)\rangle = 0,
$$

and thus $T(\boldsymbol{u},\,\pi)=\boldsymbol{0}$ in $\mathbb{R}^n_+$. In addition, according to the Green's formula (4.4), we have

$$\begin{aligned}&\forall(\boldsymbol{v},\,\vartheta)\in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\\ &\qquad\qquad\text{such that } v_n=0,\ \partial_n\boldsymbol{v'}=\boldsymbol{0} \text{ and } \operatorname{div}\boldsymbol{v}=0 \text{ on } \Gamma,\\ &\qquad\qquad\qquad \langle(\partial_n\boldsymbol{u'},u_n),\,(\boldsymbol{v'},\,\vartheta-\partial_n v_n)\rangle_\Gamma=\langle\boldsymbol{g},\,(\boldsymbol{v'},\,\vartheta-\partial_n v_n)\rangle_\Gamma\,.\end{aligned}$$

As we saw in the proof of Lemma 4.2, by Lemma 1.1, it follows that for any $\boldsymbol{\mu}\in \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma)\times W^{1-1/p',\,p'}_{-\ell+1}(\Gamma)$,

$$\langle(\partial_n\boldsymbol{u'}-\boldsymbol{g'},u_n-g_n),\,\boldsymbol{\mu}\rangle_\Gamma=0,$$

that is, $\partial_n\boldsymbol{u'}=\boldsymbol{g'}$ and $u_n=g_n$ on Γ. Hence $(\boldsymbol{u},\,\pi)$ satisfies (2.1).

Now, let us solve problem (4.7). By Theorem 3.10, we know that under hypothesis (4.1), for all $(\boldsymbol{f},\,h)\in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\perp \mathcal{S}^\sharp_{[1-\ell-n/p]}$, there exists a unique $(\boldsymbol{v},\,\vartheta)\in \boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)/\mathcal{S}^\sharp_{[1+\ell-n/p']}$ solution to

$$-\Delta\boldsymbol{v}+\nabla\vartheta=\boldsymbol{f}\quad\text{and}\quad\operatorname{div}\boldsymbol{v}=h\quad\text{in }\mathbb{R}^n_+,\qquad \partial_n\boldsymbol{v'}=\boldsymbol{0}\quad\text{and}\quad v_n=0\quad\text{on }\Gamma,$$

with the estimate

$$\|(\boldsymbol{v},\,\vartheta)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)/\mathcal{S}^\sharp_{[1+\ell-n/p']}}\ \leqslant\ C\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}+\|h\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\right).$$

Consider the linear form $\Xi:\ (\boldsymbol{f},\,h)\longmapsto\langle\boldsymbol{g},\,(\boldsymbol{v'},\,\vartheta-\partial_n v_n)\rangle_\Gamma$ defined on the product space $\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\perp \mathcal{S}^\sharp_{[1-\ell-n/p]}$. According to (4.6), we have for any $\boldsymbol{\varphi}\in\boldsymbol{\mathcal{N}}^\Delta_{[1+\ell-n/p']}\times\mathcal{A}^\Delta_{[1+\ell-n/p']}$, or equivalently, for any $(\boldsymbol{\lambda},\,\mu)\in\mathcal{S}^\sharp_{[1+\ell-n/p']}$,

$$\begin{aligned}|\Xi(\boldsymbol{f},\,h)|\ &=\ |\langle\boldsymbol{g},(\boldsymbol{v'}+\boldsymbol{\varphi'},\,\vartheta-\partial_n v_n+\partial_n\varphi_n)\rangle_\Gamma|\\ &=\ \left|\left\langle\boldsymbol{g},([\boldsymbol{v'}+\boldsymbol{\lambda'}],\,[\vartheta+\mu]-\partial_n[v_n+\lambda_n])\right\rangle_\Gamma\right|\\ &\leqslant\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)\times W^{-1/p,\,p}_{\ell-1}(\Gamma)}\,\|(\boldsymbol{v},\,\vartheta)+(\boldsymbol{\lambda},\,\mu)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}.\end{aligned}$$

Thus

$$\begin{aligned}|\Xi(\boldsymbol{f},\,h)|\ &\leqslant\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)\times W^{-1/p,\,p}_{\ell-1}(\Gamma)}\,\|(\boldsymbol{v},\,\vartheta)\|_{\boldsymbol{W}^{2,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)/\mathcal{S}^\sharp_{[1+\ell-n/p']}}\\ &\leqslant\ C\ \|\boldsymbol{g}\|_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)\times W^{-1/p,\,p}_{\ell-1}(\Gamma)}\left(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}+\|h\|_{W^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\right).\end{aligned}$$

