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# Bifurcation for a reaction-diffusion system with unilateral and Neumann boundary conditions

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# BIFURCATION FOR A REACTION-DIFFUSION SYSTEM WITH UNILATERAL AND NEUMANN BOUNDARY CONDITIONS

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Abstract. We consider a reaction-diffusion system of activator-inhibitor or substrate-depletion type which is subject to diffusion-driven instability if supplemented by pure Neumann boundary conditions. We show by a degree-theoretic approach that an obstacle (e.g. a unilateral membrane) modeled in terms of inequalities, introduces new bifurcation of spatial patterns in a parameter domain where the trivial solution of the problem without the obstacle is stable. Moreover, this parameter domain is rather different from the known case when also Dirichlet conditions are assumed. In particular, bifurcation arises for fast diffusion of activator and slow diffusion of inhibitor which is the difference from all situations which we know.

## 1. Introduction

We will study bifurcations of stationary solutions of the reaction-diffusion system

$$\begin{aligned} \frac{du}{dt} &= d_1\Delta u + b_{11}u + b_{12}v + f_1(u,v),\\ \frac{dv}{dt} &= d_2\Delta v + b_{21}u + b_{22}v + f_2(u,v) \end{aligned} \tag{1.1}$$

in a bounded domain $\Omega \subseteq \mathbb{R}^N$ with Neumann boundary conditions for $u$ and certain unilateral conditions for $v$. A typical example are Neumann-Signorini boundary conditions

$$\begin{cases} \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega,\\ v \ge 0, \quad \frac{\partial v}{\partial n} \ge 0, \quad \frac{\partial v}{\partial n}\cdot v = 0 & \text{on } \Gamma,\\ \frac{\partial v}{\partial n} = 0 & \text{on } (\partial\Omega)\setminus\Gamma, \end{cases} \tag{1.2}$$

where $\Gamma \subseteq \partial\Omega$. The diffusion coefficients $d = (d_1, d_2) \in \mathbb{R}^2_+ := (0,\infty)^2$ will be bifurcation parameters, $f_j$ are small perturbations. Our assumptions concerning the reals $b_{ij}$ will guarantee that Turing's well-known effect [25] of "diffusion-driven instability" for (1.1) with purely Neumann conditions

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.3}$$

occurs. In particular, the trivial solution of the system (1.1), (1.3) is linearly stable only if $d = (d_1, d_2) \in D_S \subset \mathbb{R}^2_+$ (domain of stability), but unstable if $d = (d_1, d_2) \in D_U = \mathbb{R}^2_+ \setminus \overline{D}_S$. The systems of activator-inhibitor type are included in our assumptions.

Our goal is to show the existence and location of bifurcations of stationary spatially nonconstant solutions (spatial patterns) of the problem (1.1), (1.2) in the domain $D_S$, where bifurcation is excluded for the problem (1.1), (1.3). Under the additional assumption that there is also a Dirichlet condition replacing the Neumann condition for $u$ and $v$ in (1.2)

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on at least a small part of the boundary, something similar was done in [1, 2, 4, 5, 6, 9, 10, 14, 15, 16, 17, 18, 23, 29]. However, such a Dirichlet condition was rather artificial from the point of view of interpretation in models in biology, but the case of the conditions of the type (1.2) without any Dirichlet part remained an open problem for many years, and to our knowledge it is solved only in the current paper. The study of this case is essentially more complicated than that of the case with Dirichlet data and, moreover, the results for the case of the conditions of the type (1.2) without any Dirichlet part surprisingly differ from those for the case with Dirichlet conditions on a part of $\partial\Omega$. Let us mention here only one basic difference.

In the case of classical Neumann or mixed (Dirichlet-Neumann) conditions the domain of stability $D_S$ has such a shape that bifurcation for the classical Neumann problem (which can take place only in $D_U$) occurs only if the diffusion $d_1$ of the activator is in some sense fast with respect to the slow diffusion $d_2$ of the inhibitor, and there is a simple heuristic explanation of this phenomenon (see e.g. [3, p. 518]). This is true also if we replace on a part of the boundary the Neumann condition by a unilateral condition and if similarly Dirichlet condition on some part of the boundary is given, even if bifurcation occurs for smaller relation $d_2/d_1$ than in the classical case. However, we will see that in the case of the boundary conditions (1.2) (without Dirichlet part on $\partial\Omega$) there are bifurcation points also with arbitrarily large $d_1$ and small $d_2$. A possible interpretation of the unilateral condition (1.2) for $v$ is that there is a unilateral membrane or some other kind of regulation on $\Gamma$ which guarantees, by allowing a possible flux into the domain, that the concentration cannot undergo a certain threshold (which is shifted to zero in our model).

**Basic assumptions.** Concerning the constant matrix $B = (b_{ij})$, we assume

$$b_{11} > 0, \quad b_{11} + b_{22} < 0, \quad |B| := b_{11}b_{22} - b_{12}b_{21} > 0. \tag{1.4}$$

The last two inequalities mean that if we consider (1.1) as a dynamical system *without* the diffusion terms, then the trivial solution is stable. This system is of an activator-inhibitor or of a substrate-depletion type (see e.g. [3, 21]) since (1.4) implies in particular

$$b_{11} > 0 > b_{22}, \qquad b_{12}b_{21} < b_{11}b_{22} < 0. \tag{1.5}$$

It is well-known that *with* the diffusion terms and pure Neumann conditions (1.3) this system is subject to Turing's effect [25] of "diffusion-driven instability" mentioned above.

We will always assume that $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are continuous and there is $c > 0$ such that

$$|f_k(u, v)| \le c \cdot (1 + |u| + |v|)^p \quad \text{for all } u, v \in \mathbb{R},\ k = 1, 2. \tag{1.6}$$

for some $p > 0$ with $p < \frac{N}{N-2}$ if $N \ge 3$, and $p > 0$ if $N = 2$ (no condition if $N = 1$), and

$$\lim_{(u,v)\to(0,0)} \frac{f_k(u, v)}{|u| + |v|} = 0 \qquad (k = 1, 2). \tag{1.7}$$

**Description of the domain of stability $D_S$.** Letting $0 < \kappa_1 \le \kappa_2 \le \cdots$ denote the nonzero eigenvalues of $-\Delta$ on $\Omega$ with Neumann boundary conditions (1.3), we define the family of hyperbolas

$$\begin{aligned} C_n &= \big\{(d_1, d_2) \in \mathbb{R}^2_+ : (\kappa_n d_1 - b_{11})(\kappa_n d_2 - b_{22}) = b_{12}b_{21}\big\} \\ &= \Big\{(d_1, d_2) \in \mathbb{R}^2_+ : d_2 = \frac{b_{12}b_{21}/\kappa_n^2}{d_1 - b_{11}/\kappa_n} + \frac{b_{22}}{\kappa_n}\Big\} \end{aligned} \tag{1.8}$$

with vertical asymptotes $\frac{b_{11}}{\kappa_n}$. One can show that the trivial solution of (1.1), (1.3) is stable if and only if $d$ lies to the right/under the common envelope of the hyperbolas $C_1, C_2, \ldots$; we denote this "domain of stability" by $D_S$, see Figure 1. Roughly speaking, by "crossing" the hyperbola $C_n$, one loses the corresponding multiplicity of "stable directions". In space dimension $N = 1$ this was shown in [20], for $N > 1$ see e.g. [2]. Nontrivial solutions of the corresponding stationary problem

$$\begin{aligned} d_1\Delta u - b_{11}u - b_{12}v - f_1(u, v) = 0,\\ d_2\Delta v - b_{21}u - b_{22}v - f_2(u, v) = 0 \end{aligned} \tag{1.9}$$

can bifurcate (and really bifurcate under additional assumptions) from trivial solutions only at the hyperbolas $C_n$ (see e.g. [20, 12]).

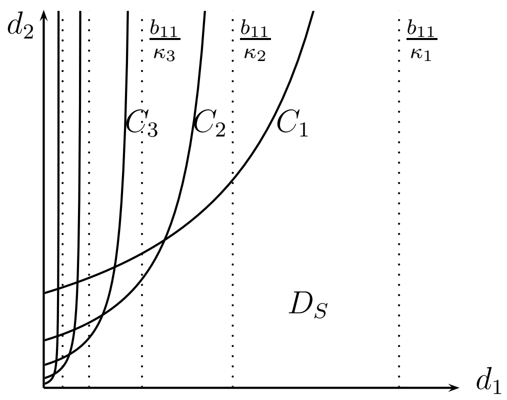


FIGURE 1. Hyperbolas $C_n$, their vertical asymptotes, and $D_S$

Let us formulate a special case of our main result. We call a point $d_0 \in \mathbb{R}^2_+$ a *bifurcation point* of the problem (1.9), (1.2) if for any neighborhood of $(d_0, 0) \in \mathbb{R}^2_+ \times W^{1,2}(\Omega, \mathbb{R}^2)$ there is a weak solution $((d_1, d_2), (u, v)) \in \mathbb{R}^2_+ \times W^{1,2}(\Omega, \mathbb{R}^2)$ of (1.9), (1.2) with $(u, v) \neq (0, 0)$, see Section 2.

Actually, in the following result, we obtain spatial patterns in the sense that for all weak solutions $((d_1, d_2), (u, v))$ in a neighborhood of $(d_0, 0) \in \mathbb{R}^2_+ \times W^{1,2}(\Omega, \mathbb{R}^2)$ the couple $(u, v)$ is spatially nonhomogeneous.

**Theorem 1.1.** *Assume that* $\mathrm{mes}_{N-1}\Gamma > 0$*. There are* $0 < d_0 < \omega_2 < \infty$*,* $\omega_1 \in (0, \infty)$ *and for every* $\varepsilon > 0$ *some* $\omega_\varepsilon \in (0, \infty)$ *such that there is a connected set* $\mathfrak{C} \subseteq D_S$ *of bifurcation points of the problem* (1.9), (1.2) *which "separates" the sets*

$$U_+ := [\omega_1, \infty) \times [\omega_2, \infty) \quad \textit{and} \quad U_- := [\omega_\varepsilon, \infty) \times [\varepsilon, d_0]$$

*in the sense that* $\mathfrak{C} \cap (U_+ \cup U_-) = \emptyset$*, and*

(1) $\mathfrak{C}$ *meets* $d_1 = \infty$ *at some* $d_2 \in (d_0, \omega_2]$*, i.e. there is a sequence* $(d_{1,n}, d_{2,n}) \in \mathfrak{C}$ *with* $d_{1,n} \to \infty$ *and* $d_{2,n} \to d_2$*.*
(2) $\mathfrak{C}$ *meets* $d_2 = 0$ *or* $d_2 = \infty$ *or* $\bigcup_{n=1}^{\infty} C_n$*, i.e. there is a sequence* $(d_{1,n}, d_{2,n}) \in \mathfrak{C}$ *which satisfies* $d_{2,n} \to 0$ *or* $d_{2,n} \to \infty$ *or which converges to some point of* $\bigcup_{n=1}^{\infty} C_n$*.*

Actually, we will obtain an estimate for $d_0$ which reminds of the characterization of the second eigenvalue of linear problems (Remark 6.3).

Hence, qualitatively, $\mathfrak{C}$ may look e.g. as sketched in Figure 2 (in the forthcoming paper [11], we will show that in space dimension $N = 1$, this figure actually describes the bifurcation points in $D_S$ completely.)

The main idea of the proof is to show that for $(d_1, d_2) \in U_\pm$ a certain associate map has the Leray-Schauder degree 0 or $-1$, respectively (in small neighborhoods of 0).

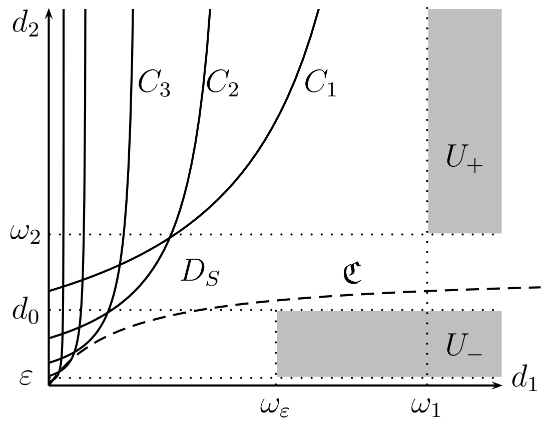


FIGURE 2. Bifurcation points of (1.9), (1.2) and the two zones $U_\pm$

**Comparison with the Dirichlet case.** As we mentioned on the beginning, the bifurcations of stationary solutions of (1.1) with boundary conditions of a type (1.2) but with Neumann condition replaced by Dirichlet condition on a part $\Gamma_D \subseteq \partial\Omega \setminus \Gamma$ were studied already in the past. In this case the domain of stability of the trivial solution of the corresponding classical problem, i.e. (1.1) with mixed boundary conditions $u = 0$ on $\Gamma_D$, $\partial u/\partial n = 0$ on $\partial\Omega \setminus \Gamma_D$, is described again as above but now $\kappa_j$ in the definition of $C_j$ are eigenvalues of $-\Delta$ with mixed boundary conditions mentioned. However, in this case there cannot be bifurcation points in the zone

$$Z_0 := (\tfrac{b_{11}}{\kappa_1}, \infty) \times (0, \infty) \tag{1.10}$$

to the right of the vertical asymptote of the right-most hyperbola $C_1$. The shape of the connected set $\mathfrak{C}$ of bifurcation points lying in $D_S$ is in this case unbounded in $d_2$-direction with $\frac{b_{11}}{\kappa_1}$ as its vertical asymptote, i.e. $\mathfrak{C}$ may look qualitatively as in Figure 3. Actually, numerical calculations suggest that in space dimension $N = 1$ it really has roughly the shape as in this figure. (We note that there are other branches of bifurcation points in

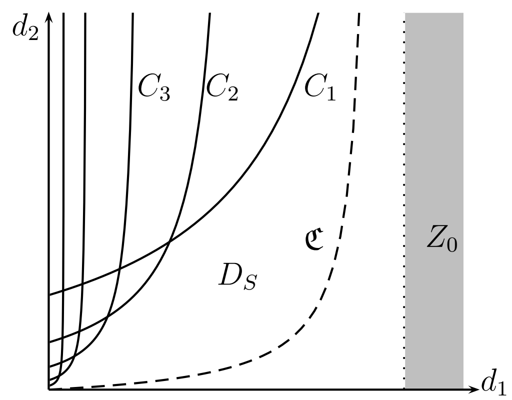


FIGURE 3. Branch of bifurcation points $\mathfrak{C}$ in the Dirichlet case and the zone (1.10)

$D_U$ [29], but we consider here only what happens in $D_S$.) In fact, it can be shown that the Leray-Schauder degree of the map associated naturally to the problem has for $(d_1, d_2) \in Z_0$ the value 1 (in small neighborhoods of 0) but for $(d_1, d_2)$ close to certain hyperbolas $C_n$ the value 0. Results of such type (with a Dirichlet part) can be found e.g. in [10, 29].

In the forthcoming paper [11], we will show that for the case of the conditions (1.2) in dimension $N = 1$ there is no bifurcation point $(d_1, d_2)$ to the right of $C_1$ with large $d_2$ so that actually the branch $\mathfrak{C}$ in Figure 2 describes all bifurcation points in $D_S$ in the sense that the existence of an additional branch as in Figure 3 is excluded.

Hence, the difference of the pure Neumann-Signorini case (1.2) from the case with a Dirichlet part is not only that we need rather different mathematical methods to attack the problem but also the location of the branch of bifurcation points is different. The branch as shaped in Figure 3 cannot occur under boundary conditions (1.2), and vice versa.

A particular case of the conditions (1.2) was touched briefly in [2] (which is devoted mainly to the case with Dirichlet conditions), but there is a mistake. The method used cannot be applied in fact and the partial result mentioned there is wrong.

