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Oseen equation**

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1 Introduction

The Dirichlet, Neumann, and Robin boundary value problems for second order partial differential equations are important model problems in mathematical physics (see [2]). Traditionally, the Dirichlet and Neumann problems for the Laplace equation in domains with smooth boundary have been studied by the method of integral equations long time ago. Later, also the Robin problem for the Laplace equation in smooth and Lipschitz domains has been investigated by this method (see [13], [14], [11], [12], [9], [8]).

Recently, also the boundary value problems for the scalar Oseen equation

$$-\Delta u + 2\lambda\partial_1 u = 0 \quad \text{in } \Omega \subset \mathbb{R}^3, \quad (1)$$

have been studied by the method of integral equations (see [15]). Here the authors study the Dirichlet problem, i.e. they prescribe the boundary condition

$$u = g \quad \text{on } \partial\Omega,$$

and the Oseen Neumann problem, prescribing the boundary condition

$$\frac{\partial u}{\partial n} - \lambda n_1 u = g \quad \text{on } \partial\Omega.$$

Here $n = n^\Omega$ is the outward (with respect to Ω) unit normal vector on $\partial\Omega$.

In the present paper we study the Robin problem for the scalar Oseen equation in $\Omega \subset \mathbb{R}^3$, i.e. the scalar Oseen problem 1 with prescribed boundary condition

$$\frac{\partial u}{\partial n} + hu = g \quad \text{on } \partial\Omega, \quad (2)$$

where h denotes a positive function, and the Robin problem corresponding to the Oseen Neumann condition studied in [15], i.e. we prescribe the boundary condition

$$\frac{\partial u}{\partial n} - \lambda n_1 u + hu = g \quad \text{on } \partial\Omega \quad (3)$$

with $h \geq 0$. We prove unique solvability of these problems, a representation of the solution in form of a scalar Oseen single layer potential, and the maximum principle for the solution of the Robin problem for the scalar Oseen equation.

2 The maximum principle for the Robin problem

Let $\Omega \subset R^3$ be an open set with boundary of class C^1 . Denote by $\bar{\Omega}$ the closure of Ω , and by $n = n^\Omega(z)$ the outward (with respect to Ω) unit normal vector in $z \in \partial\Omega$. Let $g, h \in C^0(\partial\Omega)$ and $\lambda \in R$ be given. Then we call u a classical solution of the Robin problem for the scalar Oseen equation (1), (2), if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, if there exists $\partial u(z)/\partial n$ at each $z \in \partial\Omega$, and if (1), (2) are satisfied.

The following result holds true for general $2 \leq m \in N$.

Proposition 2.1 *Let $G \subset R^m$ be an open set with bounded boundary ∂G . Let g and $a \geq 0$ be functions defined on ∂G , and let $\lambda \in R$. Suppose that θ is a unit vector defined on ∂G such that $\{z + t\theta; -\delta < t < 0\} \subset \Omega$, $\{z + t\theta; 0 < t < \delta\} \cap \Omega = \emptyset$ for some $\delta > 0$, and suppose that there exists*

$$\frac{\partial u(z)}{\partial \theta} = \lim_{t \rightarrow 0^-} \frac{u(z + t\theta) - u(z)}{t}.$$

Moreover, let $u \in C^2(G) \cap C^0(\bar{G})$, $-\Delta u + 2\lambda\partial_1 u = 0$ in G . Suppose that for $z \in \partial G$ with $a(z) \neq 0$ the vector $\theta = \theta(z)$ is a unit vector and if $a(z) = 0$ and $\partial u(z)/\partial \theta$ makes sense or not, put $a(z)(\partial u(z)/\partial \theta) = 0$. Suppose that $a(\partial u/\partial \theta) + u = g$ on ∂G . If G is unbounded suppose moreover that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then

$$\inf_{x \in \partial G} g(x) \leq \inf_{x \in G} u(x) \leq \sup_{x \in G} u(x) \leq \sup_{x \in \partial G} g(x).$$

Proof. The maximum principle ([4], Chapter 3, Theorem 3.1) gives that there exists $z \in \partial G$ such that $u \leq u(z)$. If $a(z) \neq 0$ then $\partial u(z)/\partial \theta \geq 0$ because θ is an outward pointing vector and $u \leq u(z)$. Thus

$$\sup_{x \in G} u(x) = u(z) \leq a(z) \frac{\partial u(z)}{\partial \theta} + u(z) = g(z) \leq \sup_{x \in \partial G} g(x).$$

