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Abstract: We define Oseen single layer and double layer potentials and study their properties. Using the integral equation method we prove the existence and uniqueness of an L^q-solution of the Robin problem for the Oseen system.

Keywords: Oseen equations, Robin problem, single layer potential

1 Introduction

The Oseen system is one of the basic system of equations in hydrodynamics. The most studied problem for the Oseen system is the Dirichlet problem (see [6], [1], [2], [3], [4]). We shall study another problem - the Robin problem for the Oseen system. (For the formulation of the problem see for example [14].) Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Denote by $\mathbf{n}^\Omega(\mathbf{x})$ (or shortly $\mathbf{n}$) the outward unit normal of Ω at $\mathbf{x} \in \partial\Omega$. If $\mathbf{u} = (u_1, \dots, u_m)$ is a velocity, and p is a pressure, we define by

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI \tag{1}$$

the corresponding stress tensor, where I denotes the identity matrix and

$$\hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

is the deformation tensor, with $(\nabla\mathbf{u})^T$ as the matrix transposed to $\nabla\mathbf{u}$. Let $\lambda \in R^1 \setminus \{0\}$ be given, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. We shall study the Robin problem for the Oseen system

$$-\Delta\mathbf{u} + 2\lambda\partial_1\mathbf{u} + \nabla p = 0 \quad \text{in } \Omega, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$T(\mathbf{u}, p)\mathbf{n} - \lambda n_1\mathbf{u} + h\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega. \tag{3}$$

(If $h \equiv 0$ we say about the Neumann problem for the Oseen system.) We shall study a so called L^q-solution of the problem (2), (3) for $\mathbf{g} \in L^q(\partial\Omega, R^m)$, i.e. the non-tangential maximal functions of $\mathbf{u}$, $\nabla\mathbf{u}$ and p are in $L^q(\partial\Omega)$ and the condition (3) is fulfilled in the sense of the non-tangential limit. We use the integral equation method. We define Oseen single layer and double layer potentials and prove that they have similar properties like corresponding Stokes potentials. It is a tradition to look for a solution of the Neumann and Robin problems in the form of a single layer potential. It fails for domains with holes (similarly like for the Stokes system). So, we shall look for a solution in the form of a modified single layer potential.

The integral equation method was used for the Neumann problem for the Stokes system - i.e. for $\lambda = 0$ and $h \equiv 0$ (see [22]). If Ω is bounded and q is close to 2 then the Neumann problem for the Stokes system is solvable if and only if

$$\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{w}\ \mathrm{d}\mathcal{H}_2 = 0$$

for all rigid body motions $\mathbf{w}$ (see [22]). For the Oseen system (i.e. $\lambda \in R^1 \setminus \{0\}$) we prove a totally different result:

Let Ω be bounded and $1 < q < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. If $q \neq 0$ suppose moreover that Ω has a boundary of class $\mathcal{C}^1$. If $\mathbf{g} \in L^q(\partial\Omega, R^m)$ then the Robin problem (2), (3) has a unique L^q-solution.

For the exterior Robin problem for the Stokes system we prove the following result:

Let Ω be an unbounded domain with compact Lipschitz boundary and $1 < q < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. If $q \neq 0$ suppose moreover that Ω has a boundary of class $\mathcal{C}^1$. Let $\mathbf{g} \in L^q(\partial\Omega, R^m)$. If $\mathbf{u}$, p is an L^q-solution of the Robin problem (2), (3) then there exists a constant p_∞ and a vector $\mathbf{u}_\infty$ such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. On the other hand if $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$ are given then there exists a unique L^q-solution $\mathbf{u}$, p of the Robin problem (2), (3) such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$.

2 Definition of the problem

Let $\Omega \subset R^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Fix $a > 0$. If $\mathbf{x} \in \partial\Omega$ denote the nontangential approach regions of opening a at the point $\mathbf{x}$ by

$$\Gamma(\mathbf{x}) = \Gamma_a(\mathbf{x}) = \{\mathbf{y} \in \Omega; |\mathbf{x} - \mathbf{y}| < (1 + a)\,\mathrm{dist}(\mathbf{y}, \partial\Omega)\}.$$

If now $\mathbf{v}$ is a vector function defined in Ω we denote the nontangential maximal function of $\mathbf{v}$ on $\partial\Omega$ by

$$\mathbf{v}^*(x) = \sup\{|\mathbf{v}(\mathbf{y})|; \mathbf{y} \in \Gamma(\mathbf{x})\}.$$

It is well known that if $\mathbf{v}^* \in L^q(\partial\Omega)$ for one choice of a, where $1 \leq q < \infty$, then it holds for arbitrary choice of a. (See, e.g. [11] and [26], p. 62.) Next, define the nontangential limit of $\mathbf{v}$ at $\mathbf{x} \in \partial\Omega$

$$\mathbf{v}(\mathbf{x}) = \lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Gamma(\mathbf{x})}} \mathbf{v}(\mathbf{y})$$

whenever the limit exists.

Fix $\lambda \in R^1$, $1 < q < \infty$, $\mathbf{g} \in L^q(\partial\Omega, R^m)$, $h \in L^\infty(\partial\Omega)$. *We say that $\mathbf{u} \in \mathcal{C}^\infty(\Omega, R^m)$, $p \in \mathcal{C}^\infty(\Omega)$ is an L^q-solution of the Robin problem for the Oseen system (2), (3) if (2) holds true, $|\mathbf{u}|^*, |\nabla\mathbf{u}|^*, p^* \in L^q(\partial\Omega)$, there exist the* *nontangential limits of* $\mathbf{u}$*,* $\nabla\mathbf{u}$ *and* p *at almost all points of* $\partial\Omega$ *and (3) holds in the sense of the nontangential limits at almost all points of* $\partial\Omega$*.*

Let $\mathbf{u}$, p be defined on Ω. Denote $\omega = \{\lambda\mathbf{x}; \mathbf{x} \in \Omega\}$, $\tilde{\mathbf{u}}(\mathbf{x}) = (2\lambda)^2\mathbf{u}(\mathbf{x}/(2\lambda))$, $\tilde{p}(\mathbf{x}) = 2\lambda p(\mathbf{x}/(2\lambda))$. Easy calculation yields that $\mathbf{u}$, p is an L^q-solution of the Robin problem for the Oseen system (2), (3) if and only if $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^q-solution of the Robin problem for the Oseen system

$$-\Delta\tilde{\mathbf{u}} + \partial_1\tilde{\mathbf{u}} + \nabla\tilde{p} = 0, \quad \nabla\cdot\tilde{\mathbf{u}} = 0 \quad \text{in } \omega, \tag{4}$$

$$T(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} - \frac{1}{2}n_1\tilde{\mathbf{u}} + \tilde{h}\tilde{\mathbf{u}} = \tilde{\mathbf{g}} \qquad \text{on } \partial\omega, \tag{5}$$

where

$$\tilde{h}(\mathbf{x}) = 2\lambda h(\mathbf{x}/(2\lambda)), \quad \tilde{\mathbf{g}}(\mathbf{x}) = 2\lambda\mathbf{g}(\mathbf{x}/(2\lambda)). \tag{6}$$

So, we can restrict ourselves to the case $2\lambda = 1$.

3 Stokes potentials

Let $\mathbf{x} = [x_1, \dots, x_m] \in R^m$, where $m = 2, 3$. Denote the ball $B(\mathbf{x}; r) = \{\mathbf{y} \in R^m; |\mathbf{x} - \mathbf{y}| < r\}$. For $0 \neq \mathbf{x} \in R^m$ and $j, k \in \{1, \dots, m\}$ we define the Stokes fundamental tensor by

$$E_{jk}(\mathbf{x}) = \frac{1}{8\pi}\left\{\delta_{jk}\frac{1}{|\mathbf{x}|} + \frac{x_j x_k}{|\mathbf{x}|^3}\right\}, \qquad m = 3, \tag{7}$$

$$E_{jk}(\mathbf{x}) = \frac{1}{4\pi}\left[\delta_{jk}\ \ln\frac{1}{|\mathbf{x}|} + \frac{x_j x_k}{|\mathbf{x}|^2}\right], \qquad m = 2, \tag{8}$$

$$Q_k(\mathbf{x}) = \frac{x_k}{\mathcal{H}_{m-1}(\partial B(0;1))|\mathbf{x}|^m}. \tag{9}$$

Here $\delta_{jk} = 1$ for $j = k$, $\delta_{jk} = 0$ otherwise and $\mathcal{H}_k$ denotes the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in R^k.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary and $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$, $1 < q < \infty$. Define the Stokes single layer potential with density $\mathbf{\Psi}$ by

$$(E_\Omega\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} E(\mathbf{x} - \mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y})$$

and the corresponding pressure by

$$(Q_\Omega\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} Q(\mathbf{x} - \mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y})$$

whenever it makes sense. Then the couple $(E_\Omega\mathbf{\Psi}, Q_\Omega\mathbf{\Psi}) \in C^\infty(R^m \setminus \partial\Omega, R^{m+1})$ solves the Stokes system

$$\Delta\mathbf{u} = \nabla p, \quad \nabla\cdot\mathbf{u} = 0 \tag{10}$$

in $R^m \setminus \partial\Omega$. Moreover, $E_\Omega \mathbf{\Psi}(\mathbf{x})$ is the nontangential limit of $E_\Omega \mathbf{\Psi}$ with respect to Ω and $R^m \setminus \overline{\Omega}$ at almost all $\mathbf{x} \in \Omega$. We have $(Q_\Omega \mathbf{\Psi})^* \in L^q(\partial\Omega)$, $|\nabla E_\Omega \mathbf{\Psi}|^* \in L^q(\partial\Omega)$. If Ω is bounded or $m = 2$ or $\int \mathbf{\Psi} \, \mathrm{d}\mathcal{H}_{m-1} = 0$ then $|E_\Omega \mathbf{\Psi}|^* \in L^q(\partial\Omega)$. (See [22].) (If $\Omega \subset R^2$ is unbounded and $\int \mathbf{\Psi} \, \mathrm{d}\mathcal{H}_1 \neq 0$ then $|E_\Omega \mathbf{\Psi}|^* \equiv \infty$ on $\partial\Omega$.)

