Weak solutions to certain problems in fluid mechanics

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Oscillations in conservation laws

Nonlinear conservation law

$$\partial_t \mathbf{u} + \mathrm{div}_x \mathbb{F}(\mathbf{u}) = 0$$

Linear field equation

$$\partial_t \mathbf{u} + \mathrm{div}_x \mathbb{V} = 0$$

Nonlinear "constitutive" relation

$$\mathbb{F}(\mathbf{u}) = \mathbb{V}$$

Oscillations

$$\int_B \mathbf{u}_\varepsilon \to \int_B \mathbf{u} \text{ for all } B, \ \liminf_{\varepsilon \to 0} \int_B |\mathbf{u}_\varepsilon|^2 \boxed{>} \int_B |\mathbf{u}|^2$$

Convex integration

Field equations, constitutive relations

$$\partial_t \mathbf{u} + \mathrm{div}_x \mathbb{V} = 0, \ \ \mathbb{V} = \mathbb{F}(\mathbf{u})$$

Reformulation, subsolutions

$$\mathbb{V} = \mathbb{F}(\mathbf{u}) \Leftrightarrow G(\mathbf{u}, \mathbb{V}) = E(\mathbf{u}), E(\mathbf{u}) \leq G(\mathbf{u}, \mathbb{V}) < \boxed{\overline{e}(\mathbf{u})}$$

$$E \text{ convex}, \ \ \overline{e} \text{ ``concave''}$$

Oscillatory lemma, oscillatory increments

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0, \ \ \mathbf{u}_\varepsilon \boxed{\rightharpoonup} 0$$

$$E(\mathbf{u} + \mathbf{u}_\varepsilon) \leq G\left(\mathbf{u} + \mathbf{u}_\varepsilon, \mathbb{V} + \mathbb{V}_\varepsilon\right) < \overline{e}(\mathbf{u} + \mathbf{u}_\varepsilon)$$

$$\liminf \int E(\mathbf{u}_\varepsilon) \boxed{\geq} \int \left(\overline{e}(\mathbf{u}) - E(\mathbf{u})\right)^\alpha$$

Abstract Euler system

Equation

$$\partial_t \mathbf{u} + \mathrm{div}_x \left(\frac{(\mathbf{u} + \mathbf{h}[\mathbf{u}]) \odot (\mathbf{u} + \mathbf{h}[\mathbf{u}])}{r[\mathbf{u}]} + \mathbb{H}[\mathbf{u}] \right) = 0, \ \mathrm{div}_x \mathbf{u} = 0$$

$$\mathbf{v} \odot \mathbf{v} \equiv \mathbf{v} \times \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I}$$

$$(0, T) \times \Omega, \ \Omega = \left([-1, 1]|_{\{-1;1\}} \right)^N$$

Energy constraint

$$\frac{1}{2} \frac{|\mathbf{u} + \mathbf{h}[\mathbf{u}]|^2}{r[\mathbf{u}]} = e[\mathbf{u}]$$

Boundary conditions

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0, \ \mathbf{u}(T, \cdot) = \mathbf{u}_T$$

Abstract operators

Control set Q

$$Q \subset (0,T) \times \Omega, \ |Q| = |(0,T) \times \Omega|$$

Boundedness

b maps bounded sets in $L^\infty((0,T) \times \Omega; R^N)$ on bounded sets in $C_b(Q, R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q; R^M) \text{ (uniformly for } (t,x) \in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\rm weak}([0,T]; L^2(\Omega; R^N)) \text{ and weakly-(*) in } L^\infty((0,T) \times \Omega; R^N);$$

Causality

$$\mathbf{v}(t,\cdot) = \mathbf{w}(t,\cdot) \text{ for } 0 \le t \le \tau \le T \text{ implies } b[\mathbf{v}] = b[\mathbf{w}] \text{ in } [(0,\tau] \times \Omega] \cap Q.$$