Now for $v = -u$ we have

$$\inf_{x \in \partial G} g(x) = - \sup_{x \in \partial G} (-g(x)) \leq - \sup_{x \in G} v(x) = \inf_{x \in G} u(x).$$

3 Potentials

Let $\lambda \in R$, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$, and let

$$E_{2\lambda}(x) = \frac{1}{4\pi|x|} e^{-(|\lambda x| - \lambda x_1)}$$

denote the fundamental solution of the scalar Oseen equation (1). Note that $E_0(x) = 1/(4\pi|x|)$ is the fundamental solution of the Laplace equation $-\Delta u = 0$. If $\Omega \subset R^3$ is an open set with bounded boundary of class $C^{1,\alpha}$, $0 < \alpha$, and $\varphi \in C^0(\partial\Omega)$, then the scalar Oseen single layer potential

$$E_{2\lambda}^\Omega \varphi(x) = \int_{\partial\Omega} E_{2\lambda}(x-y)\varphi(y) \, d\sigma_y$$

is well defined. Easy calculations yield $E_{2\lambda}^\Omega \varphi \in C^\infty(R^3 \setminus \partial\Omega)$ and $-\Delta E_{2\lambda}^\Omega \varphi + 2\lambda \partial_1 u = 0$ in $R^3 \setminus \partial\Omega$ (see [15]). Moreover, for $\lambda = 0$ we find

$$E_0^\Omega \varphi(x) = O(1/|x|), \quad |\nabla E_0^\Omega \varphi(x)| = O(1/|x|^2) \quad \text{as } |x| \rightarrow \infty. \quad (4)$$

If $\lambda \neq 0$, then

$$|E_{2\lambda}^\Omega \varphi(x)| + |\nabla E_{2\lambda}^\Omega \varphi(x)| = O(e^{-(|\lambda x| - \lambda x_1)}/|x|) \quad \text{as } |x| \rightarrow \infty. \quad (5)$$

Since $E_{2\lambda}^\Omega$ is an integral operator with weakly singular kernel, it is a compact linear operator on $C^0(\partial\Omega)$ (see for example [17]).

Lemma 3.1 *Let $\Omega \subset R^m$ be an open set with bounded Lipschitz boundary $\partial\Omega$. Let $k(x, y)$ be defined for $[x, y] \in R^m \times \partial\Omega$; $x \neq y$ and $|k(x, y)| \leq C|x-y|^{1-m+\beta}$ with positive constants C, β . Suppose that $k(x, \cdot)$ is measurable and $k(\cdot, y)$ is continuous. Let $f \in L^\infty(\partial\Omega)$. Then*

$$kf(x) = \int_{\partial\Omega} k(x, y)f(y) \, d\sigma_y$$

is a continuous function in R^m .

(See [6], Lemma 3.2.)

Lemma 3.2 *Let $\lambda \in R$. Define $R_{2\lambda}(x) = E_{2\lambda}(x) - E_0(x)$. Then*

$$R_{2\lambda}(x) = O(1), \quad |\nabla R_{2\lambda}(x)| = O(|x|^{-1}) \quad |x| \rightarrow 0.$$

If $\Omega \subset R^3$ is an open set with bounded boundary of class $C^{1,\alpha}$, $0 < \alpha < 1$, $\varphi \in C^0(\partial\Omega)$, then for

$$R_{2\lambda}^\Omega \varphi(x) = \int_{\partial\Omega} R_{2\lambda}(x-y)\varphi(y) \, d\sigma_y$$

we find $R_{2\lambda}^\Omega \varphi \in C^1(R^m)$.

Proof. Put $f(t) = (e^t - 1)/t$ for $t \neq 0$, $f(0) = 1$. Then f is continuous. So, there is a constant C such that $|f(t)| \leq C$ for $|t| \leq 1$. If $0 < |t| \leq 1$ then $|f'(t)| = |e^t/t - (e^t - 1)/t^2| \leq (C + e)/t$. Clearly,

$$R_{2\lambda}(x) = f(-(|\lambda x| - \lambda x_1)) \frac{-(|\lambda x| - \lambda x_1)}{|x|}.$$

Thus $|R_{2\lambda}(x)| = O(1)$ as $|x| \rightarrow 0$. Moreover,

$$\begin{aligned} |\nabla R_{2\lambda}(x)| &\leq |f'(-(|\lambda x| - \lambda x_1))| \frac{|8\lambda(|\lambda x| - \lambda x_1)|}{|x|} \\ &+ |f(-(|\lambda x| - \lambda x_1))| O(1/|x|) = O(1/|x|), \quad |x| \rightarrow 0. \end{aligned}$$

Using Lemma 3.1 for $R_{2\lambda}^\Omega$ and $\partial_j R_{2\lambda}^\Omega$ we obtain $R_{2\lambda}^\Omega \varphi \in C^1(R^m)$.