In other words, Ξ is continuous on $\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\perp \mathcal{S}^\sharp_{[1-\ell-n/p]}$, and according to the Riesz representation theorem, we can deduce that there exists a unique $(\boldsymbol{u},\,\pi)\in \boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)/\mathcal{S}^\sharp_{[1-\ell-n/p]}$, which is the dual space of $\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\perp \mathcal{S}^\sharp_{[1-\ell-n/p]}$, such that

$$\begin{aligned}&\forall(\boldsymbol{f},\,h)\in \boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)\times \overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+),\\ &\Xi(\boldsymbol{f},\,h)=\langle\boldsymbol{u},\boldsymbol{f}\rangle_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\boldsymbol{W}^{0,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}+\langle\pi,-h\rangle_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)\times\overset{\circ}{W}{}^{1,\,p'}_{-\ell+1}(\mathbb{R}^n_+)}\,.\end{aligned}$$

Then, we can conclude that the pair $(\boldsymbol{u},\,\pi)$ satisfies (4.7) and, moreover, that the kernel of the associated operator is $\mathcal{S}^\sharp_{[1-\ell-n/p]}$. $\square$

To end this study, using the method of the proof of the existence in Theorem 3.5, we can establish the existence of very weak solutions to the nonhomogeneous problem with very singular boundary conditions.

THEOREM 4.4. *Let $\ell \in \mathbb{Z}$ and assume that*

$$n/p' \notin \{1, \dots, \ell+1\} \quad \text{and} \quad n/p \notin \{1, \dots, -\ell+1\}.$$

For any $\boldsymbol{f} \in \boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)$, $h \in W^{1,\,p}_{\ell+1}(\mathbb{R}^n_+)$, $g_n \in W^{-1/p,\,p}_{\ell-1}(\Gamma)$, $\boldsymbol{g}' \in \boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)$, satisfying the compatibility condition

$$\begin{aligned} \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{N}}^{\Delta}_{[1+\ell-n/p']} \times \mathcal{A}^{\Delta}_{[1+\ell-n/p']},& \\ \int_{\mathbb{R}^n_+} (\boldsymbol{f} - \nabla h)\cdot \boldsymbol{\varphi}\,\mathrm{d}x \ + \ \langle \operatorname{div} \boldsymbol{f},\, \Pi_N \operatorname{div} \boldsymbol{\varphi}\rangle_{W^{-1,\,p}_{\ell+1}(\mathbb{R}^n_+)\times \mathring{W}^{1,\,p'}_{-\ell-1}(\mathbb{R}^n_+)}& \\ + \ \langle g_n,\, \partial_n \varphi_n\rangle_{\boldsymbol{W}^{-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{1-1/p',\,p'}_{-\ell+1}(\Gamma)}& \\ - \ \langle \boldsymbol{g}',\, \boldsymbol{\varphi}'\rangle_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)\times \boldsymbol{W}^{2-1/p',\,p'}_{-\ell+1}(\Gamma)} \ &= \ 0, \end{aligned}$$

problem $(S^\sharp)$ admits a solution $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+) \times W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)$, unique up to an element of $\mathcal{S}^\sharp_{[1-\ell-n/p]}$, with the estimate

$$\begin{aligned} \inf_{(\boldsymbol{\lambda},\,\mu)\in \mathcal{S}^\sharp_{[1-\ell-n/p]}} &\Big(\|\boldsymbol{u}+\boldsymbol{\lambda}\|_{\boldsymbol{W}^{0,\,p}_{\ell-1}(\mathbb{R}^n_+)} + \|\pi+\mu\|_{W^{-1,\,p}_{\ell-1}(\mathbb{R}^n_+)} \Big) \\ &\leqslant C \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)} + \|h\|_{W^{0,\,p}_{\ell+1}(\mathbb{R}^n_+)} + \|g_n\|_{W^{-1/p,\,p}_{\ell-1}(\Gamma)} + \|\boldsymbol{g}'\|_{\boldsymbol{W}^{-1-1/p,\,p}_{\ell-1}(\Gamma)} \Big). \end{aligned}$$

Acknowledgments. A part of the second author's work was done during her stay in Université de Pau et des Pays de l'Adour. She would like to express her gratitude to Prof. Amrouche and his colleagues for their extraordinary hospitality.

REFERENCES

[1] F. ALLIOT AND C. AMROUCHE, *The Stokes problem in $\mathbb{R}^n$: An approach in weighted Sobolev spaces*, Math. Methods Appl. Sci., 9 (1999), pp. 723–754.

[2] C. AMROUCHE, V. GIRAULT, AND J. GIROIRE, *Weighted Sobolev spaces for Laplace's equation in $\mathbb{R}^n$*, J. Math. Pures Appl., 73 (1994), pp. 579–606.

[3] C. AMROUCHE AND S. NEČASOVÁ, *Laplace equation in the half-space with a nonhomogeneous Dirichlet boundary condition*, Math. Bohem., 126 (2001), pp. 265–274.

[4] C. AMROUCHE, *The Neumann problem in the half-space*, C. R. Math. Acad. Sci. Paris, 335 (2002), pp. 151–156.

