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**Abstract**

We study the homogenization problem for the evolutionary Navier-Stokes system under the critical size of obstacles. Convergence towards the limit system of Brinkman's type is shown under very mild assumptions concerning the shape of the obstacles and their mutual distance.



**Key words:** Navier-Stokes system, homogenization, Brinkman's law


## 1 Introduction

There is a vast amount of mathematical literature devoted to the flow of an incompressible fluid in perforated domains, where the number of holes, or obstacles, tends to infinity. To be more specific, consider a bounded spatial domain $\Omega \subset R^3$, together with a family of obstacles (compact sets) $T^1_\varepsilon, \dots, T^{N(\varepsilon)}_\varepsilon$, parametrized by $\varepsilon \to 0$. The motion of an incompressible fluid is governed by the *Navier-Stokes system* of equations

$$\mathrm{div}_x \mathbf{u} = 0 \text{ in } (0,T) \times \Omega_\varepsilon, \tag{1.1}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \mathbf{f}_\varepsilon \text{ in } (0,T) \times \Omega_\varepsilon, \tag{1.2}$$

where

$$\Omega_\varepsilon = \Omega \setminus \cup_{i=1}^{N(\varepsilon)} T^i_\varepsilon. \tag{1.3}$$

The symbol $\mathbf{u}$ denotes the fluid velocity, $p$ is the pressure, $\mathbf{f}_\varepsilon$ denotes a driving force, and $\mathbb{S}$ is the viscous stress tensor given by *Newton's rheological law*

$$\mathbb{S} = \nu (\nabla_x \mathbf{u} + \nabla^t_x \mathbf{u}), \ \nu > 0. \tag{1.4}$$


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Problem (1.1 - 1.4) is supplemented by the no-slip boundary conditions for the velocity

$$\mathbf{u}|_{\partial\Omega_\varepsilon} = 0, \tag{1.5}$$

and the initial condition

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{1.6}$$

As is well-known (see Leray [13], Hopf [11], Ladyzhenskaya [12], Temam [19], and many others), problem (1.1 - 1.6) possesses at least one *weak* solution provided $\partial\Omega_\varepsilon$ is sufficiently regular, $\mathbf{f}_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon;R^3))$, and $\mathbf{u}_{0,\varepsilon} \in L^2(\Omega_\varepsilon;R^3))$, $\mathrm{div}_x\mathbf{u}_{0,\varepsilon} = 0$, $\mathbf{u}_{0,\varepsilon}\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0$. Uniqueness as well as regularity of solutions in terms of the data represent a well known outstanding open problem in mathematical fluid mechanics.

The class of admissible weak solutions may be specified by imposing the *energy inequality*

$$\begin{aligned}\int_{\Omega_\varepsilon} \frac{1}{2}|\mathbf{u}|^2(\tau,\cdot)\ \mathrm{d}x + \nu\int_0^\tau\int_{\Omega_\varepsilon} |\nabla_x\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t \\ \leq \int_{\Omega_\varepsilon}\frac{1}{2}|\mathbf{u}_{0,\varepsilon}|^2\ \mathrm{d}x + \int_0^\tau\int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon\cdot\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t\end{aligned} \tag{1.7}$$

for a.a. $\tau > 0$. As a matter of fact, the velocity field is weakly continuous, more precisely $\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^2(\Omega_\varepsilon;R^3))$, therefore we may assume that (1.7) holds for *any* $\tau > 0$. Extending $\mathbf{u}$ to be zero outside $\Omega_\varepsilon$, relation (1.7) yields a bound

$$\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;W_0^{1,2}(\Omega;R^3)) \tag{1.8}$$

in terms of the data $\mathbf{u}_{0,\varepsilon}$, $\mathbf{f}_{0,\varepsilon}$. In particular, for

$$\{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0} \text{ bounded in } L^2(\Omega;R^3), \tag{1.9}$$

$$\{\mathbf{f}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;W^{-1,2}(\Omega;R^3)), \tag{1.10}$$

the associated family of weak solutions $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of problem (1.1 - 1.7) satisfies

$$\begin{aligned}\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)) \\ \text{and weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^3)),\end{aligned} \tag{1.11}$$

at least for a suitable subsequence.

It is easy to check that the limit $\mathbf{u}$ satisfies the incompressibility constraint (1.1) a.a. in $(0,T)\times\Omega$, however, performing the passage in the momentum equation (1.2) is more delicate. The collective effect of friction forces imposed on the fluid by each obstacle results, in general, in a new term of a *Brinkman* type appearing in the limit problem. Obviously, the asymptotic size as well as shape of the obstacles plays a crucial role in this process. We consider the so-called critical case, where the diameters of the sets $T_\varepsilon^i$ do not exceed the value $\varepsilon^3$, while their mutual distances are larger than $\varepsilon$. This distribution of obstacles is called *critical* since for “larger” holes or “shorter” mutual distances the limit velocity would necessarily vanish, while in the opposite case the limit problem would be the same as (1.1), (1.2). Note, however, that suitable scaling of the velocities in the former case gives rise to a Darcy-type law as the effective equation (see Allaire [2], [4], Mikelič [15]).

More specifically, we assume that

$$T_\varepsilon^i \subset B_\varepsilon^i \equiv \{x \mid |x - x_\varepsilon^i| < r_\varepsilon^i\},\ i = 1, \dots, N(\varepsilon),$$

$$\overline{B}_\varepsilon^i \subset \Omega \text{ for } i = 1, \dots, N(\varepsilon),\ \overline{B}_\varepsilon^i \cap \overline{B}_\varepsilon^j = \emptyset \text{ whenever } i \neq j.$$

Let $d_\varepsilon^i$ be a distance between balls $B_\varepsilon^i$, $B_\varepsilon^j$, $j \neq i$, and $\partial\Omega$. Then we suppose the following conditions for the perforation:

$$r_\varepsilon^i < d_\varepsilon^i, \quad \lim_{\varepsilon \to 0} \max_{1 \le i \le N(\varepsilon)} d_\varepsilon^i = 0, \tag{1.12}$$

$$\sum_{i=1}^{N(\varepsilon)} \frac{(r_\varepsilon^i)^2}{(d_\varepsilon^i)^3} \le C_1, \tag{1.13}$$

where $C_1$ is independent on of $i$ and $\varepsilon$.

Our approach is based on the concept of *Stokes' capacity*, analogous to the classical Newtonian capacity used in the homogenization problems for elliptic equations by DalMaso and Skrypnik [7], [8], Marchenko and Khruslov [14], Skrypnik [16], [17] among others. For a compact set $Q \subset R^3$, we introduce

$$C_{k,l}(Q) = \int_{R^3 \setminus Q} \nabla_x \mathbf{v}^k : \nabla_x \mathbf{v}^l \ \mathrm{d}x, \tag{1.14}$$

where $\mathbf{v}^k$ is the unique solution of the *model problem*

$$-\Delta_x \mathbf{v}^k + \nabla_x q^k = 0,\ \mathrm{div}_x \mathbf{v}^k = 0 \text{ in } B(x_0, 1) \setminus Q, \tag{1.15}$$

$$\mathbf{v}^k|_{\partial Q} = \mathbf{e}^k,\ \mathbf{v}^k|_{\partial B(x_0,1)} = 0, \tag{1.16}$$

here $\mathbf{e}^k$, $k = 1, 2, 3$ is the canonical basis of the space $R^3$. Let $B(x_0, r)$ be a minimal ball such that $Q \subset B(x_0, r)$, $r \ll 1$. Moreover, let normalize the pressure by the following equality

$$\int_{B(x_0,1)} q^k \, dx = 0.$$

In addition to the hypotheses (1.12), (1.13), we suppose that at least for a suitable subsequence,

$$\lim_{\varepsilon \to 0} \sum_{T_\varepsilon^i \subset G} C_{k,l}(T_\varepsilon^i) = \int_G C_{k,l}(x) \ \mathrm{d}x \tag{1.17}$$

for any Borel set $G \subset \Omega$, where $\mathbb{C} = \{C_{k,l}\}_{k,l=1}^3$, $\mathbb{C} \in L^\infty(\Omega; R^{3\times 3}_{\mathrm{sym}})$, see Section 5. It can be shown that the matrix $\mathbb{C}$ is constant in the case of periodically distributed obstacles of identical (rescaled) shape, see Allaire [3]. The effective momentum equation satisfied by the limit velocity field $\mathbf{u}$ reads

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \mathbb{C}\mathbf{u} + \nabla_x p = \mathrm{div}_x \mathbb{S} + \mathbf{f}, \tag{1.18}$$

where $\mathbf{f}$ is a weak limit of the sequence $\{\mathbf{f}_\varepsilon\}_{\varepsilon>0}$.

In order to justify the limit passage from (1.2) to (1.18), it is necessary to control the pressure in the associated stationary Stokes system

$$-\Delta \mathbf{v} + \nabla_x q = \mathbf{f}_\varepsilon \text{ in } \Omega_\varepsilon,\ \mathbf{v}|_{\partial\Omega_\varepsilon} = 0. \tag{1.19}$$

As observed in the seminal work of Tartar [18], this step requires the existence of *restriction operator* $\mathcal{R}_\varepsilon$ enjoying the following properties:

- $\mathcal{R}_\varepsilon : W_0^{1,2}(\Omega; R^3) \to W_0^{1,2}(\Omega_\varepsilon; R^3)$ is a bounded linear operator,

$$\|\mathcal{R}_\varepsilon[\mathbf{v}]\|_{W_0^{1,2}(\Omega_\varepsilon;R^3)} \leq c\|\mathbf{v}\|_{W_0^{1,2}(\Omega;R^3)}, \tag{1.20}$$

  with $c$ independent of $\varepsilon$.

