INSTITUTE of MATHEMATICS

ACADEMY of SCIENCES of the CZECH REPUBLIC

Stresses in equilibrium configurations of inextensible nets with slack

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Preprint No. 15-2015

PRAHA 2015

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Abstract The paper deals with nets formed by two families of fibers (cords) which can grow shorter but not longer, in a deformation. The nets are treated as two dimensional continua in the three dimensional space. The inextensibility condition places unilateral constraint on the partial derivatives $y_{,1}$ and $y_{,2}$ of the deformation $y : \Omega \to \mathrm{R}^3$ of the form

$$|y_{,1}(x)| \le 1, \quad |y_{,2}(x)| \le 1,$$

$x = (x_1, x_2) \in \Omega \subset \mathrm{R}^2$. There is no deformation energy, the total energy reduces to the potential energy of the net under external forces. Equilibrium configurations are those of minimum energy. The stresses in equilibrium configurations thus reduce to the reactions to the constraints. Nonzero stresses occur only in tense regions where one or two constrains are satisfied with the equality sign. The paper follows [12] in treating the stress problem via the dual variational problem in the sense of convex analysis. Unlike [12], where stresses are modeled as finitely additive set functions, here a (perhaps more economic) choice of spaces is made that leads to more accessible stresses represented by (countably additive) measures. The present development is made possible by an observation, of independent value, that the space of measures with divergence measure is the dual of another Banach space, in the present context naturally interpreted as the space of strains. Our measures generalize stressfields represented by ordinary functions to account for stress concentrations along folded lines in tension, frequently occurring in equilibrium configurations of the net.

Keywords Unilateral constraints; reaction stress, measures; equilibrium equation

Contents

1 Introduction

The paper deals with nets formed by two families of fibers (cords) which can grow shorter but not longer, in a deformation. The net is treated as a continuum; thus the reference configuration is a two dimensional planar region $\Omega \subset \mathrm{R}^2$ in which the fibers of the first family are imagined as perpendicular to those of the second family and parallel to the coordinate axes. In a deformation, the angles between the fibers of the two families need not remain right. There is no deformation energy. Pipkin [13–14] calls such objects inextensible nets with slack.

The partial derivatives $y_{,1}(x)$, $y_{,2}(x) \in \mathrm{R}^3$ of a deformation $y : \Omega \to \mathrm{R}^3$ of inextensible nets with slack thus must satisfy the unilateral constraints

$$|y_{,1}(x)| \le 1, \quad |y_{,2}(x)| \le 1, \tag{1.1}$$

$x \in \Omega$, where $|\cdot|$ denotes the euclidean norm on R^3. The energy reduces to the potential energy under the external forces

$$\mathsf{E}(y) = -\int_\Omega y \cdot b \, d\mathcal{L}^2 - \int_{\mathcal{S}} y \cdot s \, d\mathcal{H}^1,$$

where the right hand side is the sum of the area and line integrals of the densities of external force over Ω and the free part of the boundary $\mathcal{S} \subset \partial\Omega$. In [14], Pipkin posed the problem of proving the existence of solutions for the energy minimization problem. The problem was solved by Paroni [12], who proved that there always exists an equilibrium state in the class of lipschitzian displacements. Moreover, he analysed the reaction stresses caused by the constraint (1.1) using the dual problem from the convex duality theory [5; Chapter III]. Given the original primal problem, there is always some freedom in the dual problem, related to the choice of the space Y in Section 3, below. While Y is the space of (virtual) strains, the dual space Y^* is the space of stresses. The choice in [12] leads to Y^* formed by finitely additive matrix valued set functions (similar choice is made in a different mechanical problem in [11; Section 5]). In contrast to countably additive set functions (measures), finitely additive set functions have a number of unusual ('paradoxical') properties [25] and their existence is based on nonconstructive methods using the axiom of choice via the Hahn–Banach theorem. This makes them inaccessible.

In this note I start from the same the primal problem as in [12], but make a different choice of the dual problem that leads to stresses represented by (countably additive) matrix valued measures with divergence a measure, called divergence measure fields. These were introduced in [3–4], studied in [21–23], and applied to masonry structures in [7–10]. Stresses represented by measures generalize regular functions, allowing concentrations (singularities) on sets of lower dimensions such as surfaces, lines and even fractal dimensions.

We note that the system of forces in Tchebychev nets (those characterized by the equalities in (1.1)) was analysed by Williams [24] using measures in a way different from here.

The present treatment is based on the crucial observation that the Banach space of divergence measure fields, denoted in the present context by $\mathfrak{S}(\Omega, \mathrm{Lin})$, is a dual of another Banach space; in other words, $\mathfrak{S}(\Omega, \mathrm{Lin})$ has a predual. This fact is of independent interest, especially since the predual is the completion of a certain space

$\mathcal{X}(\Omega, \mathrm{Lin})$ of matrix valued functions on Ω, which is interpreted as the space of virtual strains.

The convex duality theory needs extra conditions to ensure that the primal and dual problems give the same results. Two conditions to this effect are of interest here: the normality condition, which is necessary and sufficient to ensure this coincidence, and Slater's condition, which is a comfortable sufficient condition, and guarantees the existence of the solution of the dual problem also. Slater's condition cannot be satisfied in the present topology on the space of strains (see the remark after Lemma 5.5, below), so normality must be invoked to show that the infima of the primal and dual problems coincide. In the treatment in [12], the employed topology makes Slater's condition equivalent to the 'slack condition' stated below as (6.5). In [12], (6.5) guarantees the coincidence of the infima and the existence of the solution of the dual problem also. Here the coincidence of the infima is proved more generally, but for the existence of the solution of the dual problem (6.5) is still needed.

Notation. If m, n are two positive integers, we denote by Lin the set of all linear maps from R^n to R^m.

If $\Omega \subset \mathrm{R}^n$, we denote by $\mathrm{Lip}(\Omega, \mathrm{R}^m)$ the set of all lipschitzian maps y from the closure $\mathrm{cl}\,\Omega$ of Ω to R^m, i.e., those satisfying

$$|y(x_1) - y(x_2)| \le k|x_1 - x_2|$$

for all x_1, $x_2 \in \mathrm{cl}\,\Omega$ and some constant k. The smallest of all possible k is denoted by $\mathrm{Lip}(y)$ and is called the Lipschitz constant of y. By $\mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ we denote the set of all maps from $\mathrm{Lip}(\Omega, \mathrm{R}^m)$ which vanish on the boundary of Ω. If Z is a finite dimensional vectorspace then $C_c^\infty(\Omega, Z)$ [respectively, $C_0(\Omega, Z)$] is the set of all maps $\varphi : \mathrm{R}^n \to Z$ which are indefinitely differentiable whose support is compact and contained in Ω [respectively, which are continuous and vanish on the boundary of Ω]. If m is a nonnegative integer then $C^m(\mathrm{cl}\,\Omega, Z)$ is the set of all class m maps $\varphi : \Omega \to Z$ such that the map and its derivatives up to order m have continuous extensions to the closure $\mathrm{cl}\,\Omega$ of Ω.

Let $\bar{\mathrm{R}} = \mathrm{R} \cup \{\infty\}$.

2 Deformations of inextensible nets with slack; the role of convexity

Let the reference configuration $\Omega \subset \mathrm{R}^2$ be a bounded, open set with lipschitzian boundary. We shall consider deformations y from $\mathrm{Lip}(\Omega, \mathrm{R}^3)$. Recall that Rademacher's theorem [6; Theorem 3.1.6] says that the Fréchet derivative $\nabla y(x)$ exists for almost every $x \in \Omega$ with respect to the two dimensional Lebesgue measure $\mathcal{L}^2$. Also, by [5; Proposition 2.3, Section X.2.3], $\mathrm{Lip}(\Omega, \mathrm{R}^3)$ coincides with the Sobolev space $W^{1,\infty}(\Omega, \mathrm{R}^3)$ of maps with bounded generalized derivative.

Let $|\cdot|$ denote the euclidean norm on R^3 while $|\cdot|_1$ the l^1 norm on R^2, given by $|a|_1 = |a_1| + |a_2|$ for every $a \in \mathrm{R}^2$. Let Lin be the space of all linear transformations from R^2 to R^3. Let $|\cdot|_{\mathrm{K}}$ be a norm on Lin given by

$$|A|_{\mathrm{K}} = \sup\{|Av| : v \in \mathrm{R}^2,\ |v|_1 \le 1\} \equiv \max\{|Ae_1|, |Ae_2|\}, \tag{2.1}$$

$A \in \mathrm{Lin}$, and let K be a subset of Lin given by

$$\mathrm{K} = \{A \in \mathrm{Lin} : |A|_{\mathrm{K}} \le 1\} \tag{2.2}$$

where e_1, e_2 are the canonical basis vectors in R^2. We observe that K is a compact convex, equilibrated subset of Lin which contains some neighborhood of 0 in Lin. We refer to Appendix A for the basic notions of the convexity theory. We can now summarize the following alternative forms of the inextensibility condition. We shall see that (ii) is most convenient.

Remark 2.1. *Let $y : \mathrm{cl}\,\Omega \to \mathrm{R}^3$. Then the following three conditions are equivalent:*
(i) $y \in \mathrm{Lip}(\Omega, \mathrm{R}^3)$ *and* (1.1) *holds for $\mathcal{L}^2$ almost every $x \in \Omega$;*
(ii) $y \in \mathrm{Lip}(\Omega, \mathrm{R}^3)$ *and*

$$\nabla y(x) \in \mathrm{K} \quad \text{for } \mathcal{L}^2 \text{ almost every } x \in \Omega;$$

(iii) *we have* [24, 12]

$$|y(x_1) - y(x_2)| \le |x_1 - x_2|_1 \tag{2.3}$$

for every $x_1, x_2 \in \Omega$ that can be joined by a line segment lying in Ω.

