Uniqueness of rarefaction waves in compressible Euler systems

Ondřej Kreml

Institute of Mathematics, Academy of Sciences of the Czech Republic
Joint work with Eduard Feireisl (Parts I and II) and Alexis Vasseur (Part II)

Seminar on PDEs, IM AS CR, March 24, 2015

Setting of the problem

Let us first consider the compressible isentropic Euler system in the whole 2D space

$$\left\{ \begin{array}{rcl} \partial_t \rho + \mathrm{div}_x(\rho v) & = & 0 \\ \partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] & = & 0 \\ \rho(\cdot, 0) & = & \rho^0 \\ v(\cdot, 0) & = & v^0 \, . \end{array} \right. \tag{1}$$

Unknowns:

- $\rho(x, t)$ ... density
- $v(x, t)$ ... velocity

The pressure $p(\rho)$ is given.

Introduction

- It is a hyperbolic system of conservation laws
- The theory of hyperbolic conservation laws is far frome being completely understood
- Solutions develop singularities in finite time even for smooth initial data
- Admissibility comes into play due to the entropy inequality ("selector" of physical solutions in case of existence of many solutions)
- There are satisfactory results in the case of scalar conservation laws (in 1D as well as in multi-D), there is a lot of entropies: $\Rightarrow$ Kruzkov, 1970: Well–posedness theory in BV.
- There are also satisfactory results in the case of systems of conservation laws in 1D: Lax, Glimm, Bressan, Bianchini, ...

Introduction II

Back to our case, the isentropic Euler system:

- In more than 1D there is only one (entropy, entropy flux) pair, which is

$$\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}, \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\right)$$

  with the internal energy $\varepsilon(\rho)$ given through

$$p(\rho) = \rho^2\varepsilon'(\rho).$$

- Local existence of strong (and therefore admissible) solutions is proved
- On the other hand global existence of weak solutions in general (it is a system in multi D!) is still an open problem, there are only partial results
- The weak–strong uniqueness property holds for this system

Weak solution

Definition 1

By a *weak solution* of Euler system on $\mathbb{R}^2 \times [0,\infty)$ we mean a pair $(\rho, v) \in L^\infty(\mathbb{R}^2 \times [0,\infty))$ such that the following identities hold for every test functions $\psi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$, $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,\infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho \partial_t \psi + \rho v \cdot \nabla_x \psi] \, dxdt + \int_{\mathbb{R}^2} \rho^0(x) \psi(x,0) dx = 0$$

$$\int_0^\infty \int_{\mathbb{R}^2} [\rho v \cdot \partial_t \phi + \rho v \otimes v : \nabla_x \phi + p(\rho) \operatorname{div}_x \phi] \, dxdt + \int_{\mathbb{R}^2} \rho^0(x) v^0(x) \cdot \phi(x,0) dx = 0.$$

Admissible weak solution

Definition 2

A bounded weak solution (ρ, v) of Euler system is *admissible* if it satisfies the following inequality for every nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}^2 \times [0, \infty))$:

$$\int_0^\infty \int_{\mathbb{R}^2} \left[\left(\rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) \partial_t \varphi \right.$$
$$\left. + \left(\rho\varepsilon(\rho) + \rho \frac{|v|^2}{2} + p(\rho) \right) v \cdot \nabla_x \varphi \right] dxdt$$
$$+ \int_{\mathbb{R}^2} \left(\rho^0(x)\varepsilon(\rho^0(x)) + \rho^0(x) \frac{|v^0(x)|^2}{2} \right) \varphi(x, 0)\, dx \ \geq \ 0\,.$$

Riemann problem

Denote $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \tag{2}$$

where $\rho_\pm, v_\pm$ are constants.
In particular the initial data are "1D" and there is a classical theory about self-similar solutions to the Riemann problem in 1D (they are unique in the class of BV functions).
In the case of system (1), the initial singularity can resolve to at most 3 structures (rarefaction wave, admissible shock or contact discontinuity) connected by constant states.

