Weak and strong solutions to problems arising in fluid mechanics

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Navier-Stokes-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \equiv \mu \Delta \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}, \ \ \mu > 0, \ \ \lambda \geq 0$$

Internal energy equation

$$\partial_t \Big(\varrho e(\varrho \vartheta) \Big) + \mathrm{div}_x \Big(\varrho e(\varrho, \vartheta) \mathbf{u} \Big) - \mathrm{div}_x \left(\kappa \nabla_x \vartheta \right)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}, \ \ \kappa > 0$$

Existence of smooth solutions (classical theory)

Local solutions

A. Valli, W.Zajaczkowski [1982] Existence of classical *local-in-time* solutions in the class:

$$\varrho \in C([0, T_{\max}); W^{3,2}(\Omega)), \ \vartheta_0 \in C([0, T_{\max}); W^{3,2}(\Omega))$$
$$\mathbf{u} \in C([0, T_{\max}); W^{3,2}(\Omega; R^3))$$

Global solutions

A.Matsumura, T.Nishida [1980,1983] Existence of classical *global-in-time* solutions in the same class for the initial data sufficiently close to a static state

Weak solutions

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Gibbs' relation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Total energy balance

$$E(t) = \int \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = 0$$

Weak solutions - internal energy formulation

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Thermal energy balance

$$\partial_t \Big(\varrho e(\varrho \vartheta) \Big) + \mathrm{div}_x \Big(\varrho e(\varrho, \vartheta) \mathbf{u} \Big) - \mathrm{div}_x (\kappa \nabla_x \vartheta)$$

$$\boxed{\geq} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}$$

Total energy balance

$$\int \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] (\tau, \cdot) \ \mathrm{d}x \boxed{\leq} E_0$$

Weak solutions - entropy formulation

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Entropy balance

$$\partial_t \Big(\varrho s(\varrho \vartheta) \Big) + \mathrm{div}_x \Big(\varrho s(\varrho, \vartheta) \mathbf{u} \Big) - \mathrm{div}_x \left(\kappa \frac{\nabla_x \vartheta}{\vartheta} \right)$$

$$\boxed{\geq} \frac{1}{\vartheta} \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\kappa |\nabla_x \vartheta|^2}{\vartheta} \right]$$

Total energy balance

$$\int \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] (\tau, \cdot) \, \mathrm{d}x \boxed{\leq} E_0$$

Existence of weak solutions

Existence of weak solutions [E.F.2003], [E.F., A.Novotný 2009]

The Navier-Stokes-Fourier system admits a global-in-time weak solution for any finite energy initial data

Compatibility property of weak solutions

Any weak solution that possesses necessary smoothness is a classical one

Problems

- density oscillations
- temperature concentrations

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u})\Big] - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \mathbf{u}$$

Existence of weak solutions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence [E.Chiodaroli, E.F., O.Kreml 2013]

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta \ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0, T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Relative (modulated) energy

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta)\Big)$$

Coercivity of the ballistic free energy

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Dissipative solutions - summary

Global existence in the viscous case

Global-in-time weak dissipative solutions of the **Navier-Stokes-Fourier or Euler Fourier system** exist for any finite energy initial data (under some hypotheses imposed on constitutive relations)

Compatibility

Regular dissipative solutions are strong solutions

Weak-strong uniqueness

Dissipative and strong solutions emanating from the same (regular) initial data coincide as long as the latter exists. The strong solutions are unique in the class of weak solutions

Dissipative solutions to the Euler-Fourier system

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

Infinitely many dissipative weak solutions

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in $(0, T)$

Navier-Stokes-Fourier revisited

Viscosity and heat conductivity

$$\mu > 0, \ \eta \geq 0, \ \kappa = \kappa(\vartheta) \approx (1 + \vartheta^{\alpha}), \ \alpha \geq 2$$

Pressure

$$p(\varrho, \vartheta) = \varrho \vartheta + a \varrho^{\gamma}, \ \gamma > 3$$

Internal energy

$$\varrho e(\varrho, \vartheta) = c_v \varrho \vartheta + \frac{a}{\gamma - 1} \varrho^{\gamma}$$

Blow-up criterion

Blow-up of smooth solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth ($W^{2,3}$). Then the Navier-Stokes-Fourier system admits a strong solution defined on a (possibly short) time interval $(0, T)$.
If

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} \right] < \infty,$$

then the solution can be extended beyond T.

Regularity criterion

Regularity for weak solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth ($W^{2,3}$). Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system such that

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is regular.