For $\mathbf{y} \in \partial\Omega$ we define $K^\Omega(\cdot, \mathbf{y}) = T(E(\cdot - \mathbf{y}), Q(\cdot - \mathbf{y}))\, \mathbf{n}^\Omega(\mathbf{y})$ on $R^m \setminus \{\mathbf{y}\}$. We have

$$K^\Omega_{j,k}(\mathbf{x}, \mathbf{y}) = \frac{m}{\mathcal{H}_{m-1}(\partial B(0;1))} \frac{(y_j - x_j)(y_k - x_k)(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{m+2}}.$$

Denote

$$\Pi^\Omega_k(\mathbf{x}, \mathbf{y}) = \frac{2}{\mathcal{H}_{m-1}(\partial B(0;1))} \left\{ -m \frac{(y_k - x_k)(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^{m+2}} + \frac{n^\Omega_k(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^m} \right\}.$$

For $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ we define the Stokes double layer potential with density $\mathbf{\Psi}$ by

$$(D_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} K^\Omega(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}), \quad \mathbf{x} \in R^m \setminus \partial\Omega$$

and the corresponding pressure by

$$(\Pi_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} \Pi^\Omega(\mathbf{x} - \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}), \quad \mathbf{x} \in R^m \setminus \partial\Omega.$$

Then the pair $(D_\Omega \mathbf{\Psi}, \Pi_\Omega \mathbf{\Psi}) \in C^\infty(R^m \setminus \partial\Omega)^{m+1}$ solves the Stokes system (10) in $R^m \setminus \partial\Omega$. For $\mathbf{x} \in \partial\Omega$ we denote

$$(K_\Omega \mathbf{\Psi})(\mathbf{x}) = \lim_{\delta \downarrow 0} \int_{\partial\Omega \setminus B(\mathbf{x}, \delta)} K^\Omega(\mathbf{x}, \mathbf{y}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}),$$

$$(K'_\Omega \mathbf{\Psi})(\mathbf{x}) = \lim_{\delta \downarrow 0} \int_{\partial\Omega \setminus B(\mathbf{x}, \delta)} K^\Omega(\mathbf{y}, \mathbf{x}) \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}).$$

Then K_Ω, K'_Ω are bounded linear operators on $L^q(\partial\Omega, R^m)$. Moreover, there exist the non-tangential limits of $\nabla E_\Omega \mathbf{\Psi}$, $Q_\Omega \mathbf{\Psi}$ and $D_\Omega \mathbf{\Psi}$ at almost all points of $\partial\Omega$. If we denote by $[f]_+$ the non-tangential limit of f with respect to Ω and by $[f]_-$ the non-tangential limit of f with respect to $R^m \setminus \overline{\Omega}$, then

$$[D_\Omega \mathbf{\Psi}]_\pm(\mathbf{x}) = \pm \frac{1}{2} \mathbf{\Psi}(\mathbf{z}) + K_\Omega \mathbf{\Psi}(\mathbf{z}), \tag{11}$$

$$[T(E_\Omega \mathbf{\Psi}, Q_\Omega \mathbf{\Psi})]_\pm \mathbf{n}^\Omega = \pm \frac{1}{2} \mathbf{\Psi} - K'_\Omega \mathbf{\Psi}. \tag{12}$$

(See [22].)

4 Oseen fundamental tensor

If $O_{jk}(\mathbf{x})$, $Z_j(\mathbf{x})$ are tempered distributions then O_{jk}, Z_j is called a fundamental tensor for the Oseen equation (4) in R^m, $m = 2, 3$, if

$$-\Delta O_{jk} + \partial_1 O_{jk} + \partial_j Z_k(\cdot) = \delta_{jk},$$

$$\partial_1 O_{1k} + \ldots + \partial_m O_{mk} = 0$$

for $j, k = 1, \ldots, m$. We are interested in fundamental tensors such that $O_{jk}(\mathbf{x}) \to 0$, $Z_j(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. The existence of such fundamental tensor was proved in [10], §VII.3. The explicit formula of the fundamental tensor of the Oseen system is very complicated. We only gather properties of the fundamental tensor (see [10] or [24]): We have $O_{jk} = O_{kj} \in \mathcal{C}^\infty(R^m \setminus \{0\})$,

$$Z_k(\mathbf{x}) = Q_k(\mathbf{x}), \tag{13}$$

If β is a multi-index, then we have

$$\partial^\beta O_{jk}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m-|\beta|)/2}) \quad \text{as } |\mathbf{x}| \to \infty. \tag{14}$$

If $|\mathbf{z}| \neq |z_1|$ then

$$\lim_{r\to\infty} |O(r\mathbf{z})| r^{(m-1)/2} = 0. \tag{15}$$

If $r > 0$ and $q > 1 + 1/m$ then we have

$$|\nabla O_{jk}| \in L^q(R^m \setminus B(0; r)). \tag{16}$$

Denote

$$R_{jk}(\mathbf{x}) = O_{jk}(\mathbf{x}) - E_{jk}(\mathbf{x}). \tag{17}$$

If $m = 3$ then

$$|\partial^\alpha R(\mathbf{x})| = O(|x|^{-|\alpha|}) \quad \text{as } |\mathbf{x}| \to 0. \tag{18}$$

If $m = 2$ then

$$|R(\mathbf{x})| = O(1) \quad \text{as } |\mathbf{x}| \to 0, \tag{19}$$

$$|\nabla R(\mathbf{x})| = O(\ln |\mathbf{x}|) \quad \text{as } |\mathbf{x}| \to 0, \tag{20}$$

$$|\partial^\alpha R(\mathbf{x})| = O(|x|^{-|\alpha|+1}) \quad \text{as } |\mathbf{x}| \to 0 \quad \text{for } |\alpha| \geq 2. \tag{21}$$

Lemma 4.1. *If $\lambda \neq 0$ and $u_1, \ldots, u_m, p$ are tempered distributions in R^m satisfying (2) in R^m in the sense of distributions, then $u_1, \ldots, u_m, p$ are polynomials.*

Proof. For R^3 [15], Proposition 6.1. The proof is literally the same for other dimensions.

Corollary 4.2. *Let $m = 2$ or $m = 3$. Then there exists a unique fundamental tensor $O_{jk}(\mathbf{x})$, $Z_j(\mathbf{x})$ for the Oseen equation (4) in R^m such that $O_{jk}(\mathbf{x}) \to 0$, $Z_j(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.*

Proof. If $\tilde{O}_{jk}(\mathbf{x})$, $\tilde{Z}_j(\mathbf{x})$ is another such fundamental tensor then $\tilde{O}_{jk} - O_{jk}$, $\tilde{Z}_j - Z_j$ is a solution of the equation (4) in R^m. Lemma 4.1 gives that $\tilde{O}_{jk} - O_{jk} \equiv 0$, $\tilde{Z}_j - Z_j \equiv 0$.

5 Oseen potentials

Let $\Omega \subset R^m$ be an open set with Lipschitz boundary, $m = 2$ or $m = 3$. For $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ with $1 < q < \infty$ define the Oseen single layer potential with density $\mathbf{\Psi}$

$$O_\Omega \mathbf{\Psi}(\mathbf{x}) = \int_{\partial\Omega} O(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}).$$

Clearly $O_\Omega\mathbf{\Psi}$, $Q_\Omega\mathbf{\Psi}$ is a solution of the Oseen equation (4) in $R^m \setminus \partial\Omega$. Denote

$$R_\Omega \mathbf{\Psi}(\mathbf{x}) = \int_{\partial\Omega} R(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}) = O_\Omega\mathbf{\Psi}(\mathbf{x}) - E_\Omega\mathbf{\Psi}(\mathbf{x}).$$

For $\mathbf{y} \in \partial\Omega$ and $\mathbf{x} \in R^m \setminus \{\mathbf{y}\}$ define $K^{\Omega,Os}(\cdot,\mathbf{y}) = T(O(\cdot - \mathbf{y}, Q(\cdot - \mathbf{y})\mathbf{n}^\Omega(\mathbf{y}) - n_1^\Omega O(\cdot - \mathbf{y})/2$, i.e.

$$K^{\Omega,Os}_{j,k}(\mathbf{x},\mathbf{y}) \quad = \quad \mathbf{n}^\Omega(\mathbf{y})\cdot\nabla_{\mathbf{y}} O_{jk}(\mathbf{x}-\mathbf{y}) + \sum_{i=1}^m n_i^\Omega(\mathbf{y})\frac{\partial}{\partial y_k} O_{ji}(\mathbf{x}-\mathbf{y}) \quad (22)$$

$$+ n_k^\Omega(\mathbf{y}) Q_j(\mathbf{x}-\mathbf{y}) + \frac{n_1^\Omega(\mathbf{y})}{2} O_{jk}(\mathbf{x}-\mathbf{y}). \quad (23)$$

Denote

$$\Pi^{\Omega,Os}_k(\mathbf{x},\mathbf{y}) \quad = \quad \mathbf{n}^\Omega(\mathbf{y})\cdot\nabla_{\mathbf{y}} Q_k(\mathbf{x}-\mathbf{y}) + \sum_{i=1}^m n_i^\Omega(\mathbf{y})\frac{\partial}{\partial y_k} Q_i(\mathbf{x}-\mathbf{y}) \quad (24)$$

$$- n_k^\Omega(\mathbf{y}) Q_1(\mathbf{x}-\mathbf{y}) + \frac{n_1^\Omega(\mathbf{y})}{2} Q_k(\mathbf{x}-\mathbf{y}). \quad (25)$$

For $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ we define the Oseen double layer potential with density $\mathbf{\Psi}$ by

$$(D_\Omega^{Os}\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} K^{\Omega,Os}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}), \quad \mathbf{x} \in R^m \setminus \partial\Omega$$

and the corresponding pressure by

$$(\Pi_\Omega^{Os}\mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} \Pi^{\Omega,Os}(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}), \quad \mathbf{x} \in R^m \setminus \partial\Omega.$$

For $\mathbf{x} \in \partial\Omega$ we denote

$$(K_{\Omega,Os}\mathbf{\Psi})(\mathbf{x}) = \lim_{\delta\downarrow 0} \int_{\partial\Omega\setminus B(\mathbf{x},\delta)} K^{\Omega,Os}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}).$$

$$(K'_{\Omega,Os}\mathbf{\Psi})(\mathbf{x}) = \lim_{\delta\downarrow 0} \int\limits_{\partial\Omega\setminus B(\mathbf{x},\delta)} K^{\Omega,Os}(\mathbf{y},\mathbf{x})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}).$$

Lemma 5.1. *Let $m \in N$. Then there exists a constant C such that for all Borel measurable function f, and $\mathbf{x} \in R^m$, $r > 0$, $0 < \alpha < m$, $\beta > 0$*

$$\int\limits_{B(\mathbf{x};r)} \frac{|f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{m-\alpha}}\ \mathrm{d}\mathcal{H}_m(\mathbf{y}) \le Cr^\alpha Mf(x),$$

where

$$Mf(\mathbf{x}) = \sup_{r>0} \int\limits_{B(\mathbf{x};r)} \frac{|f(\mathbf{y})|}{\mathcal{H}_m(B(0;r))}\ \mathrm{d}\mathcal{H}_m(\mathbf{y}).$$

(See [28], Lemma 2.8.3.)