A partial motivation for the correct answer in the case without Dirichlet conditions given in the current paper was an unpublished numerical simulation performed by Jan Eisner some years ago, suggesting that in the one dimensional case the branches of critical points do not look like in Figure 3 but are closer to Figure 2. The authors thank him for discussions concerning those computations.

The plan of the paper is as follows. In Section 2, we formulate general bifurcation results for problems of type (1.9) with unilateral conditions and give several examples. In particular, these results contain Theorem 1.1. In Section 3, we introduce the general functional analytic framework which will be used for the remainder of the paper. After proving some auxiliary results about a "shadow system" in Section 4, we will be able to show that the earlier mentioned degree is 0 for $(d_1, d_2) \in U_+$. However, the crucial part of the paper is to show that this degree is $-1$ for $(d_1, d_2) \in U_-$. The proof of that part is divided into two sections: In Section 5, we describe a rather general approach which shows that the degree of an auxiliary map is $\pm 1$. We show in Section 6 how this can be used to show that the degree for the map we are actually interested in is $-1$. In the final Section 7, the results of the previous sections are combined to prove the bifurcation results of Section 2. Actually, Sections 4–6 contain more general results concerning properties of auxiliary systems than those necessary for the proof of our bifurcation theorems. In fact, we could have used them to formulate more general bifurcation results in a functional analytic setting, see Remark 7.2.

## 2. Main Bifurcation Results and Applications to Unilateral Problems

In the sequel, we will work with the spaces $\mathbb{H}_0 := W^{1,2}(\Omega, \mathbb{R})$ and $\mathbb{H} := \mathbb{H}_0 \times \mathbb{H}_0$.

Recall that eigenvalues $\kappa_n$ of $-\Delta$ with Neumann boundary conditions (1.3) and the corresponding eigenfunctions $u \in \mathbb{H}_0$ are characterized by the variational equality

$$\int_\Omega \nabla u \cdot \nabla \varphi = \kappa_n \int_\Omega u\varphi \, dx \qquad \text{for all } \varphi \in \mathbb{H}_0. \tag{2.1}$$

We define weak solutions of the problem (1.9), (1.3) in a standard manner as couples $(u, v) \in \mathbb{H}$ which satisfy the variational equations

$$\begin{aligned} d_1 \int_\Omega \nabla u \cdot \nabla \varphi &= \int_\Omega \big(b_{11} u + b_{12} v + f_1(u, v)\big)\varphi \, dx \quad \text{for all } \varphi \in \mathbb{H}_0, \\ d_2 \int_\Omega \nabla v \cdot \nabla \varphi &= \int_\Omega \big(b_{21} u + b_{22} v + f_2(u, v)\big)\varphi \, dx \quad \text{for all } \varphi \in \mathbb{H}_0, \end{aligned} \tag{2.2}$$

where all integrals are finite under the assumption (1.6) due to Sobolev's embedding theorems and Hölder's inequality. Similarly, considering the cone

$$K_0 := \{v \in \mathbb{H}_0 : v|_\Gamma \geq 0 \text{ (in the sense of traces)}\}, \tag{2.3}$$

we define weak solutions of (1.9), (1.2) as couples $(u, v) \in \mathbb{H}$ satisfying the variational inequality

$$\begin{aligned} & d_1 \int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega \big(b_{11} u + b_{12} v + f_1(u,v)\big)\varphi \, dx \qquad \text{for all } \varphi \in \mathbb{H}_0, \\ & v \in K_0, \quad d_2 \int_\Omega \nabla v \cdot (\nabla \varphi - \nabla v) \geq \\ & \qquad \int_\Omega \big(b_{21} u + b_{22} v + f_2(u,v)\big)(\varphi - v)\, dx \qquad \text{for all } \varphi \in K_0. \end{aligned} \tag{2.4}$$

We call $d_0 \in \mathbb{R}^2_+$ a *bifurcation point* of (2.4) if for each neighborhood of $(d_0, 0) \in \mathbb{R}^2_+ \times \mathbb{H}$ there are $((d_1, d_2), (u, v)) \in \mathbb{R}^2_+ \times \mathbb{H}$ with $(u, v) \neq (0, 0)$ satisfying (2.4). We say that the *bifurcation point $d_0$ is spatially nonhomogeneous* if there is a neighborhood $W$ of $(d_0, 0) \in \mathbb{R}^2_+$ such that $(u, v)$ is spatially nonhomogeneous (nonconstant) for every $((d_1, d_2), (u, v)) \in W \times \mathbb{H}$ satisfying (2.4) with $(u, v) \neq (0, 0)$. For the particular cone (2.3), we call these points (spatially nonhomogeneous) bifurcation points of (1.9), (1.2).

We call a point $d = (d_1, d_2) \in \mathbb{R}^2_+$ a *critical point* of (2.4) if there is a weak solution $(u, v) \neq (0, 0)$ of (2.4) with $f_1 = f_2 = 0$. A compactness argument implies that every bifurcation point of (2.4) is a critical point, see Proposition 7.1; cf. also e.g. [2].

However, our main bifurcation result does not only deal with the cone (2.3), but actually one can replace (2.3) by any closed convex cone $K_0 \subseteq \mathbb{H}_0$ with its vertex in 0 (i.e. $K_0$ is closed and convex with $0 \in K_0 + K_0 \subseteq K_0$) satisfying certain hypotheses. In order to formulate these hypotheses, we denote by $e$ either

$$e(x) := 1 \quad \text{or} \quad e(x) := -1, \tag{2.5}$$

the choice of the sign in (2.5) being arbitrary but fixed. Our main results concerning bifurcation for the problem (1.9), (1.2) are the following two theorems.

**Theorem 2.1.** *Suppose* (1.4), *and let $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy* (1.6), (1.7). *Let $K_0 \subseteq \mathbb{H}_0$ be a closed convex cone with its vertex in* 0 *with the following properties.*

$$\text{For any eigenfunction } u \text{ of } \Delta \text{ with (1.3) there is } \varepsilon > 0 \text{ with } e + \varepsilon u \in K_0, \tag{2.6}$$

*$-e \notin K_0$, and there is*

$$u_0 \in e + K_0 \quad \text{with} \quad \int_\Omega u_0 \, dx = 0 \quad \text{and} \quad \frac{1}{\operatorname{mes}_N \Omega} \int_\Omega |u_0|^2 \, dx < \Big(\frac{|B|}{b_{12} b_{21}}\Big)^2. \tag{2.7}$$

*Then there are $\omega_1, \omega_2 > 0$, $d_0 > 0$, and for each $\varepsilon > 0$ some $\omega_\varepsilon > 0$ with the following properties.*

(1) *The sets $U_+ := [\omega_1, \infty) \times [\omega_2, \infty)$ and $U_- := [\omega_\varepsilon, \infty) \times [\varepsilon, d_0]$ contain no critical point of* (2.4).
(2) *There is no sequence $(d_{1,n}, d_{2,n}) \in \mathbb{R}^2_+$ of critical points of* (2.4) *with $d_{1,n} \to \infty$ and $d_{2,n} \to d_0$.*
(3) *The set of bifurcation points of* (2.4) *in $D_S$ contains a connected set $\mathfrak{C}$ which separates $U_+$ and $U_-$ in the following sense:*
    (a) *$\mathfrak{C}$ contains a sequence $(d_{1,n}, d_{2,n})$ with $d_{1,n} \to \infty$ and $d_{2,n} \to d_\infty \in (d_0, \omega_2)$.*
    (b) *$\mathfrak{C}$ contains a sequence $(d_{1,n}, d_{2,n})$ which converges to some point of a hyperbola $C_m$ $(m = 1, 2, \dots)$ or which satisfies $d_{2,n} \to 0$ or $d_{2,n} \to \infty$.*

*All bifurcation points of* (2.4) *in $\mathbb{R}^2_+ \setminus \bigcup_{n=1}^\infty C_n$ are spatially nonhomogeneous.*

*Remark* 2.1. Our proof will show that for each $u_0$ satisfying (2.7) one can actually choose

$$d_0 := -b_{22}\frac{\left(\dfrac{|B|}{b_{12}b_{21}}\right)^2 \operatorname{mes}_N\Omega - \displaystyle\int_\Omega |u_0|^2\,dx}{\displaystyle\int_\Omega |\nabla u_0|^2\,dx} > 0 \tag{2.8}$$

in Theorem 2.1. The quantities $\omega_1, \omega_2 > 0$ in Theorem 2.1 are independent of $u_0$, but $\omega_\varepsilon > 0$ might also depend on $u_0$.

**Theorem 2.2.** *Under the hypotheses of Theorem 2.1, let $C_0 \subseteq \mathbb{R}^2_+$ denote the critical points of (2.4), and let $\widetilde{U}_\pm$ denote the component of $\mathbb{R}^2_+ \setminus C_0$ containing $U_\pm$.*

*Let $I$ be a closed (not necessarily bounded) interval, and let $\gamma\colon I \to \mathbb{R}^2_+$ be continuous such that there are two points $t_\pm \in I$, $t_- < t_+$ with $\gamma(t_\pm) \in \widetilde{U}_\pm$.*

*Then there is a global bifurcation of (2.4) on $\gamma$ in the sense that there is a connected set $\mathfrak{C}_0 \subseteq I \times \mathbb{H}$ of $(t, u, v)$, satisfying $(u, v) \neq (0, 0)$ and (2.4) with $(d_1, d_2) = \gamma(t)$, such that the following holds.*

1. *The closure $\overline{\mathfrak{C}}_0$ in $I \times \mathbb{H}$ contains a point from $(t_-, t_+) \times \{0\}$.*
2. *$\mathfrak{C}_0$ is unbounded, or $\overline{\mathfrak{C}}_0$ contains a point of the form $(s, (u, v))$ with either $s \in \partial I$ (boundary understood in $\mathbb{R}$) and $(u, v) \neq 0$ or with $s \notin [t_-, t_+]$ and $u = v = 0$.*

Actually, we will see that both results hold even for more general problems (Remarks 7.2 and 7.1).

Theorem 2.2 implies in particular, that the bifurcation of Theorem 2.1 is global in a sense along every path $\gamma$ connecting $U_-$ with $U_+$.

Theorems 2.1 and 2.2 apply to a large class of cones $K_0$. In fact, in the subsequent examples, the hypothesis (2.6) of Theorem 2.1 follows from the fact that eigenfunctions of $\Delta$ and their traces are uniformly bounded. Hence, only the existence of a function $u_0$ satisfying (2.7) needs some discussion.

*Example* 2.1. The hypotheses of Theorem 2.1 with $e(x) \equiv 1$ are satisfied for the cone (2.3), corresponding to the situation described in the introduction, if only $\operatorname{mes}_{N-1}\Gamma > 0$. Indeed, the condition (2.7) is fulfilled by any function $u_0 = u_1 - u_2$ with $u_1, u_2 \in \mathbb{H}_0$, $u_k(\Omega) \subseteq [0, 1]$, $u_1|_\Gamma = 1$, $u_2|_\Gamma = 0$, $\int_\Omega u_1\,dx = \int_\Omega u_2\,dx$, if the suppports of $u_1$, $u_2$ are sufficiently small.

In particular, the conclusion of Theorem 2.1 holds for weak solutions of (1.9), (1.2). Hence, Theorem 1.1 is a special case of Theorem 2.1.

*Example* 2.2. Let us consider finitely many pairwise disjoint sets $\Gamma_1, \dots, \Gamma_n \subseteq \partial\Omega$ with $\operatorname{mes}_{N-1}\Gamma_k > 0$ for all $k$ and the cone

$$K_0 := \{v \in \mathbb{H}_0 : \int_{\Gamma_k} v\,dx \geq 0 \text{ for all } k = 1, \dots, n\}.$$

In this case, the variational inequality (2.4) corresponds to weak solutions of (1.9) with the unilateral boundary conditions of integral type

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } (\partial\Omega) \setminus \bigcup_{k=1}^n \Gamma_k,$$

$$\int_{\Gamma_k} v\,dx \geq 0,\ \frac{\partial v}{\partial n} \equiv \mathit{const} \geq 0,\ \frac{\partial v}{\partial n} \cdot \int_{\Gamma_k} v\,dx = 0 \quad \text{on } \Gamma_k,$$

see e.g. [7, Observation 5.2]. The hypotheses of Theorem 2.1 are satisfied automatically. One can choose the same function $u_0$ as in Example 2.1 corresponding to $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$.

*Example* 2.3. We consider now a set $\Omega_0 \subseteq \Omega$, $\operatorname{mes}_N \Omega_0 > 0$, and the corresponding cone

$$K_0 := \{v \in \mathbb{H}_0 : v|_{\Omega_0} \geq 0\}.$$

For this choice of $K_0$, the variational inequality (2.4) corresponds to weak solutions of the problem

$$\begin{aligned} d_1\Delta u - b_{11}u - b_{12}v - f_1(u,v) = 0 \quad &\text{on } \Omega, \\ d_2\Delta v - b_{21}u - b_{22}v - f_2(u,v) = 0 \quad &\text{on } \Omega \setminus \Omega_0, \\ d_2\Delta v - b_{21}u - b_{22}v - f_2(u,v) \leq 0 \leq v \quad &\text{on } \Omega_0, \\ (d_2\Delta v - b_{21}u - b_{22}v - f_2(u,v))v = 0 \quad &\text{on } \Omega_0 \end{aligned}$$

with Neumann boundary conditions (1.3). Thus, roughly speaking, we require now unilateral conditions in the interior set $\Omega_0$. Assume that $\overline{\Omega_0} \subseteq \Omega$ and

$$0 < \operatorname{mes}_N \Omega_0 \leq \operatorname{mes}_N \overline{\Omega_0} < \frac{1}{2}\Big(\frac{|B|}{b_{12}b_{21}}\Big)^2 \operatorname{mes}_N \Omega. \tag{2.9}$$

Then the hypotheses of Theorem 2.1 are satisfied. To construct the required function $u_0$, let us realize that $\frac{|B|}{-b_{12}b_{21}} < 1$. Consider a closed set $\Omega_1 \subseteq \Omega \setminus \overline{\Omega_0}$ with $\operatorname{mes}_N \Omega_1 = \operatorname{mes}_N \overline{\Omega_0}$ and fix for sufficiently small $\varepsilon > 0$ a function $u \in \mathbb{H}_0$ whose support lies in a sufficiently small neighborhood of $\Omega_1 \cup \overline{\Omega_0}$ and which satisfies $|u(x)| \leq 1 + \varepsilon$ on $\Omega$, $u|_{\Omega_0} = 1 + \varepsilon$, and $u|_{\Omega_1} = -(1+\varepsilon)$. We can assume $\left|\int_\Omega u\,dx\right| < \varepsilon \operatorname{mes}_N \Omega$, and then $u_0 := u - \int_\Omega u\,dx / \operatorname{mes}_N \Omega \in e + K_0$ satisfies (2.7).

*Example* 2.4. We can similarly consider unilateral conditions of integral type on disjoint sets $\Omega_1, \ldots, \Omega_n \subseteq \Omega$ by considering the cone

$$K_0 := \{v \in \mathbb{H}_0 : \int_{\Omega_k} v\,dx \geq 0 \text{ for all } k = 1, \ldots, n\}.$$

In this case, the hypotheses of Theorem 2.1 are satisfied if (2.9) holds for $\Omega_0 = \bigcup_{k=1}^n \Omega_k$.

*Example* 2.5. It is of course also possible to combine the previous examples and e.g. consider a cone like

$$K_0 := \Big\{v \in \mathbb{H}_0 : v|_\Gamma \geq 0,\, v|_{\Omega_0} \geq 0,$$
$$\int_{\Gamma_j} v\,dx \geq 0 \text{ for } j = 1, \ldots, n,\, \int_{\Omega_k} v\,dx \geq 0 \text{ for } k = 1, \ldots, m\Big\}.$$

In this case, the hypotheses of Theorem 2.1 are satisfied if at least one of the (disjoint) sets $\Gamma, \Gamma_j, \Omega_0, \Omega_k$ has positive measure, $\overline{\Omega_k} \subseteq \Omega$ for all $k$ (including $k = 0$) and if the measure of the union of these sets $\overline{\Omega_k}$ is strictly less than

$$\frac{1}{2}\Big(\frac{|B|}{b_{12}b_{21}}\Big)^2 \operatorname{mes}_N \Omega.$$

*Example* 2.6. All above examples hold in the same manner when we reverse all inequalities in the unilateral conditions. In this case, we just have to choose the cone $-K_0$ instead of $K_0$ and consider $e(x) = -1$ instead of $e(x) \equiv 1$ in (2.5) (and invert the sign of the constructed function $u_0$ required for Theorem 2.1).