Subsolutions

Velocities, fluxes

$$\mathbf{v} \in C_{\rm weak}([0,T]; L^2(\Omega; R^N)) \cap L^\infty((0,T)\times\Omega; R^N),\ \mathbf{v}(0,\cdot) = \mathbf{u}_0,\ \mathbf{v}(T,\cdot) = \mathbf{u}_T$$

$$\mathbb{F} \in L^\infty((0,T)\times\Omega; R^{N\times N}_{{\rm sym},0})$$

Field equations, differential constraints

$$\partial_t \mathbf{v} + {\rm div}_x \mathbb{F} = 0,\ {\rm div}_x \mathbf{v} = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega; R^N)$$

Non-linear constraint

$$\mathbf{v} \in C(Q; R^N),\ \mathbb{F} \in C(Q; R^{N\times N}_{{\rm sym},0}),$$

$$\sup_{(t,x)\in Q, t>\tau} \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{h}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{h}[\mathbf{v}])}{r[\mathbf{v}]} - \mathbb{F} + \mathbb{H}[\mathbf{v}]\right] - e[\mathbf{v}] < 0$$

for any $0 < \tau < T$

Subsolution continued

“Implicit” constitutive relation

$$\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]$$

$$\frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]\geq\frac{1}{2}|\mathbf{v}|^2,\ \mathbb{U}\in R^{N\times N}_{0,\mathrm{sym}}$$

$$\boxed{\frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]=\frac{1}{2}|\mathbf{v}|^2}\Leftrightarrow\mathbb{U}=\mathbf{v}\otimes\mathbf{v}-\frac{1}{N}|\mathbf{v}|^2\mathbb{I}$$

Oscillatory lemma

Hypotheses

$$U \subset R \times R^N,\ N = 2,3 \text{ bounded open set}$$

$$\tilde{\mathbf{h}} \in C(U; R^N),\ \tilde{\mathbb{H}} \in C(U; R^{N\times N}_{\mathrm{sym},0}),\ \tilde{e},\ \tilde{r} \in C(U),\ \tilde{r} > 0,\ \tilde{e} \leq \overline{e} \text{ in } U$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{\tilde{\mathbf{h}} \otimes \tilde{\mathbf{h}}}{\tilde{r}} - \tilde{\mathbb{H}}\right] < \tilde{e} \text{ in } U.$$

Conclusion

$$\mathbf{w}_n \in C^\infty_c(U; R^N),\ \mathbb{G}_n \in C^\infty_c(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n) \otimes (\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n)\right] < \tilde{e} \text{ in } U,$$

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^N)$$

$$\liminf_{n\to\infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}}\ \mathrm{d}x\mathrm{d}t \geq \Lambda(\overline{e}) \int_U \left(\tilde{e} - \frac{1}{2}\frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}}\right)^2\ \mathrm{d}x\mathrm{d}t$$

Basic ideas of analysis

Localization

Localizing the result of DeLellis and Széhelyhidi to "small" cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the "small" cubes

Eliminating singular sets

Applying Whitney's decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{\varrho = 0\}$)

Energy and other coefficients depending on solutions

Showing boundedness and continuity of the energy $\overline{e}(\mathbf{u})$ as well as other quantities as the case may be

Expected results

Basic assumption

The set of subsolutions is non-empty

Good news

The problem admits global-in-time (finite energy) weak solutions of any (large) initial data

Bad news

There are infinitely many solutions for given initial data

More bad news

There exist data for which the problem admits infinitely many "admissible" solutions, meaning solutions that dissipate the energy

Example I, Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u})\Big] - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \mathbf{u}$$

Application of convex integration

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \left(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \right) - \Delta \vartheta = - \varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

“Energy”

$$e = \chi(t) - \boxed{\frac{3}{2} \varrho \vartheta [\mathbf{v}]}$$

Existence of weak solutions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta\ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0, T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Dissipative solutions to the Euler-Fourier system

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2,\ \ \varrho_0 > 0,\ \ \vartheta_0 > 0$$

Infinitely many dissipative weak solutions

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in $(0, T)$

Example II, Euler-Korteweg-Poisson system

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho) |\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Reformulation, Step 1

Extending the density

$$\partial_t \varrho + \mathrm{div}_x \tilde{\mathbf{J}} = 0, \ \varrho(0,\cdot) = \varrho_0$$