Proposition 5.2. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary. Let $\mathcal{K}$ be a function defined on $\overline{\Omega} \times \partial\Omega$. Suppose that $\mathcal{K}(\mathbf{x},\cdot)$ is Borel measurable, $\mathcal{K}(\cdot,\mathbf{y})$ is continuous on $\overline{\Omega}\setminus\{\mathbf{y}\}$ for all $\mathbf{y} \in \partial\Omega$ and $|\mathcal{K}(\mathbf{x},\mathbf{y})| \le C_1|\mathbf{x}-\mathbf{y}|^{\alpha+1-m}$ with $0 < \alpha < m-1$. For $f \in L^q(\partial\Omega)$, $1 < q < \infty$ define*

$$\mathcal{K}f(\mathbf{x}) = \int\limits_{\partial\Omega} \mathcal{K}(\mathbf{x},\mathbf{y})f(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}). \tag{26}$$

Then there exists a constant C_2 dependent on Ω, q and α such that

$$\|(\mathcal{K}f)^*\|_{L^q(\partial\Omega)} \le C_2\|f\|_{L^q(\partial\Omega)},$$

$\mathcal{K}f$ is finite almost everywhere on $\partial\Omega$, $\mathcal{K}f(\mathbf{x})$ is the nontangential limit of $\mathcal{K}f$ for almost all $\mathbf{x} \in \partial\Omega$ and $\|\mathcal{K}f\|_{L^q(\partial\Omega)} \le C_2\|f\|_{L^q(\partial\Omega)}$.

Proof. There are $\mathbf{z}^1,\dots,\mathbf{z}^k \in \partial\Omega$ and $\delta > 0$ such that $\partial\Omega \subset B(\mathbf{z}^1;\delta)\cup\ldots\cup B(\mathbf{z}^k;\delta)$ and for each $j \in \{1,\dots,k\}$ there is a coordinate system centered at $\mathbf{z}^j$ and a Lipschitz continuous function φ^j such that $B(0;2\delta)\cap\Omega = \{[\mathbf{x}',x_m] \in B(0;2\delta); x_m > \varphi^j(\mathbf{x}')\}$. Choose a constant L such that $|\nabla\varphi^j| \le L$. Let $\mathbf{z} \in \partial\Omega$. Choose j such that $\mathbf{z} \in B(\mathbf{z}^j;\delta)$. Let $\mathbf{x} \in \Gamma(\mathbf{z})$. If $|\mathbf{x}-\mathbf{z}| \ge \delta$ then $\mathrm{dist}(\mathbf{x},\partial\Omega) \ge \delta/(1+a)$ and

$$|\mathcal{K}f(\mathbf{x})| \le C_1\left(\frac{\delta}{1+a}\right)^{\alpha+1-m}\|f\|_{L^1(\partial\Omega)} \le C_3\|f\|_{L^q(\partial\Omega)},$$

where $C_3 = C_1[\delta/(1+a)]^{\alpha+1-m}\mathcal{H}_{m-1}(\partial\Omega)^{(p-1)/p}$. Let now $|\mathbf{x}-\mathbf{z}| < \delta$. For $0 < r \le 1$ put $f_r = f$ on $\partial\Omega\cap B(\mathbf{z}^j,2r\delta)$, $f_r = 0$ elsewhere, $g_r = f - f_r$, $\tilde{f}_1(\mathbf{x}') = f_1(\mathbf{x}',\varphi^j(x'))$. Then

$$|\mathcal{K}g_1(\mathbf{x})| \le C_1\delta^{\alpha+1-m}\|g_1\|_{L^1(\partial\Omega)} \le C_3\|f\|_{L^q(\partial\Omega)}.$$

If $\mathbf{y} \in \partial\Omega$ then $|\mathbf{z}-\mathbf{y}| \leq |\mathbf{z}-\mathbf{x}| + |\mathbf{y}-\mathbf{x}| \leq (1+a)|\mathbf{y}-\mathbf{x}| + |\mathbf{y}-\mathbf{x}|$. According to Lemma 5.1 there exists a constant C_4 such that

$$\max(|\mathcal{K} f_r(\mathbf{z})|, |\mathcal{K} f_r(\mathbf{x})|) \leq \int\limits_{B(\mathbf{z}^j; r2\delta)} C_1 \left(\frac{|\mathbf{y}-\mathbf{z}|}{2+a} \right)^{\alpha+1-m} |f(\mathbf{y})| \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y})$$

$$\leq \int\limits_{\{\mathbf{y}' \in R^{m-1}; |\mathbf{y}'| < r2\delta\}} C_1 \left(\frac{|\mathbf{y}'|}{2+a} \right)^{\alpha+1-m} |\tilde{f}_1(\mathbf{y}')| \sqrt{1+L^2} \, \mathrm{d}\mathcal{H}_{m-1} \leq C_4 r^\alpha M \tilde{f}_1(\mathbf{z}').$$

Thus $(\mathcal{K} f)^*(\mathbf{z})| \leq C_3 \|f\|_{L^q(\partial\Omega)} + C_4 M \tilde{f}_1(\mathbf{z})$. Since there exists a constant C_5 such that $\|Mg\|_{L^q} \leq C_5 \|g\|_{L^q}$ (see [28], Theorem 2.8.2), we have $\|(\mathcal{K} f)^*\|_{L^q(\partial\Omega)} \leq C_3 \|f\|_{L^q(\partial\Omega)} + C_4 C_5 \|\tilde{f}_1\|_{L^q} \leq (C_3 + C_4 C_5) \|f\|_{L^q(\partial\Omega)}$.

Let $\mathbf{z} = [\mathbf{z}', z_m]$ be as above. We use the same notation. $M\tilde{f}_1$ is finite at almost all points of $\mathbf{x}'$ with $|\mathbf{x}'| < \delta$. Suppose that $M\tilde{f}_1(\mathbf{z}') < \infty$. Fix $\epsilon > 0$. We can choose $0 < r \leq 1$ such that $C_4 r^\alpha M \tilde{f}_1(\mathbf{z}') < \epsilon/3$. Then $|\mathcal{K} f_r(\mathbf{z})| < \epsilon/3$. If $\mathbf{x} \in \Gamma(\mathbf{z})$, $|\mathbf{x}-\mathbf{z}| < \delta$ then $|\mathcal{K} f_r(\mathbf{z})| < \epsilon/3$. Since $\mathcal{K} g_r$ is continuous in $\mathbf{z}$ by the Theorem on continuity of parametrized integrals there exist $\rho \in (0,\delta)$ such that $|\mathcal{K} g_r(\mathbf{x}) - \mathcal{K} g_r(\mathbf{z})| < \epsilon/3$ for $|\mathbf{x}-\mathbf{z}| < \rho$. If $\mathbf{x} \in \Gamma(\mathbf{z})$, $|\mathbf{x}-\mathbf{z}| < \rho$ then $|\mathcal{K} f(\mathbf{x}) - \mathcal{K} f(\mathbf{z})| \leq |\mathcal{K} g_r(\mathbf{x}) - \mathcal{K} g_r(\mathbf{z})| + |\mathcal{K} f_r(\mathbf{x})| + |\mathcal{K} f_r(\mathbf{z})| < \epsilon$.

By virtue of limit

$$\|\mathcal{K} f\|_{L^q(\partial\Omega)} \leq \|(\mathcal{K} f)^*\|_{L^q(\partial\Omega)} \leq C_2 \|f\|_{L^q(\partial\Omega)}.$$

Proposition 5.3. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$, and $1 < q < \infty$. If $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ then $O_\Omega \mathbf{\Psi}(\mathbf{z})$ it the non-tangential limit of $O_\Omega \mathbf{\Psi}$ at $\mathbf{z}$ for almost all $\mathbf{z} \in \partial\Omega$. There exists a constant C such that $\|(O_\Omega \mathbf{\Psi})^*\|_{L^q(\partial\Omega)} \leq C \|\mathbf{\Psi}\|_{L^q(\partial\Omega)}$. The operator O_Ω is a compact bounded linear operator in $L^q(\partial\Omega, R^m)$.*

Proof. For $\mathbf{x} \in \partial\Omega$ denote

$$M_1(\mathbf{f})(\mathbf{x}) = \sup\{|\mathbf{f}(\mathbf{y})|; \mathbf{y} \in \Gamma(\mathbf{x}) \cap B(\mathbf{x}; 1)\}.$$

According to [22] there exists a constant C_1 such that $\|M_1(E_\Omega \mathbf{\Psi})\|_{L^q(\partial\Omega)} \leq C_1 \|\mathbf{\Psi}\|_{L^q(\partial\Omega)}$ for $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$. Moreover, if $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ then $E_\Omega \mathbf{\Psi}(\mathbf{z})$ it the non-tangential limit of $E_\Omega \mathbf{\Psi}$ at $\mathbf{z}$ for almost all $\mathbf{z} \in \partial\Omega$. Since there exists a constant C_2 such that $|R(\mathbf{y})| \leq C_2$ for $|\mathbf{y}| \leq 1 + \operatorname{diam} \partial\Omega$, Proposition 5.2 gives that $O_\Omega \mathbf{\Psi}(\mathbf{z})$ it the non-tangential limit of $O_\Omega \mathbf{\Psi}$ at $\mathbf{z}$ for almost all $\mathbf{z} \in \partial\Omega$, and there exists a constant C_3 such that $\|M_1(O_\Omega \mathbf{\Psi})\|_{L^q(\partial\Omega)} \leq C_3 \|\mathbf{\Psi}\|_{L^q(\partial\Omega)}$ for $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$. Since $O_{jk}(\mathbf{y}) \to 0$ as $|\mathbf{y}| \to \infty$, there exists a constant C_4 such that $\|(O_\Omega \mathbf{\Psi})^*\|_{L^q(\partial\Omega)} \leq C_4 \|\mathbf{\Psi}\|_{L^q(\partial\Omega)}$ for $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$.

The operator E_Ω is a compact linear operator on $L^q(\partial\Omega, R^m)$ by [22]. Since $R(\mathbf{x}-\mathbf{y})$ is bounded on $\partial\Omega \times \partial\Omega$, the operator R_Ω is a compact linear operator on $L^q(\partial\Omega, R^m)$ by [9], §4.5.2, Satz 2.

Lemma 5.4. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$, and $1 < q < \infty$. If $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ and $j \in \{1, \dots, m\}$ then*

$$\partial_j R\mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \downarrow 0} \int\limits_{\partial\Omega \setminus B(\mathbf{x};\epsilon)} \partial_j R(\mathbf{x} - \mathbf{y})\mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}) \tag{27}$$

for $\mathbf{x} \in R^m \setminus \partial\Omega$. Define $\partial_j R\mathbf{\Psi}(\mathbf{x})$ by the limit (27) whenever this limit makes sense. Then $\partial_j R$ is a compact linear operator on $L^q(\partial\Omega, R^m)$. There exists a constant C such that if $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ then

$$\|(\partial_j R\mathbf{\Psi})^*\|_{L^q(\partial\Omega)} \leq \|\mathbf{\Psi}\|_{L^q(\partial\Omega)},$$

and $\partial_j R\mathbf{\Psi}(\mathbf{x})$ is the non-tangential limit of $\partial_j R\mathbf{\Psi}$ at almost all $\mathbf{x} \in \partial\Omega$.

Proof. Since there exists a constant C_1 such that $|\partial_j R(\mathbf{x} - \mathbf{y})| \leq C_1|\mathbf{x} - \mathbf{y}|^{1-m-1/2}$, the lemma is an easy consequence of Proposition 5.2.

