Weak solutions to problems involving perfect fluids

Eduard Feireisl
joint work with E.Chiodaroli (Lausanne), D.Donatelli (L'Aquila), O.Kreml (Praha), and P.Marcati (L'Aquila)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

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Oscillations in conservation laws

Nonlinear conservation law

$$\partial_t \mathbf{u} + \mathrm{div}_x \mathbb{F}(\mathbf{u}) = 0$$

Linear field equation

$$\partial_t \mathbf{u} + \mathrm{div}_x \mathbb{V} = 0$$

Nonlinear "constitutive" relation

$$\mathbb{F}(\mathbf{u}) = \mathbb{V}$$

Oscillations

$$\int_B \mathbf{u}_\varepsilon \to \int_B \mathbf{u} \text{ for all } B, \ \liminf_{\varepsilon \to 0} \int_B |\mathbf{u}_\varepsilon|^2 \boxed{>} \int_B |\mathbf{u}|^2$$

Convex integration

Field equations, constitutive relations

$$\partial_t \mathbf{u} + \mathrm{div}_x \mathbb{V} = 0,\ \ \mathbb{V} = \mathbb{F}(\mathbf{u})$$

Reformulation, subsolutions

$$\mathbb{V} = \mathbb{F}(\mathbf{u}) \Leftrightarrow G(\mathbf{u}, \mathbb{V}) = E(\mathbf{u}), E(\mathbf{u}) \leq G(\mathbf{u}, \mathbb{V}) < \boxed{\overline{e}(\mathbf{u})}$$

E convex, $\overline{e}$ "concave"

Oscillatory lemma, oscillatory increments

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0,\ \ \mathbf{u}_\varepsilon \boxed{\rightharpoonup} 0$$

$$E(\mathbf{u} + \mathbf{u}_\varepsilon) \leq G\left(\mathbf{u} + \mathbf{u}_\varepsilon, \mathbb{V} + \mathbb{V}_\varepsilon\right) < \overline{e}(\mathbf{u} + \mathbf{u}_\varepsilon)$$

$$\liminf \int E(\mathbf{u}_\varepsilon) \boxed{\geq} \int \left(\overline{e}(\mathbf{u}) - E(\mathbf{u})\right)^\alpha$$

Convex integration - DeLellis and Shékelyhidi

Incompressible Euler system in R^3

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Reformulation

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} \right) + \nabla_x \Pi = 0$$

Linear system vs. non-linear constitutive equation

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

$$\mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I}, \ \mathbb{U} \in R^{3 \times 3}_{0,\mathrm{sym}}$$

Convex integration continued

Implicit constitutive relation

$$\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]$$

$$\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]\geq\frac{1}{2}|\mathbf{v}|^2$$

$$\boxed{\frac{3}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]=\frac{1}{2}|\mathbf{v}|^2}\Leftrightarrow\mathbb{U}=\mathbf{v}\otimes\mathbf{v}-\frac{1}{3}|\mathbf{v}|^2\mathbb{I}$$

Convex integration - subsolutions

Equations

$\mathbf{v}$, $\mathbb{U}$ smooth in $(0, T)$

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

Extremal values

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Energy

piece-wise smooth function e

Convex set

$$\frac{1}{2}|\mathbf{v}|^2 \leq \lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < e \text{ in } (0, T)$$

Oscillatory lemma

Oscillatory increments

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0, \ \partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0$$

$$\mathbf{w}_\varepsilon, \ \mathbb{V}_\varepsilon \in C_c^\infty(Q)$$

$$\mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(V)$$

$$\lambda_{\max}\left[(\mathbf{v} + \mathbf{w}_\varepsilon) \otimes (\mathbf{v} + \mathbf{w}_\varepsilon) - (\mathbb{U} + \mathbb{V}_\varepsilon) \right] < e$$

Energy

$$\liminf_{\varepsilon \to 0} \int_V \left(|\mathbf{v} + \mathbf{w}_\varepsilon|^2 \right) \geq \int_V |\mathbf{v}|^2 + c \int_V \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^\alpha$$

Applications to Euler flows

Incompressible Euler system - DeLellis, Székelyhidi [2008]

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{u} = 0$$

Compressible Euler with solenoidal data - Chiodaroli [2013]

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Possible extensions

- problems with "non-constant" coefficients
- systems coupled with the Euler equations
- maximal regularity of solutions (?)

Expected results

Good news

The problem admits global-in-time (finite energy) weak solutions of any (large) initial data

Bad news

There are infinitely many solutions for given initial data

More bad news

There exist data for which the problem admits infinitely many "admissible" solutions

Non-constant coefficients

Convex set

$$\frac{1}{2}\frac{1}{\boxed{r(t,x)}}|\mathbf{v}+\boxed{\mathbf{q}(t,x)}|^2$$

$$\leq \lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{q}(t,x))\otimes(\mathbf{v}+\mathbf{q}(t,x))}{r(t,x)}+\boxed{\mathbb{W}(t,x)}-\mathbb{U}\right] < e$$

Equations

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{q})\otimes(\mathbf{v}+\mathbf{q})}{r} - \frac{1}{3}\frac{|\mathbf{v}+\mathbf{q}|^2}{r} + \mathbb{W}\right) = 0, \mathbb{W} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{q}(t,x)|^2}{r(t,x)} = \boxed{e(t,x)}$$

Implicit variant

Velocity dependent parameters

The quantities

$$\mathbf{q}, \ e, \ \mathbb{W}$$

may depend on $\mathbf{v}$

Main issues: Boundedness and weak continuity

Basic ideas of analysis

Localization

Localizing the result of DeLellis and Széhelyhidi to "small" cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the "small" cubes