Proposition 3.3 *Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$. For $x, y \in \partial\Omega$, $x \neq y$ set*

$$L_{2\lambda}^\Omega(x, y) = n^\Omega(x) \cdot \nabla_x E_{2\lambda}(x - y).$$

For $\varphi \in C^0(\partial\Omega)$ define

$$L_{2\lambda}^\Omega \varphi(x) = \int_{\partial\Omega} L_{2\lambda}^\Omega(x, y) \varphi(y) \, d\sigma_y.$$

Then $L_{2\lambda}^\Omega$ is a compact linear operator on $C^0(\partial\Omega)$.

Proof. It is well known that L_0^Ω is a compact linear operator on $C^0(\partial\Omega)$ (see for example [17] or [10]). Since $L_{2\lambda}^\Omega - L_0^\Omega$ is an integral operator with weakly singular kernel (see Lemma 3.2), it is a compact linear operator on $C^0(\partial\Omega)$ (see for example [17]).

Proposition 3.4 *Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$, $\varphi \in C^0(\partial\Omega)$. Then $E_{2\lambda}^\Omega \varphi \in C^0(R^3)$. Put $u = E_{2\lambda}^\Omega \varphi$ in Ω . Then*

$$\frac{\partial u(x)}{\partial n} = \frac{1}{2} \varphi(x) + L_{2\lambda}^\Omega \varphi(x).$$

Proof. The proposition is well known for $\lambda = 0$ (see for example [17] or [10].) By virtue of Lemma 3.2 we obtain the result for arbitrary λ .

Corollary 3.5 *Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$, $\varphi \in C^0(\partial\Omega)$. If $E_{2\lambda}^\Omega \varphi = 0$ on $\partial\Omega$, then $\varphi = 0$.*

Proof. Put $u = E_{2\lambda}^\Omega \varphi$ in Ω , $v = E_{2\lambda}^\Omega \varphi$ in $G = R^3 \setminus \bar{\Omega}$. Proposition 3.4 gives that $E_{2\lambda}^\Omega \varphi \in C^0(R^3)$. Moreover, $E_{2\lambda}^\Omega \varphi$ is a solution of the scalar Oseen equation in $R^3 \setminus \partial\Omega$ and $E_{2\lambda}^\Omega \varphi(x) \rightarrow 0$ as $x \rightarrow \infty$. Maximum principle ([4], Chapter 3, Theorem 3.1) gives that $E_{2\lambda}^\Omega \varphi = 0$ in R^3 . Fix $x \in \partial\Omega$. Let n be the unit outward normal of Ω at x . According to Proposition 3.4 we have

$$0 = \frac{\partial u(x)}{\partial n} + \frac{\partial v(x)}{\partial(-n)} = \frac{1}{2}\varphi(x) + L_{2\lambda}^\Omega \varphi(x) + \frac{1}{2}\varphi(x) + L_{2\lambda}^G \varphi(x) = \varphi(x).$$

Definition 3.6 Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$. For $x \in \partial\Omega$, $\beta > 0$ denote the non-tangential approach region of opening β at the point x by

$$\Gamma_\beta(x) := \{y \in \Omega; |x - y| < (1 + \beta) \text{dist}(y, \partial\Omega)\}.$$

If

$$c = \lim_{\substack{y \rightarrow x \\ y \in \Gamma_\beta(x)}} u(y)$$

for each $\beta > \beta_0$, we call c the non-tangential limit of u at $x \in \partial\Omega$. We fix $\beta > 0$ large enough such that $x \in \bar{\Gamma}_\beta(x)$ for every $x \in \partial\Omega$. If now u is a function defined in Ω , we denote the non-tangential maximal function of u on $\partial\Omega$ by

$$u^*(x) = \sup\{|u(y)|; y \in \Gamma_\beta(x)\}.$$

Lemma 3.7 Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$, $\varphi \in C^0(\partial\Omega)$. Then $|\nabla E_{2\lambda}^\Omega \varphi|^* \in L^2(\partial\Omega)$. Moreover, the non-tangential limit of $\nabla E_{2\lambda}^\Omega \varphi$ exists at almost all points of $\partial\Omega$.