-
$$\mathcal{R}_\varepsilon[\mathbf{v}] = \mathbf{v} \text{ for any } \mathbf{v} \in W_0^{1,2}(\Omega_\varepsilon; R^3). \tag{1.21}$$

-
$$\mathrm{div}_x \mathcal{R}_\varepsilon[\mathbf{v}] = 0 \text{ whenever } \mathrm{div}_x \mathbf{v} = 0. \tag{1.22}$$

As we will see in Section 3 below, the operator $\mathcal{R}_\varepsilon$ can be constructed under very mild restrictions imposed on the shape of the obstacles $\{T^i_\varepsilon\}_{i=1,\varepsilon>0}^{N(\varepsilon)}$, in particular if all of them are convex.

Homogenization of the *stationary* Navier-Stokes system under distribution of identical obstacles was considered by Marchenko and Khruslov [14] and later on was analyzed in detail in the seminal papers of Allaire [3], [4]. Desvillettes et al. [9] generalized Allaire's result to the quasi-stationary problem, where the family of obstacles is formed by balls that are allowed to move with prescribed velocities. To the best of our knowledge, the only result concerning homogenization of the *evolutionary* incompressible Navier-Stokes system with the limiting behavior of a Darcy-type law was obtained by Mikelič [15]. In comparison with the previous results, we impose only very mild hypotheses concerning the spatial distribution, the size, and the specific shape of the obstacles. One possible way of the proof of this kind of homogenization result is to follow the ideas of Allaire by verifying the abstract conditions in [4]. However, we prefer to give a direct proof which we believe can be of interest. Last but not least, we handle the evolutionary Navier-Stokes system combining homogenization with the classical Lions-Aubin argument.

The paper is organized as follows. In Section 2, we specify the principal hypotheses concerning the shape of obstacles and formulate our main result. One of the main ingredients of the proof is the construction of the restriction operator $\mathcal{R}_\varepsilon$ based on the recent results of Acosta et al. [1], Diening et al.

[10], concerning the inverse of the divergence operator on John's domains, see Sections 3. The properties of solutions to the model problem are given in Section 4. In Section 5, we analyze the associated stationary Stokes problem. Finally, we examine the time dependent Navier-Stokes system in Section 6.

# 2 Hypotheses and main result

In addition to (1.12), (1.13), we assume that the obstacles satisfy the following geometrical condition:

CONDITION (G):
There exists a constant $\omega > 0$ such that at each point $x \in \partial T^i_\varepsilon$ there exists a closed cone $C_x$ with vertex at $x$ and of aperture $\omega$ such that

$$C_x \cap T^i_\varepsilon = \{x\}.$$

We say that $\mathbf{u}_\varepsilon$ is a *weak solution* of problem (1.1 - 1.7) if

- $\mathbf{u}_\varepsilon$ belongs to the class $L^\infty(0,T;L^2(\Omega_\varepsilon;R^3)) \cap L^2(0,T;W^{1,2}_0(\Omega_\varepsilon;R^3))$;
- $\mathrm{div}_x \mathbf{u}_\varepsilon = 0$ a.a. in $(0,T) \times \Omega_\varepsilon$;
- the integral identity

$$\int_0^T \int_{\Omega_\varepsilon} \Big( \mathbf{u}_\varepsilon \cdot \partial_t \mathbf{w} + (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w} \Big) \mathrm{d}x \ \mathrm{d}t = - \int_{\Omega_\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \mathbf{w}(0,\cdot) \ \mathrm{d}x \quad (2.1)$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \mathbb{S} : \nabla_x \mathbf{w} \ \mathrm{d}x \ \mathrm{d}t - \int_0^T \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{w} \ \mathrm{d}x \ \mathrm{d}t$$

  holds for any test function $\mathbf{w} \in C^\infty_c([0,T) \times \Omega_\varepsilon; R^3)$, $\mathrm{div}_x \mathbf{w} = 0$;
- the energy inequality

$$\int_{\Omega_\varepsilon} \frac{1}{2} |\mathbf{u}_\varepsilon|^2 (\tau, \cdot) \ \mathrm{d}x + \nu \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_{\Omega_\varepsilon} \frac{1}{2} |\mathbf{u}_{0,\varepsilon}|^2 \ \mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t$$

  holds for a.a. $\tau > 0$.

Our main result reads as follows.

**Theorem 2.1** *Let $\{\Omega_\varepsilon\}_{\varepsilon>0} \subset R^3$ be a family of domains given by (1.3), where $T^i_\varepsilon$, $i = 1, \dots, N(\varepsilon)$, satisfy (1.12), (1.13), (1.17) together with condition $(G)$. Assume that*

$$\left\{ \begin{array}{c} \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \textit{ weakly in } L^2(\Omega; R^3), \\ \\ \mathbf{f}_\varepsilon \to \mathbf{f} \textit{ weakly in } L^2((0,T) \times \Omega; R^3). \end{array} \right\}$$

*Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions of problem (1.1 - 1.7).*

*Then, at least for a suitable subsequence,*

$$\mathbf{u}_\varepsilon \to \mathbf{u} \textit{ in } L^2((0,T) \times \Omega; R^3)) \textit{ and weakly in } L^2(0,T; W^{1,2}(\Omega; R^3)), \tag{2.2}$$

*where $\mathbf{u}$ is a weak solution of the problem*

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \mathbb{C}\mathbf{u} + \nabla_x p = \mathrm{div}_x \mathbb{S} + \mathbf{f} \textit{ in } (0,T) \times \Omega, \tag{2.3}$$

$$\mathrm{div}_x \mathbf{u} = 0 \textit{ a.a. in } (0,T) \times \Omega, \tag{2.4}$$

*with $\mathbb{C}$ given by (1.17), supplemented with the initial condition*

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0, \tag{2.5}$$

*and the boundary condition*

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{2.6}$$

In Theorem 2.1 and hereafter, we always assume that $\mathbf{u}_\varepsilon$ were extended to be zero outside $\Omega_\varepsilon$. The weak solutions for problem (2.3 - 2.6) are defined analogously to those for problem (1.1 - 1.7). The rest of the paper is devoted to the proof of Theorem 2.1.

# 3 Extension operator

One of the main novelties of the present paper is a construction of Tartar's restriction operator under very mild hypotheses imposed on the distribution of the obstacles. Note that comparable results obtained by Allaire [3] hold only in the periodic setting and for a fixed shape of a model hole. The hypothesis of periodicity was later relaxed by Desvillettes et al. [9], where, however, the model hole is a ball. Our approach is based on recent results of Acosta et al. [1] concerning the explicit construction of the so-called Bogovskii operator on John's domains (see also Diening et al. [10]).

**Proposition 3.1** *Let $\{T^i_\varepsilon\}^{N(\varepsilon)}_{i=1,\varepsilon>0}$ satisfy hypotheses (1.12), (1.13), together with condition $(G)$.*

*Then there exists a restriction operator $\mathcal{R}_\varepsilon$ enjoying properties (1.20 - 1.22).*

**Proof:**
The proof will be carried over in several steps.

**Step 1:**

For a given function $\mathbf{u}$, we introduce $\mathbf{w}^i_\varepsilon$, $i = 1, \dots, N(\varepsilon)$ satisfying

$$\mathbf{w}^i_\varepsilon \in W^{1,2}_0(C^i_\varepsilon; R^3),$$

$$-\Delta \mathbf{w}^i_\varepsilon + \nabla_x q^i_\varepsilon = -\Delta \mathbf{u} \text{ in } C^i_\varepsilon,$$

$$\mathrm{div}_x \mathbf{w}^i_\varepsilon = \mathrm{div}_x \mathbf{u} + \frac{1}{|B^i_\varepsilon|} \int_{T^i_\varepsilon} \mathrm{div}_x \mathbf{u} \ \mathrm{d}x \text{ in } C^i_\varepsilon,$$

$$\mathbf{w}^i_\varepsilon = \mathbf{u} \text{ on } \partial C^i_\varepsilon \setminus \partial T^i_\varepsilon, \ \mathbf{w}^i_\varepsilon = 0 \text{ on } \partial T^i_\varepsilon,$$

where $C^i_\varepsilon$ is the control volume around $T^i_\varepsilon$ that is $C^i_\varepsilon := B^i_\varepsilon \setminus T^i_\varepsilon$.

If $\mathbf{u} \in W^{1,2}_0(\Omega; R^3)$, we set

$$\mathcal{R}_\varepsilon[\mathbf{u}] = \begin{cases} 0 \text{ in } T^i_\varepsilon, \\ \\ \mathbf{w}^i_\varepsilon \text{ in } C^i_\varepsilon, \ i = 1, \dots, N(\varepsilon), \\ \\ \mathbf{u} \text{ otherwise.} \end{cases}$$

**Step 2:**

Following the arguments of Allaire [3], we observe that it is enough to verify that

$$\|\nabla_x \mathbf{w}^i_\varepsilon\|^2_{L^2(C^i_\varepsilon; R^{3\times 3}))} \leq c\|\mathbf{u}\|^2_{W^{1,2}(C^i_\varepsilon; R^3)}, \ i = 1, \dots, N(\varepsilon), \tag{3.1}$$

where $c$ is independent of $i$, $\varepsilon$.