We note that the specialization of (2.3) to $x_1 - x_2$ parallel either to the first or second axis in R^2 expresses the inextensibility of the cords from the first or second family. We say that y is an admissible deformation if it satisfies Conditions (i)–(iii).

We shall work with Dirichlet's boundary condition, i.e., restrict ourselves to deformations $y \in \mathrm{Lip}(\Omega, \mathrm{R}^3)$ such that

$$y = f \quad \text{on} \quad \partial\Omega \tag{2.4}$$

where $f : \Omega \to \mathrm{R}^3$ is a fixed map. We take into account (2.4) by writing $y = u + f$ where $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^3)$. If the net is subject to the external force represented by an R^3 valued measure b on Ω then the total energy corresponding to a deformation $y = u + f$ is

$$-\int_\Omega y \cdot db = -\int_\Omega u \cdot db - c_0, \quad c_0 := -\int_\Omega f \cdot db.$$

The constraint imposed by the equivalent Conditions (i)–(iii) is taken into account in the following standard way: We define the density of stored energy $w : \mathrm{Lin} \to \bar{\mathrm{R}}$ as the indicator function $I(\cdot, \mathrm{K})$ of K (see Appendix A). Then we define the total energy $\mathsf{E} : \mathrm{Lip}_0(\Omega, \mathrm{R}^3) \to \bar{\mathrm{R}}$ by

$$\mathsf{E}(u) = \int_\Omega w(\nabla u + \nabla f)\, d\mathcal{L}^2 - \int_\Omega u \cdot db - c_0,$$

$y \in \mathrm{Lip}_0(\Omega, \mathrm{R}^3)$. Then $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^3)$ is an equilibrium state if and only if

$$\mathsf{E}(u) = \inf\{\mathsf{E}(u) : u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^3)\}.$$

The stress is conventionally identified with the derivative of the density of the stored energy w with respect to the deformation (strain) variable F. In our case we identify the strain with elements F of Lin, in particular with $\nabla y(x) = \nabla u(x) + \nabla f(x) \in \mathrm{Lin}$, and the stress with the subdifferential $\partial w(F)$.

Remark 2.2. *For* $w = I(\cdot, \mathrm{K})$ *with* K *given by* (2.2) *we have*

$$\partial w(F) = \begin{cases} \{0\} & \text{if } \ F \in \operatorname{int} \mathrm{K}, \\ \{tFe_1 \otimes e_1 : t \geq 0\} & \text{if } \ |Fe_1| = 1,\ |Fe_2| < 1, \\ \{tFe_2 \otimes e_2 : t \geq 0\} & \text{if } \ |Fe_2| = 1,\ |Fe_1| < 1, \\ \{t_1 Fe_1 \otimes e_1 + t_2 Fe_2 \otimes e_2 : t_1,\ t_2 \geq 0\} & \text{if } \ |Fe_2| = |Fe_2| = 1 \\ \emptyset & \text{else,} \end{cases} \tag{2.5}$$

for any $F \in \operatorname{Lin}$.

The three regimes $(2.5)_{2-4}$ correspond to F on the boundary of K, in which case the stress highly nonunique. We shall see in Theorem 6.6, below, that the pointwise value of the density of the stress measure belongs to $\partial w(F)$. It is worth mentioning that the regimes in $(2.5)_{2,3}$ show that the stress is parallel to the actual direction $Fe_i = y_{,i}$ of the fiber in tension, i.e., the one with $|y_{,i}| = 1$. The regime $(2.5)_4$ shows the stress as a mixture of the preceding two when both fibers are in tension.

Proof The subdifferential is related to the normal cone $\mathrm{N}(F, \mathrm{K})$ to K by (7.2), below. We have

$$\mathrm{N}(F, \mathrm{K}) = \begin{cases} \{tFe_1 \otimes e_1 : t \geq 0\} & \text{if } \ |Fe_1| = 1,\ |Fe_2| < 1, \\ \{tFe_2 \otimes e_2 : t \geq 0\} & \text{if } \ |Fe_2| = 1,\ |Fe_1| < 1, \\ \{t_1 Fe_1 \otimes e_1 + t_2 Fe_2 \otimes e_2 : t_1,\ t_2 \geq 0\} & \text{if } \ |Fe_2| = |Fe_2| = 1, \\ \{0\} & \text{else.} \end{cases} \tag{2.6}$$

To prove, e.g., the third regime, we note that if

$$S = t_1 Fe_1 \otimes e_1 + t_2 Fe_2 \otimes e_2 \tag{2.7}$$

where $t_1,\ t_2 \geq 0$ then for any $F' \in \mathrm{K}$,

$$\begin{aligned} (F' - F) \cdot S &= t_1 F' \cdot (Fe_1 \otimes e_1) + t_2 F' \cdot (Fe_2 \otimes e_2) - t_1 - t_2 \\ &= t_1 F' e_1 \cdot Fe_1 + t_2 F' e_2 \cdot Fe_2 - t_1 - t_2 \leq 0 \end{aligned}$$

since $F' \cdot (Fe_1 \otimes e_1) = F' e_1 \cdot Fe_1 \leq |F' e_1||Fe_1| \leq 1$ and the same for the second index. Thus any S of the form (2.7) with $t_1,\ t_2 \geq 0$ belongs to $\mathrm{N}(F, \mathrm{K})$. Conversely, any $S \in \operatorname{Lin}$ can be written in the form (2.7) with $t_1,\ t_2 \in \mathrm{R}$. Assuming that $S \in \mathrm{N}(F, \mathrm{K})$, we must have $(F' - F) \cdot S \leq 0$ for any $F' \in \mathrm{K}$. The choices $F' = -Fe_1 \otimes e_1 + Fe_2 \otimes e_2 \in \mathrm{K}$ and $F' = Fe_1 \otimes e_1 - Fe_2 \otimes e_2 \in \mathrm{K}$ then gives $t_1,\ t_2 \geq 0$, proving the third regime. The remaining regimes are similar but simpler. □

The equilibrium equation

$$\operatorname{div} S + b = 0 \quad \text{on} \quad \Omega$$

certainly reduces the ambiguity in the stress, but the nonuniqueness still remains very large. The point is that the actual stress must satisfy the minimum complementary energy principle to be formulated and employed below.

In Section 3, below, we treat this principle abstractly as the dual problem in the sense of convexity theory. In Section 4 we introduce the spaces of strains and stresses in duality appropriate for the present approach to nets. Then in Section 5 we generalize the net problem slightly by replacing the set K from (2.2) by a general closed, equilibrated convex set $\mathrm{K} \subset \mathrm{Lin}$ containing some neighborhood of 0 and analyse the primal problem, including Paroni's proof of the existence of the solution. Section 6 treats the dual poblem, proves the existence of the solution and discusses the relationship between the solutions of the primal and dual problems. Appendix A outlines basic concepts of the convexity theory. Appendix B describes the relationship between the present approach based on measures and that by Paroni using finitely additive set functions.

3 Duality for convex variational problems

This section summarizes a particular case of the Fenchel duality theory as presented in [5; Chapter III]. Our treatment of the dual problem differs by sign from that in [5] and by the corresponding replacement of the supremum by infimum. This is, of course, entirely equivalent, but more suitable for mechanical applications.

Let V, V^* and Y, Y^* be two pairs consisting of a normed space and its dual, let $\Lambda : \mathrm{V} \to \mathrm{Y}$ be a continuous linear operator and $\Lambda^* : \mathrm{V}^* \to \mathrm{Y}^*$ be its adjoint. Let $C : \mathrm{V} \to \bar{\mathrm{R}}$ and $D : \mathrm{Y} \to \bar{\mathrm{R}}$ be two convex functions and define $\mathsf{E} : \mathrm{V} \to \bar{\mathrm{R}}$ by

$$\mathsf{E}(u) = C(u) + D(\Lambda u)$$

for each $u \in \mathrm{V}$. In this context, the primal problem is

$$(\mathcal{P}) \qquad \text{Find } u \in \mathrm{V} \text{ which minimizes } \mathsf{E} \text{ on } \mathrm{V}.$$

We denote

$$\inf \mathcal{P} = \inf \{\mathsf{E}(u) : u \in \mathrm{V}\}$$

and say that $u \in \mathrm{V}$ is a solution of $\mathcal{P}$ if $\mathsf{E}(u) = \inf \mathcal{P}$.

Assume that both C and D are not identically ∞ and bounded from below by an affine continuous function. Let $\mathsf{G} : \mathrm{Y}^* \to \bar{\mathrm{R}}$ be defined by

$$\mathsf{G}(S) = C^*(-\Lambda^* S) + D^*(S),$$

$S \in \mathrm{Y}^*$, where $C^* : \mathrm{V}^* \to \bar{\mathrm{R}}$ and $D : \mathrm{Y}^* \to \bar{\mathrm{R}}$ are conjugate functions, and recall from Appendix A that C^* and D^* are not identically ∞, lowersemicontinuous, and bounded from below by an affine continuous function.

By definition, the dual problem reads

$$(\mathcal{P}^*) \qquad \text{Find } S \in \mathrm{Y}^* \text{ which minimizes } \mathsf{G} \text{ on } \mathrm{Y}^*.$$

We denote

$$\inf \mathcal{P}^* = \inf \{\mathsf{G}(S) : S \in \mathrm{Y}^*\}$$

and say that $S \in \mathrm{Y}^*$ is a solution of $\mathcal{P}^*$ if $\mathsf{G}(S) = \inf \mathcal{P}^*$.

Generally,

$$\inf \mathcal{P} + \inf \mathcal{P}^* \geq 0,$$

[5; Proposition III.1.1], and there are nonpathological situations where the strict inequality holds. Nevertheless, one is interested in conditions under which

$$\inf \mathcal{P} + \inf \mathcal{P}^* = 0. \tag{3.1}$$

To discuss these conditions, note that the main idea of the duality theory is to consider a family of perturbations of $\mathcal{P}$ with E replaced by a slightly different function. In the present case, one takes $E \in \mathrm{Y}$ close to 0 and replaces E by $\Phi(u, E)$ given by

$$\Phi(u, E) = C(u) + D(\Lambda u + E),$$

$u \in \mathrm{V}$, $E \in \mathrm{Y}$, and one examines the problem

$$h(E) = \inf \{C(u) + D(\Lambda u + E) : u \in \mathrm{V}\}. \tag{3.2}$$

The function $h : \mathrm{Y} \to \bar{\mathrm{R}}$ is convex [5; Lemma III.2.1] and $\inf \mathcal{P} = h(0)$.