However, it is not possible by our approach to invert only *some* but not *all* inequalities in the unilateral conditions (i.e. if we have unilateral conditions acting in opposite directions simultaneously): In this case, the first two hypotheses of Theorem 2.1 are not satisfied.

## 3. Functional Analytic Setting

3.1. **Considered Operators and their Basic Properties.** Throughout this paper, we assume that $b_{ij}$ are constants satisfying (1.4). We consider the usual Sobolev space $\mathbb{H}_0 := W^{1,2}(\Omega)$ with the scalar product

$$\langle u, v\rangle := \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega u(x)v(x)\,dx,$$

and the corresponding norm $\|\cdot\|$, and put $\mathbb{H} := \mathbb{H}_0\times\mathbb{H}_0$.

We define $A_0\colon \mathbb{H}_0 \to \mathbb{H}_0$ by the duality formula

$$\langle A_0 u, \varphi\rangle := \int_\Omega u(x)\varphi(x)\,dx \quad \text{for all } u,\varphi\in\mathbb{H}_0$$

and we define $e$ by (2.5) (the sign being fixed). We always assume that the functions $f_k\colon \mathbb{R}^2\to\mathbb{R}$ $(k=1,2)$ are continuous and satisfy (1.6). We define operators $F_k\colon \mathbb{R}^2_+\times\mathbb{H}\to\mathbb{H}_0$ $(k=1,2)$ and $F\colon \mathbb{R}^2_+\times\mathbb{H}\to\mathbb{H}$ by the duality

$$\langle F_k(d_1,d_2,u,v),\varphi\rangle := \int_\Omega d_k^{-1} f_k(u(x),v(x))\varphi(x)\,dx \quad \text{for all } \varphi\in\mathbb{H}_0,$$

and $F=(F_1,F_2)$, respectively.

**Proposition 3.1.** *The operator $A_0\colon \mathbb{H}\to\mathbb{H}$ is compact, symmetric and positive. $F_k$ and $F$ are well-defined, continuous and compact in the sense that for compact $D\subseteq\mathbb{R}^2_+$ and bounded $M\subseteq\mathbb{H}$ the images $F_k(D\times M)$ $(k=1,2)$ and $F(D\times M)$ are precompact. Moreover, if* (1.7) *holds, then we have for each $\widetilde{d}\in\mathbb{R}^2_+$*

$$\lim_{\substack{(d,U)\to(\widetilde{d},0)\\ U\neq 0}} \frac{F(d,U)}{\|U\|} = 0. \tag{3.1}$$

*Proof.* See e.g. [8, Proposition 3.2] or [26]. □

It follows that $A_0$ has a sequence of eigenvalues $\lambda_0\ge\lambda_1\ge\cdots>0$ (counting with multiplicities) and a corresponding system of eigenfunctions $(e_0,e_1,\dots)$ forming an orthonormal base of $\mathbb{H}_0$. Let us set

$$\kappa_n := \frac{1}{\lambda_n} - 1 \ge 0, \quad \text{i.e.} \quad \lambda_n = \frac{1}{1+\kappa_n} \qquad (n=0,1,2,\dots). \tag{3.2}$$

**Proposition 3.2.** *The numbers $\kappa_n$ are the eigenvalues of $-\Delta$ (in the weak sense) with Neumann boundary conditions, and $e_n$ are corresponding eigenfunctions. In particular, $1=\lambda_0>\lambda_1$, and $e$ and $e_0$ differ only by a nonzero factor.*

*Proof.* Note that $A_0u=\lambda u$ means that for all $\varphi\in\mathbb{H}_0$ we have

$$\int_\Omega u(x)\varphi(x)\,dx = \langle A_0u,\varphi\rangle = \langle\lambda u,\varphi\rangle = \int_\Omega \lambda\nabla u(x)\cdot\nabla\varphi(x)\,dx + \int_\Omega \lambda u(x)\varphi(x)\,dx.$$

This is just (2.1) with $\kappa_n=(1-\lambda)/\lambda=\lambda^{-1}-1$, i.e. $\lambda>0$ is an eigenvalue of $A_0$ (with corresponding eigenfunction $u$) if and only if $\lambda=\frac{1}{\mu+\kappa_n}=\lambda_n$ for some $n\in\{0,1,\dots\}$. □

For $d=(d_1,d_2)\in\mathbb{R}^2_+$, we define a linear operator $A(d)\colon\mathbb{H}\to\mathbb{H}$ by

$$A(d)\begin{pmatrix}u\\ v\end{pmatrix} := \begin{pmatrix}\frac{b_{11}+d_1}{d_1}A_0u + \frac{b_{12}}{d_1}A_0v\\ \frac{b_{21}}{d_2}A_0u + \frac{b_{22}+d_2}{d_2}A_0v\end{pmatrix}. \tag{3.3}$$

Proposition 3.1 implies that $A\colon \mathbb{R}^2_+ \times \mathbb{H} \to \mathbb{H}$ is compact in the sense that for compact $D \subseteq \mathbb{R}^2_+$ and bounded $M \subseteq \mathbb{H}$ the image $A(D \times M)$ is precompact.

Also, we assume that $K_0 \subseteq \mathbb{H}_0$ is some closed convex cone with its vertex in 0 (i.e. $0 \in K_0 + K_0 \subseteq K_0$). We denote by $P_{K_0}$ the canonical projection onto $K_0$, i.e. $P_{K_0}u$ is the unique element of $K_0$ with closest distance to $u$. It is well-known that $P_{K_0}$ is a well-defined continuous positively homogeneous operator, and that $v = P_{K_0}u$ is characterized by the variational inequality

$$v \in K_0, \qquad \langle v - u, \varphi - v\rangle \ge 0 \quad \text{for all } \varphi \in K_0,$$

see e.g. [13, Section 1.2]. We associate to $K_0$ the cone

$$K := \mathbb{H}_0 \times K_0 \subseteq \mathbb{H},$$

and let $P_K$ denote the canonical projection onto $K$; then $P_K(u, v) = (u, P_{K_0}v)$.

**Observation 3.1.** *For $d = (d_1, d_2) \in \mathbb{R}^2_+$, the couple $U = (u, v) \in \mathbb{H}$ is a weak solution of* (1.9), (1.3) *if and only if*

$$U = A(d)U + F(d, U),$$

*and a solution of* (2.4) *if and only if*

$$U = P_K\big(A(d)U + F(d, U)\big). \tag{3.4}$$

*In particular, $U = (u, v)$ is a weak solution of* (1.9), (1.2) *if and only if the equality* (3.4) *holds with the cone* (2.3).

*Proof.* The first claim follows by just inserting the definitions into (2.2). For the second claim observe in addition that (3.4) is equivalent to the variational inequality

$$U \in K, \qquad \langle U - \big(A(d)U + F(d, U)\big), \Phi - U\rangle \ge 0 \quad \text{for all } \Phi \in K$$

which is equivalent to (2.4). □

*Remark* 3.1. All results from here until Remark 6.2 hold also in the following more general situation: $\mathbb{H}_0$ is a real Hilbert space, $\mathbb{H} := \mathbb{H}_0 \times \mathbb{H}_0$, and $A_0\colon \mathbb{H}_0 \to \mathbb{H}_0$ is a compact positive symmetric linear operator with the simple largest eigenvalue $\lambda_0 = 1$. In this general setting, we let $\lambda_0 = 1 > \lambda_1 \ge \cdots > 0$ denote the eigenvalues with a corresponding orthonormal base of eigenfunctions $(e_0, e_1, \dots)$, and we define $\kappa_n$ by (3.2) (in particular, $0 = \kappa_0 < \kappa_1 \le \cdots$). In this abstract setting, we assume about $e \in \mathbb{H}_0$ that it is a nonzero multiple of $e_0$, i.e. an eigenvector to the eigenvalue $\lambda_0 = 1$. Also, in this abstract setting, we define $A$ by (3.3), and we assume that $F$ is any map with the properties described in Proposition 3.1. Finally, we assume that $K_0 \subseteq \mathbb{H}_0$ is a closed convex cone with vertex in 0, $K := \mathbb{H}_0 \times K_0$, and we let $P_K$ and $P_{K_0}$ be the corresponding projections.

The only change for this abstract setting will be that one has to replace the hypothesis (2.6) throughout by the condition

$$\text{for each } n = 1, 2, \dots \text{ there is } \delta_n > 0 \text{ with } \{e + \delta_n e_n, e - \delta_n e_n\} \subseteq K_0. \tag{3.5}$$

**Lemma 3.1.** *Suppose*

$$e \in K_0,\ -e \notin K_0, \text{ and there is } u_- \in K_0 \text{ with } \langle u_-, e\rangle < 0. \tag{3.6}$$

*Then $\alpha e = P_{K_0}(\beta e)$, $\alpha, \beta \in \mathbb{R}$, if and only if $\alpha = \beta \ge 0$.*

The assumption (3.6) means that $P_{K_0}(-e) \ne 0$, and is satisfied under the hypotheses of Theorem 2.1. Indeed, (2.6) implies $e \in K_0$, and if $u_0$ is from (2.7), $u_- := u_0 - e$ then $u_- \in K_0$ and $\langle u_-, e\rangle = -\operatorname{mes}\Omega < 0$.

*Proof.* The equation $\alpha e = P_K(\beta e)$ is equivalent to the variational inequality

$$\alpha e \in K_0, \quad \langle \alpha e - \beta e, \varphi - \alpha e \rangle \geq 0 \quad \text{for all } \varphi \in K_0.$$

Choosing $\varphi := \alpha e + u_- \in K_0 + K_0 \subseteq K_0$ with $u_-$ as in (3.6), we obtain $(\alpha - \beta) \langle e, u_- \rangle \geq 0$, and choosing $\varphi = e + \alpha e \in K_0 + K_0 \subseteq K_0$, we obtain $(\alpha - \beta) \|e\|^2 \geq 0$. Both together implies $\alpha = \beta$. Finally, since $e \in K_0 \setminus (-K_0)$, we have $\alpha e \in K_0$ if and only if $\alpha \geq 0$. □

We denote by

$$P_0 u := \frac{\langle u, e \rangle}{\|e\|^2} e$$

the orthogonal projection onto the subspace spanned by $e$. Using either a straightforward calculation or observing that $P_0$ is the spectral projection onto the eigenspace of $A_0$ to the eigenvalue 1, one sees that $P_0$ satisfies

$$P_0 A_0 = A_0 P_0 = P_0. \tag{3.7}$$

3.2. **The meaning of $C_n$.** The role of the hyperbolas (1.8) in our functional analytic framework is explained by the following observation, cf. [20] for the case $N = 1$.

**Proposition 3.3.** *For $d \in \mathbb{R}^2_+$, the equation $U = A(d)U$ has a solution $U \neq 0$ if and only if $d \in \bigcup_{n=1}^{\infty} C_n$.*

Recall that Observation 3.1 implies in particular that the solutions $U = (u, v)$ of $U = A(d)U$ are the weak solutions of (1.9), (1.3) with $f_1 = f_2 = 0$.

*Proof.* Since $(e_n)$ is an orthonormal basis, every $u \in \mathbb{H}_0$ can be written as a series $u = \sum_{n=0}^{\infty} \langle u, e_n \rangle e_n$. We thus have $U - A(d_1, d_2)U = 0$ with $U = (u, v) \in \mathbb{H}$ if and only if

$$\Big(1 - \frac{b_{11} + d_1}{d_1} \lambda_n\Big) \langle u, e_n \rangle - \frac{b_{12}}{d_1} \lambda_n \langle v, e_n \rangle = 0$$
$$\frac{-b_{21}}{d_2} \lambda_n \langle u, e_n \rangle + \Big(1 - \frac{b_{22} + d_2}{d_2} \lambda_n\Big) \langle v, e_n \rangle = 0$$

for all $n = 0, 1, \ldots$. We have $(u, v) \neq 0$ if and only if the above system has a nontrivial solution for some $n$, i.e. $A(d_1, d_2)U = U$ has a nontrivial solution if and only if for some $n$ the determinant of the above system vanishes, i.e. if and only if

$$\Big(1 - \frac{b_{11} + d_1}{d_1} \lambda_n\Big)\Big(1 - \frac{b_{22} + d_2}{d_2} \lambda_n\Big) = \frac{b_{12}}{d_1} \lambda_n \frac{b_{21}}{d_2} \lambda_n.$$

Multiplying by $\lambda_n^{-2} d_1 d_2 = (1 + \kappa_n)^2 d_1 d_2$, we see that this does not happen when $n = 0$ (since $|B| \neq 0$) and for $n \geq 1$, it means exactly $(d_1, d_2) \in C_n$. □

By deg, we will denote the classical Leray-Schauder degree (in the space $\mathbb{H}$ or $\mathbb{H}_0$). Moreover, for $r > 0$ and $U_0 \in \mathbb{H}$, we use the notation

$$B_r(U_0) := \{U \in \mathbb{H} : \|U - U_0\| < r\}.$$

**Corollary 3.1.** *For $d \in D_S$ and any $r > 0$, we have*

$$\deg\big(id - A(d), B_r(0), 0\big) = -1.$$

*Proof.* By Proposition 3.3, the above degree is defined for all $d = (d_1, d_2) \in \mathbb{R}^2_+ \setminus \bigcup_{n=1}^{\infty} C_n$, in particular, for all $d \in D_S$. By the homotopy invariance, the degree is independent of $d \in D_S$. Hence, without loss of generality, we can assume $d_1 > \frac{b_{11}}{\kappa_1}$ and $d_2 < -b_{22}$. Consider now the homotopy

$$H(t, \begin{pmatrix} u \\ v \end{pmatrix}) := \begin{pmatrix} \frac{b_{11}+d_1}{d_1} A_0 u + \frac{t b_{12}}{d_1} A_0 v \\ \frac{t b_{21}}{d_2} A_0 u + \frac{b_{22}+d_2}{d_2} A_0 v \end{pmatrix}.$$

Applying Proposition 3.3 for the case that $b_{12}$ and $b_{21}$ is replaced by $tb_{12}$ and $tb_{21}$, respectively, since $d$ lies in view of $d_1 > \frac{b_{11}}{\kappa_1}$ for any value of $t \in (0,1]$ in the corresponding zone (1.10) and thus not on any of the corresponding hyperbolas for $t \in (0,1]$, we find that $H(t,U) \neq U$ for $t \in (0,1]$. Moreover, since

$$\frac{1}{1+\kappa_1} < \frac{d_1}{b_{11}+d_1} < 1$$

implies that $\mu_1 := \frac{d_1}{b_{11}+d_1} \in (\lambda_1, \lambda_0)$, and since $\mu_2 := \frac{d_2}{b_{22}+d_2} < 0 = \inf_n \lambda_n$, the operators $id - \mu_1^{-1}A_0$ and $id - \mu_2^{-1}A_0$ are invertible, and thus also $id - H(0,\,\cdot\,)$ is invertible. Hence, if $M \subseteq \mathbb{H}_0$ denotes an open neighborhood of 0 satisfying $M \times M \subseteq B_r(0)$, the homotopy invariance, excision, and Cartesian product properties of the degree thus imply

$$\begin{aligned} \deg\bigl(id - A(d), B_r(0), 0\bigr) = \deg\bigl(id - H(0,\,\cdot\,), M \times M, 0\bigr) = \\ \deg\bigl(id - \mu_1^{-1}A_0, M, 0\bigr) \cdot \deg\bigl(id - \mu_2^{-1}A_0, M, 0\bigr). \end{aligned} \tag{3.8}$$

The Leray-Schauder index formula for a compact linear operator implies that $\deg\bigl(id - \mu_k^{-1}A_0, M, 0\bigr) = (-1)^{m_k}$ $(k = 1,2)$, where $m_k$ denotes the number (counted according to multiplicities) of the real eigenvalues of $\mu_k^{-1}A_0$ which are larger than 1, see e.g. [30, Proposition 14.5]. Since $\mu_2 < 0$, $\mu_1 \in (\lambda_1, \lambda_0)$, and the eigenvalues of $\mu_k^{-1}A_0$ are $\bigl\{\mu_k^{-1}\lambda_0, \mu_k^{-1}\lambda_1, \dots\bigr\}$, the operator $\mu_2^{-1}A_0$ has only negative eigenvalues while $\mu_1^{-1}A_0$ has exactly one eigenvalue which is larger than 1 (namely $\mu_1^{-1}\lambda_0$), and this eigenvalue has multiplicity 1. Hence, we have $m_1 = 1$ and $m_2 = 0$ which implies that the first factor in the product (3.8) is $-1$, and the second factor is 1. □

## 4. Some auxiliary results

The aim of this section will be to provide lemmas which allow to calculate the Leray-Schauder degree for a map associated to the family of systems

$$\begin{aligned} d_1u - \bigl((b_{11}+d_1)A_0u + b_{12}A_0v + h_1e\bigr) = 0, \\ d_2v - P_{K_0}\bigl(b_{21}A_0u + (b_{22}+d_2)A_0v + h_2e\bigr) = 0 \end{aligned} \tag{4.1}$$

with $d_1, d_2 \in \mathbb{R}^2_+$, $h_1, h_2 \in \mathbb{R}$. The terms $h_ke$ actually will help us to calculate the degree also with $h_1 = h_2 = 0$.