Flux ansatz

$$\tilde{\mathbf{J}} = \varrho(\mathbf{U}_0 - Z), \ Z = Z(t)$$

$$\partial_t \int_{\mathbb{T}^3} \mathbf{H}[\tilde{\mathbf{J}}] \ \mathrm{d}x + \int_{\mathbb{T}^3} \mathbf{H}[\tilde{\mathbf{J}}] \ \mathrm{d}x = 0$$

$\mathbf{H}$ – standard Helmholtz projection

$$\mathrm{meas} \left\{ x \in \mathbb{T}^3 \ \Big| \ \varrho(t,x) = 0 \right\} = 0 \text{ for any } t \in [0,T]$$

Reformulation, Step 2

Flux ansatz

$$\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{w},\ \ \mathrm{div}_x \mathbf{w} = 0,\ \ \mathbf{w}(0, \cdot) = 0$$

$$\mathbf{w} \in \boxed{C_{\rm weak}([0,T], L^2(\Omega; R^3))} \cup L^\infty((0,T) \times \Omega; R^3)$$

Equations

$$\partial_t \left(\mathbf{w} + \tilde{\mathbf{J}} \right) + \mathrm{div}_x \left(\frac{(\mathbf{w} + \tilde{\mathbf{J}}) \otimes (\mathbf{w} + \tilde{\mathbf{J}})}{\varrho} \right) + \nabla_x p(\varrho) + \left(\mathbf{w} + \tilde{\mathbf{J}} \right) =$$

$$\nabla_x \left(\chi(\varrho) \Delta_x \varrho \right) + \frac{1}{2} \nabla_x \left(\chi'(\varrho) |\nabla_x \varrho|^2 \right) - 4 \mathrm{div}_x \left(\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} \right)$$

$$+ \varrho \nabla_x V$$

Reformulation, Step 3

Final flux ansatz

$$\tilde{\mathbf{J}} = \mathbf{H}[\tilde{\mathbf{J}}] + \nabla_x M, \ \ \mathbf{v} = e^t \left(\mathbf{w} + \mathbf{H}[\tilde{\mathbf{J}}] \right),$$

Equations

$$\mathrm{div}_x \mathbf{v} = 0, \ \ \mathbf{v}(0, \cdot) = \mathbf{H}[\mathbf{J}_0]$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{r} + \mathbb{H} \right) + \nabla_x \Pi = 0$$

Coefficients

$$r = e^t \varrho, \ \ \mathbf{h} = e^t \nabla_x M$$

Driving terms

Convective term

$$\mathbb{H}(t,x) = 4e^t \left(\chi(\varrho) \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} - \frac{1}{3} \chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 \mathbb{I} \right)$$

$$4e^t \left(\frac{1}{3} |\nabla_x V|^2 \mathbb{I} - \nabla_x V \otimes \nabla_x V \right), \ \mathbb{H} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Pressure term

$$\Pi(t,x) = e^t \Big(p(\varrho) + \partial_t M + M - \chi(\varrho) \Delta_x \varrho \Big)$$

$$-e^t \left(\frac{1}{2} \chi'(\varrho) |\nabla_x \varrho|^2 - \frac{4}{3} \chi(\varrho) |\nabla_x \sqrt{\varrho}|^2 + \overline{\varrho} V + \frac{1}{3} |\nabla_x V|^2 \right) + \boxed{\Lambda}$$

$\Lambda -$ a suitable constant

Example III, Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$

Energy functional

Energy in the convex integration ansatz

$$\frac{1}{2}\frac{|\mathbf{v}+\nabla_x\Phi|^2}{\varrho} = \overline{E}[\mathbf{v}]$$

$$\equiv \Lambda(t) - \frac{3}{2}\left(\frac{1}{6}\boxed{|\nabla_x c[\mathbf{v}]|^2} + p_0(\varrho, c[\mathbf{v}]) + \partial_t \nabla_x \Phi\right)$$

Uniform estimates

$$|\nabla_x c| \approx |\mathbf{u}| \text{ needed!}$$

Maximal regularity - Denk, Hieber, Pruess [2007]

$$\partial_t c + \frac{1}{\varrho}\Delta\left(\frac{1}{\varrho}\mathrm{div}_x\left(\varrho\nabla_x c\right)\right) = h,$$