Definition 3.1 ([5; Definition III.2.1 and Theorem III.4.1])**.**
(i) The problem $\mathcal{P}$ is said to be normal if h is finite and lowersemicontinuous at 0.
(ii) The problem $\mathcal{P}$ is said to satisfy Slater's condition if there exists an $u_0 \in \mathrm{V}$ such that $C(y_0) < \infty$, $D(\Lambda u_0) < \infty$ and D is continuous at Λu_0.

Proposition 3.2 ([5; Proposition III.2.1 and Theorem III.4.1])**.** *Assume that the functions C and D are lowersemicontinuous and each of them minorized by an affine continuous function. Then*
(i) *$\mathcal{P}$ is normal if and only if* $\inf \mathcal{P}$ *and* $\inf \mathcal{P}^*$ *are finite and* (3.1) *holds;*
(ii) *if $\mathcal{P}$ satisfies Slater's condition then $\mathcal{P}$ is normal and $\mathcal{P}^*$ has a solution.*

The normality is equivalent to (3.1) while Slater's condition is only a sufficient condition, often easier to verify, giving also the extra information about he solution of $\mathcal{P}^*$. Both the normality and Slater's condition depend on the choice of the space Y. The reader is referred to the introduction for the discussion of the role of these conditions in the present treatment and in [12].

We conclude this section with the extremality conditions.

Theorem 3.3 ([5; Proposition III.4.1 and Remark III.4.2])**.** *Let $u \in \mathrm{V}$ and $S \in \mathrm{Y}^*$. Then the following conditions are equivalent:*
(i) *We have* (3.1)*, u is a solution of $\mathcal{P}$ and S is a solution of $\mathcal{P}^*$*;
(ii) $\mathsf{E}(u) + \mathsf{G}(S) = 0$;
(iii) $\Lambda^* S \in \partial C(u)$ *and* $S \in \partial D(\Lambda u)$;
(iv) $u \in \partial C^*(-\Lambda^* S)$ *and* $\Lambda u \in \partial D^*(S)$.

4 Spaces of deformations and stresses in duality

We now introduce the space $\mathcal{X}(\Omega, \mathrm{Lin})$ of virtual strains that is suitable for the treatment below. We shall show that its dual can be identified with the space of stresses represented by (countably additive) measures with values in Lin whose weak divergence is a vector valued measure.

Let m, n be two positive integers, let $\Omega \subset \mathrm{R}^n$ be open bounded and let $C_0(\Omega, \mathrm{Lin})$ the set of all continuous maps $G : \mathrm{R}^n \to \mathrm{Lin}$ vanishing outside Ω.

Definition 4.1. We denote by $\mathfrak{X}(\Omega, \mathrm{Lin})$ the subset of $L^\infty(\Omega, \mathrm{Lin})$ consisting of all E of the form

$$E = \nabla p + G \tag{4.1}$$

$$\text{where} \quad p \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m), \quad G \in C_0(\Omega, \mathrm{Lin}), \tag{4.2}$$

and put

$$|E|_{\mathfrak{X}} = \inf \left\{ |p|_\infty + |G|_\infty : \text{ where } (p, G) \text{ satisfy (4.1) and (4.2)} \right\}$$

with $|\cdot|_\infty$ the maximum norm. $|\cdot|_{\mathfrak{X}}$ is a norm which makes the set $\mathfrak{X}(\Omega, \mathrm{Lin})$ an incomplete normed linear space. (Its completion is the quotient space X/X_0 defined in the proof of Theorem 4.6, below.)

Clearly, $E_k \to E$ in $\mathfrak{X}(\Omega, \mathrm{Lin})$ if and only if

$$\left.\begin{array}{c} E_k = \nabla p_k + G_k, \quad E = \nabla p + G \\ \text{where } p_k, p \in \mathrm{Lip}_0(\Omega, \mathrm{Lin}) \text{ and } G_k, G \in C_0(\Omega, \mathrm{Lin}) \text{ satisfy} \\ |p_k - p|_\infty \to 0, \quad |G_k - G|_\infty \to 0. \end{array}\right\} \tag{4.3}$$

As an example, let $m = n = 1$, $\Omega = (0, 2\pi)$, $E_k = p_k'$, where $p_k \in \mathrm{Lip}_0(\Omega, \mathrm{R})$ is given by $p_k = k^{-1} \sin(k^2 x)$, $x \in (0, 2\pi)$. Then $E_k \to 0$ in $\mathfrak{X}(\Omega, \mathrm{R})$; but note that E_k does not converge to E pointwise $\mathscr{L}^1$ almost everywhere in Ω; the sequence $|E_k|_{L^\infty}$ is not even bounded.

From now on we make the standing assumption that $\Omega \subset \mathrm{R}^n$ is a bounded open set with lipschitzian boundary and derive some properties of $\mathfrak{X}(\Omega, \mathrm{Lin})$.

Remarks 4.2.

(i) *The set* $C_0(\Omega, \mathrm{Lin})$ *is dense in* $\mathfrak{X}(\Omega, \mathrm{Lin})$; *in fact for every* $E \in \mathfrak{X}(\Omega, \mathrm{Lin})$ *there exists a sequence* $E_k \in C_0(\Omega, \mathrm{Lin})$ *such that*

$$\left.\begin{array}{c} E_k \to E \quad \text{in} \quad \mathfrak{X}(\Omega, \mathrm{Lin}), \\ E_k \to E \quad \text{for } \mathscr{L}^n \text{ almost every point of } \Omega, \\ |E_k|_{L^\infty} \le |E|_{L^\infty}; \end{array}\right\} \tag{4.4}$$

(ii) *if* $E \in \mathfrak{X}(\Omega, \mathrm{Lin})$ *and* $\varphi \in C^1(\mathrm{cl}\,\Omega)$ *then* $\varphi E \in \mathfrak{X}(\Omega, \mathrm{Lin})$;

(iii) *if* $E_k \to E$ *in* $\mathfrak{X}(\Omega, \mathrm{Lin})$ *and the sequence* $|E_k|_{L^\infty}$ *is bounded, then*

$$E_k \rightharpoonup^* E \quad \text{in} \quad L^\infty(\Omega, \mathrm{Lin}).$$

Proof (i): Represent E in the form (4.1), (4.2). Let $\varphi : \mathrm{Lin} \to \bar{\mathrm{R}}$ be given by

$$\varphi(E) = \begin{cases} 0 & \text{if } |E| \le |\nabla p|_{L^\infty}, \\ \infty & \text{else,} \end{cases}$$

$E \in \mathrm{Lin}$. The considerations similar to, but simpler than, those in [5; Proof of Proposition 2.6, Section X.2.3] show that there exists a sequence $p_k \in C_c^\infty(\Omega, \mathrm{R}^m)$ such that

$$|p - p_k|_\infty \to 0,$$

$$\nabla p_k \to \nabla p \quad \text{for } \mathscr{L}^n \text{ almost every point of } \Omega,$$

$$|\nabla p_k|_{L^\infty} \le |\nabla p|_{L^\infty}.$$

Setting $E_k = \nabla p_k + G$ we obtain the assertion.

(ii): Representing E in the form (4.1), (4.2), we find $\varphi E = \nabla q + G$ where $q = \varphi p \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ and $G = -p \otimes \nabla\varphi + \varphi E \in C_0(\Omega, \mathrm{Lin})$.

(iii): Since the sequence $|E_k|_{L^\infty}$ is bounded, it suffices to verify

$$\int_\Omega E_k \cdot T \, d\mathcal{L}^n \to \int_\Omega E \cdot T \, d\mathcal{L}^n$$

for testfunctions T from any dense subset of $L^1(\Omega, \mathrm{Lin})$; e.g., from $C_c^\infty(\Omega, \mathrm{Lin})$. Using the characterization (4.3), we have

$$\begin{aligned}\int_\Omega E_k \cdot T \, d\mathcal{L}^n &= \int_\Omega (\nabla p_k + G_k) \cdot T \, d\mathcal{L}^n \\ &= -\int_\Omega p_k \cdot \operatorname{div} T \, d\mathcal{L}^n + \int_\Omega G_k \cdot T \, d\mathcal{L}^n \\ &\to -\int_\Omega p \cdot \operatorname{div} T \, d\mathcal{L}^n + \int_\Omega G \cdot T \, d\mathcal{L}^n \\ &= \int_\Omega E \cdot T \, d\mathcal{L}^n. \quad \square\end{aligned}$$

Definitions and Propositions 4.3 ([2; Chapter 1])**.**

(i) If Z is a finite dimensional inner product space, we denote by $\mathcal{M}(\Omega, Z)$ the set of all Z valued Borel measures on R^n which vanish outside Ω.

(ii) For any $S \in \mathcal{M}(\Omega, Z)$ we denote by $|S|$ the total variation measure of S, i.e., the smallest nonnegative measure in $\mathcal{M}(\Omega, \mathrm{R})$ such that $|S(A)| \le |S|(A)$ for any Borel subset A of R^n; alternatively, for any Borel subset A of R^n,

$$|S|(A) = \sup\Big\{\sum_{i=1}^p |S(B_i)| : B_1, \dots, B_p \text{ is a Borel partition of } A\Big\}.$$

The total variation $|S|_{\mathcal{M}}$ is defined by

$$|S|_{\mathcal{M}} = |S|(\Omega).$$

(iii) If $\phi \in \mathcal{M}(\Omega, \mathrm{R})$ and $\alpha : \Omega \to Z$ a ϕ integrable map, then $\alpha\phi$ is the measure in $\mathcal{M}(\Omega, Z)$ given by $(\alpha\phi)(A) = \int_A \alpha \, d\phi$ for any Borel subset A of R^n.