### 4.1. Particular Solutions of (4.1).

Consider for fixed $d_1, d_2 \in \mathbb{R}^2_+$, $h_1, h_2 \in \mathbb{R}$ besides (4.1) the same system without the operator $P_{K_0}$:

$$\begin{aligned} d_1u = (b_{11}+d_1)A_0u + b_{12}A_0v + h_1e, \\ d_2v = b_{21}A_0u + (b_{22}+d_2)A_0v + h_2e \in K_0. \end{aligned} \tag{4.2}$$

Note that we added in (4.2) the requirement $v \in K_0$. By that requirement, every solution $(u,v)$ of (4.2) is automatically a solution of (4.1). These are in a sense the simplest solutions of (4.1), and the following result characterizes these almost completely.

**Lemma 4.1.** *Suppose* (3.6) *holds. Let $d_1, d_2 \in \mathbb{R}^2_+$, $h_1, h_2 \in \mathbb{R}$ be fixed.*

(1) *If $(u,v)$ satisfies* (4.1) *but not* (4.2), *then $u$ and $v$ are not both multiples of $e$.*
(2) *If $d = (d_1,d_2) \notin \bigcup_{n=1}^{\infty} C_n$, then* (4.2) *has a solution if and only if*

$$b_{21}h_1 \geq b_{11}h_2. \tag{4.3}$$

*Then the solution $(u,v)$ is unique, $(u,v) = (\alpha e, \beta e)$ with some $\alpha, \beta \in \mathbb{R}$, and $\beta \neq 0$ if and only if the inequality in* (4.3) *is strict.*

*Proof.* If $(u, v) = (\alpha e, \beta e)$ are solutions of (4.1), then the second equation of (4.1) means

$$d_2\beta e = P_{K_0}\Big(\big(b_{21}\alpha + (b_{22} + d_2)\beta + h_2\big)e\Big),$$

and by Lemma 3.1 the expression after $P_{K_0}$ is in $K_0$ and therefore $P_{K_0}$ can be removed which means that (4.2) holds. For the second claim, we observe that the couple

$$u = \frac{b_{12}h_2 - b_{22}h_1}{|B|}e, \quad v = \frac{b_{21}h_1 - b_{11}h_2}{|B|}e \tag{4.4}$$

satisfies (4.2) if and only if $v \in K_0$; since $e \in K_0 \setminus (-K_0)$, the latter is the case if and only if (4.3) holds. If $(d_1, d_2) \notin \bigcup_{n=1}^{\infty} C_n$, then the solution of (4.2) (without the requirement $v \in K_0$) is unique by Proposition 3.3, and so there cannot be other solutions of (4.2) besides (4.4). □

4.2. **The Shadow System.** In order to calculate the degree for large $d_1$, we study first what happens for solutions of (4.1) when $d_1 \to \infty$. It will be more convenient to consider sequences of solutions and to consider the quantity $c_i = h_i/d_i$ instead of $h_i$. This leads us to the study of the system which occurs in the following lemma.

**Lemma 4.2.** *Suppose that $(u_n, v_n) \in \mathbb{H}$ is a bounded sequence of solutions of*

$$u_n = \frac{b_{11} + d_{1,n}}{d_{1,n}}A_0u_n + \frac{b_{12}}{d_{1,n}}A_0v_n + c_{1,n}e, \tag{4.5}$$

$$v_n = P_{K_0}\Big(\frac{b_{21}}{d_{2,n}}A_0u_n + \frac{b_{22} + d_{2,n}}{d_{2,n}}A_0v_n + c_{2,n}e\Big), \tag{4.6}$$

*$d_{1,n} \to \infty$, $d_{2,n} \to d_\infty \in (0, \infty]$, and $c_{1,n}, c_{2,n} \in \mathbb{R}$. Then $c_{1,n} \to 0$ and the sequence $c_{2,n}$ is bounded from above. If additionally $c_{2,n}$ is bounded, the sequence $(c_{2,n}, u_n, v_n)$ contains a convergent subsequence.*

*Proof.* Solving (4.5) for $c_{1,n}e$, we see that $c_{1,n}$ is bounded. Hence, there is a subsequence such that $c_{1,n_k} \to \hat{c}_1$. However, passing to a further subsequence if necessary, we can assume that $A_0u_{n_k}$ converges in norm. Hence, (4.5) implies that also $u_{n_k} \to u$ for some $u \in \mathbb{H}_0$. Passing to the limit in (4.5), we find $u = A_0u + \hat{c}_1e$ and thus

$$\langle u, e\rangle = \langle A_0u + \hat{c}_1e, e\rangle = \langle u, A_0e\rangle + \hat{c}_1\langle e, e\rangle = \langle u, e\rangle + \hat{c}_1\|e\|^2.$$

This implies $\hat{c}_1 = 0$. Since this holds for every convergent subsequence, it follows that $c_{1,n} \to 0$.

The equality (4.6) means

$$v_n \in K_0, \quad \langle v_n - \Big(\frac{b_{21}}{d_{2,n}}A_0u_n + \frac{b_{22} + d_{2,n}}{d_{2,n}}A_0v_n + c_{2,n}e\Big), \varphi - v_n\rangle \geq 0 \quad \text{for all } \varphi \in K_0.$$

Choosing $\varphi := v_n + e \in K_0 + K_0 \subseteq K_0$, we obtain

$$c_{2,n}\|e\|^2 \leq \langle v_n - \Big(\frac{b_{21}}{d_{2,n}}A_0u_n + \frac{b_{22} + d_{2,n}}{d_{2,n}}A_0v_n\Big), e\rangle.$$

Hence, the sequence $c_{2,n}$ is automatically bounded from above. If it is also bounded from below, we find a subsequence such that $c_{2,n_k} \to c_2 \in \mathbb{R}$ and that $A_0u_{n_k}$ and $A_0v_{n_k}$ converge in norm. It follows from (4.5) and (4.6) that also $u_{n_k}$ and $v_{n_k}$ converge in norm. □

**Lemma 4.3** (Shadow system). *Suppose that* $(u_n, v_n) \in \mathbb{H}$ *is a sequence of solutions of* (4.5) *and* (4.6) *where* $d_{1,n} \to \infty$, $d_{2,n} \to d_\infty \in (0, \infty]$, *and* $(c_{2,n}, u_n, v_n) \to (c_2, u, v)$. *Then there is some* $C \in \mathbb{R}$ *with*

$$u = \frac{-b_{12}}{b_{11}} Ce, \tag{4.7}$$

$$v = P_{K_0}\left(\left(c_2 - \frac{b_{12}b_{21}}{b_{11}d_\infty}C\right)e + \frac{b_{22} + d_\infty}{d_\infty}A_0 v\right), \tag{4.8}$$

*the first and the second fraction being understood as* 0 *and* 1, *respectively, if* $d_\infty = \infty$. *Moreover,* $d_{1,n}c_{1,n} \to c_1$ *with*

$$C = \frac{\langle v, e\rangle}{\|e\|^2} + \frac{c_1}{b_{12}}. \tag{4.9}$$

Equation (4.7) means that $u$ is constant. However, in view of (4.9) it is more convenient for us to write the constant in the form (4.7).

For later calculation, we point out a slight unsymmetry in the notation which however will be convenient: We have $c_{2,n} \to c_2$ but $d_{1,n}c_{1,n} \to c_1$.

Actually, we can even rewrite (4.7)–(4.9) equivalently as the single equation

$$v = P_{K_0}\left(\left(\frac{b_{22} + d_\infty}{d_\infty}A_0 - \frac{b_{12}b_{21}}{b_{11}d_\infty}P_0\right)v + \left(c_2 - \frac{b_{21}}{b_{11}d_\infty}c_1\right)e\right) \tag{4.10}$$

in the sense that if $(u, v, C)$ is a solution of (4.7)–(4.9), then $v$ satisfies (4.10), and conversely if $v$ satisfies (4.10) and we calculate $u$ and $C$ by (4.7) and (4.9), then $(u, v, C)$ satisfy (4.7)–(4.9).

The notion "shadow system" was used for a similar situation (in dimension $N = 1$ for the corresponding Neumann problem) as $d_2 \to \infty$ in [22] (see also [12]), and in [2] for the particular case $c_1 = c_2 = 0$ (if the nonlinearities in [2] vanish).

*Proof of Lemma* 4.3. Since $c_{1,n} \to 0$, passing to the limit in (4.5), we obtain that $u = A_0 u$. Hence, $u$ is an eigenvector of $A_0$ to the eigenvalue 1 and thus (4.7) holds with some $C \in \mathbb{R}$. Moreover, applying $P_0$ on both sides of (4.5), we find by (3.7) that

$$d_{1,n}P_0 u_n = (b_{11} + d_{1,n})P_0 u_n + b_{12}P_0 v_n + d_{1,n}c_{1,n}e.$$

Hence, using (4.7),

$$-d_{1,n}c_{1,n}e = b_{11}P_0 u_n + b_{12}P_0 v_n \to b_{11}P_0 u + b_{12}P_0 v = b_{11}u + b_{12}P_0 v.$$

This shows that $d_{1,n}c_{1,n} \to c_1$ for some $c_1$, and moreover $-c_1 e = b_{11}u + b_{12}P_0 v$ which by the definition of $P_0 v$ and (4.7) means (4.9). Finally, (4.8) is obtained by passing to the limit in (4.6) and inserting (4.7). □

**Lemma 4.4** (Properties of the shadow system). *Assume* (3.6), *and that* $(u, v) \in \mathbb{H}_0 \times \mathbb{H}_0$ *are solutions of* (4.7), (4.8) *with some* $d_\infty \in (0, \infty]$, $c_1, c_2 \in \mathbb{R}$ *and* $C \in \mathbb{R}$ *from* (4.9).

(1) *If* $v = \alpha e$, $\alpha \in \mathbb{R}$, *then* $\alpha = C - b_{12}^{-1}c_1 \geq 0$ *and*

$$\frac{|B|}{b_{11}d_\infty}C + c_2 - \frac{b_{22}}{b_{12}d_\infty}c_1 = 0. \tag{4.11}$$

*In case* $d_\infty = \infty$, *we understand* (4.11) *as* $c_2 = 0$. *In case* $d_\infty < \infty$, *we have*

$$c_2 \leq \frac{b_{21}}{b_{11}d_\infty}c_1, \tag{4.12}$$

*the inequality* (4.12) *being strict if and only if* $\alpha > 0$.

(2) *If $v \neq \alpha e$ for all $\alpha \in \mathbb{R}$ and $d_\infty = \infty$ then*

$$c_2 \langle v, e \rangle > 0. \tag{4.13}$$

*Proof.* Let $v = \alpha e$, $\alpha \in \mathbb{R}$. By (4.9), we have then $\alpha = C - b_{12}^{-1} c_1$, and (4.8) means

$$\alpha e = P_{K_0}\bigg(\Big(c_2 - \frac{b_{12}b_{21}}{b_{11}d_\infty}C\Big)e + \frac{b_{22}+d_\infty}{d_\infty}\alpha e\bigg).$$

By Lemma 3.1 and the form of $\alpha$ this is equivalent to $\alpha \geq 0$ and (4.11). Inserting the inequality $\alpha \geq 0$ into (4.11), we obtain (4.12) with equality if and only if $\alpha = 0$.

Assume now that $v \neq \alpha e$ for all $\alpha \in \mathbb{R}$ and $d_\infty = \infty$. Then the equation (4.8) is equivalent to the variational inequality

$$v \in K_0, \quad \langle v - (c_2 e + A_0 v), \varphi - v\rangle \geq 0 \quad \text{for all } \varphi \in K_0.$$

For the choice $\varphi = 0 \in K_0$, this implies

$$\|v\|^2 \leq c_2 \langle v, e\rangle + \langle A_0 v, v\rangle .$$

Since $A_0$ is selfadjoint and compact with the largest eigenvalue 1 and corresponding eigenfunction $e$, and since $v \neq \alpha e$ for all $\alpha \in \mathbb{R}$, we have $\langle A_0 v, v\rangle < \|v\|^2$. Hence, (4.13) must hold. □

4.3. **Solutions of** (4.1)**.** In the previous section we have shown that the solutions of (4.1) converge (as $d_1 \to \infty$) in a sense to solutions of the shadow system. Now we want to make some observations about these solutions for large $d_1$ *without* referring to the shadow system. Later, we will combine both observations.

To study (4.1) for large $d_1$, we will frequently use that we are able to reduce the system (4.1) to a single operator equation if $d_1 > b_{11}/\kappa_1$ by the following result.

**Lemma 4.5.** *Let $d = (d_1, d_2) \in \mathbb{R}^2_+$ lie on no vertical asymptote of the hyperbolas $C_1, C_2, \dots$, i.e.*

$$d_1 \notin \Big\{\frac{b_{11}}{\kappa_1}, \frac{b_{11}}{\kappa_2}, \dots\Big\}. \tag{4.14}$$

*Then for any map $P\colon \mathbb{H}_0 \to \mathbb{H}_0$ and any $h_1, h_2 \in \mathbb{R}$ the system*

$$d_1 u = (b_{11} + d_1)A_0 u + b_{12} A_0 v + h_1 e, \tag{4.15}$$

$$v = P\Big(\frac{b_{21}}{d_2}A_0 u + \frac{b_{22}+d_2}{d_2}A_0 v + \frac{h_2}{d_2}e\Big), \tag{4.16}$$

*is equivalent to the system*

$$u = \big(d_1 - (b_{11}+d_1)A_0\big)^{-1}(b_{12}A_0 v + h_1 e) \tag{4.17}$$

$$v = P\big(f_d(A_0)v + he\big), \tag{4.18}$$

*where $f_d(A_0)$ is understood in the sense of symmetric operator calculus with the function*

$$f_d(\lambda) := \frac{b_{12}b_{21}}{d_2} \cdot \frac{\lambda^2}{d_1 - (b_{11}+d_1)\lambda} + \frac{b_{22}+d_2}{d_2}\lambda,$$

*and*

$$h := \frac{h_2}{d_2} - \frac{b_{21}h_1}{b_{11}d_2}. \tag{4.19}$$

In particular, since $P = P_{K_0}$ and $P = id$ are positively homogeneous, it follows under the hypothesis (4.14) that the system (4.1) is equivalent to (4.17), (4.18) with $P = P_{K_0}$, and that the system (4.2) is for $v \in K_0$ equivalent to (4.17), (4.18) with $P = id$.