Proposition 5.5. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$, and $1 < q < \infty$. Then $K'_{\Omega,Os}$ is a bounded linear operator on $L^q(\partial\Omega, R^m)$. If $\mathbf{\Psi} \in L^q(\partial\Omega)$ then $\|(\nabla O_\Omega \mathbf{\Psi})^*\|_{L^q(\partial\Omega)} \leq C\|\mathbf{\Psi}\|_{L^q(\partial\Omega)}$ with C dependent only on Ω and q, $\nabla O_\Omega \mathbf{\Psi}$ has a non-tangential limit at almost all points of $\partial\Omega$, and*

$$[T(O_\Omega \mathbf{\Psi}, Q_\Omega \mathbf{\Psi})]_\pm \mathbf{n}^\Omega - \frac{1}{2} n_1^\Omega O_\Omega \mathbf{\Psi} = \pm\frac{1}{2}\mathbf{\Psi} - K'_{\Omega,Os}\mathbf{\Psi}.$$

Proof. The proposition is an easy consequence of (12), Lemma 5.4 and Lemma 5.3.

Lemma 5.6. $\nabla \cdot Q = 0$, $-\Delta Q + \partial_1 Q - \nabla Q_1 = 0$ *in $R^m \setminus \{0\}$ in the sense of distributions.*

Proof. Denote $h_{Lap}(\mathbf{x}) = -(2\pi)^{-1} \ln |\mathbf{x}|$ for $m = 2$, $h_{Lap}(\mathbf{x}) = (4\Pi)^{-1}|\mathbf{x}|$ for $m = 3$. Then h_{Lap} is a fundamental solution for the Laplace equation. We have $Q = -\nabla h_{Lap}$. Thus

$$\nabla \cdot Q = -\Delta h_{Lap} = 0,$$

$$-\Delta Q_j + \partial_1 Q_j = \Delta \partial_j h_{Lap} - \partial_1 \partial_j h_{Lap} = \partial_j(\Delta h_{Lap} - \partial_1 h_{Lap}) = \partial_j Q_1.$$

Proposition 5.7. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$, and $1 < q < \infty$. If $\mathbf{\Psi} \in L^q(\partial\Omega, R^m)$ then $D_\Omega^{Os}\mathbf{\Psi}$, $\Pi_\Omega^{Os}\mathbf{\Psi}$ is a solution of the Oseen system (4) in $R^m \setminus \partial\Omega$.*

Proof. If $\mathbf{y} \in \partial\Omega$, $k \in \{1, \dots, m\}$ then $[K_{1,k}^{\Omega,Os}(\mathbf{x}, \mathbf{y}), \dots, K_{m,k}^{\Omega,Os}(\mathbf{x}, \mathbf{y}), \Pi_k(\mathbf{x}, \mathbf{y})]$ is a solution of the Oseen system (4) in $R^m \setminus \{\mathbf{y}\}$ by Lemma 5.6. So, $D_\Omega^{Os}\mathbf{\Psi}$, $\Pi_\Omega^{Os}\mathbf{\Psi}$ is a solution of the Oseen system (4) in $R^m \setminus \partial\Omega$.

Proposition 5.8. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$, and $1 < q < \infty$. Then $K_{\Omega,Os}$ is a bounded linear operator on $L^q(\partial\Omega, R^m)$. If $\mathbf{\Psi} \in L^q(\partial\Omega)$ then $\|(D_\Omega^{Os}\mathbf{\Psi})^*\|_{L^q(\partial\Omega)} \le C\|\mathbf{\Psi}\|_{L^q(\partial\Omega)}$ with C dependent only on Ω and q, $D_\Omega^{Os}\mathbf{\Psi}$ has a non-tangential limit at almost all points of $\partial\Omega$, and*

$$[D_\Omega^{Os}\mathbf{\Psi})]_\pm \mathbf{n}^\Omega = \pm\frac{1}{2}\mathbf{\Psi} + K_{\Omega,Os}\mathbf{\Psi}.$$

Proof. The proposition is an easy consequence of (11), Lemma 5.3 and Lemma 5.4.

Proposition 5.9. *Let $\omega \subset R^m$ be a bounded domain with Lipschitz boundary, $\tilde h \equiv 0$, $\tilde{\mathbf{g}} \in L^q(\partial\Omega, R^m)$, $1 < q < \infty$, $m = 2$ or $m = 3$. If $\tilde{\mathbf{u}}$, $\tilde p$ is an L^q-solution of the Neumann problem (4), (5) then*

$$\tilde{\mathbf{u}} = O_\omega\tilde{\mathbf{g}} + D_\omega^{Os}\tilde{\mathbf{u}}, \qquad \tilde p = Q_\omega\tilde{\mathbf{g}} + \Pi_\omega^{Os}\tilde{\mathbf{u}}. \tag{28}$$

Proof. Let $\Omega(j)$ be domains from Lemma 6.1. Green's formula gives (28) for $\Omega(j)$ (see [10], §VII.6). By virtue of Lebesgue lemma be obtain (28) for ω.

6 Regular L^2-solution of the Dirichlet problem

Let $\omega \subset R^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$, $\mathbf{g} \in W^{1,2}(\partial\omega)$. *We say that $\tilde{\mathbf{u}} \in \mathcal{C}^2(\Omega, R^m)$, $\tilde p \in \mathcal{C}^1(\Omega)$ is a regular L^2-solution of the Dirichlet problem (4),*

$$\tilde{\mathbf{u}} = \mathbf{g} \quad \text{on } \partial\omega \tag{29}$$

if $\tilde{\mathbf{u}}$, $\tilde p$ is a solution of the Oseen system (4) in ω, the non-tangential maximal functions $(|\tilde{\mathbf{u}}|)^*, (|\nabla\tilde{\mathbf{u}}|)^*, \tilde p^* \in L^2(\partial\omega)$, there exist the non-tangential limits of $\tilde{\mathbf{u}}$, $\nabla\tilde{\mathbf{u}}$, $\tilde p$ at almost all points of $\partial\omega$, and the Dirichlet condition (29) is fulfilled in the sense of the non-tangential limit at almost all points of $\partial\omega$.

If ω is a bounded open set with connected boundary we shall look for a solution in the form of an Oseen single layer potential $\tilde{\mathbf{u}} = O_\omega\mathbf{\Psi}$, $\tilde p = Q_\omega\mathbf{\Psi}$ with $\mathbf{\Psi} \in L^2(\partial\omega, R^m)$. Let now $G(1), \ldots, G(k)$ be all bounded components of $R^m \setminus \overline{\omega}$. If $k \in N$ we cannot look for a solution of this problem in this form because

$$\int_{\partial G(j)} (O_\omega\mathbf{\Psi}) \cdot \mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1} = 0 \tag{30}$$

by the Divergence theorem. But this is not a necessary condition for the solvability of the problem. Fix open balls $B(j)$ such that $\overline{B}(j) \subset G(j)$. We shall look for a solution of the Dirichlet problem (4), (29) in the form of a modified Oseen single layer potential

$$\tilde{\mathbf{u}} = O_\omega \mathbf{\Psi} + \sum_{j=1}^{k} (D^{Os}_{B(j)} \mathbf{n}^{B(j)}) \int\limits_{\partial G(j)} \mathbf{\Psi} \cdot \mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1}, \tag{31}$$

$$\tilde{p} = Q_\omega \mathbf{\Psi} + \sum_{j=1}^{k} (\Pi^{Os}_{B(j)} \mathbf{n}^{B(j)}) \int\limits_{\partial G(j)} \mathbf{\Psi} \cdot \mathbf{n}^\Omega \, \mathrm{d}\mathcal{H}_{m-1} \tag{32}$$

with $\mathbf{\Psi} \in L^2(\partial\omega, R^m)$.

Lemma 6.1. *If $\Omega \subset R^m$ is a bounded domain with Lipschitz boundary then there is a sequence of domains Ω_j with boundaries of class C^∞ such that*

- $\overline{\Omega}_j \subset \Omega$.
- *There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(\mathbf{y}) \in \Gamma_a(\mathbf{y})$ for each j and each $\mathbf{y} \in \partial\Omega$ and $\sup\{|\mathbf{y} - \Lambda_j(\mathbf{y})|; \mathbf{y} \in \partial\Omega\} \to 0$ as $j \to \infty$.*
- *There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$, $\int_E \omega_j \, d\mathcal{H}_{m-1} = \mathcal{H}_{m-1}(\Lambda_j(E))$, and so that $\omega_j \to 1$ point wise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$.*
- *The normal vectors to Ω_j, $\mathbf{n}(\Lambda_j(\mathbf{y}))$, converge point wise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$, to $\mathbf{n}(\mathbf{y})$.*

(See [27], Theorem 1.12)

Lemma 6.2. *Let $\omega \subset R^m$ be a domain with compact Lipschitz boundary, $1 < q < \infty$, $q' = q/(q-1)$, $\tilde{h} \in L^\infty(\partial\omega)$, $\tilde{\mathbf{g}} \in L^q(\partial\Omega, R^m)$. Let $\tilde{\mathbf{u}}$, $\tilde{p}$ be an L^q-solution of the Robin problem (4), (5). If ω is unbounded suppose moreover $|\tilde{\mathbf{u}}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m)/2})$, $|\nabla\tilde{\mathbf{u}}(\mathbf{x})| + |\tilde{p}(\mathbf{x})| = O(|\mathbf{x}|^{-m/2})$ as $|\mathbf{x}| \to \infty$; $r^{(m-1)/2}\tilde{\mathbf{u}}(r\mathbf{x}) \to 0$ as $r \to \infty$ for $|\mathbf{x}| \neq |x_1|$. If $\mathbf{u}^* \in L^{q'}(\partial\omega)$, then*

$$\int\limits_{\partial\omega} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{u}} \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\omega} \tilde{h} |\tilde{\mathbf{u}}|^2 \, \mathrm{d}\mathcal{H}_{m-1} + 2 \int\limits_{\omega} |\hat{\nabla}\tilde{\mathbf{u}}|^2 \, \mathrm{d}\mathcal{H}_m. \tag{33}$$

Proof. Suppose first that ω is bounded. Let $\omega(j)$ be domains from Lemma 6.1. By virtue of Green's formula and Lebesgue's lemma

$$\int\limits_{\partial\omega} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{u}} \, \mathrm{d}\mathcal{H}_{m-1} = \int\limits_{\partial\omega} \tilde{h} |\tilde{\mathbf{u}}|^2 \, \mathrm{d}\mathcal{H}_{m-1} + \lim_{j\to\infty} \int\limits_{\partial\omega(j)} \tilde{\mathbf{u}} \cdot [T(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} - n_1 \tilde{\mathbf{u}}/2] \, \mathrm{d}\mathcal{H}_{m-1}$$

$$= \int_{\partial\omega} \tilde{h}|\tilde{\mathbf{u}}|^2 \, d\mathcal{H}_{m-1} + \lim_{j\to\infty} \int_{\omega(j)} [2|\hat{\nabla}\tilde{\mathbf{u}}|^2 + \tilde{\mathbf{u}} \cdot (\Delta\tilde{\mathbf{u}} - \nabla p - \partial_1 \tilde{\mathbf{u}})] \, d\mathcal{H}_m$$

$$= \int_{\partial\omega} \tilde{h}|\tilde{\mathbf{u}}|^2 \, d\mathcal{H}_{m-1} + 2 \int_{\omega} |\hat{\nabla}\tilde{\mathbf{u}}|^2 \, d\mathcal{H}_m.$$