(iv) The polar decomposition of a measure $S \in \mathcal{M}(\Omega, Z)$ asserts that there exists a map $S^\circ : \Omega \to Z$ such that $S = S^\circ|S|$; the map S° is $|S|$ essentially unique and $S^\circ(x) = 1$ for $|S|$ almost every $x \in \Omega$. We call S° the direction vector of S.

Definitions 4.4.

(i) We denote by $\mathfrak{S}(\Omega, \mathrm{Lin})$ the set $S \in \mathcal{M}(\Omega, \mathrm{Lin})$ for which there exists a measure $\operatorname{div} S \in \mathcal{M}(\Omega, \mathrm{R}^m)$ such that

$$\int_\Omega \nabla p \cdot dS + \int_\Omega p \cdot d\operatorname{div} S = 0$$

for every continuously differentiable $p : \mathrm{R}^n \to \mathrm{R}^m$ with compact support contained in Ω. We define the norm on $\mathfrak{S}(\Omega, \mathrm{Lin})$ by

$$|S|_{\mathfrak{S}} = \max\big\{|S|_{\mathcal{M}}, |\operatorname{div} S|_{\mathcal{M}}\big\}.$$

(ii) We define the dual pairing between $\mathfrak{S}(\Omega, \mathrm{Lin})$ and $\mathfrak{X}(\Omega, \mathrm{Lin})$ by

$$\langle S, E\rangle = \int_\Omega G \cdot dS - \int_\Omega p \cdot d\operatorname{div} S$$

for any $S \in \mathfrak{S}(\Omega, \mathrm{Lin})$ and $E \in \mathfrak{X}(\Omega, \mathrm{Lin})$, where G and p is any pair satisfying (4.1), (4.2). It is easily seen that the value of $\langle S, E\rangle$ is independent of the choice of G and p.

Clearly,

$$\langle S, E\rangle \le |S|_{\mathfrak{S}} |E|_{\mathfrak{X}} \tag{4.5}$$

for every $S \in \mathfrak{S}(\Omega, \mathrm{Lin})$ and $E \in \mathfrak{X}(\Omega, \mathrm{Lin})$ and if $E \in C_0(\Omega, \mathrm{Lin})$ then $\langle S, E\rangle$ reduces to the usual integral, i.e.,

$$\langle S, E\rangle = \int_\Omega E \cdot dS. \tag{4.6}$$

In view of this, we interpret $\langle S, E\rangle$ as an extension of the above integral to the case when it does not exist classically:

Example 4.5. Let $n = 2$, $m = 1$, $\Omega = (-\pi, \pi)^2$, $p(x) = |\sin x_1 \sin x_2|$, $x \in \operatorname{cl}\Omega$, and let $S = e_1 \mathscr{H}^1 \llcorner I$ where $e_1 = (1, 0)$, $I = \{te_1 : -\pi < t < \pi\}$, and $\mathscr{H}^1 \llcorner I$ is the restriction of the 1 dimensional Hausdorff measure to I. One easily sees that $p \in \mathrm{Lip}_0(\Omega, \mathrm{R}^1)$ and hence $E := \nabla p \in \mathfrak{X}(\Omega, \mathrm{Lin})$. Also, $S \in \mathfrak{S}(\Omega, \mathrm{Lin})$ and $\operatorname{div} S = 0$. The function p is not differentiable at any point of I, which is the support of S, and hence the integral on the right hand side of (4.6) does not make sense. However, the duality $\langle S, E\rangle$ does exist and one finds from the definition that $\langle S, E\rangle = 0$. That this value is natural is seen as follows. By Remark 4.2(i) and its proof the element $E = \nabla p$ can be approximated by a sequence $E_k = \nabla p_k$, where $p_k \in C^\infty_{\mathrm{c}}(\Omega, \mathrm{R}^1)$. For these approximations, the integral is well defined and one finds that

$$\langle S, E_k\rangle = \int_\Omega E_k \cdot dS = 0.$$

On the other hand, by (4.5),

$$|\langle S, E\rangle - \langle S, E_k\rangle| \le |S|_{\mathfrak{S}} |E - E_k|_{\mathfrak{X}} \to 0$$

and thus necessarily $\langle S, E\rangle = 0$. □

Theorem 4.6. *The dual $\mathfrak{X}(\Omega, \mathrm{Lin})^*$ is isometrically isomorphic to $\mathfrak{S}(\Omega, \mathrm{Lin})$ under the identification of $L \in \mathfrak{X}(\Omega, \mathrm{Lin})^*$ with $S \in \mathfrak{S}(\Omega, \mathrm{Lin})$ by $\langle L, E\rangle = \langle S, E\rangle$ for any $E \in \mathfrak{X}(\Omega, \mathrm{Lin})$.*

Note that an incomplete normed space X_0 and its completion X have essentially identical duals since each continuous functional on X_0 can be extended by continuity to X. Thus $\mathfrak{S}(\Omega, \mathrm{Lin})$ is also the dual of the completion of $\mathfrak{X}(\Omega, \mathrm{Lin})$ under the norm $|\cdot|_{\mathfrak{X}}$.

Proof Consider the Banach space $\mathrm{X} = C_0(\Omega, \mathrm{Lin}) \times C_0(\Omega, \mathrm{R}^m)$ with the norm

$$|(G, p)| = |G|_\infty + |p|_\infty$$

for any $(G, p) \in \mathrm{X}$. By the double application of the Riesz representation theorem [18; Theorem 6.19], the dual X^* is isometrically isomorphic to $\mathscr{M} = \mathscr{M}(\Omega, \mathrm{Lin}) \times \mathscr{M}(\Omega, \mathrm{R}^m)$ with the norm

$$|(S,\delta)| = \max\{|S|_{\mathcal{M}}, |\delta|_{\mathcal{M}}\}.$$

The identification with the dual is

$$\langle (G,p),(S,\delta)\rangle = \int_\Omega G\cdot dS + \int_\Omega p\cdot d\delta$$

for any $(G,p)\in \mathrm{X}$ and $(S,\delta)\in\mathcal{M}$. Let X_0 be the closed linear subspace of X defined by

$$\mathrm{X}_0 = \{(G,p)\in \mathrm{X} : p\in C_0(\Omega,\mathrm{R}^m) \text{ is of class } 1 \text{ and } G=\nabla p\}.$$

The quotient space $\mathrm{X}/\mathrm{X}_0 = \{\xi = (G,p)+\mathrm{X}_0 : (G,p)\in \mathrm{X}\}$ has the norm

$$|\xi| = \inf\{|(G,p)| : (G,p)\in\xi\}$$

for any $\xi\in \mathrm{X}/\mathrm{X}_0$. If $E\in\mathfrak{X}(\Omega,\mathrm{Lin})$ then all pairs (G,p) as in (4.1) belong to the same class $\xi\in\mathrm{X}/\mathrm{X}_0$ and we can therefore define the map $T:\mathfrak{X}(\Omega,\mathrm{Lin})\to\mathrm{X}/\mathrm{X}_0$ by setting TE = the class determined by (G,p). Then T is an isometric map which maps $\mathfrak{X}(\Omega,\mathrm{Lin})$ onto a dense set of X/X_0. By the general theory [19; Subsections 4.8 & 4.9], the dual of X/X_0 is the set of all $(S,\delta)\in\mathrm{X}^*$ such that $\langle (G,p),(S,\delta)\rangle = 0$ for all $(G,p)\in\mathrm{X}_0$. One easily finds that the last condition is satisfied if and only if $S\in\mathfrak{S}(\Omega,\mathrm{Lin})$ and $\delta = \operatorname{div} S$. □

Proposition 4.7. *Let $S\in\mathfrak{S}(\Omega,\mathrm{Lin})$ and $E\in\mathfrak{X}(\Omega,\mathrm{Lin})$. Then*

(i) *we have*

$$\langle S,E\rangle \le |S|_{\mathcal{M}}|E|_{L^\infty};$$

(ii) *there exists a signed measure $\langle\langle S,E\rangle\rangle$ on Ω such that*

$$\langle S,\varphi E\rangle = \int_\Omega \varphi\, d\langle\langle S,E\rangle\rangle$$

for every $\varphi\in C^1(\mathrm{cl}\,\Omega)\cap C_0(\Omega)$. The total variation satisfies

$$|\langle\langle S,E\rangle\rangle|_{\mathcal{M}} \le |S|_{\mathcal{M}}|E|_{L^\infty}.$$

If $E\in C_0(\Omega,\mathrm{Lin})$ then $\langle\langle S,E\rangle\rangle = E\cdot S$ is the scalar product of the measure S with a continuous function E.

(iii) *The measure $\langle\langle S,E\rangle\rangle$ is absolutely continuous with respect to the total variation measure $|S|$ with the density δ that satisfies $|\delta|\le|E|_{L^\infty}$ at $|S|$ almost every point of Ω.*

Proof (i): If $E\in C_0(\Omega,\mathrm{Lin})$ then

$$\langle S,E\rangle = \int_\Omega E\cdot dS \le |S|_{\mathcal{M}}|E|_{L^\infty}.$$

If $E\in\mathfrak{X}(\Omega,\mathrm{Lin})$, we find the sequence $E_k\in C_0(\Omega,\mathrm{Lin})$ of approximations as in (4.4). Then

$$\langle S,E_k\rangle \le |S|_{\mathcal{M}}|E_k|_{L^\infty} \le |S|_{\mathcal{M}}|E|_{L^\infty}$$

and $\langle S,E_k\rangle\to\langle S,E\rangle$. The limit gives the result.