**Step 3:**

Furthermore, in view of Allaire [3, Section 2.2], estimate (3.1) follows as soon as we are able to solve an auxiliary problem:

*Given $f \in L^2(\mathcal{B} \setminus T^i_\varepsilon)$, $\mathcal{B} = \{x| \ |x - x^i_\varepsilon| \leq 1\}$, $\int_{\mathcal{B}\setminus T^i_\varepsilon} f \ \mathrm{d}x = 0$, find $\mathbf{v} \in \mathcal{B} \setminus T^i_\varepsilon$ such that*

$$\mathrm{div}_x \mathbf{v} = f \ \text{ in } \mathcal{B} \setminus T^i_\varepsilon, \ \mathbf{v}|_{\partial(\mathcal{B}\setminus T^i_\varepsilon)} = 0, \tag{3.2}$$

$$\|\mathbf{v}\|_{W^{1,2}_0(\mathcal{B}\setminus T^i_\varepsilon; R^3)} \leq c\|f\|_{L^2(\mathcal{B}\setminus T^i_\varepsilon)}, \tag{3.3}$$

*where $c$ is independent of $i$, $\varepsilon$.*

Since $T^i_\varepsilon$ satisfy condition (G), solutions of problem (3.2) can be constructed by the method of Acosta et al. [1]. Indeed, in notation of [1, Theorem 4.1], the reference point $x_0$ can be taken on the sphere $\{x| \ |x - x^i_\varepsilon| = 1/2\}$, where the curve connecting $x_0$ with a point $x \in \partial T^i_\varepsilon$ can be taken the axe of the cone $C_x$ up to its intersection $y$ with the sphere $\{x| \ |x - x^i_\varepsilon| = 1/2\}$, together with the geodesics connecting $y$, $x_0$ on $\{x| \ |x - x^i_\varepsilon| = 1/2\}$. In accordance with [1, Theorem 4.1], the bound (3.3) follows, where $c$ depends only on the aperture $\omega$ appearing in condition (G). Proposition 3.1 has been proved.

Q.E.D.

As a consequence of Proposition 3.1, we can construct the so-called *Bogovskii's operator* - a suitable branch of $\mathrm{div}_x^{-1}$ in $\Omega_\varepsilon$.

**Proposition 3.2** *For each $\varepsilon > 0$ there exists a linear operator $\mathcal{B}_\varepsilon$ such that $\mathbf{v} = \mathcal{B}_\varepsilon[f]$ solves the problem*

$$\mathrm{div}_x \mathbf{v} = f \ \textit{in}\ \Omega_\varepsilon, \ \mathbf{v}|_{\partial\Omega_\varepsilon} = 0 \tag{3.4}$$

*for any $f \in L^2(\Omega_\varepsilon)$, $\int_{\Omega_\varepsilon} f \ \mathrm{d}x = 0$. Moreover,*

$$\|\mathbf{v}\|_{W^{1,2}(\Omega_\varepsilon;R^3)} \leq c\|f\|_{L^2(\Omega_\varepsilon)}, \tag{3.5}$$

*with $c$ independent of $\varepsilon$.*

**Proof:**
Extending $f$ to be zero outside $\Omega_\varepsilon$ we first solve the problem

$$\mathrm{div}_x \mathbf{w}_0 = f \text{ in } \Omega, \ \mathbf{w}_0|_{\partial\Omega} = 0,$$

$$\|\mathbf{w}_0\|_{W^{1,2}(\Omega;R^3)} \leq c\|f\|_{L^2(\Omega_\varepsilon)}.$$

Now, take $\mathbf{v}_0 = \mathcal{R}_\varepsilon[\mathbf{w}_0]$, where $\mathcal{R}_\varepsilon$ is the restriction operator constructed explicitly in Proposition 3.1. Accordingly, we have

$$\mathrm{div}_x \mathbf{v}_0 = f + g_0,$$

where

$$\int_\Omega g_0 \ \mathrm{d}x = 0, \ \|g_0\|_{L^2(\Omega)} \leq c_1\|f\|_{L^2(\Omega_\varepsilon)}, \text{ with } c_1 = \sqrt{|T^i_\varepsilon|/|B^i_\varepsilon|} \leq \varepsilon^3.$$

Repeating the same construction for $g_0$ we obtain $\mathbf{v}_1$ such that

$$\mathrm{div}_x(\mathbf{v}_0 + \varepsilon^3\mathbf{v}_1) = f + g_1,$$

with

$$\|\mathbf{v}_1\|_{W_0^{1,2}(\Omega_\varepsilon;R^3)} \leq c\|f\|_{L^2(\Omega_\varepsilon)}, \ \|g^1\|_{L^2(\Omega_\varepsilon)} \leq (\varepsilon^3)^2\|f\|_{L^2(\Omega_\varepsilon)}$$

Thus by induction we can construct a function

$$\mathbf{v} = \sum_{i=0}^{\infty}(\varepsilon^3)^i\mathbf{v}_i$$

with the desired properties.

Then, using Propositions 3.1, 3.2 and following the idea of Allaire [3] about the existence of an extension operator for the pressure (Proposition 1.1.4, [3]), we have the following statement.

**Theorem 3.1** *Let the restriction operator $\mathcal{R}_\varepsilon$ satisfy properties (1.20)–(1.22). Then there exists an extension operator $\mathcal{P}_\varepsilon$ defined for every $q_\varepsilon \in L_2(\Omega_\varepsilon)$ by*

$$\int_\Omega \nabla\left[\mathcal{P}_\varepsilon(q_\varepsilon)\right] \cdot \mathbf{h}\, dx = \int_\Omega \nabla\, q_\varepsilon \cdot \mathcal{R}_\varepsilon \mathbf{h}\, dx$$

*for each $\mathbf{h} \in W_0^{1,2}(\Omega; R^3)$, and satisfying the following conditions*

- $\mathcal{P}_\varepsilon : L_2(\Omega_\varepsilon) \to L_2(\Omega)$ *is a linear continuous operator;*
- $\mathcal{P}_\varepsilon[q_\varepsilon] = q_\varepsilon$ *in* $L_2(\Omega_\varepsilon)$;
- $\|\mathcal{P}_\varepsilon(q_\varepsilon)\|_{L_2(\Omega)} \le c\|q_\varepsilon\|_{L_2(\Omega_\varepsilon)}$;
- $\|\nabla[\mathcal{P}_\varepsilon(q_\varepsilon)]\|_{W^{-1,2}(\Omega)} \le c\|\nabla q_\varepsilon\|_{W^{-1,2}(\Omega_\varepsilon)}$;

*where* $c$ *is a constant independent of* $\varepsilon$ *and* $q_\varepsilon$.

# 4 Model Stokes problem

In the next consideration we will need the following pointwise and integral estimates of the solutions to the model problem (1.15), (1.16).

**Lemma 4.1** *Let* $\mathbf{v}^k$, $q^k$ *are solutions to the model problem (1.15), (1.16) and the set* $Q$ *satisfies the condition* $(G)$. *There exist positive constants* $C_j$, $j = 1,\ldots,4$, *not depending on* $r$ *such that the following estimates are valid*

$$|D^\alpha v_i^k(x)| \le C_1 \frac{r}{|x-x_0|^{1+|\alpha|}},\ |q^k(x)| \le C_2 \frac{r}{|x-x_0|^2},\ x \in B(x_0,1)\setminus B(x_0,r), \tag{4.1}$$

*where* $|\alpha| = 0,1,2,\ i = 1,2,3,$ *and*

$$\int\limits_{B(x_0,d)} |\mathbf{v}^k|^2\,dx \le C_3 r^2 d, \qquad \int\limits_{B(x_0,d)} |\nabla_x \mathbf{v}^k|^2\,dx \le C_3 r, \tag{4.2}$$

$$\int\limits_{B(x_0,d)} |q^k|^2\,dx \le C_4 r, \tag{4.3}$$

*for every* $d > r$.

**Proof:** The proof of the estimates (4.1), (4.2) without any restriction on the set $Q$ can be found for example in [14]. The integral estimate (4.3) of the pressure is proved under the additional condition on the shape of the set $Q$. Namely, analogously to [6] we use the Bogovskii operator $B$ such that the function $\mathbf{v} = B[q^k]$ is the solution of the following problem

$$\operatorname{div}\mathbf{v} = q^k \text{ in } B(x_0,1)\setminus Q, \quad \mathbf{v} = 0,\ x \in \partial\{B(x_0,1)\setminus Q\},$$

moreover the next estimate is valid

$$\|\mathbf{v}\|_{W^{1,2}(B(x_0,1)\setminus Q;R^3)} \le c\|q^k\|_{L_2(B(x_0,1)\setminus Q)}.$$

The construction of this operator for the domain $Q$ under the condition $(G)$ was provided in Proposition 3.2. Multiply the first equation in (1.15) on $\mathbf{v}$ and integrate over $B(x_0,r)\setminus\Omega$, as a result we have:

$$\int_{B(x_0,r)\setminus Q} \nabla\mathbf{v}^k : \nabla(B[q^k])\,dx - \int_{B(x_0,r)\setminus Q} q^k\,\nabla\cdot(B[q^k])\,dx = 0.$$

Using Hölder's and Young's inequalities we derive

$$\int_{B(x_0,r)\setminus Q} |q^k|^2\, dx \leq \left( \int_{B(x_0,r)\setminus Q} |\nabla \mathbf{v}^k|^2 dx \right)^{\frac{1}{2}} \left( \int_{B(x_0,r)\setminus Q} |\nabla (B[q^k])|^2\, dx \right)^{\frac{1}{2}}$$

$$\leq C(\varepsilon) \int_{B(x_0,r)\setminus Q} |\nabla \mathbf{v}^k|^2 dx + \varepsilon \int_{B(x_0,r)\setminus Q} |q^k|^2\, dx.$$

Choosing $\varepsilon = \frac{1}{2}$ and applying (4.2) we obtain (4.3).

Q.E.D.