Eliminating singular sets

Applying Whitney's decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{\varrho = 0\}$)

Energy and other coefficients depending on solutions

Showing boundedness and continuity of the energy $\overline{e}(\mathbf{u})$ as well as other quantities as the case may be

Example I, Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \mathbf{u}$$

Application of convex integration

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \left(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \right) - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

“Energy”

$$e = \chi(t) - \boxed{\frac{3}{2} \varrho \vartheta [\mathbf{v}]}$$

Existence of weak solutions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta\ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0, T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Dissipative solutions to the Euler-Fourier system

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

Infinitely many dissipative weak solutions

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in $(0, T)$

Example II, Euler-Korteweg-Poisson system

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho)|\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Reformulation, Step 1

Extending the density

$$\partial_t \varrho + \mathrm{div}_x \tilde{\mathbf{J}} = 0, \ \ \varrho(0, \cdot) = \varrho_0$$

Flux ansatz

$$\tilde{\mathbf{J}} = \varrho(\mathbf{U}_0 - Z), \ \ Z = Z(t)$$

$$\partial_t \int_{\mathbb{T}^3} \mathbf{H}[\tilde{\mathbf{J}}] \ \mathrm{d}x + \int_{\mathbb{T}^3} \mathbf{H}[\tilde{\mathbf{J}}] \ \mathrm{d}x = 0$$

$\mathbf{H}$ – standard Helmholtz projection

$$\mathrm{meas} \left\{ x \in \mathbb{T}^3 \ \Big| \ \varrho(t,x) = 0 \right\} = 0 \text{ for any } t \in [0, T]$$

Reformulation, Step 2

Flux ansatz

$$\mathbf{J} = \tilde{\mathbf{J}} + \mathbf{w},\ \mathrm{div}_x \mathbf{w} = 0,\ \mathbf{w}(0,\cdot) = 0$$

$$\mathbf{w} \in \boxed{C_{\mathrm{weak}}([0,T], L^2(\Omega; R^3))} \cup L^\infty((0,T)\times\Omega; R^3)$$

Equations

$$\partial_t \left(\mathbf{w} + \tilde{\mathbf{J}}\right) + \mathrm{div}_x \left(\frac{(\mathbf{w}+\tilde{\mathbf{J}}) \otimes (\mathbf{w}+\tilde{\mathbf{J}})}{\varrho} \right) + \nabla_x p(\varrho) + \left(\mathbf{w} + \tilde{\mathbf{J}}\right) =$$

$$\nabla_x \Big(\chi(\varrho)\Delta_x \varrho\Big) + \frac{1}{2}\nabla_x\Big(\chi'(\varrho)|\nabla_x\varrho|^2\Big) - 4\mathrm{div}_x\Big(\chi(\varrho)\nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}\Big)$$
$$+\varrho\nabla_x V$$

Reformulation, Step 3

Final flux ansatz

$$\tilde{\mathbf{J}} = \mathbf{H}[\tilde{\mathbf{J}}] + \nabla_x M, \ \mathbf{v} = e^t \left(\mathbf{w} + \mathbf{H}[\tilde{\mathbf{J}}] \right),$$

Equations

$$\mathrm{div}_x \mathbf{v} = 0, \ \mathbf{v}(0, \cdot) = \mathbf{H}[\mathbf{J}_0]$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{r} + \mathbb{H} \right) + \nabla_x \Pi = 0$$

Coefficients

$$r = e^t \varrho, \ \mathbf{h} = e^t \nabla_x M$$

Driving terms

Convective term

$$\mathbb{H}(t,x) = 4e^t \left(\chi(\varrho)\nabla_x\sqrt{\varrho} \otimes \nabla_x\sqrt{\varrho} - \frac{1}{3}\chi(\varrho)|\nabla_x\sqrt{\varrho}|^2\mathbb{I} \right)$$

$$4e^t \left(\frac{1}{3}|\nabla_x V|^2\mathbb{I} - \nabla_x V \otimes \nabla_x V \right), \ \mathbb{H} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Pressure term

$$\Pi(t,x) = e^t \Big(p(\varrho) + \partial_t M + M - \chi(\varrho)\Delta_x\varrho \Big)$$

$$-e^t \left(\frac{1}{2}\chi'(\varrho)|\nabla_x\varrho|^2 - \frac{4}{3}\chi(\varrho)|\nabla_x\sqrt{\varrho}|^2 + \overline{\varrho}V + \frac{1}{3}|\nabla_x V|^2 \right) + \boxed{\Lambda}$$

$\Lambda -$ a suitable constant

Example III, Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$

Energy functional

Energy in the convex integration ansatz

$$\frac{1}{2}\frac{|\mathbf{v}+\nabla_x\Phi|^2}{\varrho} = \overline{E}[\mathbf{v}]$$

$$\equiv \Lambda(t) - \frac{3}{2}\left(\frac{1}{6}\boxed{|\nabla_x c[\mathbf{v}]|^2} + p_0(\varrho, c[\mathbf{v}]) + \partial_t\nabla_x\Phi\right)$$

Uniform estimates

$$|\nabla_x c| \approx |\mathbf{u}| \text{ needed!}$$

Maximal regularity - Denk, Hieber, Pruess [2007]

$$\partial_t c + \frac{1}{\varrho}\Delta\left(\frac{1}{\varrho}\mathrm{div}_x\left(\varrho\nabla_x c\right)\right) = h,$$