(ii): If $\varphi\in C^1(\mathrm{cl}\,\Omega,\mathrm{R})\cap C_0(\Omega)$ and $E\in\mathfrak{X}(\Omega,\mathrm{Lin})$ then $\varphi E\in\mathfrak{X}(\Omega,\mathrm{Lin})$ and $|\varphi E|_{L^\infty}\le|\varphi|_\infty|E|_{L^\infty}$. Hence by (i), $\langle S,\varphi E\rangle\le|S|_{\mathcal{M}}|E|_{L^\infty}|\varphi|_\infty$, and thus fixing S and E, the Riesz representation theorem provides the measure $\langle\langle S,E\rangle\rangle$ and the estimate of its total variation. In particular, if $E\in C_0(\Omega,\mathrm{Lin})$ then

$$\langle S, \varphi E \rangle = \int_\Omega \varphi E \cdot dS$$

and thus $\langle\!\langle S, E \rangle\!\rangle$ has the asserted form.

(iii): If $E \in C_0(\Omega, \mathrm{Lin})$ then $\langle\!\langle S, E \rangle\!\rangle = E \cdot S$ is absolutely continuous with respect to S and therefore for every open subset Σ of Ω we have

$$\int_\Sigma \varphi d\langle\!\langle S, E \rangle\!\rangle \le |\varphi|_\infty |E|_{L^\infty} |S|(\Sigma) \tag{4.7}$$

for every $\varphi \in C_0(\Sigma)$. If $E \in \mathcal{X}(\Omega, \mathrm{Lin})$, we find the sequence $E_k \in C_0(\Omega, \mathrm{Lin})$ of approximations as in (4.4). One finds that

$$\int_\Omega \varphi d\langle\!\langle S, E_k \rangle\!\rangle \to \int_\Omega \varphi d\langle\!\langle S, E \rangle\!\rangle$$

for any $\varphi \in C_0(\Omega)$. Moreover, if Σ is an open subset of Ω we have

$$\int_\Sigma \varphi d\langle\!\langle S, E_k \rangle\!\rangle \le |\varphi|_\infty |E_k|_{L^\infty} |S|(\Sigma) \le |\varphi|_\infty |E|_{L^\infty} |S|(\Sigma)$$

for every $\varphi \in C_0(\Sigma)$. The limit yields (4.7) for a general E. This yields, in turn, that the total variation measure satisfies

$$|\,\langle\!\langle S, E \rangle\!\rangle\,|(\Sigma) \le |E|_{L^\infty} |S|(\Sigma) \tag{4.8}$$

for any open subset Σ of Ω. The well known regularity of Borel measures [18; Theorem 2.17] allows to extend (4.8) to any Borel subset Σ of Ω and thus the absolute continuity is proved. □

5 Generalized problem of inextensible nets

We now generalize the framework of Section 2 slightly to simplify the notation.

Let m, n be two positive integers, let $\mathrm{K} \subset \mathrm{Lin}$ be a compact, convex, equilibrated set containing some neighborhood of 0 in Lin. This is a generalization of the set K given by (2.2) in the case of nets. Let $\Omega \subset \mathrm{R}^n$ be a open bounded set with lipschitzian boundary. We consider the Dirichlet boundary data given by a lipschitzian map $f : \Omega \to \mathrm{R}^m$. We thus consider all deformations $y \in \mathrm{Lip}(\Omega, \mathrm{R}^m)$ that satisfy $y = f$ on $\partial\Omega$. If we write

$$y = f + u$$

where u is the displacement, then $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$. We consider u, and not y, as the basic variable. We require that the deformation satisfies $\nabla y(x) \in \mathrm{K}$ for $\mathcal{L}^2$ almost every $x \in \Omega$ exactly as in the case of nets. In terms of u this reads

$$\nabla u + \nabla f \in \mathrm{K} \quad \text{for } \mathcal{L}^2 \text{ almost every } x \in \Omega.$$

Accordingly, we consider the following set of 'virtual gradients' of displacement:

$$\mathfrak{K} = \{E \in \mathcal{X}(\Omega, \mathrm{Lin}) : E(x) + \nabla f(x) \in \mathrm{K} \text{ for } \mathcal{L}^n \text{ almost every } x \in \Omega\}.$$

We assume that

$$\nabla f \in \mathrm{K} \quad \text{for } \mathcal{L}^2 \text{ almost every } x \in \Omega, \tag{5.1}$$

so that

$$0 \in \{u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m) : \nabla u \in \mathfrak{K}\}.$$

Lemma 5.1. *$\mathfrak{K}$ is a closed convex subset with empty interior of $\mathfrak{X}(\Omega, \mathrm{Lin})$.*

Proof The convexity is immediate. We prove that $\mathfrak{K}$ is closed as follows. Let E_k be a sequence of elements of $\mathfrak{K}$ which converges to some E in $\mathfrak{X}(\Omega, \mathrm{Lin})$. We thus have $E_k(x) + \nabla f(x) \in \mathrm{K}$ for $\mathcal{L}^n$ almost every $x \in \Omega$. Since K is bounded, the sequence $|E_k|_{L^\infty}$ is bounded and thus Remark 4.2(iii) tells us that $E_k \rightharpoonup^* E$ in $L^\infty(\Omega, \mathrm{Lin})$. Since K is closed convex, the inclusion $\mathrm{ran}(E_k + \nabla f) \subset \mathrm{K}$ is stable under the weak* convergence, i.e., $\mathrm{ran}(E + \nabla f) \subset \mathrm{K}$. Finally, to show that $\mathfrak{K}$ has empty interior, we prove that for every $E \in \mathfrak{K}$ there exists a sequence E_k which converges to E in $\mathfrak{X}(\Omega, \mathrm{Lin})$ but which is outside of $\mathfrak{K}$. Indeed, let $x_0 \in \Omega$ be any Lebesgue point of $E + \nabla f$ and let $u_0 \in C_c^\infty(\Omega, \mathrm{R}^m)$ be such that the set

$$\{x \in \Omega : \nabla u_0(x) + E(x_0) + \nabla y(x_0) \notin \mathrm{K}\}$$

has positive $\mathcal{L}^n$ measure. Let $p_k \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ be defined by $p_k(x) = k^{-1}\nabla u_0(kx)$, $x \in \Omega$, and put $E_k = \nabla p_k + E$. Since $|p_k|_\infty \to 0$, we have $E_k \to E$ in $\mathfrak{X}(\Omega, \mathrm{R}^m)$ and the set

$$\{x \in \Omega : \nabla p_k(x) + E(x) + \nabla y(x) \notin \mathrm{K}\}$$

has positive $\mathcal{L}^n$ measure. □

We define the stored energy $w : \mathrm{Lin} \to \bar{\mathrm{R}}$ as the indicator function $I(\cdot, \mathrm{K})$. If the body force is represented by an R^m valued measure b on Ω then the total energy functional $\mathsf{E} : \mathrm{Lip}(\Omega, \mathrm{R}^m) \to \bar{\mathrm{R}}$ is given by

$$\mathsf{E}(u) = \int_\Omega w(\nabla u + \nabla f)\, d\mathcal{L}^n - \int_\Omega u \cdot db - c_0,$$

$u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$, where

$$c_0 = \int_\Omega u \cdot db.$$

The primal problem is

($\mathcal{P}$) Find $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ which minimizes E on $\mathrm{Lip}_0(\Omega, \mathrm{R}^m)$.

We can write

$$\mathsf{E}(u) = C(u) + D(\nabla u), \quad y \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m), \tag{5.2}$$

where $C : \mathrm{Lip}_0(\Omega, \mathrm{R}^m) \to \bar{\mathrm{R}}$, $D : \mathfrak{X}(\Omega, \mathrm{R}^{n\times m}) \to \bar{\mathrm{R}}$ are given by

$$C(u) = -\int_\Omega u \cdot db - c_0, \quad D = I(\cdot, \mathfrak{K}),$$

$u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$.

Remark 5.2. *Let $u_k \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ be a sequence such that the sequence $|\nabla u_k|_{L^\infty}$ is bounded. Then there exists a subsequence, again denoted by u_k, such that $|u - u_k|_\infty \to 0$ for some $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$.*

Proof Under our assumptions on Ω we have $\mathrm{Lip}_0(\Omega, \mathrm{R}^m) = W_0^{1,\infty}(\Omega, \mathrm{R}^m)$. The result then follows from the compactness of the embedding $W_0^{1,\infty}(\Omega, \mathrm{R}^m) \subset C_0(\mathrm{cl}\,\Omega, \mathrm{R}^m)$ [1; Theorem 6.3, Part III]. Alternatively, one can use the Arzela–Ascoli theorem [19; Subsection A5] as in [12]. □

Lemma 5.3. *If* $u_k \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ *and* $E_k \in \mathfrak{X}(\Omega, \mathrm{Lin})$ *satisfy* $\nabla u_k + E_k \in \mathfrak{K}$ *for every* k *and* $|E_k|_{\mathfrak{X}} \to 0$ *then there is a subsequence of* u_k*, again denoted by* u_k*, such that* $|u_k - u|_\infty \to 0$ *for some* $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ *with* $\nabla u \in \mathfrak{K}$.

Proof Since $|E_k|_{\mathfrak{X}} \to 0$, we have

$$E_k = \nabla p_k + G_k,$$

where $p_k \in \mathrm{Lip}_0(\Omega, \mathrm{Lin})$ and $G_k \in C_0(\Omega, \mathrm{Lin})$ satisfy $|p_k|_\infty \to 0$, $|G_k|_\infty \to 0$. The condition $\nabla u_k + E_k \in \mathfrak{K}$ is then rewritten as

$$\nabla(u_k + p_k) + G_k \in \mathfrak{K}$$

and as K is bounded and $G_k(x)$ uniformly bounded, also $\nabla(u_k + p_k)(x)$ is uniformly bounded. Remark 5.2 implies that there exists a subsequence of $u_k + p_k$, again denoted by $u_k + p_k$, such that $|u_k + p_k - u|_\infty \to 0$ for some $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$. By the hypothesis, $\nabla u_k + E_k \in \mathfrak{K}$ for every k and the sequence $\nabla u_k + E_k \in \mathfrak{X}(\Omega, \mathrm{Lin})$ converges in $\mathfrak{X}(\Omega, \mathrm{Lin})$ to ∇u. Thus applying Lemma 5.1 we obtain the conclusion. □

The function h of (3.2) is here $h : \mathfrak{X}(\Omega, \mathrm{Lin}) \to \bar{\mathrm{R}}$ where

$$h(E) = \inf \{C(u) + D(\nabla u + E) : u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)\}$$

$E \in \mathfrak{X}(\Omega, \mathrm{Lin})$.