*Proof.* The inverse in (4.17) exists if and only if $d_1/(b_{11}+d_1) \notin \{\lambda_0, \lambda_1, \dots\}$ which in view of $d_1/(b_{11}+d_1) < 1 = \lambda_0$ and (3.2) means (4.14). Now (4.17) is just (4.15), solved for $u$; inserting this into (4.16) and observing that $e$ is an eigenvector of $A_0$ to the eigenvalue 1 and thus an eigenvector of $b_{21}A_0(d_1-(b_{11}+d_1)A_0)^{-1}$ to the eigenvalue $b_{21}(d_1-(b_{11}+d_1))^{-1} = -b_{21}/b_{11}$, we obtain (4.18). □

**Corollary 4.1.** *Let $d=(d_1,d_2)\in\mathbb{R}^2_+$ satisfy* (4.14)*, and $(u,v)\in\mathbb{H}$ be a solution of* (4.1)*. If $v=\alpha e$, $\alpha\in\mathbb{R}$, then also $u=\beta e$ with some $\beta\in\mathbb{R}$. Moreover, if additionally* (3.6) *holds, then $(u,v)$ is a solution of* (4.2)*.*

*Proof.* Apply Lemma 4.5 with $P=P_{K_0}$. Since $v=\alpha e$ is an eigenvector of $A_0$ (to the eigenvalue $\lambda_0$) and thus also an eigenvector of $(d_1-(b_{11}+d_1)A_0)^{-1}$ (to the eigenvalue $(d_1-(b_{11}+d_1)\lambda_0)^{-1}$), we conclude from (4.17) that $u=\beta e$ with some $\beta\in\mathbb{R}$. For the second claim, we observe that $A_0u=\beta e$ and $A_0 v=\alpha e$, i.e. we know that $v$ and the argument of $P_{K_0}$ in (4.1) are both multiples of $e$. Hence, it follows by using Lemma 3.1 that the second equation in (4.1) is equivalent to the second equation in (4.2). □

**Lemma 4.6.** *The function $f_d$ of Lemma* 4.5 *satisfies for any $v\in\mathbb{H}_0$*

$$\langle (id - f_d(A_0))v, e\rangle = \frac{-|B|}{b_{11}d_2}\langle v,e\rangle, \tag{4.20}$$

$$\langle (id - f_d(A_0))v, v-P_0v\rangle \ge 0 \quad \text{if } d_1 > \frac{b_{11}}{\kappa_1} \tag{4.21}$$

*with strict inequality in* (4.21) *if $v$ is not a multiple of $e$.*

*Proof.* Since $A_0e=e$, we have $e-f_d(A_0)e=(1-f_d(1))e$. Hence,

$$\langle (id - f_d(A_0))v, e\rangle = \langle v, (id-f_d(A_0))e\rangle = (1-f_d(1))\langle v,e\rangle,$$

which is (4.20). Note that $\overline{P}_0 := id - P_0$ is the spectral projection of $A_0$ corresponding to the complement of $\{\lambda_0\}=\{1\}$. An elementary calculation shows that $1-f_d$ is positive on this set if $d_1\kappa_1 > b_{11}$. Hence, the symmetric operator $id-f_d(A_0)$ is positive on the range of $\overline{P}_0$. Since the spectral projection $\overline{P}_0$ is symmetric and commutes with $A_0$ and thus with $id-f_d(A_0)$, we obtain

$$\langle (id-f_d(A_0))v, \overline{P}_0 v\rangle = \langle (id-f_d(A_0))v, \overline{P}_0^2 v\rangle = \langle (id-f_d(A_0))\overline{P}_0 v, \overline{P}_0 v\rangle \ge 0.$$

Moreover, the inequality is strict unless $\overline{P}_0 v = 0$, which means $v=\alpha e$ for some $\alpha\in\mathbb{R}$. □

**Lemma 4.7.** *Suppose $e\in K_0$. Let $d=(d_1,d_2)\in\mathbb{R}^2_+$ satisfy $d_1>\frac{b_{11}}{\kappa_1}$, and $(u,v)\in\mathbb{H}$ be a solution of* (4.1)*. Then*

$$b_{21}h_1 - b_{11}h_2 \ge \frac{\langle v,e\rangle}{\|e\|^2}|B|. \tag{4.22}$$

*Moreover, unless $v=\alpha e$, $\alpha\in\mathbb{R}$, the inequality in* (4.22) *is strict, and we have*

$$\langle v,e\rangle < 0. \tag{4.23}$$

*Proof.* With the notation of Lemma 4.5, we have (4.18) with $P=P_{K_0}$. Writing out the inequality characterizing this projection, we see that (4.18) is equivalent to

$$v\in K_0, \qquad \langle v-(f_d(A_0)v+he), \varphi - v\rangle \ge 0 \quad \text{for all } \varphi\in K_0. \tag{4.24}$$

Using the test function $\varphi := v+e \in K_0+K_0\subseteq K_0$, we thus obtain by (4.20) that

$$0 \le \langle (id-f_d(A_0))v - he, e\rangle = \frac{-|B|}{b_{11}d_2}\langle v,e\rangle - h\|e\|^2. \tag{4.25}$$

Inserting the definition (4.19), we obtain (4.22). Moreover, using the test function $\varphi = 0 \in K_0$ in (4.24), we obtain by (4.21), (4.20), and the definition of $P_0$, that

$$\begin{aligned} 0 &\le \langle (id - f_d(A_0))v, -(v - P_0v) - P_0v\rangle + h\,\langle e, v\rangle \\ &\le \langle (id - f_d(A_0))v, -P_0v\rangle + h\,\langle e, v\rangle = \frac{-|B|}{b_{11}d_2}\langle v, -P_0v\rangle + h\,\langle e, v\rangle \\ &= \frac{-|B|}{b_{11}d_2}\langle v, e\rangle \frac{\langle -v, e\rangle}{\|e\|^2} + h\,\langle v, e\rangle = \frac{1}{\|e\|^2}\Big(\frac{|B|}{b_{11}d_2}\langle v, e\rangle + h\,\|e\|^2\Big)\langle v, e\rangle\,, \end{aligned}$$

where the first inequality is strict unless $v$ is a multiple of $e$. Note now that the first factor is non-positive by (4.25). Hence, if $v$ is not a multiple of $e$, both factors must be strictly negative which means that the inequality in (4.22) is strict and (4.23) holds. □

**Proposition 4.1.** *Suppose* (3.6) *holds. Let* $(u_n, v_n)$ *satisfy* (4.5), (4.6) *where* $d_{1,n} \to \infty$, $d_{2,n} \to d_\infty = \infty$. *If the norm of* $(u_n, v_n)$ *is bounded and* $\liminf_{n\to\infty} c_{2,n} \ge 0$, *then* $c_{1,n} \to 0$, $c_{2,n} \to c_2 = 0$, *and a subsequence of* $(d_{1,n}c_{1,n}, u_n, v_n)$ *converges to some* $(c_1, u, v)$ *satisfying* (4.7)–(4.9). *Moreover,* $v = \alpha e$ *with* $\alpha = C - b_{12}^{-1}c_1 \ge 0$ *with* $C$ *from* (4.9).

*Proof.* By Lemma 4.2 we have $c_{1,n} \to 0$ and can assume, passing to a subsequence if necessary, that $c_{2,n} \to c_2 \ge 0$, and $(u_n, v_n) \to (u, v)$. By Lemma 4.3, we have $d_{1,n}c_{1,n} \to c_1 \in \mathbb{R}$, and (4.7)–(4.9) holds. If $v \ne \alpha e$ for all $\alpha \in \mathbb{R}$, then Lemma 4.4 implies (4.13), and so $c_2 > 0$ and $\langle v, e\rangle > 0$. This implies $\langle v_n, e\rangle > 0$ for large $n$ which by Lemma 4.7 means that $v_n = \alpha_n e$ for some $\alpha_n \in \mathbb{R}$. Hence, $v_n \to v$ implies that $v = \alpha e$ for some $\alpha \in \mathbb{R}$, contradicting our assumption. Thus, $v = \alpha e$ for some $\alpha \in \mathbb{R}$, and Lemma 4.4(1) implies in view of $d_\infty = \infty$ that $c_2 = 0$ and $\alpha = C - b_{12}^{-1}c_1 \ge 0$. Since the whole argument can be repeated with any subsequence, we find that actually $c_{2,n} \to 0$. □

The following result will be our crucial tool to prove that the degree of a related map vanishes if $d_1$ and $d_2$ are large.

**Theorem 4.1.** *Assume* (3.6). *Then for every* $C_0 \ge 0$ *there are* $\omega_1, \omega_2 > 0$ *such that for all* $h_1, h_2 \in \mathbb{R}$ *and all* $(d_1, d_2) \in \mathbb{R}^2_+$ *satisfying* $d_1 \ge \omega_1$, $d_2 \ge \omega_2$, *and*

$$h_2 \ge 0, \quad |d_2h_1| \le C_0h_2, \tag{4.26}$$

*the problems* (4.1) *and* (4.2) *have exactly the same solutions.*

*Proof.* Assume by contradiction that there are sequences $h_{1,n}, h_{2,n} \in \mathbb{R}$ and $d_{1,n}, d_{2,n} \in \mathbb{R}^2_+$ with $d_{1,n}, d_{2,n} \ge n$ and (4.26) such that for each $n$ there are solutions $U_n = (\widetilde{u}_n, \widetilde{v}_n) \in \mathbb{H}$ of the corresponding problem (4.1) which are not simultaneously solutions of the corresponding problem (4.2). By Lemma 4.1(1), we have $U_n \ne 0$. Hence, $(u_n, v_n) := U_n / \|U_n\|$ are solutions of (4.5), (4.6) with

$$c_{i,n} := \frac{h_{i,n}}{d_{i,n}\,\|U_n\|} \qquad (i = 1, 2).$$

Note that $c_{2,n} \ge 0$ by (4.26). Passing to a subsequence, we can assume by Proposition 4.1 that $c_{1,n} \to 0$, $c_{2,n} \to c_2 = 0$, $(u_n, v_n) \to (u, v)$, and $d_{1,n}c_{1,n} \to c_1$ such that (4.7)–(4.9) holds with $d_\infty = \infty$, and $v = \alpha e$ with $\alpha = C - b_{12}^{-1}c_1 \ge 0$. Since (4.26) shows that

$$|d_{1,n}c_{1,n}| \le C_0\,|c_{2,n}|\,,$$

and since $c_{2,n} \to 0$, we have $d_{1,n}c_{1,n} \to 0$ and thus $c_1 = 0$ which implies $C = \alpha$, hence $v = Ce$. By (4.7), we thus have either $u = v = 0$ or $v = Ce$ with $C > 0$. The former cannot happen, since $(u_n, v_n)$ are normed by construction and converge to $(u, v)$. Hence, $\langle v_n, e\rangle \to C\,\|e\|^2 > 0$. We conclude from (4.22) that there is some $\varepsilon > 0$ such that

$$b_{21}d_{1,n}c_{1,n} - b_{11}d_{2,n}c_{2,n} \geq \varepsilon$$

for $n$ large. This is a contradiction, because $d_{1,n}c_{1,n} \to c_1 = 0$ and $b_{11}d_{2,n}c_{2,n} \geq 0$. □

## 5. Degree Nonzero

Our approach for a result about nonzero degree consists of two steps. In the first step, we calculate the degree of a map *with* a right-hand side in a neighborhood of a certain zero of that map. The other step consists in showing (using the homotopy invariance and excision property of the degree) that these degrees coincide. The first step can be shown even for rather general operators, but for the second step we need a hypothesis which is surprisingly hard to verify and which we discuss later on.

This type of approach and also parts of the proof of the first step are inspired by the proof of [24, Theorem 5]. However, even for the first step (which corresponds to [24, ($\beta$) on p. 293]) we have a serious technical difficulty: The proof in [24] requires essentially the symmetry of the considered operator which we do not have in our case. As a substitute, we will use the symmetric operator $f_d(A_0)$ of Lemma 4.5. For this technical reason, we will assume that the hypothesis (4.14) of Lemma 4.5 is satisfied.

**Theorem 5.1.** *Assume* (3.6) *and* (2.6). *Suppose that* $h_1, h_2 \in \mathbb{R}$ *and* $d = (d_1, d_2) \in \mathbb{R}^2_+ \setminus \bigcup_{n=1}^{\infty} C_n$ *satisfy* (4.14) *and*

$$b_{21}h_1 > b_{11}h_2. \tag{5.1}$$

*Then* (4.2) *has a unique solution* $U_0$*, and for each* $t_0 \geq 0$ *there is* $r > 0$ *such that for all* $t \in [0, t_0]$ *the problem*

$$\begin{aligned} d_1 u &= (b_{11} + d_1)A_0 u + b_{12}A_0 v + h_1 e \\ d_2 v &= \big(tP_{K_0} + (1-t)id\big)\big(b_{21}A_0 u + (b_{22} + d_2)A_0 v + h_2 e\big) \end{aligned} \tag{5.2}$$

*has at most the solution* $U_0$ *in* $B_r(U_0)$*.*

*Proof.* The uniqueness and existence of the solution $U_0 = (u_0, v_0)$ of (4.2) is contained in Lemma 4.1(2). Moreover, Lemma 4.1 also implies in view of (5.1) that $v_0 = \alpha e$, $\alpha \neq 0$. We have $e \in K_0 \setminus (-K_0)$ and $v_0 \in K_0$, and therefore $\alpha > 0$. In particular, (2.6) implies that

$$\text{for every } n = 1, 2, \ldots \text{ there is } \delta_n > 0 \text{ with } \{v_0 - \delta_n e_n, v_0 + \delta_n e_n\} \subseteq K_0. \tag{5.3}$$

If for some $t_0 \geq 0$ there is no $r > 0$ with the required properties, we find a sequence $t_n \in [0, t_0]$ and a sequence $(u_n, v_n) \in \mathbb{H}$ with $(u_n, v_n) \neq U_0$, $\|(u_n, v_n) - U_0\| \to 0$, such that $(u_n, v_n)$ satisfies (5.2) with $t = t_n$. Applying Lemma 4.5 with $P = t_n P_{K_0} + (1 - t_n)id$, we find

$$u_n = \big(d_1 - (b_{11} + d_1)A_0\big)^{-1}(b_{12}A_0 v_n + h_1 e), \tag{5.4}$$

$$v_n = \big(t_n P_{K_0} + (1 - t_n)id\big)\big(f_d(A_0)v_n + he\big) \tag{5.5}$$

with $h$ from (4.19). Since $U_0 = (u_0, v_0)$ satisfies (4.2), we apply Lemma 4.5 also with $P = id$ and find

$$u_0 = \big(d_1 - (b_{11} + d_1)A_0\big)^{-1}\big(b_{12}A_0 v_0 + h_1 e\big), \tag{5.6}$$

$$v_0 = f_d(A_0)v_0 + he. \tag{5.7}$$

We must have $v_n \neq v_0$ for all $n$, since otherwise (5.4) and (5.6) would imply $(u_n, v_n) = (u_0, v_0) = U_0$, contradicting our choice of the sequence $(u_n, v_n)$. Using $v_n \neq v_0$, (5.5) and (5.7), we calculate

$$\frac{v_n - v_0}{\|v_n - v_0\|} = f_d(A_0)\frac{v_n - v_0}{\|v_n - v_0\|} + t_n \frac{P_{K_0} w_n - w_n}{\|v_n - v_0\|}, \tag{5.8}$$

where

$$w_n := f_d(A_0)v_n + he = v_0 + f_d(A_0)(v_n - v_0). \tag{5.9}$$

We will now show that the last term in (5.8) tends to 0 as $n \to \infty$. To this end, recall that the eigenvectors $e_0, e_1, \dots$ to the eigenvalues $\lambda_k = \frac{1}{1+\kappa_k}$ of $A_0$ form an orthonormal base of $\mathbb{H}_0$. Defining $\mu_{n,k} := \langle v_n - v_0, e_k\rangle$, i.e.

$$v_n - v_0 = \sum_{k=0}^{\infty} \mu_{n,k} e_k, \tag{5.10}$$

we have then due to (5.9)

$$w_n - v_0 = f_d(A_0)(v_n - v_0) = \sum_{k=0}^{\infty} f_d(\lambda_k)\mu_{n,k} e_k. \tag{5.11}$$

Since $\lambda_k \to 0$, the definition of $f_d$ implies $f_d(\lambda_k) \to 0$. Hence, for each $\varepsilon > 0$, we find some $k_\varepsilon$ such that $|f_d(\lambda_k)| \leq \varepsilon$ for all $k \geq k_\varepsilon$.