[5] C. AMROUCHE AND Y. RAUDIN, *Nonhomogeneous biharmonic problem in the half-space, L^p theory and generalized solutions*, J. Differential Equations, 236 (2007), pp. 57–81.

[6] C. AMROUCHE, Y. RAUDIN, *Singular boundary conditions and regularity for the biharmonic problem in the half-space*, Comm. Pure Appl. Anal., 6 (2007), pp. 957–982.

[7] C. AMROUCHE, S. NEČASOVÁ, AND Y. RAUDIN, *Very weak, generalized and strong solutions to the Stokes system in the half-space*, J. Differential Equations, 244 (2008), pp. 887–915.

[8] A. BABIN, A. MAHALOV, AND B. NICOLAENKO, *3D Navier–Stokes and Euler equations with initial data characterized by uniformly large vorticity*, Indiana Univ. Math. J., 50 (2001), pp. 1–35.

[9] A. BABIN, A. MAHALOV, AND B. NICOLAENKO, *Global regularity of 3D rotating Navier–Stokes equations for resonant domains*, Indiana Univ. Math. J., 48 (1999), pp. 1133–1176.

[10] T. Z. BOULMEZAOUD, *On the Stokes system and the biharmonic equation in the half-space: An approach via weighted Sobolev spaces*, Math. Methods Appl. Sci., 25 (2002), pp. 373–398.

[11] L. CATTABRIGA, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Sem. Mat. Univ. Padova, 31 (1961), pp. 308–340.

[12] D. Einzel, P. Panzer, and M. Liu, *Boundary condition for fluid flow: Curved or rough surfaces*, Phys. Rev. Lett., 64 (1990), pp. 2269–2272.

[13] R. Farwig, *A note on the reflection principle for the biharmonic equation and the Stokes system*, Acta Appl. Math., 25 (1994), pp. 41–51.

[14] R. Farwig, and H. Sohr, *On the Stokes and Navier–Stokes system for domains with noncompact boundary in L^q-spaces*, Math. Nachr., 170 (1994), pp. 53–77.

[15] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. I. *Linearized Steady Problems*, Springer Tracts Nat. Philos. 39, Springer-Verlag, New York, 1994.

[16] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Vol. II. *Nonlinear Steady Problems*, Springer Tracts Nat. Philo. 39, Springer-Verlag, New York, 1994.

[17] B. Hanouzet, *Espaces de Sobolev avec poids. Application au problème de Dirichlet dans un demi-espace*, Rend. Sem. Mat. Univ. Padova, 46 (1971), pp. 227–272.

[18] W. Jäger and A. Mikelić, *On the roughness-induced effective boundary conditions for an incompressible viscous flow*, J. Differential Equations, 170 (2001), pp. 96–122.

[19] O. A. Ladyzhenskaya and V. A. Solonnikov, *Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid*, Trudy Mat. Inst. Steklov, 59 (1960), pp. 115–173.

[20] E. Lauga, M. Brenner, and H. Stone, *Microfluidics: The no-slip boundary condition*, in Springer Handbook of Experimental Fluid Dynamics, C. Tropea, A Yarin, and J. F. Foss, eds., Springer, 2007, pp. 1219–1240.

[21] V. G. Maz'ya, B. A. Plamenevskiĭ, and L. I. Stupyalis, *The three-dimensional problem of steady-state motion of a fluid with a free boundary*, in Elliptic Boundary Value Problems, V. G. Maz'ya et al., eds., Amer. Math. Soc. Transl. Ser. 2, Vol. 123, AMS, Providence, RI, 1984, pp. 171–268.

[22] N. V. Priezjev, A. A. Darhuber, and S. M. Troian, *Slip behaviour in liquid films on surfaces of patterned wettability: Comparison between continuum and molecular dynamics simulations*, Phys. Rev. E., 71 (2005), article 041608.

[23] N. V. Priezjev and S. M. Troian, *Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces: Molecular-scale simulations versus continuum predictions*, J. Fluid Mech., 554 (2006), pp. 25–46.

[24] T. Qian, X.-P. Wang, and P. Sheng, *Hydrodynamic slip boundary condition at chemically patterned surfaces: A continuum deduction from molecular dynamics*, Phys. Rev. E., 72 (2005), article 022501.

[25] N. Tanaka, *On the boundary value problem for the stationary Stokes system in the half-space*, J. Differential Equations, 115 (1995), pp. 70–74.

[26] W. M. Zajączkowski, *Global special regular solutions to the Navier-Stokes equations in axially symmetric domains under boundary slip conditions*, Dissertationes Math. (Rozprawy Mat.), 432 (2005), 138 pp.

[27] Y. Zhu and S. Granick, *Limits of the hydrodynamic no-slip boundary condition*, Phys. Rev. Lett., 88 (2002), article 106102.