Theorem 5.4. *Assume that* f *satisfies* (5.1). *Then the problem* $\mathcal{P}$ *has a solution. More generally, for any* $E \in \mathfrak{X}(\Omega, \mathrm{Lin})$ *such that* $h(E) < \infty$ *there exists a* $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ *such that*

$$h(E) = C(u) + D(\nabla u + E). \tag{5.3}$$

Proof Clearly, the first part of the assertion is a spacial case of the second, noting that (5.1) guarantees $h(0) < \infty$. Thus we prove only the second part. Let $E \in \mathfrak{X}(\Omega, \mathrm{Lin})$ and $h(E) < \infty$. Let $u_k \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ be the minimizing sequence, i.e., a sequence satisfying

$$C(u_k) + D(\nabla u_k + E) \to h(E).$$

The finiteness of $C(u_k) + D(\nabla u_k + E)$ implies $\nabla u_k + E \in \mathfrak{K}$ for all k sufficiently large. The boundedness of K gives that the sequence $|\nabla u_k|_{L^\infty}$ is uniformly bounded. By Remark 5.2 there exists a subsequence, again denoted by u_k, such that $|u_k - u|_\infty \to 0$ for some $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$. Hence $\nabla u_k + E \to \nabla u + E$ in $\mathfrak{X}(\Omega, \mathrm{Lin})$. Lemma 5.1 then gives $\nabla u + E \in \mathfrak{K}$. Thus $D(\nabla u_k + E) = D(\nabla u + E) = 0$ for all k sufficiently large. As also $C(u_k) \to C(u)$, we see that (5.3) holds. □

Lemma 5.5.
(i) *The functions* C *and* D *are lowersemicontinuous.*
(ii) *The problem* $\mathcal{P}$ *is normal.*

Slater's condition cannot be satisfied since the existence of u_0 such that $D(\nabla u_0) < \infty$ and D continuous at ∇u_0 would require that ∇u_0 be an interior point of $\mathfrak{K}$, but the interior of $\mathfrak{K}$ is empty by Lemma 5.1

Proof (i): The lowersemicontinuity of C is immediate. Since D is the indicator function of the closed set $\mathfrak{K}$ (see Lemma 5.1), it is automatically lowersemicontinuous.

(ii): Suppose, on the contrary, that there exists a sequence E_k converging to 0 in $\mathfrak{X}(\Omega, \mathrm{Lin})$ such that

$$\lim_{k\to\infty} h(E_k) < h(0). \tag{5.4}$$

Thus $h(E_k) < \infty$ for all k sufficiently large and hence by Theorem 5.4 there exists an $u_k \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ such that

$$h(E_k) = C(u_k) + D(\nabla u_k + E_k).$$

Thus $\nabla u_k + E_k \in \mathfrak{K}$ for every k. By Lemma 5.3 there is a subsequence of u_k, again denoted by u_k, and $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ such that $|u_k - u|_\infty \to 0$ and $\nabla u \in \mathfrak{K}$. Then $C(u_k) \to C(u)$ and $D(\nabla u_k + E_k) = D(\nabla u) = 0$; hence

$$C(u_k) + D(\nabla u_k + E_k) \to C(u) + D(\nabla u) \geq C(u^\circ) + D(\nabla u^\circ) = h(0);$$

where u° is the minimizer of $\mathscr{P}$. This violation of (5.4) proves that $\mathscr{P}$ is normal. □

6 The dual of the generalized problem $\mathscr{P}$

For the primal problem of Section 5 we determine the dual problem, prove the equality

$$\inf \mathscr{P} + \inf \mathscr{P}^* = 0, \tag{6.1}$$

show that under additional hypothesis it has a solution, and determine the relationships between the solutions of the primal and dual problems.

To evaluate the dual energy, we note [15; Proposition I.6] that corresponding to each compact, convex, equilibrated set $\mathrm{K} \subset \mathrm{Lin}$ containing some neighborhood of 0 in Lin there exists a unique norm $|\cdot|_{\mathrm{K}}$ on Lin such that

$$\mathrm{K} = \{A \in \mathrm{Lin} : |A|_{\mathrm{K}} \leq 1\}.$$

The norm $|\cdot|_{\mathrm{K}}$ is given by

$$|A|_{\mathrm{K}} = \inf\{t \geq 0 : A \in t\mathrm{K}\}, \quad A \in \mathrm{Lin}.$$

We also introduce the dual norm $|\cdot|_{\mathrm{K}}^*$ on Lin by

$$|B|_{\mathrm{K}}^* = \sup\{A \cdot B : A \in \mathrm{Lin}, |A|_{\mathrm{K}} \leq 1\},$$

$B \in \mathrm{Lin}$. In view of finite dimensionality of Lin, the norms $|\cdot|_{\mathrm{K}}$, $|\cdot|_{\mathrm{K}}^*$ and the euclidean norm $|\cdot|$ are mutually equivalent.

Example 6.1. *The dual norm of the norm* (2.1) *is* $|B|_{\mathrm{K}}^* = |Be_1| + |Be_2|$, $B \in \mathrm{Lin}$.

Proof Any $A \in \mathrm{Lin}$ with $|A|_{\mathrm{K}} \leq 1$ is of the form $A = f_1 \otimes e_1 + f_2 \otimes e_2$ where $f_i \in \mathrm{R}^3$ satisfy $|f_i| \leq 1$, $i = 1, 2$. Then $B \cdot A = f_1 \cdot Be_1 + f_2 \cdot Be_2 \leq |Be_1| + |Be_2|$. On the other hand, the right hand side is achieved when $f_i = Be_i/|Be_i|$. □

We identify $\mathfrak{S}(\Omega, \mathrm{Lin})$ with the dual of $\mathfrak{X}(\Omega, \mathrm{R}^m)$.

Proposition 6.2. *The functions* $C^* : \mathrm{Lip}_0(\Omega, \mathrm{R}^m)^* \to \bar{\mathrm{R}}$ *and* $D^* : \mathfrak{S}(\Omega, \mathrm{R}^{n\times m}) \to \bar{\mathrm{R}}$ *are given by*

$$C^*(z) = \begin{cases} c_0 & \text{if} \quad \langle z, p\rangle + \int\limits_\Omega p \cdot db = 0 \ \text{ for all } \ p \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m), \\ \infty & \text{else} \end{cases} \tag{6.2}$$

for all $z \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)^*$ *and*

$$D^*(S) = \sup\{\langle S, E\rangle : E \in \mathfrak{K}\}, \tag{6.3}$$

$S \in \mathcal{M}(\Omega, \mathrm{Lin})$. *If* $\nabla f \in C(\mathrm{cl}\,\Omega)$, *if* S *has the form* $S = \sigma\phi$ *where* ϕ *is a finite nonnegative measure on* Ω *and* $\sigma \in L^1(\phi, \mathrm{Lin})$ *then*

$$D^*(S) = \int\limits_\Omega |\sigma|_{\mathrm{K}}^* \, d\phi - \langle S, \nabla f\rangle. \tag{6.4}$$

Proof We have

$$C^*(z) = \sup\{\langle z, p\rangle + \int\limits_\Omega p \cdot db + c_0 : p \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)\}$$

which gives (6.2). Equation (6.3) is immediate.

To prove the second part, note that by (6.3) and by the density of $C := C_0(\Omega, \mathrm{Lin})$ in $\mathfrak{X}(\Omega, \mathrm{Lin})$ we have

$$\begin{aligned} D^*(S) &= \sup\{\langle S, E\rangle : E \in C,\ E + \nabla f \in \mathrm{K} \ \text{ for } \ \mathcal{L}^n \ \text{ almost every point of } \ \Omega\} \\ &= \sup\{\langle S, F - \nabla f\rangle : E \in C,\ F \in \mathrm{K} \ \text{ on } \ \Omega\} \\ &= \sup\{\langle S, F\rangle : F \in C,\ F \in \mathrm{K} \ \text{ on } \ \Omega\} - \langle S, \nabla f\rangle \end{aligned}$$

and

$$\begin{aligned} &\sup\{\int\limits_\Omega F \cdot \sigma \, d\phi : F \in C,\ |F|_{\mathrm{K}} \le 1 \ \text{ on } \ \Omega\} \\ &\quad \le \sup\{\int\limits_\Omega |F|_{\mathrm{K}} |\sigma|_{\mathrm{K}}^* \, d\phi : F \in C,\ |F|_{\mathrm{K}} \le 1 \ \text{ on } \ \Omega\} \end{aligned}$$

which proves the inequality "$\le$" in (6.4). The converse inequality is proved by showing that for every $\varepsilon > 0$ there exists $F \in C$ such that $|F|_{\mathrm{K}}|\sigma|_{\mathrm{K}}^* - \varepsilon < F \cdot \sigma$. □

Corollary 6.3. *For* E *of the form* (5.2) *the dual energy is given by*

$$\mathsf{G}(S) = \begin{cases} D^*(S) + c_0 & \text{if} \quad \operatorname{div} S + b = 0, \\ \infty & \text{else,} \end{cases}$$

$S \in \mathfrak{X}(\Omega, \mathrm{Lin})$. *The dual problem reads*

$$(\mathcal{P}^*) \qquad \begin{cases} \textit{Find } S \in \mathfrak{S}(\Omega, \mathrm{Lin}) \textit{ which minimizes } D^*(S) + c_0 \\ \textit{over the set of all solutions of } \operatorname{div} S + b = 0. \end{cases}$$

Theorem 6.4.