Now we use (5.3). We thus find some $\delta > 0$ such that $v_0 + \mu e_k = \alpha e + \mu e_k \in K_0$ whenever $|\mu| < \delta$ and $k < k_\varepsilon$. By Bessel's inequality, we have $|\mu_{n,k}|^2 \leq \|v_n - v_0\|^2$ for all $k$, and since $\|v_n - v_0\| \to 0$, we conclude that there is some $n_\varepsilon$ such that $|k_\varepsilon f_d(\lambda_k)\mu_{n,k}| < \delta$ for all $n \geq n_\varepsilon$ and all $k$. In particular, for $n \geq n_\varepsilon$ the vector

$$s_{n,k_\varepsilon} := v_0 + \sum_{k=0}^{k_\varepsilon - 1} f_d(\lambda_k)\mu_{n,k} e_k = \frac{1}{k_\varepsilon}\sum_{k=0}^{k_\varepsilon - 1} (v_0 + k_\varepsilon f_d(\lambda_k)\mu_{n,k} e_k)$$

is a convex combination of elements from $K_0$ and thus belongs to $K_0$. Since $P_{K_0} w_n$ is that element of $K_0$ with the closest distance to $w_n$, we conclude for $n \geq n_\varepsilon$, using (5.10), (5.11), and Parseval's identity, that

$$\|w_n - P_{K_0} w_n\|^2 \leq \|w_n - s_{n,k_\varepsilon}\|^2 = \Big\|\sum_{k=k_\varepsilon}^{\infty} f_d(\lambda_k)\mu_{n,k} e_k\Big\|^2 = \sum_{k=k_\varepsilon}^{\infty} |f_d(\lambda_k)\mu_{n,k}|^2$$
$$\leq \varepsilon^2 \sum_{k=0}^{\infty} |\mu_{n,k}|^2 = \varepsilon^2 \Big\|\sum_{k=0}^{\infty} \mu_{n,k} e_k\Big\|^2 = \varepsilon^2 \|v_n - v_0\|^2 .$$

Thus, we have seen that the last term in (5.8) tends to 0 as $n \to \infty$. Hence, it follows from (5.8) that 1 belongs to the spectrum of $f_d(A_0)$. By the spectral mapping theorem, we thus find some $\lambda \in \{\lambda_0, \lambda_1, \dots\}$ with $f_d(\lambda) = 1$. Hence, we have $f_d(1/(1+\kappa_n)) = 1$ for some $n \in \{0, 1, \dots\}$. However, elementary calculation shows that $f_d(1) = 1 + \frac{|B|}{b_{11}d_2} \neq 1$, and $f_d(1/(1+\kappa_n)) = 1$ for $n \geq 1$ if and only if $(d_1, d_2) \in C_n$ which contradicts our hypothesis. □

**Corollary 5.1.** *Assume* (3.6) *and* (2.6). *Suppose that* $d = (d_1, d_2) \in \mathbb{R}^2_+ \setminus \bigcup_{n=1}^{\infty} C_n$ *satisfy* (4.14), *and* $h_1, h_2 \in \mathbb{R}$ *satisfy* (5.1). *Then* (4.2) *has a unique solution* $U_0$, *and there is some* $r > 0$ *with*

$$\deg\Big(id - P_K\Big(A(d) + \begin{pmatrix} h_1 e \\ h_2 e\end{pmatrix}\Big), B_r(U_0), 0\Big) = \deg\big(id - A(d), B_r(0), 0\big) \in \{\pm 1\} .$$

*Proof.* From Theorem 5.1, we obtain that there is some $r > 0$ such that the homotopy

$$H(t,U) := U - (tP_K + (1-t)id)\Big(A(d)U - \begin{pmatrix} h_1 e \\ h_2 e \end{pmatrix}\Big)$$

satisfies $H(t,U) = 0$ for $(t,U) \in [0,1] \times \overline{B}_r(U_0)$ only if $U = U_0$. Hence, the homotopy invariance and topological invariance of the degree imply

$$\deg\big(H(1,\,\cdot\,), B_r(U_0), 0\big) = \deg\big(H(0,\,\cdot\,), B_r(U_0), 0\big) = \\ \deg\big(H(0,\,\cdot\,+U_0), B_r(0), 0\big) = \deg\big(id - A(d), B_r(0), 0\big).$$

Since $id - A(d)$ is linear (and the degree is defined, hence $id - A(d)$ is even an isomorphism), it follows that the degree is 1 or $-1$. □

Corollary 5.1 is the announced first step in the calculation of the degree. The second step is easily carried out if one makes an assumption about the nonexistence of nontrivial solutions of an auxiliary problem:

**Theorem 5.2.** *Assume* (3.6) *and* (2.6). *Let* $d = (d_1, d_2) \in \mathbb{R}^2_+ \setminus \bigcup_{n=1}^{\infty} C_n$ *satisfy* (4.14), *and let* $\alpha, \beta \in \mathbb{R}$ *satisfy*

$$b_{21}\alpha > b_{11}\beta. \tag{5.12}$$

*Suppose that there is some* $\varepsilon > 0$ *such that for all* $t \in [0,\varepsilon]$ *and* $h_1 := t\alpha$, $h_2 := t\beta$, *all solutions of the problem* (4.1) *satisfy* (4.2). *Then for each* $r > 0$ *we have*

$$\deg\big(id - P_K A(d), B_r(0), 0\big) = \deg\big(id - A(d), B_r(0), 0\big) \in \{\pm 1\}\,.$$

*Proof.* Problem (4.2) with $h_1 := t\alpha$, $h_2 := t\beta$ has for each $t \in [0,\varepsilon]$ a unique solution $U_0(t)$ by Corollary 5.1. Since by hypothesis there are no further solutions of (4.1), the homotopy invariance and excision property of the degree implies that

$$\deg\Big(id - P_K\Big(A(d) + \begin{pmatrix} t\alpha e \\ t\beta e \end{pmatrix}\Big), B_r(U_0(t)), 0\Big)$$

is independent of $t \in [0,\varepsilon]$ and of $r > 0$. Hence, the claim follows by applying Corollary 5.1 with $h_1 = \varepsilon\alpha$ and $h_2 = \varepsilon\beta$. □

We discuss in the next section how the hypothesis of Theorem 5.2 can be verified. That discussion will also give a new method to prove that the degree is 0 for certain $d \in \mathbb{R}^2_+$.

## 6. Degree Calculations based on the Shadow System

Note that (4.10) can be written as

$$v = P_{K_0}\Big(\Big(\frac{b_{22} + d_\infty}{d_\infty} A_0 - \frac{b_{12}b_{21}}{b_{11}d_\infty} P_0\Big) v + \lambda e\Big) \tag{6.1}$$

with

$$\lambda = c_2 - \frac{b_{21}}{b_{11}d_\infty} c_1. \tag{6.2}$$

One may ask whether this problem has a solution $v$ which is not a multiple of $e$. Since $P_{K_0}$, $A_0$, and $P_0$ are positively homogeneous, the answer to this question depends only on the sign of $\lambda$, i.e. we have only to distinguish the three cases $\lambda > 0$, $\lambda < 0$, and $\lambda = 0$.

In fact, only the cases $\lambda = 0$ and $\lambda < 0$ and the corresponding sets $\mathbb{E}_0$ and $\mathbb{E}_-$ introduced below are really used in the proof of Theorems 2.1 and 2.2. If one is interested only in these proofs, then everything related to $\mathbb{E}_+$ can be skipped. Therefore, the case $\mathbb{E}_+$ will always occur on the last place. The set $\mathbb{E}_+$ is discussed only because the corresponding assertions are of independent interest. For example, in the forthcoming paper [11], we will use the cases containing $\mathbb{E}_+$ in the subsequent Theorem 6.1 to obtain an explicit formula for the best possible constant $\omega_2$ of Theorem 1.1 in space dimension $N = 1$. We define

$$\mathbb{E}_0 := \{d_\infty \in (0,\infty) : \text{for } \lambda = 0 \text{ all solutions of (6.1) are multiples of } e\},$$
$$\mathbb{E}_- := \{d_\infty \in (0,\infty) : \text{for all } \lambda < 0 \text{ all solutions of (6.1) are multiples of } e\},$$
$$\mathbb{E}_+ := \{d_\infty \in (0,\infty) : \text{for all } \lambda > 0 \text{ all solutions of (6.1) are multiples of } e\}.$$

We point out that e.g. $d_\infty \in \mathbb{E}_+$ does not imply that (6.1) *has* a solution. In fact, using Lemma 3.1, one can show that if (3.6) holds and $\lambda > 0$ then (6.1) has no solution $v = \alpha e$.

We will discuss later in this section how to verify that $d_\infty \in (0,\infty)$ belongs to some of these sets. For the moment, we just make some trivial observations.

*Remark* 6.1. By the above observations, one could in the definition of $\mathbb{E}_-$ equivalently replace "for all $\lambda < 0$" by "for some $\lambda < 0$"; analogously for $\mathbb{E}_+$. Moreover, $d_\infty \in \mathbb{E}_-$ is by (6.2) equivalent to the fact that $v$ is a multiple of $e$ for any $(u,v,C)$ satisfying (4.7)–(4.9) with $c_1 = 0$ and $c_2 < 0$. Another equivalent characterization is that $v$ is a multiple of $e$ for any $(u,v,C)$ satisfying (4.7)–(4.9) with $c_2 = 0$ and $b_{12}c_1 < 0$ (recall that $b_{12}b_{21} < 0$ by (1.5)). Analogous equivalent characterizations hold for $\mathbb{E}_+$ and $\mathbb{E}_0$ (with opposite inequalities and with $c_1 = c_2 = 0$, respectively).

We use the following notation for a set $U \subseteq \mathbb{R}^2_+$:

$$U(\infty) := \{d_\infty \in [0,\infty] : \text{There are } (d_{1,n}, d_{2,n}) \in U \text{ with } d_{1,n} \to \infty,\ d_{2,n} \to d_\infty\}. \tag{6.3}$$

**Theorem 6.1.** *Suppose that* (3.6) *holds. Let* $U \subseteq \mathbb{R}^2_+$ *be fixed.*

(1) *If* $U(\infty) \setminus \{\infty\} \subseteq \mathbb{E}_0$ *then there is some* $\omega > 0$ *such that for each* $d = (d_1, d_2) \in U$ *with* $d_1 \geq \omega$ *the systems* (4.1) *and* (4.2) *have the same solutions if* $h_1 = h_2 = 0$.

(2) *If* $U(\infty) \subseteq \mathbb{E}_0 \cap \mathbb{E}_-$ *or* $U(\infty) \subseteq \mathbb{E}_0 \cap \mathbb{E}_+$, *then for each* $C_0 \geq 0$ *there is some* $\omega > 0$ *such that for each* $d = (d_1, d_2) \in U$ *with* $d_1 \geq \omega$ *the systems* (4.1) *and* (4.2) *have the same solutions if*

$$b_{12}h_1 \leq 0, \quad d_1 |h_2| \leq C_0 d_2 |h_1|, \tag{6.4}$$

*or*

$$b_{12}h_1 \geq 0, \quad d_1 |h_2| \leq C_0 d_2 |h_1|, \tag{6.5}$$

*respectively.*

(3) *If* $U(\infty) \setminus \{\infty\} \subseteq \mathbb{E}_0 \cap \mathbb{E}_+$ *then for each function* $g\colon (0,\infty) \to \mathbb{R}$ *satisfying*

$$\lim_{d_1 \to \infty} g(d_1) = 0 \tag{6.6}$$

*there is some* $\omega > 0$ *such that for each* $d = (d_1, d_2) \in U$ *with* $d_1 \geq \omega$ *the systems* (4.1) *and* (4.2) *have the same solutions if*

$$h_2 \geq 0, \quad |d_2 h_1| \leq |g(d_1) h_2|. \tag{6.7}$$

*Proof.* Assume by contradiction that for each $n$ there are $h_{1,n}, h_{2,n} \in \mathbb{R}$ $(d_{1,n}, d_{2,n}) \in U$ with $d_{1,n} \geq n$, without loss of generality $d_{1,n} > \frac{b_{11}}{\kappa_1}$, such that one of the three additional hypotheses hold and the corresponding problem (4.1) (with $(d_1, d_2, h_1, h_2)$ replaced by $(d_{1,n}, d_{2,n}, h_{1,n}, h_{2,n})$) has a solution $U_n := (\widetilde{u}_n, \widetilde{v}_n)$ which does not satisfy (4.2) (with $(d_1, d_2, h_1, h_2)$ replaced by $(d_{1,n}, d_{2,n}, h_{1,n}, h_{2,n})$). Lemma 4.1(1) implies $U_n \neq 0$.

Passing to a subsequence if necessary, we can assume that $d_{2,n} \to d_\infty \in (0,\infty]$, because $d_\infty = 0$ is excluded by $U(\infty) \setminus \{\infty\} \subseteq \mathbb{E}_0$. Theorem 4.1 excludes $d_\infty = \infty$ in the cases (1) and (3), because (4.26) is satisfied if $h_1 = h_2 = 0$ or if (6.6), (6.7) holds. In case (2), $d_\infty = \infty$ is excluded, because $U(\infty) \subseteq \mathbb{E}_0$ implies $\infty \notin U(\infty)$. Hence, in all cases, we only need to discuss $d_\infty \in (0,\infty)$.

Using that $A_0$ and $P_{K_0}$ are positively homogeneous, we have that $(u_n, v_n) := U_n/\|U_n\|$ are solutions of (4.5) and (4.6), where

$$c_{i,n} := \frac{h_{i,n}}{d_{i,n}\|U_n\|} \qquad (i = 1, 2).$$

Since $\|(u_n, v_n)\| = 1$ by construction, Lemma 4.2 implies that $c_{1,n} \to 0$. The relations (6.4) or (6.5) (with $h_i$ replaced by $h_{i,n}$) both imply

$$|c_{2,n}| \le C_0 |c_{1,n}|,$$

and so $c_{1,n} \to 0$ implies $c_{2,n} \to 0$ in case (2). In case (6.7), we have by hypothesis $c_{2,n} \ge 0$. Hence, in all cases $c_{2,n}$ is bounded from below. By Lemma 4.2 we conclude, passing to a subsequence if necessary, that $c_{2,n} \to c_2 \in [0, \infty)$ and that $(u_n, v_n) \to (u, v)$, in particular $\|(u, v)\| = 1$. Moreover, $c_2 = 0$ in the cases (1) or (2). In case (3), we have

$$|d_{1,n}c_{1,n}| \le g(d_{1,n})c_{2,n},$$

which implies by the boundedness of $c_{2,n}$ and (6.6) that $d_{1,n}c_{1,n} \to 0 = c_1$ in the notation of Lemma 4.3.