Let now ω be unbounded. Define $\tilde{h} = 0$ on $R^m \setminus \partial\omega$.

$$\int_{\partial\omega} \tilde{h}|\tilde{\mathbf{u}}|^2 \, d\mathcal{H}_{m-1} + 2 \int_{\omega} |\hat{\nabla}\tilde{\mathbf{u}}|^2 \, d\mathcal{H}_m = \lim_{r\to\infty} \Bigg[\int_{\partial(\omega\cap B(0;r))} \tilde{h}|\tilde{\mathbf{u}}|^2 \, d\mathcal{H}_{m-1}$$

$$+2 \int_{\omega\cap B(0;r)} |\hat{\nabla}\tilde{\mathbf{u}}|^2 \, d\mathcal{H}_m \Bigg] = \int_{\partial\omega} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{u}} \, d\mathcal{H}_{m-1} + \lim_{r\to\infty} \int_{\partial B(0;r)} \tilde{\mathbf{u}} \cdot [T(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} - n_1 \tilde{\mathbf{u}}/2]$$

$$= \int_{\partial\omega} \tilde{\mathbf{g}} \cdot \tilde{\mathbf{u}} \, d\mathcal{H}_{m-1} + \lim_{r\to\infty} \int_{\partial B(0;1)} r^{m-1} n_1 |\tilde{\mathbf{u}}(r\mathbf{x})|^2/2 \, d\mathcal{H}_{m-1}(\mathbf{x}).$$

There exists a constant C such that $|r^{m-1} n_1 |\tilde{\mathbf{u}}(r\mathbf{x})|^2/2| \leq C$ for $\mathbf{x} \in \partial B(0;1)$. Since $r^{m-1} n_1 |\tilde{\mathbf{u}}(r\mathbf{x})|^2/2 \to 0$, Lebesgue's lemma yields (33).

Proposition 6.3. *Let $\omega \subset R^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Let $\tilde{\mathbf{u}}$, $\tilde{p}$ be a regular L^2-solution of the Dirichlet problem (4), (29) with $\mathbf{g} \equiv 0$. If Ω is unbounded suppose moreover $|\tilde{\mathbf{u}}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m)/2})$, $|\nabla\tilde{\mathbf{u}}(\mathbf{x})| + |\tilde{p}(\mathbf{x})| = O(|\mathbf{x}|^{-m/2})$ as $|\mathbf{x}| \to \infty$; $r^{(m-1)/2}\tilde{\mathbf{u}}(r\mathbf{x}) \to 0$ as $r \to \infty$ for $|\mathbf{x}| \neq |x_1|$. Then $\tilde{\mathbf{u}} \equiv 0$ and $\tilde{p}$ is constant. If ω is unbounded then $\tilde{p} \equiv 0$.*

Proof. Put $h \equiv 0$. By virtue of Lemma 6.2

$$2 \int_{\omega} |\nabla\tilde{\mathbf{u}}|^2 = 0.$$

Since $\hat{\nabla}\tilde{\mathbf{u}} \equiv 0$ there exist an anti-symmetric matrix A and a vector $\mathbf{b}$ such that $\tilde{\mathbf{u}}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ (see [20], Lemma 3.1). Therefore $\tilde{u}_j$ is a harmonic function on ω, $\tilde{u}_j = 0$ on $\partial\omega$. If ω is unbounded then $\tilde{u}_j(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Thus $\tilde{u}_j \equiv 0$ by the maximum principle. Since $\nabla\tilde{p} \equiv 0$ by (4), the function $\tilde{p}$ is constant. If ω is unbounded then $\tilde{p} \equiv 0$ because $\tilde{p}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.

Lemma 6.4. *Let $\omega \subset R^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Let G be a bounded component of $R^m \setminus \overline{\omega}$. Fix an open ball B such that $\overline{B} \subset G$. Set $\mathbf{u} = D_B^{Os}\mathbf{n}^B$ in $R^m \setminus \overline{B}$. Then*

$$\int_{\partial G} \mathbf{u} \cdot \mathbf{n}^\omega \, d\mathcal{H}_{m-1} = \mathcal{H}_{m-1}(\partial G) \neq 0. \tag{34}$$

If $\tilde{G}$ is another bounded component of $R^m \setminus \overline{\omega}$ then

$$\int_{\partial \tilde{G}} \mathbf{u} \cdot \mathbf{n}^\omega \ \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{35}$$

Proof. Denote $\tilde{\mathbf{u}} = D_B^{Os} \mathbf{n}^B$, $\tilde{p} = \Pi_B^{Os} \mathbf{n}^B$ in B. Then there are the non-tangential limits of $\mathbf{u}$ and $\tilde{\mathbf{u}}$ on ∂B and it hods $\tilde{\mathbf{u}} - \mathbf{u} = \mathbf{n}^B$ (see Proposition 5.8). Since $\nabla \cdot \tilde{\mathbf{u}} = 0$, $\nabla \cdot \mathbf{u} = 0$, the divergence theorem gives

$$0 = \int_{\partial(G \setminus B)} \mathbf{u} \cdot \mathbf{n}^{G \setminus B} \ \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial B} \tilde{\mathbf{u}} \cdot \mathbf{n}^B \ \mathrm{d}\mathcal{H}_{m-1} = - \int_{\partial G} \mathbf{u} \cdot \mathbf{n}^\omega \ \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int_{\partial B} \mathbf{n}^B \cdot \mathbf{n}^B \ \mathrm{d}\mathcal{H}_{m-1} = - \int_{\partial G} \mathbf{u} \cdot \mathbf{n}^\omega \ \mathrm{d}\mathcal{H}_{m-1} + \mathcal{H}_{m-1}(\partial G).$$

If $\tilde{G}$ is another bounded component of $R^m \setminus \overline{\omega}$ then (35) is a consequence of the divergence theorem.

Lemma 6.5. *Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $m = 2$ or $m = 3$. Suppose that $\mathbf{\Psi} \in L^2(\partial \Omega, R^m)$ and $O_\Omega \mathbf{\Psi} = 0$ on $\partial \Omega$. If S is a component of $\partial \Omega$ then there exists a constant c_S such that $\mathbf{\Psi} = c_S \mathbf{n}^\Omega$ on $\partial \Omega$.*

Proof. Let ω be a component of $R^m \setminus \partial \Omega$. Then $O_\Omega \mathbf{\Psi}$, $Q_\Omega \mathbf{\Psi}$ is a regular L^2-solution of the Dirichlet problem for the Oseen equation with the zero boundary condition (see Proposition 5.3 and Proposition 5.4). Taking in mind behavior of $O_\Omega \mathbf{\Psi}$ and $Q_\Omega \mathbf{\Psi}$ at infinity, Proposition 6.3 gives that there exists a constant b_ω such that $O_\Omega \mathbf{\Psi} = 0$, $Q_\Omega \mathbf{\Psi} = b_\omega$ in ω. If S is a component of $\partial \Omega$ we choose two components ω and G of $R^m \setminus \partial \Omega$ such that $S \subset \partial \omega \cap \partial G$. According to Proposition 5.5 we have on S

$$\mathbf{\Psi} = [\mathbf{\Psi}/2 - K'_{\Omega,Os} \mathbf{\Psi}] - [-\mathbf{\Psi}/2 - K'_{\Omega,Os} \mathbf{\Psi}] = [T(O_\Omega \mathbf{\Psi}, Q_\Omega \mathbf{\Psi}) \mathbf{n}^\Omega]_+$$

$$-[T(O_\Omega \mathbf{\Psi}, Q_\Omega \mathbf{\Psi}) \mathbf{n}^\Omega]_- = (-b_\omega \mathbf{n}^\Omega) - (-b_G \mathbf{n}^\Omega).$$

Proposition 6.6. *Let $\omega \subset R^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Fix $\mathbf{\Psi} \in L^2(\partial \omega, R^m)$. If ω is a bounded domain with connected boundary define $U \mathbf{\Psi} = O_\omega \mathbf{\Psi}$. In other cases $U \mathbf{\Psi} = \tilde{\mathbf{u}}$, where $\tilde{\mathbf{u}}$ is given by (31). Then $U : L^2(\partial \omega, R^3) \to W^{1,2}(\partial \omega, R^3)$ is a Fredholm operator with index* 0.

- *If ω is unbounded then U is an isomorphism.*
- *If ω is bounded then $U(L^2(\partial \omega, R^m)) = \{\mathbf{u} \in W^{1,2}(\partial \omega); \int_{\partial \omega} \mathbf{u} \cdot \mathbf{n}^\omega = 0\}$. If G is the unbounded component of $R^m \setminus \overline{\omega}$ then the kernel of U is $\{c\mathbf{n}^\omega \chi_{\partial G}; c \in R^1\}$. (Here $\chi_{\partial G}$ denotes the characteristic function of ∂G.)*

Proof. $E_\Omega : L^2(\partial\omega, R^3) \to W^{1,2}(\partial\omega, R^3)$ is a Fredholm operator with index 0 by [22], Theorem 5.4.1. Since $U - E_\Omega$ is a compact operator by Proposition 5.3 and Lemma 5.4, the operator $U : L^2(\partial\omega, R^3) \to W^{1,2}(\partial\omega, R^3)$ is a Fredholm operator with index 0.

Let now $U\mathbf{\Psi} = 0$. Let $G(j)$ be a bounded component of $R^m \setminus \overline{\omega}$. According to (30) and Lemma 6.4 we have

$$0 = \int\limits_{\partial G(j)} \mathbf{n}^\omega \cdot U\mathbf{\Psi} \, \mathrm{d}\mathcal{H}_{m-1} = \mathcal{H}_{m-1}(\partial G(j)) \int_{\partial G(j)} \mathbf{\Psi} \cdot \mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1}.$$

Therefore

$$\int_{\partial G(j)} \mathbf{\Psi} \cdot \mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{36}$$

It means that $0 = U\mathbf{\Psi} = O_\omega \mathbf{\Psi}$. If V is a component of $R^m \setminus \overline{\omega}$ then there exists a constant c_V such that $\mathbf{\Psi} = c_V \mathbf{n}^\omega$ on ∂V (see Lemma 6.5). If V is bounded then $c_V = 0$ by (36).

If ω is unbounded then the kernel of U is trivial. Since U is of index 0, it must be surjective. Thus U is an isomorphism.