# 5 Stationary Stokes problem

The heart of the paper is the analysis of the associated *stationary* Stokes problem in the form

$$-\Delta \mathbf{v} + \nabla_x q = \mathbf{f}_\varepsilon \text{ in } \Omega_\varepsilon,\ \mathbf{v}|_{\partial\Omega_\varepsilon} = 0,\ \mathrm{div}_x \mathbf{v} = 0 \text{ in } \Omega_\varepsilon. \tag{5.1}$$

The problem (5.1) has the following weak formulation. We say that $(\mathbf{v}_\varepsilon, q_\varepsilon) \in W_0^{1,2}(\Omega_\varepsilon; R^3) \times L_2(\Omega_\varepsilon)$ is a weak solution of problem (5.1) if the following integral identities hold:

$$\int_{\Omega_\varepsilon} \nabla\, \mathbf{v}_\varepsilon : \nabla\, \mathbf{h}_\varepsilon\, dx - \int_{\Omega_\varepsilon} q_\varepsilon \nabla \cdot \mathbf{h}_\varepsilon\, dx = \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{h}_\varepsilon\, dx \quad \forall\ \mathbf{h}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon; R^3) \tag{5.2}$$

$$\int_{\Omega_\varepsilon} r_\varepsilon \nabla \cdot \mathbf{v}_\varepsilon\, dx = 0 \quad \forall\ r_\varepsilon \in L_2(\Omega_\varepsilon). \tag{5.3}$$

As is well known, problem (5.1) admits a unique solution $(\mathbf{v}_\varepsilon, q_\varepsilon)$ for any $\mathbf{f}_\varepsilon \in W^{-1,2}(\Omega; R^3)$, $\varepsilon > 0$ fixed. In accordance with our agreement, all functions defined in $\Omega_\varepsilon$ are extended to be zero in $\Omega \setminus \Omega_\varepsilon$. In particular, $\mathbf{f}_\varepsilon$ can be viewed as a functional in $W^{-1,2}(\Omega_\varepsilon; R^3)$.

**Proposition 5.1** *Let a family of domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$ satisfy the hypotheses of Theorem 2.1. Assume that*

$$\mathbf{f}_\varepsilon \to \mathbf{f} \text{ in } W^{-1,2}(\Omega; R^3). \tag{5.4}$$

*Let $(\mathbf{v}_\varepsilon,\, q_\varepsilon)$ be the unique solution of the Stokes problem (5.1) in $\Omega_\varepsilon$, $\int_{\Omega_\varepsilon} q_\varepsilon\ \mathrm{d}x = 0$. Then, at least for a suitable subsequence,*

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } W_0^{1,2}(\Omega; R^3),\ q_\varepsilon \to q \text{ weakly in } L^2(\Omega),$$

*where $(\mathbf{v}, q)$ is the unique solution of the problem*

$$-\Delta \mathbf{v} + \mathbb{C}\mathbf{v} + \nabla_x q = \mathbf{f} \text{ in } \Omega,\ \mathbf{v}|_{\partial\Omega} = 0,\ \mathrm{div}_x \mathbf{v} = 0 \text{ in } \Omega, \tag{5.5}$$

*with a matrix $\mathbb{C}$ determined by (1.17).*

**Proof:**

In view of the existing results (Allaire [3], Desvillettes et al [9]), the most difficult part of the proof is the existence of the restriction operator $\mathcal{R}_\varepsilon$ established in Proposition 3.1. The remaining part of the proof is nowadays standard. By virtue of the properties of solutions of *model problem* (1.15), (1.16), the problem may be shown to fit the abstract framework developed by Allaire [3, Section 1]. In particular, we could show that hypotheses (H1) - (H6) of [3, Section 1.1] are satisfied; whence Proposition 5.1 follows from [3, Theorem 1.1.8]. Here, we prefer to give a proof using different arguments that we believe may be of independent interest.

We extend the solution $\mathbf{v}_\varepsilon$ in problem (5.1) to $x \in \Omega \setminus \Omega_\varepsilon$ by setting $\mathbf{v}_\varepsilon = 0$ then $\mathbf{v}_\varepsilon \in W_0^{1,2}(\Omega; R^3)$. Using for the redefined function the same notation and applying the Poincaré inequality in $\Omega$, we have:

$$\|\nabla \mathbf{v}_\varepsilon\|_{L_2(\Omega;R^3)} \leq C\|\mathbf{f}_\varepsilon\|_{W^{-1,2}(\Omega;R^3)} \tag{5.6}$$

with the constant $C$ depending only on $\Omega$. Then the set of functions $\{\mathbf{v}_\varepsilon\}$ is bounded and weakly compact in $W_0^{1,2}(\Omega; R^3)$. Therefore, there exists a subsequence converging to some function $\mathbf{v} \in W_0^{1,2}(\Omega; R^3)$ as $\varepsilon \to 0$.

To show the existence of the limit pressure $q$ we need the a priori estimate for the unknown pressure $q_\varepsilon$ of problem (5.1). The operator of extension $\mathcal{P}_\varepsilon$ on the set $\Omega \setminus \Omega_\varepsilon$ was defined in Section 3 with the help of the restriction operator $\mathcal{R}_\varepsilon$ given by Proposition 3.1. It is easy to prove that the constructed extension of the pressure is bounded in $L_2(\Omega)$ and there exists a subsequence converging weakly in $L_2(\Omega)$ to some function $q \in L_2(\Omega)$ as $\varepsilon \to 0$ (see [3]).

We show that the limit functions $(\mathbf{v}, q)$ represent a solution of the homogenized problem in $\Omega$.

Denote by $\rho_\varepsilon^i$ the following numbers:

$$\rho_\varepsilon^i = \max\left\{\frac{3}{2} r_\varepsilon^i, \frac{(d_\varepsilon^i)^3 \ln^2 \frac{1}{d_\varepsilon^i}}{2\tilde{C}}\right\},$$

where $\tilde{C} = \max\limits_{0<t\leq \operatorname{diam}\Omega} t^2 \ln^2 \frac{1}{t}$. It is easy to see that

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} (\rho_\varepsilon^i)^3 = 0. \tag{5.7}$$

Let $\chi \in C_0^\infty(R^1)$ be the cut-off function with the following properties: $(i)$ $0 \leq \chi \leq 1$; $(ii)$ $\chi(\zeta) = 1$ if $\zeta \leq \lambda_1$, $\chi(\zeta) = 0$ if $\zeta \geq \lambda_2$, where the numbers $\lambda_1$, $\lambda_2$ satisfy inequalities: $\frac{2}{3} < \lambda_1 < \lambda_2 < 1$; $(iii)$ $\left|\frac{d\chi}{d\zeta}\right| \leq 2$. We define

$$\varphi_i^{(\varepsilon)}(x) = \chi\left(\frac{|x - x_\varepsilon^i|}{\rho_\varepsilon^i}\right)$$

where $x_\varepsilon^i$ is the center of ball $B_\varepsilon^i$ defined by (1.12).

Let $(\mathbf{v}_{i,\varepsilon}^k, q_{i,\varepsilon}^k)$ be a solution of the model problem (1.15), (1.16) with $Q = T_i^{(\varepsilon)}$. To describe the asymptotic behaviour of the velocity $\mathbf{v}_\varepsilon$ we need the approximations of its weak limit $\mathbf{v}$ and the function $\mathbf{f}$ from (5.4). Let fix the parameter $\kappa > 0$ and represent $\mathbf{v}$, $\mathbf{f}$ as

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_\kappa,\ \mathbf{f} = \mathbf{f}_0 + \mathbf{f}_\kappa,$$

where $\mathbf{v}_0$, $\mathbf{f}_0 \in C_0^\infty(\Omega; R^3)$ and

$$\|\mathbf{v}_\kappa\|_{W^{1,2}(\Omega;R^3)} \leq \kappa,\ \|\mathbf{f}_\kappa\|_{W^{1,2}(\Omega;R^3)} \leq \kappa.$$

Then the following asymptotic expansion is constructed:

$$\mathbf{v}_\varepsilon(x) = \mathbf{v}_0(x) + \mathbf{z}_\varepsilon^{(1)}(x) + \mathbf{z}_\varepsilon^{(2)}(x) + \mathbf{z}_\kappa^{(\varepsilon)}(x), \tag{5.8}$$

where

$$\mathbf{z}_\varepsilon^{(1)}(x) = \sum_{i=1}^{N(\varepsilon)} (\mathbf{v}_0(x_\varepsilon^i) - \mathbf{v}_0(x))\varphi_i^{(\varepsilon)}(x)$$

$$\mathbf{z}_\varepsilon^{(2)}(x) = -\sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \mathbf{v}_{i,\varepsilon}^k(x)\varphi_i^{(\varepsilon)}(x) v_0^k(x_\varepsilon^i),$$

and $\mathbf{z}_\kappa^{(\varepsilon)} \in W_0^{1,2}(\Omega_\varepsilon; R^3)$ is the remainder term of the asymptotic expansion.

Taking into account the properties of solutions to the model problem we obtain the following behaviour of terms in the asymptotic expansion for the velocity.

**Lemma 5.1** *Let $\Omega_\varepsilon$ satisfy hypotheses (1.12), (1.13), and condition $(G)$. Then $\mathbf{z}_\varepsilon^{(1)}$, $\mathbf{z}_\kappa^{(\varepsilon)}$ converge strongly to zero, and $\mathbf{z}_\varepsilon^{(2)}$ converges weakly to zero in $W^{1,2}(\Omega_\varepsilon; R^3)$ as $\varepsilon \to 0$.*

**Proof:** Consider the $L_2$-norm of the $\mathbf{z}_\varepsilon^{(1)}$:

$$\lim_{\varepsilon\to 0} \int_\Omega |\mathbf{z}_\varepsilon^{(1)}|^2\, dx = \lim_{\varepsilon\to 0} \int_\Omega \left| \sum_{i=1}^{N(\varepsilon)} (\mathbf{v}_0(x_\varepsilon^i) - \mathbf{v}_0(x))\varphi_i^{(\varepsilon)} \right|^2 dx$$

$$\leq \lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} \int_{B(x_\varepsilon^i, \lambda_2\rho_\varepsilon^i)} |\mathbf{v}_0(x_\varepsilon^i) - \mathbf{v}_0(x)|^2 dx \leq c \lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} \operatorname{meas} B(x_\varepsilon^i, \lambda_2\rho_\varepsilon^i) = 0,$$

where $c = c(\kappa)$. This proves the strong convergence of $\mathbf{z}_\varepsilon^{(1)}$ in $L_2(\Omega)$ to zero. Applying Poincaré's inequality we estimate the $L_2$-norm of the gradient of this function:

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla \mathbf{z}_\varepsilon^{(1)}|^2\, dx = \lim_{\varepsilon\to 0} \int_\Omega \left| \nabla \left( \sum_{i=1}^{N(\varepsilon)} (\mathbf{v}_0(x_\varepsilon^i) - \mathbf{v}_0(x))\varphi_i^{(\varepsilon)} \right) \right|^2 dx$$

$$\leq c \lim_{\varepsilon\to 0}\left(\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\nabla \mathbf{v}_0|^2|\varphi_i^{(\varepsilon)}|^2dx+\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\mathbf{v}_0(x_\varepsilon^i)-\mathbf{v}_0|^2|\nabla\varphi_i^{(\varepsilon)}|^2dx\right)$$

$$\leq c\lim_{\varepsilon\to 0}\int_{\bigcup_{i=1}^{N(\varepsilon)}B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\nabla\mathbf{v}_0|^2dx=0,$$

from the absolute continuity of the integral, since

$$\lim_{\varepsilon\to 0}\operatorname{meas}\bigcup_{i=1}^{N(\varepsilon)}B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)=c\lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)}(\rho_\varepsilon^i)^3=0,$$

here $c$ is independent of $\varepsilon$.