First observation

If $v_{-1} = v_{+1}$, then any self-similar solution to (1), (2) has to satisfy $v_1(t,x) = v_{-1} = v_{+1}$ and in particular there is no contact discontinuity in the self-similar solution.
The initial singularity then resolves into at most 2 structures (rarefaction waves or admissible shocks) connected by constant states.

Classification of self-similar solutions I

1) If

$$v_{+2} - v_{-2} \geq \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and a 3−rarefaction wave. The intermediate state is vacuum, i.e. $\rho_m = 0$.

2) If

$$\left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau \right| < v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and a 3−rarefaction wave. The intermediate state has $\rho_m > 0$.

Classification of self-similar solutions II

3) If $\rho_- > \rho_+$ and

$$-\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_-\rho_+}} < v_{+2} - v_{-2} < \int_{\rho_+}^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and an admissible 3−shock.

4) If $\rho_- < \rho_+$ and

$$-\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}} < v_{+2} - v_{-2} < \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of an admissible 1−shock and a 3−rarefaction wave.

Classification of self-similar solutions III

5) If

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}}$$

then the self-similar solution consists of an admissible $1-$shock and an admissible $3-$shock.

Nonuniqueness results

Theorem 3 (Chiodaroli, De Lellis, K.)

There exist Riemann initial data of the case 3) for which there exist infinitely many admissible weak solutions of the isentropic Euler system. Moreover such Riemann data are generated by a compression wave (backwards rarefaction wave), in particular this implies nonuniqueness for Lipschitz initial data.

Theorem 4 (Chiodaroli, K.)

For any Riemann initial data of the case 5) there exist infinitely many admissible weak solutions of the isentropic Euler system.

Specification of the problem I

First we define the appropriate class of weak solutions. Consider

$$\Omega = \mathcal{T}^1 \times (-a, a),$$

with $a > 0$ sufficiently large and $\mathcal{T}^1$ is a 1D torus. We will consider weak solutions periodic in x_1 and having the same boundary fluxes on $x_2 = \pm a$ as has the self-similar solution.
We consider Riemann data satisfying $v_{\pm 1} = 0$.

Specification of the problem II

More specifically we work with weak solutions satisfying:

$$\rho v_2(t, x_1, -a) = \rho_- v_{-2}, \;\; \rho v_2(t, x_1, a) = \rho_+ v_{+2};$$

$$(\rho v_j v_2 + p(\rho))\,(t, x_1, -a) = (\rho_- v_{-j} v_{-2} + p(\rho_-))$$

$$\left(\frac{1}{2}\rho|v|^2 + \rho\varepsilon(\rho) + p(\rho)\right) v_2(t, x_1, -a) =$$
$$\left(\frac{1}{2}\rho_-|v_-|^2 + \rho_-\varepsilon(\rho_-) + p(\rho_-)\right) v_{-2}$$

and similarly for $x_2 = a$.

Specification of the problem III

This means in particular that in the weak formulation of the Euler system appear additional boundary integrals on $x_2 = \pm a$, for example the equation of continuity in the weak formulation looks as follows:

$$\begin{aligned}
&\int_\Omega [\rho(\tau,x)\varphi(\tau,x) - \rho_0(x)\varphi(0,x)]\,\mathrm{d}x \\
&+ \int_0^\tau \int_{\mathcal{T}^1} \rho_+ v_{+2}\varphi(t,x_1,a)\ \mathrm{d}x_1\mathrm{d}t \\
&- \int_0^\tau \int_{\mathcal{T}^1} \rho_- v_{-2}\varphi(t,x_1,-a)\ \mathrm{d}x_2\mathrm{d}t \\
&= \int_0^\tau \int_\Omega [\rho(t,x)\partial_t\varphi(t,x) + \rho v(t,x)\cdot\nabla\varphi(t,x)]\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Main Theorem in the isentropic case

Theorem 5 (Feireisl, K.)