Summarizing, the hypotheses of Lemma 4.3 are satisfied, and in the case (1), we have $c_1 = c_2 = 0$, in the two cases of (2), we have $c_2 = 0$ and $b_{12}c_1 \le 0$ or $b_{12}c_1 \ge 0$, respectively, and in the case (3), we have $c_1 = 0 \le c_2$. In particular, it follows from Lemma 4.3 that $(u, v, C)$ satisfy (4.7)–(4.9). Since $d_\infty \in U(\infty)$, our hypothesis on $U(\infty)$ thus implies in view of Remark 6.1 in all cases that $v$ is a multiple of $e$.

By Lemma 4.4(1), we have $v = (C - b_{12}^{-1}c_1)e$. Moreover, if we had $c_1 = c_2 = 0$, then (4.11) would imply $C = 0$, and by using (4.7), we would get $u = v = 0$, contradicting $\|(u, v)\| = 1$. In particular, $(c_1, c_2) \ne (0, 0)$, and the case (1) leads to a contradiction.

In case (3), we must have $c_2 > 0 = c_1$ which contradicts (4.12). In the remaining case (2), we have $c_2 = 0$. Since the inequality (4.12) gives a contradiction for $b_{12}c_1 > 0$, the only case which remains to be considered is $b_{12}c_1 < 0$ and $d_\infty \in \mathbb{E}_-$. In this case, we have strict inequality in (4.12), and so also Lemma 4.4 implies $v \ne 0$, i.e. $v = \alpha e$, $\alpha > 0$, and so $\langle v, e\rangle > 0$. Since $v_n \to v$, we find $\langle v_n, e\rangle > 0$ for all large $n$. Lemma 4.7 thus implies that $v_n$ is a multiple of $e$ for all large $n$, and so Corollary 4.1 implies that $(u_n, v_n)$ satisfy the corresponding system (4.2) (with $(d_1, d_2, h_1, h_2)$ replaced by $(d_{1,n}, d_{2,n}, h_{1,n}, h_{2,n})$). This contradicts our choice of $(u_n, v_n)$. □

**Theorem 6.2.** *Suppose that* (3.6) *holds. Let $U \subseteq \mathbb{R}^2_+$ be fixed such that with the notation* (6.3) *we have $U(\infty) \setminus \{\infty\} \subseteq \mathbb{E}_0$. Then there is some $\omega > 0$ such that for any $d = (d_1, d_2) \in U$ with $d_1 \ge \omega$ we have $V \ne P_K A(d)V$ for all $V \ne 0$, and moreover*

$$\deg\big(id - P_K A(d), B_r(0), 0\big) = \begin{cases} -1 & \text{if } U(\infty) \subseteq \mathbb{E}_0 \cap \mathbb{E}_- \text{ and } (2.6), \\ 0 & \text{if } U(\infty) \setminus \{\infty\} \subseteq \mathbb{E}_0 \cap \mathbb{E}_+. \end{cases}$$

*Proof.* Theorem 6.1(1) guarantees the existence of $\omega > 0$ such that for all $d = (d_1, d_2) \in U$ with $d_1 \ge \omega$ and for $h_1 = h_2 = 0$, the problem (4.1) has the same solutions as (4.2). We can assume $\omega > b_{11}/\kappa_1$, and then $d \notin \bigcup_{n=1}^{\infty} C_n$ if $d_1 \ge \omega$. Hence, Proposition 3.3 implies that (4.2), and consequently also (4.1), has only the trivial solution for $h_1 = h_2 = 0$. This means $V \ne P_K A(d)V$ for all $V \ne 0$.

In case $U(\infty) \subseteq \mathbb{E}_0 \cap \mathbb{E}_-$, we have in particular $\infty \notin U(\infty)$, and it follows in view of (6.3) that $U(\infty)$ is bounded. Hence, there is $\omega > b_{11}/\kappa_1$ such that $-d_1 b_{12} b_{21} > d_2 b_{11}$ for all $d = (d_1, d_2) \in U$ with $d_1 > \omega$, i.e. (5.12) holds with $\alpha = -d_1 b_{12}$, $\beta = d_2$ for such $d$. The assumption (6.4) is fulfilled with $h_1 = \alpha t$, $h_2 = \beta t$, $t \ge 0$, $C_0 = b_{12}^{-1}$.. Hence, Theorem 6.1(2) implies that $\omega$ could be chosen such that (4.1) has the same solutions as (4.2) for all such for all such $h_1, h_2$, $d \in U$, $d_1 > \omega$. where $t \geq 0$ and $\alpha = -d_1 b_{12}$, $\beta = d_2$. Hence, if (2.6) holds, the first formula for the degree follows from Theorem 5.2 and Corollary 3.1.

To prove the second formula for the degree, we can assume $U(\infty) \setminus \{\infty\} \subseteq \mathbb{E}_0 \cap \mathbb{E}_+$. Hence, Theorem 6.1(3) implies that there is $\omega > b_{11}/\kappa_1$ such that (4.1) has the same solutions as (4.2) for all $d = (d_1, d_2) \in U$ with $d_1 > \omega$ when $h_1 = 0$ and $h_2 = t \geq 0$. Lemma 4.1 implies that these problems are only solvable if $t = 0$ and $u = v = 0$, and so

$$\deg\big(id - P_K A(d), B_r(0), 0\big) = \deg\Big(id - P_K\Big(A(d) + \begin{pmatrix} e \\ 0 \end{pmatrix}\Big), B_r(0), 0\Big) = 0$$

by the homotopy invariance and existence property of the degree. □

For our main result, the last case of Theorem 6.2 will be only used with $U(\infty) = \{\infty\}$. Note that for this special case, one could replace Theorem 6.1(3) by Theorem 4.1 in the proof, so that the consideration of $\mathbb{E}_+$ is actually not necessary to show this special case of Theorem 6.2.

For the rest of this section we aim to give easy sufficient criteria to verify that a point $d_\infty \in (0, \infty)$ belongs to the set $\mathbb{E}_0$, $\mathbb{E}_-$, or $\mathbb{E}_+$.

**Proposition 6.1.** *For every $d_\infty \in (0, \infty)$ and $\mu \in \mathbb{R}$ the problem*

$$v = P_{K_0}\Big(\frac{b_{22} + d_\infty}{d_\infty} A_0 v + \mu e\Big) \tag{6.8}$$

*has exactly one solution $v$. If $e \in K_0$ and $\mu \geq 0$, then $v = \alpha e$ for some $\alpha \in \mathbb{R}$. If $\mu < 0$ and (3.6) holds, then $v \neq \alpha e$ for all $\alpha \in \mathbb{R}$.*

*Proof.* The problem (6.8) is equivalent to the variational inequality

$$v \in K_0, \quad \langle Bv - \mu e, \varphi - v\rangle \geq 0 \quad \text{for all } \varphi \in K_0,$$

where

$$B := id - \frac{b_{22} + d_\infty}{d_\infty} A_0.$$

For the proof of the existence and unicity of the solution for any $\mu \in \mathbb{R}$, it is sufficient to show that $\langle Bu, u\rangle \geq \delta \|u\|^2$ for all $u \in \mathbb{H}_0$ with some $\delta > 0$ (see e.g. [19, Theorem 8.2 and 8.3]). Let us write $u \in \mathbb{H}_0$ in the form $u = u_0 + u_1$ with $u_0 = P_0 u$ and $u_1 = u - u_0$. Then $u_1 \in \{e\}^\perp$ (the orthogonal complement of the span of $e$). The restriction of $A_0$ to $\{e\}^\perp$ has the spectrum $\{\lambda_1, \lambda_2, \dots\}$, and so $\langle A_0 u_1, u_1\rangle \leq \lambda_1 \|u_1\|^2$. Since $A_0$ is positive, we obtain

$$\langle Bu_1, u_1\rangle = \|u_1\|^2 + \frac{-b_{22}}{d_\infty} \langle A_0 u_1, u_1\rangle - \langle A_0 u_1, u_1\rangle \geq \big(1 - \lambda_1\big) \|u_1\|^2.$$

Moreover, by $A_0 u_0 = u_0$, we calculate

$$\langle Bu_0, u_0\rangle = \frac{-b_{22}}{d_\infty} \|u_0\|^2.$$

Hence, putting $\delta := \min\{1 - \lambda_1, -b_{22}/d_\infty\} > 0$, we obtain, since by (3.7) the selfadjoint projections $P_0$ and $\overline{P}_0 := id - P_0$ commute with $B$, that

$$\begin{aligned}\langle Bu, u\rangle &= \langle Bu, P_0^2 u\rangle + \langle Bu, \overline{P}_0^2 u\rangle = \langle P_0 Bu, u_0\rangle + \langle \overline{P}_0 Bu, u_1\rangle \\ &= \langle Bu_0, u_0\rangle + \langle Bu_1, u_1\rangle \geq \delta(\|u_0\|^2 + \|u_1\|^2) = \delta \|u\|^2.\end{aligned}$$

For the second claim, note that $v = \alpha e$ satisfies (6.8) if and only if

$$\alpha e = P_{K_0}\Big(\frac{b_{22} + d_\infty}{d_\infty} \alpha e + \mu e\Big).$$

It follows that if $e \in K_0$ then $v = \alpha e$ with $\alpha = b_{22}^{-1}\mu \geq 0$ is the unique solution. If (3.6) holds then it follows from Lemma 3.1 that any solution $v = \alpha e$ must satisfy $\alpha = b_{22}^{-1}\mu \geq 0$, which is not true for $\mu < 0$. □

**Proposition 6.2.** *Let $e \in K_0$ and $d_\infty \in (0, \infty)$. For $\mu < 0$, the problem* (6.8) *has a unique solution $v = v_\mu$, and with this notation we have*

$$\Big(\mu + \frac{b_{12}b_{21}}{b_{11}d_\infty}\frac{\langle v_\mu, e\rangle}{\|e\|^2} \neq 0 \text{ for some } \mu < 0\Big) \implies d_\infty \in \mathbb{E}_0, \tag{6.9}$$

$$\Big(\mu + \frac{b_{12}b_{21}}{b_{11}d_\infty}\frac{\langle v_\mu, e\rangle}{\|e\|^2} \geq 0 \text{ for some } \mu < 0\Big) \implies d_\infty \in \mathbb{E}_-, \tag{6.10}$$

$$\Big(\mu + \frac{b_{12}b_{21}}{b_{11}d_\infty}\frac{\langle v_\mu, e\rangle}{\|e\|^2} \leq 0 \text{ for some } \mu < 0\Big) \implies d_\infty \in \mathbb{E}_+. \tag{6.11}$$

*Equivalently, one can replace "some" by "all" in* (6.9)–(6.11)*. If* (3.6) *holds, then $v_\mu$ is not a multiple of $e$, and the implications in* (6.9)–(6.11) *are even equivalences.*

*Proof.* The existence and unicity of $v_\mu$ are contained in Proposition 6.1. Moreover, since $A_0$ and $P_{K_0}$ are positively homogeneous, also $v_\mu$ depends positively homogeneous on $\mu$. Hence, if some of the inequalities (6.9)–(6.11) holds for some $\mu < 0$, then it holds for all $\mu < 0$.

Assume that $d_\infty \notin \mathbb{E}_-$. Then there is some $\lambda < 0$ and a solution $v$ of (6.1) which is not a multiple of $e$. Then $v$ satisfies (6.8) with

$$\mu = \lambda - \frac{b_{12}b_{21}}{b_{11}d_\infty}\frac{\langle v, e\rangle}{\|e\|^2}, \tag{6.12}$$

i.e. $v = v_\mu$. We must have $\mu < 0$, since otherwise Proposition 6.1 would imply $v_\mu = \alpha e$ with $\alpha \in \mathbb{R}$. Since $\lambda < 0$, we obtain from (6.12) that the inequality in (6.10) is not satisfied for the particular $\mu$ given by (6.12), and consequently for no $\mu < 0$, as we proved above.

The proof of the implications in (6.9) and (6.11) is analogous, only with "$\lambda < 0$" replaced by "$\lambda > 0$" or "$\lambda = 0$", respectively.

Assume now that (3.6) holds. Recall that for fixed $\mu < 0$ the function $v = v_\mu$ satisfies (6.8). Hence, Proposition 6.1 implies that $v = v_\mu$ is not a multiple of $e$, and moreover, defining $\lambda$ by (6.12), we obtain that $v$ satisfies (6.1). Hence, if $d_\infty \in \mathbb{E}_-$, we cannot have $\lambda < 0$ which by (6.12) implies that the inequality in (6.10) must hold for every $\mu < 0$. Similarly, if $d_\infty \in \mathbb{E}_0$ or $d_\infty \in \mathbb{E}_+$, we must have the inequality in (6.9) and (6.11) for every $\mu < 0$. □

Choosing $\mu = b_{22}/d_\infty$ and multiplying (6.8) by $d_\infty$, we obtain as a special case of Proposition 6.2 the following criterion.

**Corollary 6.1.** *Suppose that* (3.6) *holds, and let $d_\infty \in (0, \infty)$. Then the problem*

$$d_\infty v = P_{K_0}\big((b_{22} + d_\infty)A_0 v + b_{22}e\big) \tag{6.13}$$

*has a unique solution $v$. This solution $v$ is not a multiple of $e$, and we have*

$$\frac{-\langle v, e\rangle}{\|e\|^2} \neq \frac{b_{11}b_{22}}{b_{12}b_{21}} \iff d_\infty \in \mathbb{E}_0,$$

$$\frac{-\langle v, e\rangle}{\|e\|^2} > \frac{b_{11}b_{22}}{b_{12}b_{21}} \iff d_\infty \in \mathbb{E}_0 \cap \mathbb{E}_-,$$

$$\frac{-\langle v, e\rangle}{\|e\|^2} < \frac{b_{11}b_{22}}{b_{12}b_{21}} \iff d_\infty \in \mathbb{E}_0 \cap \mathbb{E}_+.$$

Note that (6.13) is equivalent to the variational inequality

$$v \in K_0, \qquad \langle d_\infty v - d_\infty A_0 v - b_{22} A_0 v - b_{22} e, \varphi - v \rangle \geq 0 \quad \text{for all } \varphi \in K_0. \tag{6.14}$$

**Theorem 6.3.** *Suppose that $e \in K_0 \setminus (-K_0)$. If $d_0 \in (0, \infty)$ is such that there is $u_0 \in e + K_0$ with $\langle u_0, e \rangle = 0$ and*

$$\langle d_0 u_0 - d_0 A_0 u_0 - b_{22} A_0 u_0, u_0 \rangle \leq -b_{22} \Big( \frac{|B|}{b_{12} b_{21}} \Big)^2 \|e\|^2 , \tag{6.15}$$

*then $(0, d_0] \subseteq \mathbb{E}_0 \cap \mathbb{E}_-$.*

*Proof.* Let $d_\infty \in (0, d_0]$. We will apply Corollary 6.1. Note that (3.6) follows with $u_- := u_0 - e$. Thus, let $v$ be a solution of (6.13). Choosing $\varphi := (u_0 - e) + v \in K_0 + K_0 \subseteq K_0$ in (6.14), we obtain by using $\langle u_0, e \rangle = 0$, the symmetry of $A_0$, and $A_0 e = e$ that

$$S := \langle d_\infty v - d_\infty A_0 v - b_{22} A_0 v, u_0 \rangle$$

satisfies

$$S \geq \langle d_\infty v - d_\infty A_0 v - b_{22} A_0 v - b_{22} e, e \rangle = -b_{22} \langle v + e, e \rangle .$$

Hence,

$$\langle v, e \rangle \leq \frac{1}{-b_{22}} S - \|e\|^2 .$$

Since $A_0$ has its spectrum in $(0, 1]$, we have $0 < d_\infty \langle A_0 u, u \rangle \leq d_\infty \langle u, u \rangle$ for $u \neq 0$, and so

$$\langle d_\infty u - d_\infty A_0 u - b_{22} A_0 u, u \rangle \geq -b_{22} \langle A_0 u, u \rangle > 0 \quad \text{for all } u \neq 0.$$