Now we show the strong convergence of the second term in the asymptotic expansion in $L_2(\Omega;R^3)$. Applying the integral estimate (4.2) of the velocity for the model problem, it is easy to see that

$$\lim_{\varepsilon\to 0}\int_\Omega|\mathbf{z}_\varepsilon^{(2)}|^2\,dx\leq c\lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}\left(\varphi_i^{(\varepsilon)}\sum_{k=1}^{3}v_0^k(x_\varepsilon^i)(\mathbf{e}_k-\mathbf{v}_{i,\varepsilon}^k)\right)^2dx$$

$$\leq c\lim_{\varepsilon\to 0}\sum_{k=1}^{3}\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}(1+|\mathbf{v}_{i,\varepsilon}^k|^2)dx\leq c\lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)}(\rho_\varepsilon^i)^3+c\lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)}\rho_\varepsilon^i(r_\varepsilon^i)^2=0$$

where $c$ is dependent of $\kappa$. Next we prove boundedness to the norm of the gradient $\mathbf{z}_\varepsilon^{(2)}$ in $L_2(\Omega)$ by a constant not depending on $\varepsilon$.

$$\int_\Omega|\nabla\mathbf{z}_\varepsilon^{(2)}|^2\,dx$$

$$\leq c\sum_{k=1}^{3}\sum_{i=1}^{N(\varepsilon)}(v_0^k(x_\varepsilon^i))^2\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}\left(|\nabla\varphi_i^{(\varepsilon)}|^2(\mathbf{e}^k-\mathbf{v}_{i,\varepsilon}^k)^2+|\varphi_i^{(\varepsilon)}|^2|\nabla\mathbf{v}_{i,\varepsilon}^k|^2\right)dx$$

$$\leq c\sum_{k=1}^{3}\sum_{i=1}^{N(\varepsilon)}\left(\frac{1}{(\rho_\varepsilon^i)^2}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\setminus B(x_\varepsilon^i,\lambda_1\rho_\varepsilon^i)}(1+|\mathbf{v}_{i,\varepsilon}^k|^2)dx+\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\nabla\mathbf{v}_{i,\varepsilon}^k|^2dx\right)$$

$$\leq c\left(\sum_{i=1}^{N(\varepsilon)}\rho_\varepsilon^i+\sum_{i=1}^{N(\varepsilon)}\frac{(r_\varepsilon^i)^2}{\rho_\varepsilon^i}+\sum_{i=1}^{N(\varepsilon)}r_\varepsilon^i\right)\leq c,$$

where $c$ is dependent of $\kappa$.

The weak convergence of the reminder term of the asymptotic expansion to zero is a direct consequence of its definition and the previous consideration, so that $\mathbf{z}_\kappa^{(\varepsilon)}$ converges weakly to zero in $W^{1,2}(\Omega;R^3)$ as $\varepsilon\to 0$.

The next step is to show the strong convergence of the gradient $\nabla\mathbf{z}_\kappa^{(\varepsilon)}(x)$ in $L_2(\Omega_\varepsilon;R^3)$ as $\varepsilon\to 0$. Testing the integral identity (5.2) by $\mathbf{h}_\varepsilon=\mathbf{z}_\kappa^{(\varepsilon)}(x)$ and using representation (5.8), we derive:

$$\int_{\Omega_\varepsilon}\nabla\left(\mathbf{v}_0(x)+\mathbf{z}_\varepsilon^{(1)}(x)+\mathbf{z}_\varepsilon^{(2)}(x)+\mathbf{z}_\kappa^{(\varepsilon)}(x)\right):\nabla\,\mathbf{z}_\kappa^{(\varepsilon)}(x)\,dx-\int_{\Omega_\varepsilon}q_\varepsilon\nabla\cdot\mathbf{z}_\kappa^{(\varepsilon)}(x)\,dx$$

$$= \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{z}_\kappa^{(\varepsilon)}(x)\, dx. \tag{5.9}$$

Taking into account the following properties

$$\mathbf{z}_\varepsilon^{(1)} \to 0, \quad \mathbf{z}_\kappa^{(\varepsilon)} \to 0 \text{ wealky } \text{ in } W^{1,2}(\Omega; R^3) \text{ as } \varepsilon \to 0,$$

it is easy to see that

$$\int_{\Omega_\varepsilon} \nabla \left( \mathbf{v}_0(x) + \mathbf{z}_\varepsilon^{(1)}(x) \right) : \nabla \mathbf{z}_\kappa^{(\varepsilon)}(x)\, dx \to 0 \text{ as } \varepsilon \to 0.$$

The next step is to show the convergence

$$\int_{\Omega_\varepsilon} \nabla \mathbf{z}_\varepsilon^{(2)}(x) : \nabla \mathbf{z}_\kappa^{(\varepsilon)}(x)\, dx \to 0 \text{ as } \varepsilon \to 0. \tag{5.10}$$

From the definition of $\mathbf{z}_\varepsilon^{(2)}(x)$ we have

$$\int_{\Omega_\varepsilon} \nabla \mathbf{z}_\varepsilon^{(2)}(x) : \nabla \mathbf{z}_\kappa^{(\varepsilon)}(x)\, dx$$

$$= \int_{\Omega_\varepsilon} \nabla \left( \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \mathbf{v}_{i,\varepsilon}^k(x) \varphi_i^{(\varepsilon)}(x) v_0^k(x_\varepsilon^i) \right) : \nabla \mathbf{z}_\kappa^{(\varepsilon)}(x)\, dx \le c\, I_1^{(\varepsilon,\kappa)} + c\, I_2^{(\varepsilon,\kappa)}, \tag{5.11}$$

where

$$I_1^{(\varepsilon,\kappa)} := \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\setminus B(x_\varepsilon^i,\lambda_1\rho_\varepsilon^i)} |\nabla \varphi_i^{(\varepsilon)}(x)|\, |\mathbf{v}_{i,\varepsilon}^k(x)|\, |\nabla \mathbf{z}_\kappa^{(\varepsilon)}(x)|\, dx,$$

$$I_2^{(\varepsilon,\kappa)} := \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \left| \int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)} \nabla \mathbf{v}_{i,\varepsilon}^k(x) : (\varphi_i^{(\varepsilon)}(x)\, \nabla \mathbf{z}_\kappa^{(\varepsilon)}(x))\, dx \right|.$$

Applying the Hölder inequality, we derive

$$I_1^{(\varepsilon,\kappa)} \le c \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \left( \frac{1}{(\rho_i^\varepsilon)^2} \int_{B(x_\varepsilon^i,\lambda_2\rho_i^\varepsilon)\setminus B(x_\varepsilon^i,\lambda_1\rho_i^\varepsilon)} |\mathbf{v}_{i,\varepsilon}^k(x)|^2\, dx \right)^{\frac{1}{2}}$$

$$\times \left( \int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)} |\nabla \mathbf{z}_\kappa^{(\varepsilon)}|^2\, dx \right)^{\frac{1}{2}}.$$

Applying to the right-hand side the pointwise estimate (4.1) for the solutions of the model problems near the small sets, we have

$$I_1^{(\varepsilon,\kappa)} \le c \sum_{i=1}^{N(\varepsilon)} \left( \frac{(r_\varepsilon^i)^2}{\rho_\varepsilon^i} \right)^{\frac{1}{2}} \left( \int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)} |\nabla \mathbf{z}_\kappa^{(\varepsilon)}|^2\, dx \right)^{\frac{1}{2}}$$

$$\leq c \sum_{i=1}^{N(\varepsilon)} \left( \frac{(r_\varepsilon^i)^2}{(d_\varepsilon^i)^3} \right)^{\frac{1}{2}} \left( \frac{(d_\varepsilon^i)^3}{\rho_\varepsilon^i} \int_{B(x_\varepsilon^i, \lambda_2 \rho_\varepsilon^i)} |\nabla \mathbf{z}_\kappa^{(\varepsilon)}|^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq c \left( \sum_{i=1}^{N(\varepsilon)} \frac{(r_\varepsilon^i)^2}{(d_\varepsilon^i)^3} \right)^{\frac{1}{2}} \left( \sum_{i=1}^{N(\varepsilon)} \frac{(d_\varepsilon^i)^3}{\rho_\varepsilon^i} \int_{B(x_\varepsilon^i, \lambda_2 \rho_\varepsilon^i)} |\nabla \mathbf{z}_\kappa^{(\varepsilon)}|^2 \, dx \right)^{\frac{1}{2}}.$$

It is easy to see that

$$\lim_{\varepsilon \to 0} \max_{1 \leq i \leq N(\varepsilon)} \frac{(d_\varepsilon^i)^3}{\rho_\varepsilon^i} \leq c \lim_{\varepsilon \to 0} \max_{1 \leq i \leq N(\varepsilon)} \frac{1}{\ln^2 \frac{1}{d_\varepsilon^i}} = 0, \tag{5.12}$$

where the constant $c$ does not depend on $\varepsilon$. Taking into account (1.13), (5.12), we have

$$\lim_{\varepsilon \to 0} I_1^{(\varepsilon,\kappa)} \leq c \lim_{\varepsilon \to 0} \max_{1 \leq i \leq N(\varepsilon)} \frac{1}{\ln^2 \frac{1}{d_\varepsilon^i}} \left( \int_\Omega |\nabla \mathbf{z}_\kappa^{(\varepsilon)}|^2 \, dx \right)^{\frac{1}{2}} = 0. \tag{5.13}$$