Let now ω be bounded. We have proved that the kernel of U is a subset of $\{c\mathbf{n}^\omega \chi_{\partial G}; c \in R^1\}$. So, the dimension of the kernel of U is at most 1. If $\tilde{\mathbf{u}}$ is given by (31) then the divergence theorem gives $\int_{\partial\omega} \mathbf{n}^\omega \cdot \tilde{\mathbf{u}} \, \mathrm{d}\mathcal{H}_{m-1} = 0$. So, the range of U is a subset of $\{\mathbf{u} \in W^{1,2}(\partial\omega); \int_{\partial\omega} \mathbf{u} \cdot \mathbf{n}^\omega = 0\}$. Hence the co dimension of the range of U is at least 1. Since U is a Fredholm operator of index 0, the dimension of the kernel of U and the co dimension of the range of U are equal to 1.

Theorem 6.7. *Let $\omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m = 2$ or $m = 3$. Fix $\mathbf{g} \in W^{1,2}(\partial\omega, R^m)$. Then there exists a regular L^2-solution of the Dirichlet problem (4), (29) if and only if*

$$\int\limits_{\partial\omega} \mathbf{g} \cdot \mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{37}$$

If $\mathbf{u}$, p and $\tilde{\mathbf{u}}$, $\tilde{p}$ are two solutions of the problem, then $\mathbf{u} = \tilde{\mathbf{u}}$ and $p - \tilde{p}$ is constant.

Proof. If there exists a regular L^2-solution of the problem (4), (29), then the divergence theorem gives (37).

Let now (37) holds true. According to Proposition 6.6 there exists $\mathbf{\Psi} \in L^2(\partial\omega, R^m)$ such that $\tilde{\mathbf{u}}$, $\tilde{p}$ given by (31), (32) is a regular L^2-solution of the problem (4), (29). Let now $\mathbf{u}$, p be another solution of the problem. Then $\mathbf{u} - \tilde{\mathbf{u}} \equiv 0$, $p - \tilde{p}$ is constant by Proposition 6.3.

Theorem 6.8. *Let $\Omega \subset R^m$ be an open set, $R^m \setminus \Omega$ be compact, $m = 2$ or $m = 3$. Let $\mathbf{u}$, p be a bounded solution of the Oseen system (4) in Ω. Then* *there exist a number p_∞ and a vector $\mathbf{u}_\infty$ such that $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$, $p(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$. If α is a multi index then $|\partial^\alpha[\mathbf{u}(\mathbf{x}) - \mathbf{u}_\infty]| = O(|\mathbf{x}|^{(1-m-|\alpha|)/2})$, $|\partial^\alpha[p(\mathbf{x}) - p_\infty]| = O(|\mathbf{x}|^{1-m-|\alpha|})$ as $|\mathbf{x}| \to \infty$. Moreover, $r^{(m-1)/2}\mathbf{u}(r\mathbf{x}) \to 0$ as $r \to \infty$ for $|\mathbf{x}| \neq |x_1|$.*

Proof. Fix $r > 0$ such that $R^m \setminus \Omega \subset B(0;r)$ and denote $\omega = R^m \setminus \overline{B(0;r)}$, $\mathbf{g} = \mathbf{u}$ on $\partial\omega$. According to Proposition 6.6 there exists $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ such that $\tilde{\mathbf{u}}$, $\tilde{p}$ given by (31), (32) is a regular L^2-solution of the problem (4), (29). Remark that $\tilde{p} \in L^2(\omega \cap B(0;2r))$, $\tilde{\mathbf{u}} \in W^{1,2}(\omega \cap B(0;2r))$ (see [19], Lemma 2). If α is a multi index then $|\partial^\alpha \tilde{\mathbf{u}}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m-|\alpha|)/2})$, $|\partial^\alpha \tilde{p}(\mathbf{x})| = O(|\mathbf{x}|^{1-m-|\alpha|})$ as $|\mathbf{x}| \to \infty$. Moreover, $r^{(m-1)/2}\tilde{\mathbf{u}}(r\mathbf{x}) \to 0$ as $r \to \infty$ for $|\mathbf{x}| \neq |x_1|$. Denote $\mathbf{v} = \mathbf{u} - \tilde{\mathbf{u}}$, $q = p - \tilde{p}$ in ω, $\mathbf{v} = 0$, $q = 0$ elsewhere. Then $\mathbf{v} \in W^{1,2}_{loc}(R^m)$, $q \in L^2_{loc}(R^m)$, $\nabla \cdot \mathbf{v} = 0$. Moreover, $\mathbf{v}$, q is a solution of the Oseen equation (4) in $R^m \setminus \partial\omega$. Denote $\mathbf{f} = -\Delta\mathbf{v} + \partial_1\mathbf{v} + \nabla q$. Then $\mathbf{f}$ is a compactly supported distribution. Denote $\mathbf{w} = O * \mathbf{f}$, $\eta = Q * \mathbf{f}$. Then $\mathbf{v} - \mathbf{w}$, $q - \eta$ is a solution of the Oseen equation (4) in the whole R^m. If α is a multi index then $|\partial^\alpha \mathbf{w}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m-|\alpha|)/2})$, $|\partial^\alpha \eta(\mathbf{x})| = O(|\mathbf{x}|^{1-m-|\alpha|})$ as $|\mathbf{x}| \to \infty$. Moreover, $r^{(m-1)/2}\mathbf{w}(r\mathbf{x}) \to 0$ as $r \to \infty$ for $|\mathbf{x}| \neq |x_1|$. Since $\mathbf{v} - \mathbf{w}$, $q - \eta$ are bounded solutions of the Oseen equation (2) in R^m, they are constant by Lemma 4.1.

Theorem 6.9. *Let $\omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Let $\mathbf{g} \in W^{1,2}(\partial\omega, R^m)$ be fixed. If $\mathbf{u}$, p is a regular L^2-solution of the Dirichlet problem (4), (29) then there exist a constant p_∞ and a vector $\mathbf{u}_\infty$ such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. On the other hand, if p_∞, $\mathbf{u}_\infty$ are given then there exists a unique regular L^2-solution $\mathbf{u}$, p of the Dirichlet problem (4), (29) such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. Moreover,*

$$\|(\mathbf{u})^* + (\nabla\mathbf{u})^* + (p)^*\|_{L^2(\partial\Omega)} \leq C[|\mathbf{u}_\infty| + |p_\infty| + \|\mathbf{g}\|_{W^{1,2}(\partial\omega,R^m)}] \tag{38}$$

where C depends only on Ω.

Proof. If $\mathbf{u}$, p is a regular L^2-solution of the Dirichlet problem (4), (29) then there exist a constant p_∞ and a vector $\mathbf{u}_\infty$ such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. (See Theorem 6.8.)

Let now $\mathbf{u}_\infty$, p_∞ be given. According to Proposition 6.6 the operator U is an isomorphism from $L^2(\partial\omega, R^m)$ onto $W^{1,2}(\partial\omega, R^m)$. Put $\mathbf{\Psi} = U^{-1}\mathbf{g} - \mathbf{u}_\infty$. Then $\tilde{\mathbf{u}}$, $\tilde{p}$ given by (31), (32) satisfy $\tilde{\mathbf{u}} = \mathbf{g} - \mathbf{u}_\infty$ on $\partial\omega$. Put $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u}_\infty$, $p = \tilde{p} + p_\infty$. Then $\mathbf{u}$, p is a regular L^2-solution of the Dirichlet problem (4), (29) such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. According to properties of Oseen potentials (38) holds true with C depending only on Ω.

If $\mathbf{v}$, q is another solution of that problem then $|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m)/2})$, $|\nabla\mathbf{u}(\mathbf{x}) - \nabla\mathbf{v}(\mathbf{x})| + |p(\mathbf{x}) - q(\mathbf{x})| = O(|\mathbf{x}|^{-m/2})$, $r^{(m-1)/2}|\mathbf{u}(r\mathbf{x}) - \mathbf{v}(r\mathbf{x})| \to 0$ as $r \to \infty$ for $|\mathbf{x}| \neq |x_1|$ (see Theorem 6.8). Proposition 6.3 gives that $\mathbf{u} - \mathbf{v} \equiv 0$, $p - q \equiv 0$.

7 L^2-solutions of the Robin problem

Let $\omega \subset R^m$ be a domain with compact Lipschitz boundary, $m = 2$ or $m = 3$. Let now $G(1), \dots, G(k)$ be all bounded components of $R^m \setminus \overline{\omega}$. If $\tilde{\mathbf{g}} \in L^q(\partial\omega, R^m)$ we shall look for an L^q-solution of the Robin problem (4), (5) in the form of a modified Oseen single layer potential (31), (32) with $\mathbf{\Psi} \in L^q(\partial\omega, R^m)$. According to Proposition 5.3 and Proposition 5.5 the vector functions $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^q-solution of the Robin problem (4), (5) if and only if

$$\tau_{\tilde{h}} \mathbf{\Psi} = \tilde{g},$$

where

$$\tau_{\tilde{h}} \mathbf{\Psi} = \frac{1}{2} \mathbf{\Psi} - K'_{\omega,Os} \mathbf{\Psi} + \tilde{h} O_\omega \mathbf{\Psi} + L_{\tilde{h}} \mathbf{\Psi},$$

$$L_{\tilde{h}} \mathbf{\Psi} = \sum_{j=1}^{m} \left(\int_{\partial G(j)} \mathbf{\Psi} \cdot \mathbf{n} \right) \left[T(D^{Os}_{B(j)} \mathbf{n}^{B(j)}, \Pi^{Os}_{B(j)} \mathbf{n}^{B(j)}) \mathbf{n} + (\tilde{h} - n_1/2) D^{Os}_{B(j)} \mathbf{n}^{B(j)} \right].$$

Proposition 7.1. *Let $\omega \subset R^m$ be an open set with compact Lipschitz boundary, $1 < q < \infty$, $m = 2$ or $m = 3$. Suppose that $q = 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$. If $\tilde{h} \in L^\infty(\partial\omega)$ then $\tau_{\tilde{h}}$ is a Fredholm operator with index 0 in $L^q(\partial\omega, R^m)$.*

Proof. $\frac{1}{2} I - K'_\omega$ is a Fredholm operator with index 0 in $L^2(\partial\omega, R^m)$ by [22], Theorem 5.3.6. If $\partial\omega$ is of class $\mathcal{C}^1$, then K_ω is a compact operator on $L^{q'}(\partial\omega, R^m)$ where $q' = q/(q-1)$ (see [17], p. 232). Therefore K'_ω is a compact operator in $L^q(\partial\omega, R^m)$ and $\frac{1}{2} I - K'_\omega$ is a Fredholm operator with index 0 in $L^q(\partial\omega, R^m)$. Since $\tau_{\tilde{h}} - [\frac{1}{2} I - K'_\omega]$ is a compact operator by Proposition 5.3 and Lemma 5.4, we deduce that $\tau_{\tilde{h}}$ is a Fredholm operator with index 0 in $L^q(\partial\omega, R^m)$.