Main tools

- Relative energy functional
- Entropy formulation
- Weak strong uniqueness

Numerical solution

FV framework

$$\text{regular tetrahedral mesh}, \ \ Q_h = \{ v \mid v = \text{piece-wise constant} \}$$

FE framework - Crouzeix - Raviart

$$V_h = \left\{ v \ \middle| \ v = \text{piece-wise affine}, \ \tilde{v}_\Gamma \text{ continuous on face } \Gamma \right\}$$

$$\tilde{v}_\Gamma \equiv \frac{1}{|\Gamma|} \int_\Gamma v \ \mathrm{dS}_x$$

Upwind discretization of convective terms

$$\langle h\mathbf{u}; \nabla_x \varphi \rangle_E \approx \sum_\Gamma \int_\Gamma h(\cdot - \tilde{\mathbf{u}}_\Gamma \cdot \mathbf{n}) \tilde{\mathbf{u}}_\Gamma \cdot \mathbf{n} [[\varphi]] \ \mathrm{dS}_x$$

$$\equiv \sum_\Gamma \int_\Gamma \mathrm{Up}[h, \mathbf{u}] [[\varphi]] \ \mathrm{dS}_x$$

Numerical scheme [Karlsen-Karper], I

Equation of continuity

$$\int_\Omega D_t \varrho_h^k \varphi_h \ \mathrm{d}x \equiv \int_\Omega \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \varphi_h \ \mathrm{d}x = \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, u_h^k] \ [[\varphi_h]] \ \mathrm{dS}_x$$

$$- h^\alpha \boxed{\sum_{\Gamma \in \Gamma_h} \int_\Gamma [[\varrho_h^k]][[\varphi_h]] \ \mathrm{dS}_x} \text{ for all } \varphi_h \in Q_h$$

Numerical scheme [Karlsen-Karper], II

Momentum equation

$$\int_\Omega D_t\left(\varrho_h^k \widehat{\mathbf{u}}_h^k\right)\cdot\varphi_h \,\mathrm{d}x - \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k\widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \cdot [[\widehat{\varphi}_h]]\mathrm{dS}_x$$

$$-\int_\Omega p(\varrho_h^k,\vartheta_h^k)\mathrm{div}_x\varphi_h \,\mathrm{d}x - h^\alpha \boxed{\sum_{\Gamma\in\Gamma_h}\int_\Gamma [[\varrho_h^k]][[\widehat{\varphi}_h]]\left\{\widehat{\mathbf{u}}_h^k\right\}\mathrm{dS}_x}$$

$$= -\int_\Omega \left(\mu\nabla_h\mathbf{u}_h : \nabla_h\varphi + \lambda\mathrm{div}_h\mathbf{u}_h^k\mathrm{div}_h\varphi_h\right)\,\mathrm{d}x$$

for all $\varphi_h \in V_h(\Omega)$

Numerical scheme [Karlsen-Karper], III

Energy equation

$$\int_\Omega D_t \left(\varrho_h^k \vartheta_h^k\right) \varphi_h \, \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k] \, [[\varphi_h]] \mathrm{dS}_x$$

$$+ \sum_{\Gamma \in \Gamma_h} \int_\Gamma \frac{1}{d_h} [[K(\vartheta_h^k)]] \, [[\varphi_h]] \mathrm{dS}_x$$

$$= \int_\Omega \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2\right) \varphi_h \, \mathrm{d}x$$

$$- \int_\Omega \vartheta_h^k \partial_\vartheta p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_x \mathbf{u}_h^k \varphi_h \, \mathrm{d}x$$

for all $\varphi_h \in Q_h(\Omega)$

Convergence of the numerical scheme

Convergence of the numerical scheme [E.F., T.Karper, A.Novotný 2014]

The numerical solutions converge, up to a subsequence, to a weak solution of the Navier-Stokes-Fourier system

Synergy analysis - numerics, assumptions

Numerical solutions with regular initial data

Suppose that $[\varrho_h, \vartheta_h, \mathbf{u}_h]$ is a sequence of numerical solutions for regular initial data

Boundedness

Suppose that

$$\varrho_h^k, \ \vartheta_h^k, \ \mathbf{u}_h^k, \ \mathrm{div}_h \mathbf{u}_h^k$$

are bounded independently of the order of discretization h.

Synergy analysis - numerics, conclusion

Conclusion

The numerical solutions converge to a weak solution with

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Consequently:

- the limit solution is smooth
- the limit solution is unique
- the numerical scheme converges unconditionally
- error estimates (?)