Let $p(\rho) = \rho^\gamma$, $\gamma > 1$. *Let* $\tilde{\rho}(t,x) = R(x_2/t)$, $\tilde{v}(t,x) = (0, V(x_2/t))$ *be the self-similar solution to the Riemann problem consisting of rarefaction waves (locally Lipschitz for* $t > 0$*) and such that*

$$\operatorname{ess\,inf}_{(0,t)\times\Omega} \tilde{\rho} > 0. \tag{3}$$

Let (ρ, v) *be a bounded admissible weak solution such that*

$$\rho \geq 0 \ a.a. \ in \ (0,T) \times \Omega.$$

Then

$$\rho \equiv \tilde{\rho}, \ v \equiv \tilde{v} \ in \ (0,T) \times \Omega.$$

Main Theorem in the isentropic case II

Note that according to our earlier study the self-similar solution to the Riemann problem consists only of rarefaction waves and satisfies (3) if and only if the initial Riemann data satisfy

$$\left|\int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau\right| \leq v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau.$$

Relative entropy inequality

The proof is based on the relative entropy inequality. Define the relative entropy functional

$$\mathcal{E}\left(\rho, v \middle| r, V\right) = \frac{1}{2}\rho|v - V|^2 + \left(H(\rho) - H'(r)(\rho - r) - H(r)\right),$$

where $H(s) = s\varepsilon(s)$.
Concept of relative entropies goes back to DiPerna and Dafermos. Similarly as in papers by Feireisl, Novotný and others in the case of Navier-Stokes equations we first prove that any bounded admissible weak solution satisfies the relative entropy inequality with any couple of functions (r, V) such that

$$r \in C^1([0, T] \times \overline{\Omega}),\ V \in C^1([0, T] \times \overline{\Omega}),\ r > 0.$$

Relative entropy inequality II

$$\int_\Omega \mathcal{E}\left(\rho, v \middle| r, V\right)(\tau, x)\mathrm{d}x - \int_\Omega \mathcal{E}\left(\rho_0, v_0 \middle| r(0,x), V(0,x)\right)\mathrm{d}x$$
$$+ \text{boundary terms} \leq$$
$$\int_0^\tau \int_\Omega \left[\rho\left(\partial_t V + v\cdot\nabla V\right)\cdot(V - v) + \Big(p(r) - p(\rho)\Big)\operatorname{div} V\right]\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \Big[(r-\rho)\partial_t H'(r) + (rV - \rho v)\cdot\nabla H'(r)\Big](t,x)\mathrm{d}x\mathrm{d}t$$

Relative entropy inequality III

Observe that the rarefaction wave solution $(\tilde{\rho}, \tilde{v})$ may be taken as the test couple (r, V) in the relative entropy inequality as

- $\rho, \tilde{\rho}, v, \tilde{V}$ bounded,
- $\partial_t \tilde{\rho}, \partial_t \tilde{v}_2, \partial_{x_2} \tilde{\rho}, \partial_{x_2} \tilde{v}_2 \in L^\infty(0, T; L^1(\Omega))$

and such step thus can be justified by a density argument and Lebesgue dominated convergence theorem.
Therefore the initial term and the boundary terms in the relative entropy inequality vanish.

Relative entropy inequality IV

Thus we get

$$\int_\Omega \mathcal{E}\left(\rho, v \middle| \tilde\rho, \tilde v\right)(\tau, x)\mathrm{d}x \le$$
$$\int_0^\tau \int_\Omega \left[\rho\left(\partial_t \tilde v_2 + v_2 \partial_{x_2} \tilde v_2\right)\left(\tilde v_2 - v_2\right) + \Big(p(\tilde\rho) - p(\rho)\Big)\partial_{x_2}\tilde v_2\right](t,x)\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \Big[(\tilde\rho - \rho)\partial_t H'(\tilde\rho) + (\tilde\rho\tilde v_2 - \rho v_2)\partial_{x_2} H'(\tilde\rho)\Big](t,x)\mathrm{d}x\mathrm{d}t$$