(i) *We have* (6.1).

(ii) *If*

$$\operatorname{sup\,ess}\{|\nabla f(x)|_{\mathrm{K}} : x \in \Omega\} < 1 \tag{6.5}$$

then the problem $\mathcal{P}^*$ has a solution.

Proof (i): By Lemma 5.5(ii) the problem $\mathcal{P}$ is normal and by Proposition 3.2(i) the normality is equivalent to (6.1).

(ii): If (6.5) holds then there is an $\varepsilon > 0$ such that

$$\{E \in C_0(\Omega, \mathrm{Lin}) : |E|_\infty \le \varepsilon\} \subset \mathfrak{K}.$$

It follows that

$$D^*(S) \ge \varepsilon |S|_{\mathcal{M}}. \tag{6.6}$$

By [20; Example 3.3(i)], we have $\mathfrak{E} := \{S \in \mathfrak{S}(\Omega, \mathrm{Lin}) : \operatorname{div} S + b = 0\} \neq \emptyset$. Let now $S_k \in \mathfrak{E}$ be a minimizing sequence, i.e., a sequence such that

$$D^*(S_k) + c_0 \to \inf \mathcal{P}^*.$$

In particular, the sequence $D^*(S_k) \ge 0$ is bounded. By (6.6) also the sequence $|S_k|_{\mathcal{M}}$ is bounded; from $\operatorname{div} S_k + b = 0$ we see that also the sequence $|\operatorname{div} S_k|_{\mathcal{M}}$ is bounded. consequently, also the sequence $|S_k|_{\mathfrak{S}} = |S_k|_{\mathcal{M}} + |\operatorname{div} S_k|_{\mathcal{M}}$. By the Bourbaki–Alaoglu theorem [19; Theorem 4.3(c)] then there exists a subsequence, again denoted S_k, such that

$$S_k \rightharpoonup^* S \quad \text{in} \quad \mathfrak{S}(\Omega, \mathrm{Lin});$$

for some $S \in \mathfrak{S}(\Omega, \mathrm{Lin})$ and hence

$$\langle S_k, E \rangle \to \langle S, E \rangle \quad \text{for each} \quad E \in \mathfrak{X}(\Omega, \mathrm{Lin}).$$

By taking $E = \nabla p$, $p \in \mathrm{Lin}_0(\Omega, \mathrm{R}^m)$, one finds that $\operatorname{div} S + b = 0$ and hence $S \in \mathfrak{E}$. We now claim that S is a solution of $\mathcal{P}^*$. for this, it suffices to show that

$$\liminf_{k \to \infty} D^*(S_k) \ge D^*(S).$$

But this follows from the fact that by (6.3) the function D^* is weak* lowersemi-continuous, since it is the supremum of a family of weak* continuous function [5; Subsection I.2.2]. □

Remark 6.5. *Let $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ and let $y_\rho, \rho > 0$, be the sequence of mollifications of $y = u + f$, defined on the set $\Omega_\rho := \{x \in \Omega : B(x, \rho) \subset \Omega\}$, where $B(x, \rho)$ is the open ball of center x and radius ρ. Put*

$$\omega(x) = \limsup_{\rho \to 0} |\nabla y_\rho(x)|_{\mathrm{K}} \tag{6.7}$$

for any $x \in \Omega$. If $\nabla u \in \mathfrak{K}$ then

$$\left.\begin{array}{c} 0 \le \omega(x) \le 1 \quad \text{for every} \quad x \in \Omega \quad \text{and} \\ \omega(x) = |\nabla y(x)|_{\mathrm{K}} \quad \text{for} \quad \mathcal{L}^n \quad \text{almost every} \quad x \in \Omega \quad \text{and} \\ \text{under} \quad (6.8)_2 \quad \text{the equality} \quad \omega(x) = 1 \quad \text{holds only if} \quad \nabla y(x) \in \partial \mathrm{K}. \end{array}\right\} \tag{6.8}$$

Proof We have

$$y_\rho(x) = \int_{\mathrm{R}^n} y(x+h) \psi_\rho(h) \, d\mathcal{L}^n(h)$$

for every $x \in \Omega_\rho$, where $\psi_\rho(h) = \rho^{-n} \psi(h/\rho)$ where ψ is a mollifier and as a consequence,

$$\nabla y_\rho(x) = \int_{\mathrm{R}^n} \nabla y(x+h)\psi_\rho(h)\, d\mathcal{L}^n(h).$$

If $y \in \mathfrak{K}$, i.e., if $|\nabla y(x)|_{\mathrm{K}} \le 1$ for $\mathcal{L}^n$ almost every $x \in \Omega$ then Jensen's inequality provides

$$|\nabla y_\rho(x)|_{\mathrm{K}} = \Big|\int_{\mathrm{R}^n} \nabla y(x+h)\psi_\rho(h)\, d\mathcal{L}^n(h)\Big|_{\mathrm{K}} \le \int_{\mathrm{R}^n} |\nabla y(x+h)|_{\mathrm{K}}\psi_\rho(h)\, d\mathcal{L}^n(h) \le 1.$$

The definition (6.7) then gives $(6.8)_1$. Assertion $(6.8)_2$ follows from the well known fact that $\nabla y_\rho(x) \to \nabla y(x)$ for $\mathcal{L}^n$ almost every $x \in \Omega$ and $(6.8)_3$ is a consequence of $\partial\mathrm{K} = \{F \in \mathrm{Lin} : |F|_{\mathrm{K}} = 1\}$. □

Theorem 6.6. *Let $u \in \mathrm{Lip}_0(\Omega, \mathrm{R}^m)$ and $S \in \mathfrak{S}(\Omega, \mathrm{Lin})$ be any solutions of the primal and dual problems, respectively. Let ω be defined by* (6.7) *where $y = u + f$. Then we have the following statements:*
(i) *the measure S is supported by the set*

$$M = \{x \in \Omega : \omega(x) = 1\},$$

i.e., $S(A) = 0$ for any Borel subset A of $\Omega \sim M$;
(ii) *the measure S is normal to $\mathfrak{K}$ at the map ∇u, i.e.,*

$$\langle E - \nabla u, S\rangle \le 0 \tag{6.9}$$

for all $E \in \mathfrak{K}$;
(iii) *for any $E \in \mathfrak{K}$ the scalar measure $\langle\!\langle S, E - \nabla u\rangle\!\rangle$ on Ω is nonpositive.*
In particular, if $u, f \in C^1(\mathrm{cl}\,\Omega, \mathrm{R}^m)$ then

$$\mathrm{spt}\, S \subset \{x \in \Omega : \nabla y(x) \in \partial\mathrm{K}\},$$
$$S^\circ(x) \in \mathrm{N}(\nabla y(x), \mathrm{K}) \quad \text{for } \phi \text{ almost every } x \in \mathrm{spt}\, S.$$

Recall the formula (2.6) for the normal cone to K for the case of nets and its interpretation.

Proof (ii): By Theorem 3.3(iv) we have $S \in \partial D(\nabla u)$ and since D is the indicator function of $\mathfrak{K}$; by (7.2) we have $S \in \mathrm{N}(\nabla u, \mathfrak{K})$, which is (6.9).

(iii): If $E \in \mathfrak{K}$ then since $\nabla u \in \mathfrak{K}$, we have $\lambda E + (1-\lambda)\nabla u$ for any $\lambda \in C_0^1(\mathrm{cl}\,\Omega)$ satisfying $0 \le \lambda(x) \le 1$ for any $x \in \Omega$. (Here we require λ to be continuously differentiable and not merely continuous to ensure that $\lambda E + (1-\lambda)\nabla y \in \mathfrak{X}(\Omega, \mathrm{Lin})$, see Remark 4.2(ii).) Replacing E by $\lambda E + (1-\lambda)\nabla u$ in (6.9), we obtain

$$\langle S, \lambda(E - \nabla u)\rangle = \int_\Omega \lambda\, d\,\langle\!\langle E - \nabla u, S\rangle\!\rangle \le 0. \tag{6.10}$$

As this must be satisfied for each λ with the indicated properties, the conclusion follows.

(i): If λ is as in the proof of (iii), we have

$$\int_\Omega \lambda \nabla y_\rho \cdot dS \to \int_\Omega \lambda\, d\,\langle\!\langle \nabla y, S\rangle\!\rangle\,. \tag{6.11}$$

On the other hand,

$$\int_\Omega \lambda \nabla y_\rho \cdot dS = \int_\Omega \lambda \nabla y_\rho \cdot S^\circ d|S| \le \int_\Omega \lambda |\nabla y_\rho|_{\mathrm{K}} |S^\circ|_{\mathrm{K}}^* d|S|. \tag{6.12}$$

Putting $\omega_\rho(x) = \sup\{|\nabla y_\sigma(x)|_K : 0 < \sigma \le \rho\}$ and noting that $\omega_\rho(x) \to \omega(x) \le 1$ for any $x \in \Omega$, we invoke Lebesgue's dominated convergence theorem to obtain

$$\int_\Omega \lambda |\nabla y_\rho|_K |S^\circ|_K^* d|S| \le \int_\Omega \lambda \omega_\rho |S^\circ|_K^* d|S| \to \int_\Omega \lambda \omega |S^\circ|_K^* d|S|. \qquad (6.13)$$

Taking the limits in (6.12) using (6.11) and (6.13) we obtain

$$\int_\Omega \lambda \, d \, \langle\langle \nabla y, S \rangle\rangle \le \int_\Omega \lambda \omega |S^\circ|_K^* d|S|.$$

Inequality (6.10) can be rewritten as

$$\int_\Omega \lambda \, d \, \langle\langle F, S \rangle\rangle \le \int_\Omega \lambda \, d \, \langle\langle \nabla y, S \rangle\rangle$$

for any $F \in C_0(\Omega, \mathrm{Lin})$ with $F(x) \in K$ for every $x \in \Omega$ and hence

$$\int_\Omega \lambda \, d \, \langle\langle E, S \rangle\rangle \le \int_\Omega \lambda \omega |S^\circ|_K^* d|S|.$$

The arbitrariness of λ provides

$$F \cdot S^\circ(x) \le \omega(x) |S^\circ(x)|_K^* \qquad (6.14)$$

for every $F \in K$ and $|S|$ almost every $x \in \Omega$. On the other hand, by the definition of dual norms $|\cdot|_K$ and $|\cdot|_K^*$ there exists an F with $|F|_K = 1$ such that $F \cdot S^\circ(x) = |S^\circ(x)|_K^*$. As this F is in K, we see that (6.14) can hold only if $\omega(x) = 1$. Thus Assertion (i) holds. □

7 Appendix A: Convexity

This section outlines some basic concepts of the convexity theory. The reader is referred to [5; Chapter I], [16], and [17] for details.