Let us consider the second term in (5.11)

$$I_2^{(\varepsilon,\kappa)} = \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \left| \int_{B(x_\varepsilon^i, \lambda_2 \rho_\varepsilon^i)} \nabla \mathbf{v}_{i,\varepsilon}^k : \left( \nabla (\varphi_i^{(\varepsilon)} \mathbf{z}_\kappa^{(\varepsilon)}) - (\nabla \varphi_i^{(\varepsilon)}) \cdot \mathbf{z}_\kappa^{(\varepsilon)} \right) dx \right|$$

$$\leq \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \left| \int_{B(x_i^\varepsilon, \lambda_2 \rho_\varepsilon^i)} \nabla \mathbf{v}_{i,\varepsilon}^k : \nabla (\varphi_i^{(\varepsilon)} \mathbf{z}_\kappa^{(\varepsilon)}) \, dx \right|$$

$$+ \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} \left| \int_{B(x_\varepsilon^i, \lambda_2 \rho_\varepsilon^i)} \nabla \mathbf{v}_{i,\varepsilon}^k : (\nabla \varphi_i^{(\varepsilon)}) \cdot \mathbf{z}_\kappa^{(\varepsilon)} \, dx \right| = I_3^{(\varepsilon,\kappa)} + I_4^{(\varepsilon,\kappa)}. \tag{5.14}$$

To estimate $I_4^{(\varepsilon,\kappa)}$ we use the following generalized Poincaré inequality. The scalar version of this inequality was proved in [17] (Lemma 1.4, Chapter 8).

**Lemma 5.2** *For every function $\mathbf{u} \in W^{1,2}(B(0,r); R^n)$ there exists a positive constant $C$ depending on $n$ ($n$ denotes a dimension) such that the following inequality holds*

$$\frac{1}{\rho^{n-1}} \int_{B(0,\rho)} |\mathbf{u}|^2 \, dx \leq C \left( \frac{1}{r^n} \rho^{n-2} \int_{B(0,r) \setminus B(0,r \setminus 2)} |\mathbf{u}|^2 \, dx + \int_{B(0,r)} |\nabla \mathbf{u}|^2 \, dx \right)$$

*for every $\rho < \frac{r}{2}$.*

Applying this lemma, we obtain

$$\frac{1}{(\rho_\varepsilon^i)^2} \int_{B(x_\varepsilon^i, \lambda_2 \rho_\varepsilon^i)} |\mathbf{z}_\kappa^{(\varepsilon)}|^2 \, dx$$

$$\leq c\frac{\rho_\varepsilon^i}{(d_\varepsilon^i)^3}\int_{B(x_\varepsilon^i,d_\varepsilon^i)\setminus B(x_\varepsilon^i,\frac{d_\varepsilon^i}{2})}|\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx + c\int_{B(x_\varepsilon^i,d_i^\varepsilon)}|\nabla\,\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx.$$

Using the last inequality and the following estimate

$$\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\setminus B(x_\varepsilon^i,\lambda_1\rho_\varepsilon^i)}|\nabla\,\mathbf{v}_{i,\varepsilon}^k|^2\,dx \leq c\,\frac{(r_\varepsilon^i)^2}{\rho_\varepsilon^i}, \tag{5.15}$$

we have

$$I_4^{(\varepsilon,\kappa)} \leq c\sum_{i=1}^{N(\varepsilon)}\left(\frac{1}{(\rho_\varepsilon^i)^2}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx\right)^{\frac{1}{2}}$$

$$\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\setminus B(x_\varepsilon^i,\lambda_1\rho_\varepsilon^i)}|\nabla\,\mathbf{v}_{i,\varepsilon}^k|^2\,dx\right)^{\frac{1}{2}} \leq c\left(\sum_{i=1}^{N(\varepsilon)}\frac{(r_\varepsilon^i)^2}{(d_\varepsilon^i)^3}\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{i=1}^{N(\varepsilon)}\frac{(d_\varepsilon^i)^3}{\rho_\varepsilon^i}\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\nabla\,\mathbf{z}_\kappa^{(\varepsilon)}(x))|^2\,dx + \sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\mathbf{z}_\kappa^{(\varepsilon)}(x))|^2\,dx\right)^{\frac{1}{2}},$$

since the integrating is on the nonintersecting balls. Taking into account the strong convergence to zero of $\mathbf{z}_\kappa^{(\varepsilon)}(x)$ in $L_2(\Omega)$ as $\varepsilon \to 0$ and (1.12), (5.12), we have that

$$\lim_{\varepsilon\to 0} I_4^{(\varepsilon,\kappa)} = 0. \tag{5.16}$$

Now we consider $I_3^{(\varepsilon,\kappa)}$ using that $\mathbf{v}_{i,\varepsilon}^k$ is a solution to the model problem

$$I_3^{(\varepsilon,\kappa)} = \sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\left|\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}\nabla\,\mathbf{v}_{i,\varepsilon}^k : \nabla\,(\varphi_i^{(\varepsilon)}\,\mathbf{z}_\kappa^{(\varepsilon)})\,dx\right|$$

$$= \sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\left|\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}\nabla\,q_{i,\varepsilon}^k\cdot(\varphi_i^{(\varepsilon)}\,\mathbf{z}_\kappa^{(\varepsilon)})\,dx\right|$$

$$\leq c\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_{i,\varepsilon}^k|\,\left(|\mathbf{z}_\kappa^{(\varepsilon)}\,\nabla\,\varphi_i^{(\varepsilon)}| + |\varphi_i^{(\varepsilon)}\,\nabla\,\cdot\mathbf{z}_\kappa^{(\varepsilon)}|\right)\,dx = I_5^{(\varepsilon,\kappa)} + I_6^{(\varepsilon,\kappa)}.$$

First consider $I_5^{(\varepsilon,\kappa)}$ taking into account (4.1):

$$I_5^{(\varepsilon,\kappa)} \leq c\sum_{i=1}^{N(\varepsilon)}\frac{r_\varepsilon^i}{(\rho_\varepsilon^i)^2}\frac{1}{\rho_\varepsilon^i}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\setminus B(x_\varepsilon^i,\lambda_1\rho_\varepsilon^i)}|\mathbf{z}_\kappa^{(\varepsilon)}|\,dx$$

$$\leq c\sum_{i=1}^{N(\varepsilon)}\frac{r_\varepsilon^i}{(\rho_\varepsilon^i)^2}(\rho_\varepsilon^i)^{\frac{3}{2}}\left(\frac{1}{(\rho_\varepsilon^i)^2}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx\right)^{\frac{1}{2}}$$

$$\leq c\sum_{i=1}^{N(\varepsilon)}\frac{r_\varepsilon^i}{(\rho_\varepsilon^i)^2}(\rho_\varepsilon^i)^{\frac{3}{2}}\left(\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\nabla\,\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx+\frac{\rho_\varepsilon^i}{(d_\varepsilon^i)^3}\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx\right)^{\frac{1}{2}}$$

$$\leq c\sum_{i=1}^{N(\varepsilon)}\frac{r_\varepsilon^i}{(d_\varepsilon^i)^{\frac{3}{2}}}\left(\frac{(d_\varepsilon^i)^3}{\rho_\varepsilon^i}\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\nabla\,\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx+\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx\right)^{\frac{1}{2}}$$

$$\leq c\left(\sum_{i=1}^{N(\varepsilon)}\frac{(r_\varepsilon^i)^2}{(d_\varepsilon^i)^3}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{N(\varepsilon)}\frac{(d_\varepsilon^i)^3}{\rho_\varepsilon^i}\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\nabla\,\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx+\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,d_\varepsilon^i)}|\mathbf{z}_\kappa^{(\varepsilon)}|^2\,dx\right)^{\frac{1}{2}}.$$

Analogously to the previous investigations we use (1.13), (5.12), and it proves that

$$\lim_{\varepsilon\to 0}I_5^{(\varepsilon,\kappa)}=0. \tag{5.17}$$

Let us consider $I_6^{(\varepsilon,\kappa)}$. Taking into account the asymptotic expansion for the velocity (5.8), we derive

$$\operatorname{div}\mathbf{z}_\kappa^{(\varepsilon)}=\operatorname{div}\mathbf{v}_\varepsilon(x)-\operatorname{div}\mathbf{v}_0(x)-\operatorname{div}\mathbf{z}_\varepsilon^{(1)}(x)-\operatorname{div}\mathbf{z}_\varepsilon^{(2)}(x)=-\operatorname{div}\mathbf{v}_0(x)$$

$$-\operatorname{div}\left(\sum_{i=1}^{N(\varepsilon)}(\mathbf{v}_0(x_\varepsilon^i)-\mathbf{v}_0(x))\varphi_i^{(\varepsilon)}\right)-\operatorname{div}\left(\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\mathbf{v}_{i,\varepsilon}^k(x)\varphi_i^{(\varepsilon)}v_0^k(x_\varepsilon^i)\right)$$

$$=-\operatorname{div}\mathbf{v}_0-\sum_{i=1}^{N(\varepsilon)}(\mathbf{v}_0(x_\varepsilon^i)-\mathbf{v}_0(x))\nabla\,\varphi_i^{(\varepsilon)}-\sum_{i=1}^{N(\varepsilon)}\varphi_i^{(\varepsilon)}\operatorname{div}\mathbf{v}_0$$

$$+\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\mathbf{v}_{i,\varepsilon}^k(x)\nabla\,\varphi_i^{(\varepsilon)}v_0^k(x_\varepsilon^i). \tag{5.18}$$