We rewrite the terms as follows. First:

$$\rho\left(\partial_t \tilde{v}_2 + v_2 \partial_{x_2} \tilde{v}_2\right)\left(\tilde{v}_2 - v_2\right) =$$
$$\rho(\partial_t \tilde{v}_2 + \tilde{v}_2 \partial_{x_2} \tilde{v}_2)(\tilde{v}_2 - v_2) - \rho \partial_{x_2} \tilde{v}_2 (\tilde{v}_2 - v_2)^2 =$$
$$-(\rho/\tilde{\rho}) \partial_{x_2} p(\tilde{\rho})(\tilde{v}_2 - v_2) - \rho \partial_{x_2} \tilde{v}_2 (\tilde{v}_2 - v_2)^2.$$

Next:

$$\Big(p(\tilde{\rho}) - p(\rho)\Big) \partial_{x_2} \tilde{v}_2 =$$
$$-\Big(p(\rho) - p'(\tilde{\rho})(\rho - \tilde{\rho}) - p(\tilde{\rho})\Big) \partial_{x_2} \tilde{v}_2 - p'(\tilde{\rho})(\rho - \tilde{\rho}) \partial_{x_2} \tilde{v}_2$$

Finally just using the property $\partial_z H'(\tilde{\rho}) = (p'(\tilde{\rho})/\tilde{\rho}) \partial_z \tilde{\rho}$ we have

$$(\tilde{\rho} - \rho) \partial_t H'(\tilde{\rho}) + (\tilde{\rho}\tilde{v}_2 - \rho v_2) \partial_{x_2} H'(\tilde{\rho}) =$$
$$\frac{\tilde{\rho} - \rho}{\tilde{\rho}} p'(\tilde{\rho}) \partial_t \tilde{\rho} + \frac{\tilde{\rho}\tilde{v}_2 - \rho v_2}{\tilde{\rho}} p'(\tilde{\rho}) \partial_{x_2} \tilde{\rho}$$

Final step

Summing up all the terms and using again the fact that $\tilde{\rho}, \tilde{v}_2$ solve the continuity equation we end up with

$$\int_\Omega \mathcal{E}\left(\rho, v \middle| \tilde{\rho}, \tilde{v}\right)(\tau, x)\mathrm{d}x \leq$$
$$-\int_0^\tau \int_\Omega \left[\rho\left(\tilde{v}_2 - v_2\right)^2 + \left(p(\rho) - p'(\tilde{\rho})(\rho - \tilde{\rho}) - p(\tilde{\rho})\right)\right] \partial_{x_2}\tilde{v}_2(t, x)\mathrm{d}x\mathrm{d}t$$

Since $p(\rho)$ is convex the theorem follows from the fact that $\partial_{x_2}\tilde{v}_2(t, x) \geq 0$ which is a consequence of the classical theory of the self-similar solutions in the case of rarefaction waves.

Full Euler system

Now we consider the full compressible Euler system in the whole 2D space

$$
\begin{cases}
\partial_t \rho + \operatorname{div}_x(\rho v) \ = \ 0 \\
\partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x(\rho\theta) \ = \ 0 \\
\partial_t(\frac{1}{2}\rho\,|v|^2 + c_v\rho\theta) + \operatorname{div}_x((\frac{1}{2}\rho\,|v|^2 + c_v\rho\theta + \rho\theta)v) \ = \ 0 \\
\rho(\cdot,0) \ = \ \rho^0 \\
v(\cdot,0) \ = \ v^0 \\
\theta(\cdot,0) \ = \ \theta^0 \,.
\end{cases}
\tag{4}
$$

Here θ is the temperature of the gas and $c_v > 0$ is constant called specific heat at constant volume