Proof. The proposition is well known for $\lambda = 0$ (see [5]). Since $R_{2\lambda}^\Omega \varphi \in C^1(R^m)$ by Lemma 3.2, we obtain the proposition for arbitrary λ .

Proposition 3.8 . Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$, $\varphi \in C^0(\partial\Omega)$. Then

$$\int_{\partial\Omega} (E_{2\lambda}^\Omega \varphi) \left(\frac{1}{2}\varphi + L_{2\lambda}^\Omega \varphi - \lambda n_1 E_{2\lambda}^\Omega \varphi \right) d\sigma_y = \int_{\Omega} |\nabla E_{2\lambda}^\Omega \varphi|^2 dy.$$

Proof. If $\lambda = 0$ see [7]. Let now $\lambda \neq 0$. Suppose first that Ω is bounded. According to [16], Theorem 1.12 there exists a sequence of open sets $G(j)$ with boundary of class C^∞ with the following properties:

1. $\bar{G}(j) \subset \Omega$.
2. There exist homeomorphisms $\Lambda_j : \partial\Omega \rightarrow \partial G(j)$ and $\beta > 0$, such that $\Lambda_j(y) \in \Gamma_\beta(y)$ for every j and every $y \in \partial\Omega$,

$$\sup\{|y - \Lambda_j(y)|; y \in \partial\Omega\} \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

3. There are positive functions f_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $M \subset \partial\Omega$,

$$\int_M f_j \, d\sigma_y = \int_{\Lambda_j(M)} \, d\sigma_y,$$

and so that $f_j \rightarrow 1$ point-wise a.e..

4. The normal vectors $n^j(\Lambda_j(y))$ to $G(j)$ converge point-wise almost everywhere to $\mathbf{n}^\Omega(y)$.

By virtue of Proposition 3.4, Lemma 3.7, the Green lemma and the Lebesgue lemma

$$\begin{aligned} \int_{\partial\Omega} (E_{2\lambda}^\Omega \varphi) \left(\frac{1}{2} \varphi + L_{2\lambda}^\Omega \varphi - \lambda n_1 E_{2\lambda}^\Omega \varphi \right) d\sigma_y &= \lim_{j \rightarrow \infty} \int_{\partial G(j)} \left\{ (E_{2\lambda}^\Omega) \frac{\partial E_{2\lambda}^\Omega}{\partial n} - \lambda n_1 (E_{2\lambda}^\Omega)^2 \right\} d\sigma_y \\ &= \lim_{j \rightarrow \infty} \int_{G(j)} \left\{ (E_{2\lambda}^\Omega) (\Delta E_{2\lambda}^\Omega) + |\nabla E_{2\lambda}^\Omega|^2 - 2\lambda (E_{2\lambda}^\Omega) \partial_1 E_{2\lambda}^\Omega \right\} dy \\ &= \lim_{j \rightarrow \infty} \int_{G(j)} \left\{ |\nabla E_{2\lambda}^\Omega|^2 + (E_{2\lambda}^\Omega) (\Delta E_{2\lambda}^\Omega - 2\lambda \partial_1 E_{2\lambda}^\Omega) \right\} dy = \int_{\Omega} |\nabla E_{2\lambda}^\Omega \varphi|^2 dy. \end{aligned}$$

Let now Ω be unbounded. Put $G(R) = \{x \in \Omega; |x| < R\}$. Put $\varphi = 0$ outside of $\partial\Omega$. Then

$$\begin{aligned} \int_{\Omega} |\nabla E_{2\lambda}^\Omega \varphi|^2 dy &= \lim_{R \rightarrow \infty} \int_{G(R)} |\nabla E_{2\lambda}^\Omega \varphi|^2 dy \\ &= \lim_{R \rightarrow \infty} \int_{\partial G(R)} (E_{2\lambda}^{G(R)} \varphi) \left(\frac{1}{2} \varphi + L_{2\lambda}^{G(R)} \varphi - \lambda n_1 E_{2\lambda}^{G(R)} \varphi \right) d\sigma_y \\ &= \int_{\partial\Omega} (E_{2\lambda}^\Omega \varphi) \left(\frac{1}{2} \varphi + L_{2\lambda}^\Omega \varphi - \lambda n_1 E_{2\lambda}^\Omega \varphi \right) d\sigma_y \\ &\quad + \lim_{R \rightarrow \infty} \int_{\{|x|=R\}} \left\{ (E_{2\lambda}^\Omega \varphi) \frac{\partial E_{2\lambda}^\Omega \varphi}{\partial n} - \lambda n_1 (E_{2\lambda}^\Omega \varphi)^2 \right\} d\sigma_y. \end{aligned}$$