Proposition 7.2. *Let $\omega \subset R^m$ be a bounded domain with Lipschitz boundary, $1 < q < \infty$, $q' = q/(q-1)$, $\tilde{h} \in L^\infty(\partial\omega)$, $\tilde{h} \geq 0$. Let $\tilde{\mathbf{u}}$, $\tilde{p}$ be an L^q-solution of the Robin problem (4), (5) with $\tilde{\mathbf{g}} \equiv 0$. If $(\tilde{\mathbf{u}})^* \in L^{q'}(\partial\omega)$ then $\tilde{\mathbf{u}} \equiv 0$, $\tilde{p} \equiv 0$.*

Proof. Lemma 6.2 gives that $|\hat{\nabla} \tilde{\mathbf{u}}| = 0$ in ω, $\tilde{h} \tilde{\mathbf{u}} = 0$ on $\partial\omega$. Since $\hat{\nabla} \tilde{\mathbf{u}} \equiv 0$ there exist an anti-symmetric matrix A and a vector $\mathbf{b}$ such that $\tilde{\mathbf{u}}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ (see [20], Lemma 3.1). If $\int_{\partial\omega} \tilde{h} \, d\mathcal{H}_{m-1} > 0$ then $\tilde{h} \tilde{\mathbf{u}} = 0$ gives $\tilde{\mathbf{u}} \equiv 0$ (see [21], Lemma 5.1. Since $\nabla \tilde{p} = \Delta \tilde{\mathbf{u}} - \partial_1 \tilde{\mathbf{u}} = 0$ we infer that $\tilde{p}$ is constant. Since $0 = T(\tilde{\mathbf{u}}, \tilde{p}) \mathbf{n}^\omega - n_1 \tilde{\mathbf{u}}/2 + \tilde{h} \tilde{\mathbf{u}} = -\tilde{p} \mathbf{n}^\omega$ we deduce that $\tilde{p} \equiv 0$.

Let now $\tilde{h} \equiv 0$. If $j \neq 1$ then

$$\partial_j \tilde{p}(\mathbf{x}) = \Delta \tilde{u}_j(\mathbf{x}) - \partial_1 \tilde{u}_j(\mathbf{x}) = -a_{j1},$$

$$\partial_1 \tilde{p}(\mathbf{x}) = \Delta \tilde{u}_1(\mathbf{x}) - \partial_1 \tilde{u}_1(\mathbf{x}) = 0.$$

Thus there exists a constant c such that

$$\tilde{p}(\mathbf{x}) = -\sum_{j=2}^{m} a_{j1} x_j + c.$$

We have

$$0 = T(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}^\omega - n_1\tilde{\mathbf{u}}/2 = -\tilde{p}\mathbf{n}^\omega - n_1\tilde{\mathbf{u}}/2. \tag{39}$$

Thus $n_1^\omega(\tilde{p}+\tilde{u}_1/2) = 0$. The function $\tilde{p}+\tilde{u}_1/2$ is a polynomial of the first order. If $\tilde{p}+\tilde{u}_1/2 \not\equiv 0$ then $M = \{\mathbf{x}; \tilde{p}(\mathbf{x}) + \tilde{u}_1(\mathbf{x})/2 = 0\}$ is a subset of a hyperplane. So, $n_1 = 0$ outside this hyperplane. It is not possible. Hence $\tilde{p}+\tilde{u}_1/2 \equiv 0$ and

$$\sum_{j=2}^m a_{j1}x_j - c = -\tilde{p}(\mathbf{x}) = \frac{\tilde{u}_1(\mathbf{x})}{2} = \sum_{j=2}^m \frac{a_{1j}}{2}x_j + \frac{b_1}{2} = \sum_{j=2}^m \frac{-a_{j1}}{2}x_j + \frac{b_1}{2}.$$

This forces that $a_{1j} = a_{j1} = 0$ and $\tilde{p} = c = -b_1/2$, $\tilde{u}_1 = b_1 = -2c$.

Suppose first that $c = 0$. Then $\tilde{p} = \tilde{u}_1 = 0$. If $j \neq 1$ then (39) gives $n_1\tilde{u}_j = 0$. The function $\tilde{u}_j$ is a polynomial of the first order. If $\tilde{u}_j \not\equiv 0$ then $M_j = \{\mathbf{x}; \tilde{u}_j(\mathbf{x}) = 0\}$ is a subset of a hyperplane. So, $n_1 = 0$ outside this hyperplane. It is not possible. Hence $\tilde{u}_j \equiv 0$.

Let now $c \neq 0$. Fix $\mathbf{z} \in \partial\omega$. We can choose a coordinate system in a such way that $\mathbf{z} = 0$. Denote $p_j = \tilde{u}_j - b_j$. Then $p_j(\mathbf{x}) \to 0$ as $\mathbf{x} \to 0 = \mathbf{z}$. From (39) we get $n_j^\omega = n_1^\omega(p_j + b_j)/b_1$. Since $p_j(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{z}$ we deduce that $\mathbf{n}^\omega(\mathbf{x}) \to \mathbf{b}/|\mathbf{b}|$ or $\mathbf{n}^\omega(\mathbf{x}) \to -\mathbf{b}/|\mathbf{b}|$ as $\mathbf{x} \to \mathbf{z}$. (Since $\partial\omega$ is Lipschitz, it is not possible $\mathbf{n}^\omega(\mathbf{x^k}) \to \mathbf{b}/|\mathbf{b}|$ and $\mathbf{n}^\omega(\mathbf{y^k}) \to \mathbf{b}/|\mathbf{b}|$ for some sequences $\mathbf{y}^k \to \mathbf{z}$, $\mathbf{x}^k \to \mathbf{z}$.) This gives that $\partial\omega$ is of class $\mathcal{C}^1$. Now fix $\mathbf{z} \in \partial\omega$ such that $z_2 = \max\{x_2; \mathbf{x} \in \partial\omega\}$. Then $\mathbf{n}^\omega(\mathbf{z}) = [0, 1, 0, \dots, 0]$. But (39) forces $1 = n_2^\Omega(\mathbf{z}) = n_1^\omega\tilde{u}_j(\mathbf{z})/b_1 = 0$, what is a contradiction.

Theorem 7.3. *Let $\omega \subset R^m$ be a bounded domain with Lipschitz boundary, $m = 2$ or $m = 3$, $\tilde{h} \in L^\infty(\partial\omega)$, $\tilde{h} \geq 0$. Then $\tau_{\tilde{h}}$ is an isomorphism on $L^2(\partial\omega, R^m)$. Fix $\tilde{\mathbf{g}} \in L^2(\partial\omega, R^m)$. Denote $\mathbf{\Psi} = \tau_{\tilde{h}}^{-1}\mathbf{g}$. Let $\tilde{\mathbf{u}}$, $\tilde{p}$ be given by (31), (32). Then $\tilde{\mathbf{u}}$, $\tilde{p}$ is a unique L^2-solution of the Robin problem (4), (5). Moreover,*

$$\|(|\tilde{\mathbf{u}}| + |\nabla\tilde{\mathbf{u}}| + |\tilde{p}|)^*\|_{L^2(\partial\omega)} \leq C\|\tilde{\mathbf{g}}\|_{L^2(\partial\omega)} \tag{40}$$

where C depends only on ω and $\tilde{h}$.

Proof. Let $\mathbf{\Psi} \in L^2(\partial\omega, R^m)$ and $\tau_{\tilde{h}}\mathbf{\Psi} = 0$. Let $\tilde{\mathbf{u}}$, $\tilde{p}$ be given by (31), (32). Then $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the Robin problem (4), (5) with $\tilde{\mathbf{g}} \equiv 0$. Proposition 7.2 gives that $\tilde{\mathbf{u}} \equiv 0$, $\tilde{p} \equiv 0$. According to (30) and Lemma 6.4

$$0 = \int_{\partial G(j)} \tilde{\mathbf{u}} \cdot \mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1} = \mathcal{H}_{m-1}(\partial G(j)) \int_{\partial G(j)} \mathbf{\Psi} \cdot \mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1}.$$

So, (36) holds and $\tilde{\mathbf{u}} = O_\omega\mathbf{\Psi}$, $\tilde{p} = Q_\omega\mathbf{\Psi}$. Let G be an unbounded component of $R^m \setminus \overline{\omega}$. By virtue of Lemma 6.5 and (36) there exists a constant c such that $\mathbf{\Psi} = c\chi_G$. Therefore $0 = \tilde{p} = -c$ (see [20]). This forces that $\mathbf{\Psi} = 0$. Since $\tau_{\tilde{h}}$ is a Fredholm operator with index 0 by Proposition 7.1, it is an isomorphism.

Let now $\tilde{\mathbf{g}} \in L^2(\partial\omega, R^m)$. If $\mathbf{\Psi} = \tau_{\tilde{h}}^{-1}\mathbf{g}$ and $\tilde{\mathbf{u}}$, $\tilde{p}$ are given by (31), (32) then $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the Robin problem (4), (5). The uniqueness follows from Proposition 7.2. The estimate (40) is a consequence of Proposition 5.3 and Proposition 5.5.

Proposition 7.4. *Let $\omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m = 2$ or $m = 3$, $\tilde{h} \in L^\infty(\partial\omega)$, $\tilde{h} \geq 0$, $\tilde{\mathbf{g}} \equiv 0$. If $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the Robin problem (4), (5) such that $\tilde{\mathbf{u}}(\mathbf{x}) \to 0$, $\tilde{p}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, then $\tilde{\mathbf{u}} \equiv 0$, $\tilde{p} \equiv 0$.*

Proof. If α is a multi index then $|\partial^\alpha \tilde{\mathbf{u}}(\mathbf{x})| = O(|\mathbf{x}|^{(1-m-\alpha|)/2})$, $|\partial^\alpha p(\mathbf{x})| = O(|\mathbf{x}|^{1-m-\alpha|})$ as $|\mathbf{x}| \to \infty$, and $r^{(m-1)/2}\mathbf{u}(r\mathbf{x}) \to 0$ as $r \to \infty$ for $|\mathbf{x}| \neq |x_1|$ (see Theorem 6.8). By virtue of Lemma 6.2

$$\int\limits_{\partial\omega} \tilde{h}|\tilde{\mathbf{u}}|^2 \, \mathrm{d}\mathcal{H}_{m-1} + 2 \int\limits_{\omega} |\hat{\nabla}\tilde{\mathbf{u}}|^2 \, \mathrm{d}\mathcal{H}_m = 0.$$

Since $\hat{\nabla}\tilde{\mathbf{u}} \equiv 0$ there exist an anti-symmetric matrix A and a vector $\mathbf{b}$ such that $\tilde{\mathbf{u}}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ (see [20], Lemma 3.1). The relation $\tilde{\mathbf{u}}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ forces $\tilde{\mathbf{u}} \equiv 0$. Since $\nabla\tilde{p} \equiv 0$ by (4), the function $\tilde{p}$ is constant. Hence $\tilde{p} \equiv 0$ because $\tilde{p}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.