Entropy inequality

The associated entropy inequality to the system reads as follows

$$\partial_t(\rho s) + \mathrm{div}_x(\rho s v) \ \geq \ 0,$$

where

$$s(\rho, \theta) = \log\left(\frac{\theta^{c_v}}{\rho}\right)$$

Domain and weak solutions

Similarly as in the isentropic case we consider the domain

$$\Omega = \mathcal{T}^1 \times \mathbb{R}^1, \text{ where } \mathcal{T}^1 \equiv [0,1]_{\{0,1\}} \text{ is the "flat" sphere,}$$

Again we consider solutions periodic with respect to x_1.
In the variable x_2 we will prescribe far field conditions in order to prove uniqueness of solutions to the Riemann problem.

Riemann Problem

The Riemann initial data are as follows

$$(\rho^0(x), v^0(x), \theta^0(x)) := \begin{cases} (\rho_-, v_-, \theta_-) & \text{if } x_2 < 0 \\ (\rho_+, v_+, \theta_+) & \text{if } x_2 > 0, \end{cases} \tag{5}$$

and again we avoid the contact discontinuity formed by the higher dimension by assuming

$$v_\pm = (0, v_{\pm 2}).$$

Of course $\rho_\pm, \theta_\pm$ and $v_{\pm 2}$ are constants.

General form of self-similar solutions

The Riemann problem admits a solution

$$\rho(t,x) = R(t,x_2) = R(\xi), \ \theta(t,x) = \Theta(t,x_2) = \Theta(\xi),$$
$$v(t,x) = (0, V(t,x_2)) = (0, V(\xi))$$

depending solely on the self-similar variable $\xi = \frac{x_2}{t}$. Such a solution is unique in the class of BV solutions of the 1-D problem. The initial singularity resolves to at most 3 structures connected by constant states, where the first and the last structures are always either admissible shocks or rarefaction waves, whereas the middle structure is always a contact discontinuity. In special cases some of the structures can disappear.

Shock-free solutions

We consider special Riemann initial data such that the contact discontinuity does not appear and both remaining structures are rarefaction waves, more precisely:

- the entropy S is *constant* in $[0, T] \times \Omega$;
- the density R and the temperature Θ components of the Riemann solutions are interrelated through

$$\Theta = R^{\frac{1}{c_v}} \exp\left(\frac{1}{c_v} S\right);$$

- the density $R = R(t, x_2)$ and the velocity $V = V(t, x_2)$ represent a rarefaction wave solution of the 1-D *isentropic* system

$$\partial_t R + \partial_{x_2}(RV) = 0, \; R\left[\partial_t V + V \partial_{x_2} V\right] + \exp\left(\frac{1}{c_v} S\right) \partial_{x_2} R^{\frac{c_v+1}{c_v}} = 0,$$

Far field conditions I

We consider weak solutions satisfying the following far field conditions

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |\rho(t, x_1, x_2) - \rho_-| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$
$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |\rho(t, x_1, x_2) - \rho_+| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |\theta(t, x_1, x_2) - \theta_-| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$
$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |\theta(t, x_1, x_2) - \theta_+| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

and similarly

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |v_1(t, x_1, x_2)| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |v_2(t, x_1, x_2) - v_{-2}| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |v_1(t, x_1, x_2)| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |v_2(t, x_1, x_2) - v_{+2}| \, \mathrm{d}x_1 \, \mathrm{d}t = 0.$$

Weak solutions

We say that a trio $[\rho, \theta, v]$ is a weak solution of the Euler system if the following is satisfied:

- Positivity:

$$0 < \rho(t,x) \leq \overline{\rho},\ 0 < \theta(t,x) \leq \overline{\theta},\ |s(\rho,\theta)| < \overline{s},$$
$$|v(t,x)| < \overline{v} \text{ for a.a. } (t,x) \in (0,T) \times \Omega.$$

- Continuity equation

$$\int_\Omega \left[\rho(\tau,x)\varphi(\tau,x) - \rho^0(x)\varphi(0,x)\right] \mathrm{d}x$$
$$= \int_0^\tau \int_\Omega \Big[\rho(t,x)\partial_t\varphi(t,x) + \rho v(t,x) \cdot \nabla_x \varphi(t,x)\Big] \mathrm{d}x\mathrm{d}t$$

for any $0 \leq \tau \leq T$, and any test function $\varphi \in C_c^1([0,T] \times \Omega)$.