Using this and the properties of the cut-off functions, we have

$$I_6^{(\varepsilon,\kappa)}\leq c\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\int_{B(x_\varepsilon^i,\lambda_2\rho_i^\varepsilon)}|q_{i,\varepsilon}^k|\,|\operatorname{div}\mathbf{v}_0|\,dx$$

$$+\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_{i,\varepsilon}^k|\,|\mathbf{v}_0(x_\varepsilon^i)-\mathbf{v}_0(x)|\,|\nabla\,\varphi_i^{(\varepsilon)}|\,dx$$

$$+\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\setminus B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_{i,\varepsilon}^k|\,|\mathbf{v}_{i,\varepsilon}^k(x)|\,|\nabla\,\varphi_i^{(\varepsilon)}|\,|v_0^k(x_\varepsilon^i)|\,dx$$

$$\leq c\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_{i,\varepsilon}^k|^2dx\right)^{\frac{1}{2}}\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\operatorname{div}\mathbf{v}_0|^2dx\right)^{\frac{1}{2}}$$

$$+c\sum_{k=1}^{3}\left(\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\backslash B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_{i,\varepsilon}^k|^2dx\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{i=1}^{N(\varepsilon)}\frac{1}{(\rho_\varepsilon^i)^2}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\mathbf{v}_0(x_\varepsilon^i)-\mathbf{v}_0(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$+c\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\mathbf{v}_{i,\varepsilon}^k(x)|^2\,|\nabla\,\varphi_i^{(\varepsilon)}|^2\,dx\right)^{\frac{1}{2}}$$

$$\times\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)\backslash B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_{i,\varepsilon}^k|^2\,dx\right)^{\frac{1}{2}}$$

The first term on the right-hand side tends to zero as $\varepsilon\to 0$ because of the absolutely continuity of the integral. The second term also goes to zero by the same reason after application of the Poincaré inequality. The crucial point here is using the integral estimate (4.3) of the pressures to the model-type problems. The third integral on the right-hand side of the last inequality is estimated analogously to $I_1^{(\varepsilon,\kappa)}$. These arguments give us the following statement:

$$\lim_{\varepsilon\to 0} I_6^{(\varepsilon,\kappa)}=0. \tag{5.19}$$

Finally, from (5.11), (5.13)–(5.19) we derive (5.10).

Coming back to the integral identity (5.9) we have to estimate the following integral

$$\int_{\Omega_\varepsilon} q_\varepsilon\nabla\cdot\mathbf{z}_\kappa^{(\varepsilon)}(x)\,dx.$$

Using the expression for divergence of $\mathbf{z}_\kappa^{(\varepsilon)}(x)$ given in (5.18) we have

$$\int_{\Omega_\varepsilon} q_\varepsilon\nabla\cdot\mathbf{z}_\kappa^{(\varepsilon)}(x)\,dx\le c\int_{\Omega} q_\varepsilon\,\nabla\cdot\,\mathbf{v}_0\,dx+c\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_\varepsilon|\,|\nabla\cdot\,\mathbf{v}_0|\,dx$$

$$+\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\int_{B(x_i^\varepsilon,\lambda_2\rho_\varepsilon^i)}|q_\varepsilon|\,|\mathbf{v}_0(x_\varepsilon^i)-\mathbf{v}_0(x)|\,|\nabla\,\varphi_i^{(\varepsilon)}|\,dx$$

$$\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_\varepsilon|\,|\mathbf{v}_{i,\varepsilon}^k(x)|\,|\nabla\,\varphi_i^{(\varepsilon)}|\,|v_0^k(x_\varepsilon^i)|\,dx$$

$$\le c\sum_{i=1}^{N(\varepsilon)}\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_\varepsilon|^2dx\right)^{\frac{1}{2}}\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\nabla\cdot\,\mathbf{v}_0|^2dx\right)^{\frac{1}{2}}$$

$$+c\sum_{k=1}^{3}\left(\sum_{i=1}^{N(\varepsilon)}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_\varepsilon|^2dx\right)^{\frac{1}{2}}\left(\sum_{i=1}^{N(\varepsilon)}\frac{1}{(\rho_\varepsilon^i)^2}\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|\mathbf{v}_0(x_\varepsilon^i)-\mathbf{v}_0(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$+c\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_i^\varepsilon)}|\mathbf{v}_{i,\varepsilon}^k(x)|^2\,|\nabla\,\varphi_i^{(\varepsilon)}|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B(x_\varepsilon^i,\lambda_2\rho_\varepsilon^i)}|q_\varepsilon|^2\,dx\right)^{\frac{1}{2}}.$$

Applying the same arguments as in the previous investigation we get

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} q_\varepsilon\nabla\cdot\mathbf{z}_\kappa^{(\varepsilon)}(x)\,dx=0.$$

This proves the strong convergence to zero of the reminder term to the asymptotic expansion in $W_0^{1,2}(\Omega;R^3)$.

Q.E.D.

Now we will pass to the limit as $\varepsilon\to 0$ into the integral identities (5.2), (5.3). For an arbitrary $\mathbf{h}\in C_0^\infty(\Omega;R^3)$ we construct the following test function

$$\mathbf{h}_\varepsilon(x)=\mathbf{h}(x)+\mathbf{h}_\varepsilon^{(1)}(x)+\mathbf{h}_\varepsilon^{(2)}(x)\in W_0^{1,2}(\Omega_\varepsilon;R^3),$$

where

$$\mathbf{h}_\varepsilon^{(1)}(x)=\sum_{i=1}^{N(\varepsilon)}(\mathbf{h}(x_\varepsilon^i)-\mathbf{h}(x))\varphi_i^{(\varepsilon)}(x)$$

$$\mathbf{h}_\varepsilon^{(2)}(x)=-\sum_{i=1}^{N(\varepsilon)}\sum_{k=1}^{3}\mathbf{v}_{i,\varepsilon}^k(x)\varphi_i^{(\varepsilon)}(x)h^k(x_\varepsilon^i),$$

It is easy to check that the following statement is true.

**Lemma 5.3** *Let $\Omega_\varepsilon$ satisfy hypotheses (1.12), (1.13), and condition $(G)$. Then*

$$\mathbf{h}_\varepsilon^{(1)}\to 0,\quad \mathbf{h}_\varepsilon^{(2)}\to 0\quad \textit{weakly in } W^{1,2}(\Omega;R^3)\ \textit{as}\ \ \varepsilon\to 0.$$

Testing the integral identity (5.2) by $\mathbf{h}_\varepsilon$ we have

$$\int_{\Omega_\varepsilon}\nabla\,(\mathbf{v}+\mathbf{z}_\varepsilon^{(1)}+\mathbf{z}_\varepsilon^{(2)}+\mathbf{z}_\varepsilon):\nabla\,(\mathbf{h}+\mathbf{h}_\varepsilon^{(1)}+\mathbf{h}_\varepsilon^{(2)})\,dx$$

$$-\int_{\Omega_\varepsilon} q_\varepsilon\nabla\cdot(\mathbf{h}+\mathbf{h}_\varepsilon^{(1)}+\mathbf{h}_\varepsilon^{(2)})\,dx=\int_{\Omega_\varepsilon}\mathbf{f}_\varepsilon\cdot(\mathbf{h}+\mathbf{h}_\varepsilon^{(1)}+\mathbf{h}_\varepsilon^{(2)})\,dx$$

Consider every integral in this identity. From the strong convergence of $(\mathbf{z}_\varepsilon^{(1)}+\mathbf{z}_\varepsilon)$ and $\mathbf{h}_\varepsilon^{(1)}$ in $W_0^{1,2}(\Omega;R^3)$ to zero we obtain that

$$\int_{\Omega_\varepsilon}\nabla\,(\mathbf{z}_\varepsilon^{(1)}+\mathbf{z}_\varepsilon):\nabla\,(\mathbf{h}+\mathbf{h}_\varepsilon^{(1)}+\mathbf{h}_\varepsilon^{(2)})\,dx\to 0,$$

$$\int_{\Omega_\varepsilon} \nabla\left(\mathbf{v}+\mathbf{z}_\varepsilon^{(2)}\right) : \nabla\, \mathbf{h}_\varepsilon^{(1)}\, dx \to 0, \quad \int_{\Omega_\varepsilon} q_\varepsilon \nabla \cdot \mathbf{h}_\varepsilon^{(1)}\, dx \to 0$$

as $\varepsilon \to 0$. Using that $\mathbf{h}_\varepsilon^{(2)}$ and $\mathbf{z}_\varepsilon^{(2)}$ converge weakly to zero in $W_0^{1,2}(\Omega; R^3)$, we derive:

$$\int_{\Omega_\varepsilon} \nabla\, \mathbf{v} : \nabla\, \mathbf{h}_\varepsilon^{(2)}\, dx \to 0, \quad \int_{\Omega_\varepsilon} \nabla\, \mathbf{z}_\varepsilon^{(2)} : \nabla\, \mathbf{h}\, dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Since $q_\varepsilon \to q$ weakly in $L_2(\Omega)$ we obtain:

$$\int_{\Omega_\varepsilon} q_\varepsilon \nabla \cdot \mathbf{h}\, dx \to \int_{\Omega_\varepsilon} q \nabla \cdot \mathbf{h}\, dx \quad \text{as} \quad \varepsilon \to 0.$$

Consider

$$\int_{\Omega_\varepsilon} q_\varepsilon \nabla \cdot \mathbf{h}_\varepsilon^{(2)}\, dx = \int_{\Omega_\varepsilon} q_\varepsilon \sum_{i=1}^{N(\varepsilon)} \sum_{k=1}^{3} h^k(x_\varepsilon^i) \left( \mathbf{v}_{i,\varepsilon}^k \cdot \nabla \varphi_i^{(\varepsilon)} - \varphi_i^{(\varepsilon)} \nabla \cdot \mathbf{v}_{i,\varepsilon}^k(x) \right)\, dx.$$