Hence, the symmetry of $A_0$ implies that we can define a scalar product in $\mathbb{H}_0$ by

$$\langle u, \varphi \rangle_* := \langle d_\infty u - d_\infty A_0 u - b_{22} A_0 u, \varphi \rangle \quad \text{for all } u, \varphi \in \mathbb{H}_0.$$

Using the Cauchy-Schwarz inequality for this scalar product and the corresponding norm $\| \cdot \|_*$, we obtain

$$S = \langle v, u_0 \rangle_* \leq \|v\|_* \|u_0\|_* .$$

Choosing $\varphi = 0$ in (6.14), we obtain, since $v$ is not a multiple of $e$ by Corollary 6.1, that

$$\|v\|_*^2 \leq b_{22} \langle e, v \rangle = - \langle e, v \rangle_* < \|e\|_* \|v\|_* = \sqrt{-b_{22}} \|e\| \|v\|_* ,$$

and so $0 < \|v\|_* < \sqrt{-b_{22}} \|e\|$. Furthermore, we get by using $\langle A_0 u_0, u_0 \rangle \leq \langle u_0, u_0 \rangle$, $d_\infty \leq d_0$, and (6.15) that

$$\|u_0\|_*^2 = \langle d_0 u_0 - d_0 A_0 u_0 - b_{22} A_0 u_0, u_0 \rangle + (d_\infty - d_0) \langle u_0 - A_0 u_0, u_0 \rangle \leq -b_{22} \Big( \frac{|B|}{b_{12} b_{21}} \Big)^2 \|e\|^2 .$$

Summarizing, we obtain

$$\frac{- \langle v, e \rangle}{\|e\|^2} \geq \frac{S}{b_{22} \|e\|^2} + 1 > 1 - \frac{\|u_0\|_*}{-\sqrt{-b_{22}} \|e\|} \geq 1 + \frac{|B|}{b_{12} b_{21}} = \frac{b_{11} b_{22}}{b_{12} b_{21}}.$$

Hence, Corollary 6.1 implies $d_\infty \in \mathbb{E}_0 \cap \mathbb{E}_-$. □

*Remark* 6.2. All results, starting from Remark 3.1 up to now, hold also in the more general setting described in Remark 3.1. However, for the following application of Theorem 6.3 we make use of the particular definition of $A_0$ and $\mathbb{H}_0$. In this case (6.15) means

$$d_0 \int_\Omega |\nabla u_0|^2 \, dx - b_{22} \int_\Omega |u_0|^2 \, dx \leq -b_{22} \Big( \frac{|B|}{b_{12} b_{21}} \Big)^2 \operatorname{mes} \Omega,$$

which follows from the inequality in the assumption (2.7).

**Theorem 6.4.** *Assume $e \in K_0 \setminus (-K_0)$. Suppose that there is $u_0$ satisfying (2.7), and let $d_0 > 0$ be correspondingly given by (2.8). Then $(0, d_0] \subseteq \mathbb{E}_0 \cap \mathbb{E}_-$.*

*Proof.* The claim follows from Theorem 6.3 by using the second part of Remark 6.2. □

Combining Theorem 6.4 with Theorem 6.2 where we choose $U = \{d_n : n = 1, 2, \dots\}$, we obtain the following consequence.

**Corollary 6.2.** *Assume (2.6) and $-e \notin K_0$. Suppose that there is $u_0$ satisfying (2.7), and let $d_0 > 0$ be correspondingly given by (2.8). Then for any $d_\infty \in (0, d_0]$ and any sequence $d_n = (d_{1,n}, d_{2,n}) \in \mathbb{R}^2_+$ with $d_{1,n} \to \infty$ and $d_{2,n} \to d_\infty$ there is some $n_0$ such that for all $n \geq n_0$ we have $U \neq P_K A(d_n)U$ for all $U \neq 0$, and*

$$\deg\big(id - P_K A(d_n), B_r(0), 0\big) = -1 \quad \text{for all } r > 0.$$

*Remark* 6.3. In the previous corollary the natural bound for $d_\infty$ is thus the supremum of the numbers $d_0$ when $u_0$ varies over all functions $u_0$ satisfying (2.7). This supremum might be considered as a nonlinear analogon to the variational characterization of the second eigenvalue of a linear operator (when $e$ is the unique eigenfunction to the first eigenvalue). In this sense the previous results might be considered as an extension of this linear variational theory to cones. It is unknown to the authors whether such a characterization generalizes to more general settings: As observed in Remark 6.2, it is unclear whether such a result holds in the more general setting described in Remark 3.1. Indeed, the above described supremum was only obtained using the particular definition of $A_0$ and of $\mathbb{H}_0$ and not by means of more general abstract considerations.

## 7. Proof of Theorems 2.1 and 2.2

Recall that a point $d \in \mathbb{R}^2_+$ is a *critical point* of (2.4) if the equation $U = P_K A(d)U$ has a solution $U \neq 0$.

**Proposition 7.1.** *If any neighborhood of $(d_0, 0) \in \mathbb{R}^2_+ \times \mathbb{H}$ contains some $(d, U)$ satisfying $U \neq 0$ and*

$$U = P_K\big(A(d)U + tF(d, U)\big) \tag{7.1}$$

*with some $t \in [0, 1]$ then $d_0$ is a critical point of (2.4). If any neighborhood mentioned contains even $(d, U)$ satisfying (7.1) with $U \neq 0$ being constant then there is a constant solution $V \neq 0$ of $V = P_K A(d_0)V$.*

*Proof.* By hypothesis, there is a sequence $(d_n, U_n, t_n) \in \mathbb{R}^2_+ \times \mathbb{H} \times [0, 1]$ satisfying (7.1), $(d_n, U_n) \to (d_0, 0)$, and $U_n \neq 0$. Putting $V_n := U_n / \|U_n\|$, we thus have

$$V_n = P_K\Big(A(d_n)V_n + t_n \frac{F(d_n, U_n)}{\|U_n\|}\Big). \tag{7.2}$$

By (3.1), the compactness of $A$ and the continuity of $P_K$, we conclude that the right-hand side of (7.2) has a convergent subsequence. Hence, without loss of generality, we can assume $V_n \to V$. In view of $\|V_n\| = 1$, we have $\|V\| = 1$, and passing to the limit in (7.2), we obtain by (3.1) and the continuity of $A$ and $P_K$ that $V = P_K A(d_0)V$. Hence, $d_0$ is a critical point of (2.4). Moreover, if the functions $U_n$ can be chosen to be constant, then also $V$ is constant. □

For the proof of Theorem 2.2 we will use the following Rabinowitz type result.

**Theorem 7.1.** *Let $I$ be a closed interval and $\varphi\colon I\times\mathbb{H}\to\mathbb{H}$ be continuous and compact,*

$$S := \{(t,U)\in I\times\mathbb{H} : U=\varphi(t,U)\}\,.$$

*Let $t_-,t_+\in I$, $t_-<t_+$, be such that there are $r>0$ and $\varepsilon>0$ satisfying*

$$S\cap\Big(\big([t_- -\varepsilon,t_-]\cup[t_+,t_+ +\varepsilon]\big)\times\big(\overline{B}_r(0)\setminus\{0\}\big)\Big)=\emptyset \tag{7.3}$$

*and*

$$\deg\big(id-\varphi(t_-,\,\cdot\,),B_r(0),0\big)\neq\deg\big(id-\varphi(t_+,\,\cdot\,),B_r(0),0\big).$$

*Then $S\setminus(I\times\{0\})$ contains a connected set $\mathfrak{C}_0$ such that $\overline{\mathfrak{C}}_0\cap([t_-,t_+]\times\{0\})\neq\emptyset$ and at least one of the following holds:*

1. *$\mathfrak{C}_0$ is unbounded or contains a point from $(\partial I)\times\mathbb{H}$ (the boundary understood in $\mathbb{R}$).*
2. *$\overline{\mathfrak{C}}_0$ contains a point from $(I\setminus[t_- -\varepsilon,t_+ +\varepsilon])\times\{0\}$.*

This theorem is a special case of a general abstract bifurcation result from [27]; see also [9] for details how to derive Theorem 7.1 as a special case.

*Proof of Theorems* 2.1 *and* 2.2. Let $C_0\subseteq\mathbb{R}^2_+$ denote the critical points of (2.4). Let $u_0$ be from the assumption (2.7) and $d_0$ the corresponding number from (2.8). It follows from the assumptions of Theorem 2.1 that the condition (3.6) is fulfilled (see the text after Lemma 3.1), and the set $U=\big\{(d_1,d_2)\in\mathbb{R}^2_+ : d_2\geq d_1\big\}$ satisfies $U(\infty)\setminus\{\infty\}=\emptyset$. Applying Theorem 6.2 to this $U$ we obtain the existence of $\omega_1,\omega_2>0$ such that $U_+ := [\omega_1,\infty)\times[\omega_2,\infty)\subseteq\mathbb{R}^2_+\setminus C_0$. Applying Corollary 6.2 we see that for any $\varepsilon>0$ there is $\omega_\varepsilon>0$ such that $U_- := [\omega_\varepsilon,\infty)\times[\varepsilon,d_0]\subseteq\mathbb{R}^2_+\setminus C_0$. Hence, claim (1) of Theorem 2.1 holds. Moreover, claim (2) of Theorem 2.1 follows from Corollary 6.2. Furthermore, it follows from Theorem 6.2 and Corollary 6.2 that for each $d_\pm\in U_\pm$ we have

$$\deg\big(id-P_KA(d_+),B_r(0),0\big)=0\neq-1=\deg\big(id-P_KA(d_-),B_r(0),0\big) \tag{7.4}$$

for all $r>0$. Since $C_0$ is closed (e.g. by Proposition 7.1 with $F=0$), the components of $\mathbb{R}^2_+\setminus C_0$ are open and thus actually path-connected. The degree $\deg\big(id-P_KA(d),B_r(0),0\big)$ is constant on paths in $\mathbb{R}^2_+\setminus C_0$ by the homotopy invariance property, and thus constant on the components of $\mathbb{R}^2_+\setminus C_0$. Hence, (7.4) holds even for $d_\pm\in\widetilde{U}_\pm$ if $\widetilde{U}_\pm$ denotes the components of $\mathbb{R}^2_+\setminus C_0$ containing $U_\pm$.

Fixing $d_\pm\in\widetilde{U}_\pm$ and applying the homotopy invariance of the degree with $H(t,U):= U-P_K(A(d_\pm)U+tF(d_\pm,U))$, we find by Proposition 7.1 that there is some $r>0$ with

$$\begin{aligned}&\deg\big(id-P_K\big(A(d_+)+F(d_+,\,\cdot\,)\big),B_r(0),0\big)=0\neq\\ &-1=\deg\big(id-P_K\big(A(d_-)+F(d_-,\,\cdot\,)\big),B_r(0),0\big).\end{aligned}$$

Now if $\gamma$ is a path as in Theorem 2.2, it follows that the hypotheses of Theorem 7.1 are satisfied with

$$\varphi(t,U):=P_K\big(A(\gamma(t))U+F(\gamma(t),U)\big).$$

In particular, the assumption (7.3) follows from Proposition 7.1. The set $\mathfrak{C}_0$ of Theorem 7.1 has exactly the properties stated in Theorem 2.2, and so Theorem 2.2 is proved.

Proposition 3.3 implies that (4.2) with $h_1=h_2=0$ has only the trivial solution for $d=\mathbb{R}^2_+\setminus\bigcup_{n=1}^\infty C_n$. It follows by using Lemma 4.1(1) that the problem (4.1) has no nonzero constant solution. Hence, Proposition 7.1 implies that for any bifurcation point $d_0$ of (2.4) in $\mathbb{R}^2_+\setminus\bigcup_{n=1}^\infty C_n$ all solutions $(d,U)$ of (2.4) with $d$ sufficiently close to $d_0$ and $U\neq0$ sufficiently close to 0 are automatically nonconstant, i.e. the last assertion of Theorem 2.1 is proved, and it remains to prove the assertion (3) of Theorem 2.1.

We consider $D_S$ as a subset of the compact space $X := [0,\infty]\times[0,\infty]$. It follows from the already proved statements (1) and (2) of Theorem 2.1 (and using Proposition 7.1) that the set $S_0$ of bifurcation points of (2.4) lying in $D_S$ satisfies $S_0\cap(U_+\cup U_-)=\emptyset$ and $\overline{S}_0\cap A_\infty=\emptyset$, where $\overline{S}_0$ denotes the closure of $S_0$ (in $X$),

$$A_\infty := \{(\infty, d_\infty) : 0 < d_\infty \leq d_0\}.$$

Now we identify (by means of a homeomorphism) the set $Q := D_S\setminus U$ with the disc-interior $\{x\in\mathbb{R}^2 : \|x\| < 1\}$, and the boundary of $Q$ (in $X$) with the boundary of that disc. We put $A_2 := \partial Q\cup U_+$ and $A_4 := \partial Q\cup U_-$. Then $\partial Q\setminus(A_2\cup A_4)$ consists of two components which we denote by $A_1$ and $A_3$. Theorem 2.2 already proved implies in particular that any continuous path in $D_S$ connecting a point from $U_+$ with a point from $U_-$ contains a bifurcation point of (2.4), i.e. a point from $S_0$. Since the definition of bifurcation points implies that $S_0$ is closed in $D_S$, we thus verified the hypotheses of the subsequent Theorem 7.2. This result implies the claim, since for any subset $\mathfrak{C}\subseteq S_0$ we have automatically $\overline{\mathfrak{C}}\cap A_\infty=\emptyset$, because $\overline{S}\cap A_\infty=\emptyset$. □

**Theorem 7.2** (Disc-Cutting). *Let $X$ and $Q$ be as in the previous proof. Suppose that the boundary of $Q$ in $X$ is divided into four connected sets $A_1,\dots,A_4$ with $A_2$ and $A_4$ consisting of at least two points and $\overline{A}_1\cap\overline{A}_3=\emptyset$. Let $S_0\subseteq Q$ be closed in $Q$ such that each compact continuous path $P$ in $Q\cup A_2\cup A_4$ with $P\cap A_i\neq\emptyset$ $(i=2,4)$ contains some point from $S_0$. Then there is a connected subset $\mathfrak{C}\subseteq S_0$ such that $\overline{\mathfrak{C}}\cap\overline{A}_i\neq\emptyset$ for $i=1,3$.*

*Proof.* This is a special case of [28, Theorem 3.1]. □

*Remark* 7.1. Our proof shows that Theorems 2.1 and 2.2 hold for every nonlinearity $F$ for which the conclusion of Proposition 3.1 is true. For instance, one can also formulate similar results when $f_k$ in (1.1) depends also on $d_1, d_2, \nabla u(x), \nabla v(x)$, and $x$. Moreover, (3.1) is even only required for $\widetilde{d}=\gamma(t_\pm)$ for Theorem 2.2 resp. for all $\widetilde{d}\in U_+\cup U_-$ for Theorem 2.1. In particular, it is even admissible that $F(\widetilde{d},0)\neq 0$ for other values of $\widetilde{d}$.

*Remark* 7.2. Actually Theorems 2.1 and Theorem 2.2 (and even the second part of Remark 7.1) hold with the obvious modifications in the claims and proofs for the abstract setting considered in Remark 3.1 if one replaces the hypothesis (2.6) by (3.5), and the hypothesis about $u_0$ by the assumption that there is $u_0\in K$ with $\langle u_0, e\rangle = 0$ and (6.15) for some $d_0\in(0,\infty)$. It is then this $d_0$ which occurs in the general form of Theorem 2.1. Alternatively, the hypothesis about $u_0$ and $d_0$ can be replaced by the more general assumption (3.6) and $(0,d_0]\subseteq\mathbb{E}_0\cap\mathbb{E}_-$. Moreover, the latter can even be relaxed to $d_0\in M\subseteq\mathbb{E}_0\cap\mathbb{E}_-$ for any set $M\subseteq(0,\infty)$ if one replaces $U_-$ in the claims of Theorem 2.1 and 2.2 by $[\omega_\varepsilon,\infty)\times(M\cap[\varepsilon,\infty))$.

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