Weak solutions II

- Momentum equation

$$\int_\Omega \left[\rho v(\tau,x)\cdot\varphi(\tau,x) - \rho^0 v^0(x)\cdot\varphi(0,x)\right] \mathrm{d}x$$
$$= \int_0^\tau \int_\Omega \Big[\rho v(t,x)\cdot\partial_t\varphi(t,x) + \rho[v\otimes v](t,x) : \nabla_x\varphi(t,x)$$
$$+\rho(t,x)\theta(t,x)\mathrm{div}_x\varphi(t,x)\Big]\mathrm{d}x\mathrm{d}t$$

for any $0 \leq \tau \leq T$, and any $\varphi \in C^1_c([0,T]\times\Omega;\mathbb{R}^2)$.

Weak solutions III

- Energy equation

$$\int_\Omega \Bigg[\left(\frac{1}{2}\rho|v|^2 + c_v\rho\theta\right)(\tau,x)\varphi(\tau,x)$$
$$-\left(\frac{1}{2}\rho^0|v^0|^2 + c_v\rho^0\theta^0\right)(x)\varphi(0,x)\Bigg]\,\mathrm{d}x$$
$$= \int_0^\tau\int_\Omega \Bigg[\left(\frac{1}{2}\rho|v|^2 + c_v\rho\theta\right)(t,x)\partial_t\varphi(t,x)$$
$$+\left(\frac{1}{2}\rho|v|^2 + c_v\rho\theta + \rho\theta\right)v(t,x)\cdot\nabla_x\varphi(t,x)\Bigg]\mathrm{d}x\mathrm{d}t$$

for any $0 \le \tau \le T$, and any test function $\varphi \in C_c^1([0,T]\times\Omega)$.

Weak solutions IV

- Entropy inequality

$$\int_\Omega [\rho b(s(\rho,\theta))(\tau,x)\varphi(\tau,x) - \rho_0 b(s(\rho_0,\theta_0))(x)\varphi(0,x)]\,\mathrm{d}x$$
$$\geq \int_0^\tau \int_\Omega \Big[(\rho b(s(\rho,\theta))\partial_t\varphi(t,x) + \rho b(s(\rho,\theta))v\cdot\nabla_x\varphi(t,x)\Big]\mathrm{d}x\mathrm{d}t$$

for any $0 \leq \tau \leq T$, any test function $\varphi \in C_c^1([0,T]\times\Omega)$, $\varphi \geq 0$, and any $b \in C^1$, $b' \geq 0$.

Theorem 6 (Feireisl, K., Vasseur)

Let $[\rho, \theta, v]$ be a weak solution of the Euler system in $(0, T) \times \Omega$ originating from the Riemann data and satisfying the far field conditions. Suppose in addition that the Riemann data give rise to the shock-free solution $[R, \Theta, V]$ of the 1-D Riemann problem specified above.
Then

$$\rho = R,\ \ \theta = \Theta,\ \ v = (0, V) \text{ a.a. in } (0, T) \times \Omega.$$

Relative entropy inequality

First we define the ballistic free energy $H_{\tilde{\theta}}$

$$H_{\tilde{\theta}}(\rho,\theta) = \rho\left(c_v\theta - \tilde{\theta}s(\rho,\theta)\right).$$

The relative entropy functional (actually it is again the relative energy) is defined as

$$\mathcal{E}\left(\rho,\theta,v\;\middle|\tilde{\rho},\tilde{\theta},\tilde{v}\right) =$$
$$\int_\Omega \left[\frac{1}{2}\rho|v-\tilde{v}|^2 + H_{\tilde{\theta}}(\rho,\theta) - \frac{\partial H_{\tilde{\theta}}(\tilde{\rho},\tilde{\theta})}{\partial\rho}(\rho-\tilde{\rho}) - H_{\tilde{\theta}}(\tilde{\rho},\tilde{\theta})\right]\mathrm{d}x,$$

This form of the relative entropy functional was introduced by Feireisl and Novotný.