Proposition 7.5. *Let $\omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $1 < q < \infty$, $m = 2$ or $m = 3$. Suppose that $q = 2$ or $\partial\omega$ is of class $\mathcal{C}^1$. If $\tilde{h} \in L^\infty(\partial\omega)$, $\tilde{h} \geq 0$ then τ_h is an isomorphism on $L^q(\partial\omega, R^m)$.*

Proof. Let $\mathbf{\Psi} \in L^q(\partial\omega, R^m)$, $\tau_{\tilde{h}}\mathbf{\Psi} = 0$. Since $\tau_{\tilde{h}}$ is a Fredholm operator with index 0 in $L^q(\partial\omega, R^m)$ and in $L^2(\partial\omega, R^m)$ (see Proposition 7.1), we have $\mathbf{\Psi} \in L^2(\partial\omega, R^m)$ by [18], Lemma 9. If $\tilde{\mathbf{u}}$, $\tilde{p}$ are given by (31), (32), then $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the Robin problem (4), (5) with $\tilde{\mathbf{g}} \equiv 0$. Proposition 7.4 gives that $\tilde{\mathbf{u}} \equiv 0$, $\tilde{p} \equiv 0$. So, $\mathbf{\Psi} = 0$ by Proposition 6.6. Since $\tau_{\tilde{h}}$ is a Fredholm operator with index 0 by Proposition 7.1, it is an isomorphism.

Theorem 7.6. *Let $\omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m = 2$ or $m = 3$, $\tilde{h} \in L^\infty(\partial\omega)$, $\tilde{h} \geq 0$. Fix $\tilde{\mathbf{g}} \in L^2(\partial\omega, R^m)$. If $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the Robin problem (4), (5) then there exists a constant p_∞ and a vector $\mathbf{u}_\infty$ such that $p(\mathbf{x}) \to p_\infty$, $\tilde{\mathbf{u}}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. Let now $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$ be given. Denote $\mathbf{\Psi} = \tau_{\tilde{h}}^{-1}[\mathbf{g} + p_\infty \mathbf{n}^\omega + (n_1^\omega - \tilde{h})\mathbf{u}_\infty]$. Let $\tilde{\mathbf{u}}$, $\tilde{p}$ be given by (31), (32). Then $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u}_\infty$, $p = \tilde{p} + p_\infty$ is a unique L^2-solution of the Robin problem (4), (5) such that $p(\mathbf{x}) \to p_\infty$, $\tilde{\mathbf{u}}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. Moreover,*

$$\|(|\mathbf{u}| + |\nabla\mathbf{u}| + |p|)^*\|_{L^2(\partial\omega)} \leq C[\|\tilde{\mathbf{g}}\|_{L^2(\partial\omega)} + |p_\infty| + |\mathbf{u}_\infty] \tag{41}$$

where C depends only on ω and $\tilde{h}$.

Proof. If $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the Robin problem (4), (5) then there exists a constant p_∞ and a vector $\mathbf{u}_\infty$ such that $p(\mathbf{x}) \to p_\infty$, $\tilde{\mathbf{u}}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. (See Theorem 6.8.)

Let now $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$ be given. The operator $\tau_{\tilde h}$ is invertible by Proposition 7.5. Clearly, $\mathbf{u}$, p is an L^2-solution of the Robin problem such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$. The uniqueness follows from Proposition 7.4. The estimate (41) is a consequence of Proposition 5.3 and Proposition 5.5.

8 L^q-solution of the Robin problem

In this section we prove the existence of an L^q-solution of the Robin problem for ω with boundary of class $\mathcal{C}^1$.

Theorem 8.1. *Let $\omega \subset R^m$ be a bounded domain with boundary of class $\mathcal{C}^1$, $m = 2$ or $m = 3$, $1 < q < \infty$, $\tilde h \in L^\infty(\partial\omega)$, $\tilde h \geq 0$. Then $\tau_{\tilde h}$ is an isomorphism on $L^q(\partial\omega, R^m)$. Fix $\tilde{\mathbf{g}} \in L^q(\partial\omega, R^m)$. Denote $\mathbf{\Psi} = \tau_{\tilde h}^{-1}\mathbf{g}$. Let $\tilde{\mathbf{u}}$, $\tilde p$ be given by (31), (32). Then $\tilde{\mathbf{u}}$, $\tilde p$ is a unique L^q-solution of the Robin problem (4), (5). Moreover,*

$$\|(|\tilde{\mathbf{u}}| + |\nabla\tilde{\mathbf{u}}| + |\tilde p|)^*\|_{L^q(\partial\omega)} \leq C\|\tilde{\mathbf{g}}\|_{L^q(\partial\omega)} \tag{42}$$

where C depends only on ω, $\tilde h$ and q.

Proof. $\tau_{\tilde h}$ is a Fredholm operator with index 0 in $L^2(\partial\omega, R^m)$ and in $L^q(\partial\omega, R^m)$ by Proposition 7.1. Since $\tau_{\tilde h}$ is injective in $L^2(\partial\omega, R^m)$ it is injective in $L^q(\partial\omega, R^m)$ (see [18], Lemma 9). Since $\tau_{\tilde h}$ is a Fredholm operator with index 0 in $L^q(\partial\omega, R^m)$ it is an isomorphism.

Let $\mathbf{\Psi} = \tau_{\tilde h}^{-1}\mathbf{g}$, $\tilde{\mathbf{u}}$, $\tilde p$ be given by (31), (32). Clearly, $\tilde{\mathbf{u}}$, $\tilde p$ is an L^q-solution of the Robin problem (4), (5).

Now we show the uniqueness. Let $\tilde{\mathbf{g}} \equiv 0$, $\tilde{\mathbf{u}}$, $\tilde p$ be an L^q-solution of the Robin problem (4), (5). Then $T(\tilde{\mathbf{u}}, \tilde p)\mathbf{n}^\omega - n_1\tilde{\mathbf{u}}/2 = -\tilde h\tilde{\mathbf{u}}$. Proposition 5.9 gives $\tilde{\mathbf{u}} = D^{Os}_\omega\tilde{\mathbf{u}} - O_\omega h\tilde{\mathbf{u}}$ in ω. By virtue of Proposition 5.3 Proposition 5.8 we have $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}/2 + K_{\omega,Os}\tilde{\mathbf{u}} - O_\omega h\tilde{\mathbf{u}}$ in $\partial\omega$. Put $q' = q/(q-1)$. The operator $\tilde{\mathbf{u}} \mapsto K_{\omega,Os}\tilde{\mathbf{u}} - O_\omega h\tilde{\mathbf{u}}$ is compact in $L^q(\partial\omega, R^m)$ and in $L^{q'}(\partial\omega, R^m)$ by Proposition 5.3, Proposition 5.4 and [17], p. 232. Since $\tilde{\mathbf{u}} - K_{\omega,Os}\tilde{\mathbf{u}} + O_\omega h\tilde{\mathbf{u}} = 0$, [18], Lemma 9 gives that $\tilde{\mathbf{u}} \in L^{q'}(\partial\omega, R^m)$. Since $\tilde{\mathbf{u}} = D^{Os}_\omega\tilde{\mathbf{u}} - O_\omega h\tilde{\mathbf{u}}$, Proposition 5.3 and Proposition 5.8 give $(\tilde{\mathbf{u}})^* \in L^{q'}(\partial\omega)$. So, $\tilde{\mathbf{u}} \equiv 0$ by Proposition 7.2.

The estimate (42) is a consequence of Proposition 5.3 and Proposition 5.5.

Theorem 8.2. *Let $\omega \subset R^m$ be an unbounded domain with compact Lipschitz boundary, $m = 2$ or $m = 3$, $\tilde h \in L^\infty(\partial\omega)$, $\tilde h \geq 0$, $1 < q < \infty$. Fix $\tilde{\mathbf{g}} \in L^q(\partial\omega, R^m)$. If $\tilde{\mathbf{u}}$, $\tilde p$ is an L^q-solution of the Robin problem (4), (5) then there exists a constant p_∞ and a vector $\mathbf{u}_\infty$ such that $p(\mathbf{x}) \to p_\infty$, $\tilde{\mathbf{u}}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. Let now $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$ be given. Denote $\mathbf{\Psi} = \tau_{\tilde h}^{-1}[\mathbf{g} + p_\infty\mathbf{n}^\omega + (n_1^\omega - \tilde h)\mathbf{u}_\infty]$. Let $\tilde{\mathbf{u}}$, $\tilde p$ be given by (31), (32). Then $\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u}_\infty$, $p = \tilde p + p_\infty$ is a unique L^q-solution of the Robin problem (4), (5) such that $p(\mathbf{x}) \to p_\infty$, $\tilde{\mathbf{u}}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. Moreover,*

$$\|(|\mathbf{u}| + |\nabla\mathbf{u}| + |p|)^*\|_{L^q(\partial\omega)} \leq C[\|\tilde{\mathbf{g}}\|_{L^q(\partial\omega)} + |p_\infty| + |\mathbf{u}_\infty|] \tag{43}$$

where C depends only on ω, p and $\tilde{h}$.

Proof. If $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^q-solution of the Robin problem (4), (5) then there exists a constant p_∞ and a vector $\mathbf{u}_\infty$ such that $p(\mathbf{x}) \to p_\infty$, $\tilde{\mathbf{u}}(\mathbf{x}) \to \mathbf{u}_\infty$ as $|\mathbf{x}| \to \infty$. (See Theorem 6.8.)

Let now $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$ be given. The operator $\tau_{\tilde{h}}$ is invertible in $L^q(\partial\omega, R^m)$ by Proposition 7.5. Clearly, $\mathbf{u}$, p is an L^q-solution of the Robin problem such that $p(\mathbf{x}) \to p_\infty$, $\mathbf{u}(\mathbf{x}) \to \mathbf{u}_\infty$.

Let now $\tilde{\mathbf{g}} \equiv 0$ and $\tilde{\mathbf{u}}$, $\tilde{p}$ be an L^q-solution of the Robin problem (4), (5) such that $p(\mathbf{x}) \to 0$, $\tilde{\mathbf{u}}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. If $p \geq 2$ then $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the problem (4), (5). Let now $p < 2$. Fix $r > 0$ such that $\partial\omega \subset B(0;r)$ and set $\Omega = \omega \cap B(0;r)$. Define $\tilde{h} = 0$ on $\partial B(0;r)$, $\tilde{\mathbf{g}} = T(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}^\Omega - n_1^\Omega \tilde{\mathbf{u}}/2$ on $\partial B(0;r)$. Then $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^q-solution of the Robin problem (4), (5) in Ω. Theorem 8.1 gives that $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of this problem. Hence $\tilde{\mathbf{u}}$, $\tilde{p}$ is an L^2-solution of the problem (4), (5) in ω. Proposition 7.4 gives that $\tilde{\mathbf{u}} \equiv 0$, $\tilde{p} \equiv 0$.

The estimate (43) is a consequence of Proposition 5.3 and Proposition 5.5.

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