According to (5), (4) and the Lebesgue lemma

$$\begin{aligned} \left| \int_{\{|x|=R\}} \left\{ (E_{2\lambda}^\Omega \varphi) \frac{\partial E_{2\lambda}^\Omega \varphi}{\partial n} - \lambda n_1 (E_{2\lambda}^\Omega \varphi)^2 \right\} d\sigma_y \right| &\leq \int_{\{|x|=R\}} C e^{-2(|\lambda x| - \lambda x_1)} |x|^{-2} d\sigma_y \\ &= \int_{\{|x|=1\}} C e^{-2R(|\lambda x| - \lambda x_1)} d\sigma_y \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$. This gives the proposition.

4 The boundary condition (2) with $h > 0$

We shall look for a solution of the problem (1), (2) in the form of a single layer potential $E_{2\lambda}^\Omega \varphi$ with $\varphi \in C^0(\partial\Omega)$. According to Proposition 3.4 the function $E_{2\lambda}^\Omega \varphi$ is a classical solution of the problem (1), (2) if and only if

$$\frac{1}{2}\varphi + L_{2\lambda}^\Omega \varphi + hE_{2\lambda}^\Omega \varphi = g. \quad (6)$$

Theorem 4.1 *Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$. If $h \in C^0(\partial\Omega)$, $h > 0$, then $T = (1/2)I + L_{2\lambda}^\Omega + hE_{2\lambda}^\Omega$ is a continuously invertible operator in $C^0(\partial\Omega)$. Fix $g \in C^0(\partial\Omega)$.*

- *If Ω is bounded then there exists unique classical solution u of the problem (1), (2).*
- *If Ω is unbounded then there exists unique classical solution u of the problem (1), (2) such that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$.*

Moreover, $u = E_{2\lambda}^\Omega T^{-1}g$ and

$$\inf_{x \in \partial\Omega} \frac{g(x)}{h(x)} \leq \inf_{x \in \Omega} u(x) \leq \sup_{x \in \Omega} u(x) \leq \sup_{x \in \partial\Omega} \frac{g(x)}{h(x)}. \quad (7)$$

Proof. If u is a solution of the problem then u is a solution of the scalar Oseen equation (1) with the boundary condition $h^{-1}(\partial u/\partial n) + u = g/h$ on $\partial\Omega$. Proposition 2.1 gives uniqueness and the estimate (7).

Let now $\varphi \in C^0(\partial\Omega)$ be such that $T\varphi = 0$. Then $u = E_{2\lambda}^\Omega \varphi$ is a classical solution of the problem (1), (2) with $g = 0$. We have proved that $E_{2\lambda}^\Omega \varphi = u = 0$ on $\bar{\Omega}$. Corollary 3.5 gives that $\varphi = 0$.

$L_{2\lambda}^\Omega$ is a compact linear operator on $C^0(\partial\Omega)$ by Proposition 3.3. Since $E_{2\lambda}^\Omega$ is an integral operator with weakly singular kernel, it is a compact operator on $C^0(\partial\Omega)$ (see for example [17]). Thus the operator $T - (1/2)I$ is compact. Since T is one to one, the Riesz-Schauder theory gives that T is a continuously invertible operator in $C^0(\partial\Omega)$. Clearly, $u = E_{2\lambda}^\Omega T^{-1}g$ is a classical solution of the Robin problem (1), (2).

Corollary 4.2 *Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, there exists $\partial u(z)/\partial n$ at each $z \in \partial\Omega$, $\partial u/\partial n \in C^0(\partial\Omega)$ and (1) hold true. If Ω is unbounded suppose moreover that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then there exists $\varphi \in C^0(\partial\Omega)$ such that $u = E_{2\lambda}^\Omega \varphi$. Moreover,*

$$\int_{\partial\Omega} u \left(\frac{\partial u}{\partial n} - \lambda n_1 u \right) d\sigma_y = \int_{\Omega} |\nabla u|^2 dy < \infty. \quad (8)$$

Proof. Put $h = 1$, $g = \partial u/\partial n + u$. Then u is a classical solution of the Robin problem (1), (2). Theorem 4.1 gives that there exists $\varphi \in C^0(\partial\Omega)$ such that $u = E_{2\lambda}^\Omega \varphi$. Proposition 3.4 and Proposition 3.8 give (8).