The second term in the last integral is equal to zero from the solenoidality of the $\mathbf{v}_{i,\varepsilon}^k$. The reminder integral tends to zero as $\varepsilon \to 0$ due to the properties of the cut-off functions and solutions of the model problem (1.15), (1.16). Finally, from the strong convergence of $(\mathbf{h}_\varepsilon^{(1)} + \mathbf{h}_\varepsilon)$ to zero in $L_2(\Omega; R^3)$ and from the assumptions of Proposition 5.1 we have:

$$\int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot (\mathbf{h}_\varepsilon^{(1)} + \mathbf{h}_\varepsilon^{(2)})\, dx \to 0, \quad \int_{\Omega_\varepsilon} \mathbf{f}_\varepsilon \cdot \mathbf{h}\, dx \to \int_{\Omega_\varepsilon} \mathbf{f} \cdot \mathbf{h}\, dx \quad \text{as} \quad \varepsilon \to 0.$$

By the standard calculations we derive for the remainder integral from the integral identity:

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \nabla\, \mathbf{z}_\varepsilon^{(2)} : \nabla\, \mathbf{h}_\varepsilon^{(2)}\, dx = \lim_{\varepsilon \to 0} \sum_{i=1}^{N(\varepsilon)} \sum_{k,l=1}^{3} v_0^k(x_\varepsilon^i) h^l(x_\varepsilon^i) \int_{B(x_\varepsilon^i, \lambda_2 \rho_\varepsilon^i)} \nabla\, \mathbf{v}_{i,\varepsilon}^k : \nabla\, \mathbf{v}_{i,\varepsilon}^k dx$$

$$= \int_\Omega C_{k,l}(x) \mathbf{v}\, \mathbf{h}\ \mathrm{d}x.$$

Passing to the limit into the integral identities (5.2), (5.3) as $\varepsilon \to 0$ we get the main result of the Section.

Q.E.D.

# 6 Evolutionary Navier-Stokes system

In order to complete the proof of Theorem 2.1, we consider the *evolutionary* Navier-Stokes system in the form

$$\mathrm{div}_x \mathbf{u} = 0 \text{ in } (0,T) \times \Omega_\varepsilon, \tag{6.1}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \mathbf{f}_\varepsilon \text{ in } (0,T)\times\Omega_\varepsilon, \tag{6.2}$$

supplemented with the no-slip boundary condition

$$\mathbf{u}|_{\partial\Omega_\varepsilon} = 0, \tag{6.3}$$

and the initial datum

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}. \tag{6.4}$$

## 6.1 Uniform bounds

It follows easily from the energy inequality (1.7) that

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{u}_\varepsilon(t,\cdot)\|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^T \|\mathbf{u}_\varepsilon\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}t \\ \le c\Big(\|\mathbf{u}_{0,\varepsilon}\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^3)} + \int_0^T \|\mathbf{f}_\varepsilon\|^2_{L^2(\Omega_\varepsilon;\mathbb{R}^3)}\,\mathrm{d}t\Big), \end{aligned} \tag{6.5}$$

for any weak solution $\mathbf{u}_\varepsilon$ of (6.1), (6.4), where the constant $c$ is independent of $\varepsilon$.

## 6.2 Weak sequential stability

By virtue of the hypotheses of Theorem 2.1 and the uniform bounds established in (6.5), we immediately get

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)) \text{ and weakly in } L^2(0,T;W_0^{1,2}(\Omega;R^3)), \tag{6.6}$$

passing to suitable subsequences as the case may be. Moreover, $\mathrm{div}_x\mathbf{u} = 0$ a.a. in $(0,T)\times\Omega$.

Our goal is to show strong (a.a. pointwise) convergence of the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ necessary to perform the limit passage in the convective term $\mathbf{u}\otimes\mathbf{u}$. To this end, consider the unique solution $\mathbf{w}$ of the modified Stokes problem (5.5) for a given function $\mathbf{f}\in L^2(\Omega;R^3)$. In accordance with Proposition 5.1, there is a uniquely determined sequence $\{\mathbf{w}_\varepsilon\}_{\varepsilon>0}$ of solutions to the Stokes problem (5.1) in $\Omega_\varepsilon$, with $\mathbf{f}_\varepsilon = 1_{\Omega_\varepsilon}\mathbf{f}$, such that

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } W_0^{1,2}(\Omega;R^3). \tag{6.7}$$

Since the quantities

$$\varphi(t,x) = \psi(t)\mathbf{w}_\varepsilon(x),\ \psi\in C_c^\infty(0,T),$$

represent admissible test functions in the weak formulation (2.1) of the momentum equation, we may infer that the family of functions

$$t\mapsto \int_\Omega \mathbf{u}_\varepsilon(t,\cdot)\cdot\mathbf{w}_\varepsilon\,\mathrm{d}x \text{ is precompact in } C([0,T]).$$

Moreover, the Sobolev space $W^{1,2}(\Omega)$ is compactly embedded into $L^2(\Omega)$, therefore

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ in } L^2(\Omega; R^3),$$

and the functions

$$t \mapsto \int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{w} \, \mathrm{d}x = \left[ t \mapsto \int_\Omega \mathbf{u}_\varepsilon \cdot (\mathbf{w} - \mathbf{w}_\varepsilon) \, \mathrm{d}x \right] + \left[ t \mapsto \int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{w}_\varepsilon \, \mathrm{d}x \right] \tag{6.8}$$

form a precompact set in $C[0,T]$.

Finally, since the domain of definition of the modified Stokes operator (5.5) is dense in the space of square integrable solenoidal functions, we conclude that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } C_{\text{weak}}([0,T]; L^2(\Omega; R^3)). \tag{6.9}$$

Relations (6.6), (6.9) yield the desired conclusion

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2((0,T) \times \Omega; R^3). \tag{6.10}$$

## 6.3 Homogenization of the evolutionary Navier-Stokes system

To begin, we regularize equation (6.2) with respect to the time variable. After a straightforward manipulation, we obtain

$$\int_{\Omega_\varepsilon} \nabla_x [\mathbf{u}_\varepsilon(t,\cdot)]^\delta : \nabla_x \varphi \, \mathrm{d}x = \int_{\Omega_\varepsilon} [\mathbf{f}_\varepsilon(t,\cdot)]^\delta \cdot \varphi \, \mathrm{d}x \tag{6.11}$$

$$+ \int_{\Omega_\varepsilon} [\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon(t,\cdot)]^\delta : \nabla_x \varphi \, \mathrm{d}x - \int_{\Omega_\varepsilon} [\partial_t \mathbf{u}_\varepsilon(t,\cdot)]^\delta \cdot \varphi \, \mathrm{d}x$$

for any $\varphi \in W_0^{1,2}(\Omega_\varepsilon; R^3)$, $\mathrm{div}_x \varphi = 0$, and any $t \in [\delta, T-\delta]$, where $[\cdot]^\delta$ denotes the time convolution with a smoothing kernel supported in $(-\delta/2, \delta/2)$.

Now, Proposition 5.1, together with (6.6), (6.10), allow us to pass to the limit for $\varepsilon \to 0$ to obtain

$$\int_\Omega \nabla_x [\mathbf{u}(t,\cdot)]^\delta : \nabla_x \varphi \, \mathrm{d}x + \int_\Omega (\mathbb{C}[\mathbf{u}(t,\cdot)]^\delta) \cdot \varphi \, \mathrm{d}x = \int_\Omega [\mathbf{f}(t,\cdot)]^\delta \cdot \varphi \, \mathrm{d}x \tag{6.12}$$

$$+ \int_\Omega [\mathbf{u} \otimes \mathbf{u}(t,\cdot)]^\delta : \nabla_x \varphi \, \mathrm{d}x - \int_\Omega [\partial_t \mathbf{u}(t,\cdot)]^\delta \cdot \varphi \, \mathrm{d}x$$

for any $\varphi \in W_0^{1,2}(\Omega; R^3)$, $\mathrm{div}_x \varphi = 0$, and any $t \in [\delta, T-\delta]$.

Finally, we let $\delta \to 0$ in (6.12)

$$\int_0^T \int_\Omega \left( \mathbf{u} \cdot \partial_t \varphi + \overline{[\mathbf{u} \otimes \mathbf{u}]} : \nabla \varphi - \mathbf{D}[\mathbf{u}] : \mathbf{D}[\varphi] \right) \, dx \, dt + \int_\Omega (\mathbb{C}[\mathbf{u}(t,\cdot)]) \cdot \varphi \, \mathrm{d}x \tag{6.13}$$

$$= - \int_\Omega \mathbf{u}_0 \cdot \varphi(0,\cdot) \, dx - \int_0^T \int_\Omega \mathbf{f} \cdot \varphi \, dx \, dt$$

for any test function

$$\varphi \in C_c^1([0,T) \times \overline{\Omega}; R^N), \ \mathrm{div}_x \varphi = 0, \ \varphi|_{\partial\Omega} \in \mathcal{V}, \tag{6.14}$$

where $\overline{\mathbf{u} \otimes \mathbf{u}}$ denotes a weak $L^1-$limit of $\mathbf{u} \otimes \mathbf{u}$. From [5] it follows that

$$\int_0^T \int_\Omega [\overline{\mathbf{u} \otimes \mathbf{u}}] : \nabla\varphi \, dx \, dt = \int_0^T \int_\Omega [\mathbf{u} \otimes \mathbf{u}] : \nabla\varphi \, dx \, dt \tag{6.15}$$

for any test function $\varphi$ satisfying (6.14).

So we have

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \mathbb{C}\mathbf{u} + \nabla_x p = \Delta\mathbf{u} + \mathbf{g}, \tag{6.16}$$

$$\mathrm{div}_x \mathbf{u} = 0 \tag{6.17}$$

in $\Omega$,

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{6.18}$$

supplemented with the initial condition

$$\mathbf{u}(0, \cdot) = \mathbf{u}_0. \tag{6.19}$$

Theorem 2.1 has been proved.

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