Let X be a normed vectorspace and X^* its normed dual; if $\xi \in X$ and $\eta \in X^*$, we denote by $\langle \xi, \eta \rangle \equiv \langle \eta, \xi \rangle$ the value of the linear function η on ξ.

A subset K of X is said to be convex if $t_1\xi_1 + t_2\xi_2 \in K$ for any $\xi_i \in K$, $t_i \ge 0$, $i = 1, 2$, such that $t_1 + t_2 = 1$. If $\xi \in X$, then $\eta \in X^*$ is said to be a normal to K at ξ if

$$\langle \xi' - \xi, \eta \rangle \le 0$$

for every $\xi' \in K$. For a fixed ξ, the set $N(\xi, K) \subset X^*$ of all normals to K at ξ is a closed convex cone with $0 \in N(\xi, K)$. One has $N(\xi, K) \ne \{0\}$ only if ξ belongs to the boundary of K.

A subset K of X is said to be equilibrated if $\xi \in K$ implies $-\xi \in K$.

If $\Phi : X \to R \cup \{-\infty, \infty\}$ then the effective domain of Φ is defined by $\mathrm{dom}\,\Phi = \{\xi \in X : \Phi(\xi) < \infty\}$. The function Φ is said to be convex if

$$\Phi(t_1\xi_1 + t_2\xi_2) \le t_1\Phi(\xi_1) + t_2\Phi(\xi_2) \qquad (7.1)$$

for any $\xi_i \in \mathrm{dom}\,\Phi$, $t_i \ge 0$, $i = 1, 2$, such that $t_1 + t_2 = 1$. Similarly, Φ is said to be affine if its values are finite and (7.1) holds with the equality sign.

A convex function Φ on X is said to be a norm if it is nonnegative, finite valued, $\Phi(t\xi) = |t|\Phi(\xi)$ for any $\xi \in X$ and $t \in R$, and $\Phi(\xi) = 0$ only if $\xi = 0$. Two norms

Φ_1 and Φ_2 are said to be equivalent if $c_1\Phi_1(\xi) \le \Phi_2(\xi) \le c_2\Phi_1(\xi)$ for all $\xi \in X$ and some positive constants c_1, c_2.

A function $\Phi : X \to R$ is affine and continuous if and only if it is of the form $\Phi(\xi) = \langle \eta, \xi \rangle + c$ for some $\eta \in X^*$, $c \in R$ and all $\xi \in X$.

Recall that $\Phi : X \to R \cup \{-\infty, \infty\}$ is said to be lowersemicontinuous at $\xi \in X$ if $\liminf_{k\to\infty} \Phi(\xi_k) \ge \Phi(\xi)$ for every sequence ξ_k converging to ξ.

If $\xi \in X$ then $\eta \in X^*$ is said to be a subgradient of Φ at ξ if $\Phi(\xi) \in R$ and

$$\Phi(\xi') - \Phi(\xi) \ge \langle \xi' - \xi, \eta \rangle$$

for all $\xi' \in X$. The set $\partial\Phi(\xi)$ of all subgradients of Φ at ξ is a (possibly empty) closed convex subset of X^*. Clearly, if $\partial\Phi(\xi) \neq \emptyset$ then Φ is lowersemicontinuous at ξ.

The indicator function $I(\cdot, K)$ of a subset K of X is defined by

$$I(\xi, K) = \begin{cases} 0 & \text{if } \xi \in K, \\ \infty & \text{else,} \end{cases}$$

$\xi \in X$. Then $I(\cdot, K)$ is convex if and only if K is convex and $I(\cdot, K)$ is lowersemicontinuous if and only if K is closed. If K is closed convex then

$$\partial I(\xi, K) = \begin{cases} N(\xi, K) & \text{if } \xi \in K, \\ \emptyset & \text{else,} \end{cases} \tag{7.2}$$

for any $\xi \in X$.

If $\Phi : X \to R \cup \{-\infty, \infty\}$ then the conjugate function $\Phi^* : X^* \to R \cup \{-\infty, \infty\}$ is defined by

$$\Phi^*(\eta) = \sup\{\langle \xi, \eta \rangle - \Phi(\xi) : \xi \in X\},$$

$\eta \in X^*$. One has

$$\Phi(\xi) + \Phi^*(\eta) \ge \langle \xi, \eta \rangle$$

for any $\xi \in X$ and $\eta \in X^*$. If Φ is not identically ∞ and bounded from below by an affine continuous function then Φ^* is not identically ∞, lowersemicontinuous, and bounded from below by an affine continuous function.

8 Appendix B: Countable additivity inside finite additivity

As mentioned in the introduction, in [12] the stresses in the inextensible nets with slack are modeled by finitely additive set functions. This section briefly describes the relationship between the present description and that of [12].

Both the two approaches use the convex duality theory of Section 3 to construct the dual problem and to identify the stress with the solution of the dual problem. The difference lies in different choices of the duality between strains (from the space Y) and stresses (from the dual Y^*). Indeed, a single primal problem can have several nonequivalent dual problems, as already mentioned.

While the details in [12] are slightly different, one can say that the strains, interpreted as gradients F of Lipschitz Ian deformations $y : \Omega \to R^m$, are inserted in the space $L^\infty(\Omega, \mathrm{Lin})$; i.e., the choice

$$\mathrm{Y} = L^{\infty}(\Omega, \mathrm{Lin})$$

has been made. In accordance with the general theory, the stresses live in the dual Y^*. By the Yoshida–Hewitt representation theorem [25; Theorem 2.3], Y^* can be identified with the space $\mathrm{ba}_0(\Omega, \mathrm{Lin})$ of all finitely additive set functions of finite total variation that are absolutely continuous with respect to $\mathscr{L}^n$: the functionals $\Psi \in L^{\infty}(\Omega, \mathrm{Lin})^*$ are identified with $T \in \mathrm{ba}_0(\Omega, \mathrm{Lin})$ via

$$\Psi(F) = \int_{\Omega} F \cdot dT, \tag{8.1}$$

$F \in L^{\infty}(\Omega, \mathrm{Lin})$, where the integral with respect to a finitely additive function T is defined in essentially the same way as in the countably additive case.

The solutions of the primal problem here and in [12] are identical; thus $\inf \mathscr{P}$ are the same for both approaches. Under Condition (6.5), the dual problem is solvable both here and in [12], with solutions in different spaces, here $S \in \mathfrak{S}(\Omega, \mathrm{Lin})$ and in [12] $T \in \mathrm{ba}_0(\Omega, \mathrm{Lin})$, but in both cases satisfying

$$\operatorname{div} S + b = 0, \quad \operatorname{div} T + b = 0,$$

$$\inf \mathscr{P} + \inf \mathscr{P}^* = 0,$$

so that also $\inf \mathscr{P}^*$ are the same for the two solutions.

What is the relationship between S and T?

The answer lies in the observation that each $T \in \mathrm{ba}_0(\Omega, \mathrm{Lin})$ contains in itself an $S \in \mathcal{M}(\Omega, \mathrm{Lin})$ and each $S \in \mathcal{M}(\Omega, \mathrm{Lin})$ can be extended, in a nonunique way, but with the preservation of the variation, into $T \in \mathrm{ba}_0(\Omega, \mathrm{Lin})$. Indeed, the space of continuous functions vanishing on the boundary $C_0(\Omega, \mathrm{Lin})$ is a closed subspace of $L^{\infty}(\Omega, \mathrm{Lin})$. By the Riesz representation theorem, the continuous functionals on $C_0(\Omega, \mathrm{Lin})$ are represented by the integration with respect to countably additive measures. Thus if $T \in \mathrm{ba}_0(\Omega, \mathrm{Lin})$, then the restriction of (8.1) to $C_0(\Omega, \mathrm{Lin})$ induces a measure $S \in \mathcal{M}(\Omega, \mathrm{Lin})$ such that

$$\int_{\Omega} F \cdot dS = \int_{\Omega} F \cdot dT \tag{8.2}$$

for each $F \in C_0(\Omega, \mathrm{Lin})$. Conversely, given S, one can extend the functional on $C_0(\Omega, \mathrm{Lin})$ determined by S into a functional on $L^{\infty}(\Omega, \mathrm{Lin})$ with the same norm, thus obtaining $T \in \mathrm{ba}_0(\Omega, \mathrm{Lin})$ of the same total variation. However, generally, there is no direct relationship between S and T, despite the relation (8.2) between the induced functionals. Indeed, one can have $S = 0$ and yet $T \neq 0$. For this it suffices the extend the null functional on $C_0(\Omega, \mathrm{Lin})$ into a nonzero functional on $L^{\infty}(\Omega, \mathrm{Lin})$ by a simple application of the Hahn–Banach theorem. In this way one obtains a nonzero finitely additive function T such that

$$\int_{\Omega} F \cdot dT = 0$$

for each continuous function $F \in C_0(\Omega, \mathrm{Lin})$. (See [25; Theorem 3.4] for a refined version of this.)

The same restriction/extension relationship holds between the solution of the dual problem in the present sense and that in the sense of [12].

Acknowledgment The research was supported by the RVO: 67985840.

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