5 The boundary condition (3) with $h \geq 0$

We shall look for a solution of the problem (1), (3) in the form of a single layer potential $E_{2\lambda}^\Omega \varphi$ with $\varphi \in C^0(\partial\Omega)$. According to Proposition 3.4 the function $E_{2\lambda}^\Omega \varphi$ is a classical solution of the problem (1), (3) if and only if

$$\frac{1}{2}\varphi + L_{2\lambda}^\Omega \varphi - \lambda n_1 E_{2\lambda}^\Omega \varphi + h E_{2\lambda}^\Omega \varphi = g. \quad (9)$$

Theorem 5.1 *Let $\Omega \subset R^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R \setminus \{0\}$. If $h \in C^0(\partial\Omega)$, $h \geq 0$, then $T = (1/2)I + L_{2\lambda}^\Omega - \lambda n_1 E_{2\lambda}^\Omega \varphi + h E_{2\lambda}^\Omega$ is continuously invertible operator in $C^0(\partial\Omega)$. Fix $g \in C^0(\partial\Omega)$.*

- *If Ω is bounded then there exists unique classical solution u of the problem (1), (3).*
- *If Ω is unbounded then there exists unique classical solution u of the problem (1), (3) such that $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$.*

Moreover, $u = E_{2\lambda}^\Omega T^{-1}g$ and

$$\sup_{x \in \Omega} |u(x)| \leq C \sup_{x \in \partial\Omega} |g(x)|,$$

where a constant C depends only on Ω and λ .

Proof. Suppose first that u is a classical solution of the problem (1), (3) with $g = 0$. According to Corollary 4.2

$$0 = \int_{\partial\Omega} u \left(\frac{\partial u}{\partial n} - \lambda n_1 u + hu \right) d\sigma_y = \int_{\Omega} |\nabla u|^2 dy + \int_{\partial\Omega} hu^2 d\sigma_y.$$

Thus $\nabla u = 0$ in Ω and $hu = 0$ on $\partial\Omega$. Since $\nabla u = 0$ in Ω , the function u is constant on each component of Ω and $0 = \partial u / \partial n - \lambda n_1 u + hu = -\lambda n_1 u$ on $\partial\Omega$. Since u is constant on each component of $\partial\Omega$, we infer that $u = 0$ on $\partial\Omega$. Since u is constant on each component of $\bar{\Omega}$, we deduce that $u \equiv 0$.

If $\varphi \in C^0(\partial\Omega)$, $T\varphi = 0$, then $E_{2\lambda}^\Omega \varphi$ is a classical solution of the problem (1), (3) with $g = 0$. We have proved that $E_{2\lambda}^\Omega \varphi = 0$ on $\partial\Omega$. Corollary 3.5 gives that $\varphi = 0$.

$L_{2\lambda}^\Omega$ is a compact linear operator on $C^0(\partial\Omega)$ by 3.3. Since $E_{2\lambda}^\Omega$ is an integral operator with weakly singular kernel, it is a compact operator on $C^0(\partial\Omega)$ (see for example [17]). Thus the operator $T - (1/2)I$ is compact. Since T is one to one, the Riesz-Schauder theory gives that T is a continuously invertible operator in $C^0(\partial\Omega)$. If $g \in C^0(\partial\Omega)$ then $u = E_{2\lambda}^\Omega T^{-1}g$ is a classical solution of the Robin problem (1), (3).

The operator $E_{2\lambda}^\Omega T^{-1}$ is a linear operator from $C^0(\partial\Omega)$ to $C^0(\bar{\Omega})$. Suppose that $\varphi_n \rightarrow \varphi$ in $C^0(\partial\Omega)$, $E_{2\lambda}^\Omega T^{-1}\varphi_n \rightarrow \psi$ in $C^0(\bar{\Omega})$. If $x \in \Omega$, easy calculation

gives that $E_{2\lambda}^\Omega T^{-1}\varphi_n(x) \rightarrow E_{2\lambda}^\Omega T^{-1}\varphi(x)$. Hence $E_{2\lambda}^\Omega T^{-1}\varphi = \psi$ and the operator $E_{2\lambda}^\Omega T^{-1}$ is closed. The closed graph theorem ([3], Theorem II.1.9) gives that the operator $E_{2\lambda}^\Omega T^{-1}$ is bounded. So, there exists a constant C such that

$$\sup_{x \in \Omega} |E_{2\lambda}^\Omega T^{-1}g(x)| \leq C \sup_{x \in \partial\Omega} |g(x)|$$

for each $g \in C^0(\partial\Omega)$.

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