Relative entropy inequality II

The relative entropy inequality reads as

$$
\begin{aligned}
&\left[\mathcal{E}\left(\rho,\theta,v\middle|\tilde\rho,\tilde\theta,\tilde v\right)\right]_{t=0}^{t=\tau}\\
&\leq \int_0^\tau\int_\Omega \left[\rho(\tilde v - v)\cdot\partial_t\tilde v + \rho(\tilde v - v)\otimes v : \nabla_x\tilde v + (\tilde\rho\tilde\theta - \rho\theta)\mathrm{div}_x\tilde v\right]\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_\Omega \left[\rho\Big(s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big)\partial_t\tilde\theta + \rho\Big(s(\rho,\theta) - s(\tilde\rho,\tilde\theta)\Big)v\cdot\nabla_x\tilde\theta\right]\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_\Omega \left[\left(1 - \frac{\rho}{\tilde\rho}\right)\partial_t(\tilde\rho\tilde\theta) + \left(\tilde v - \frac{\rho v}{\tilde\rho}\right)\cdot\nabla_x(\tilde\rho\tilde\theta)\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Relative entropy inequality III

This inequality holds for any weak solution $[\rho, \theta, v]$ of the Euler system satisfying the far field conditions and any trio $[\tilde{\rho}, \tilde{\theta}, \tilde{v}]$ of continuously differentiable functions such that

$$\tilde{\rho} > 0,\ \tilde{\theta} > 0,\ \left\{ \begin{array}{c} \tilde{\rho} = \rho_-, \tilde{\theta} = \theta_-,\ \ \tilde{v}_1 = 0,\ \ \tilde{v}_2 = v_{-2} \text{ for } x_2 < -A, \\ \\ \tilde{\rho} = \rho_+, \tilde{\theta} = \theta_+,\ \ \tilde{v}_1 = 0,\ \ \tilde{v}_2 = v_{+2} \text{ for } x_2 > A \end{array} \right\}$$

for some $A > 0$.
Finally, using a simple density argument, we check without difficulty that the Riemann solution $[R, \Theta, (0, V)]$ can be taken as test functions.

Relative entropy inequality IV

Thus we deduce

$$
\begin{aligned}
&\left[\mathcal{E}\left(\rho,\theta,v \middle| R,\Theta,(0,V)\right)\right]_{t=0}^{t=\tau} \\
&\leq \int_0^\tau \int_\Omega \Big[\rho(V-v_2)\partial_t V + \rho(V-v_2)v_2\partial_{x_2}V + (R\Theta-\rho\theta)\partial_{x_2}V\Big]\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \Big[\rho\Big(s(\rho,\theta)-S\Big)\partial_t\Theta + \rho\Big(s(\rho,\theta)-S\Big)v_2\partial_{x_2}\Theta\Big]\,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(1-\frac{\rho}{R}\right)\partial_t(R\Theta) + \left(V-\frac{\rho v_2}{R}\right)\partial_{x_2}(R\Theta)\right]\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Using the fact, that $[R, \Theta, (0, V)]$ solve the equations we can moreover simplify the right hand side to the following:

$$
\begin{aligned}
&\left[\mathcal{E}\left(\rho, \theta, v \middle| R, \Theta, (0, V)\right)\right]_{t=0}^{t=\tau} \\
&\leq \int_0^\tau \int_\Omega \left[-\rho(V - v_2)^2 \partial_{x_2} V + (R\Theta - \rho\theta)\partial_{x_2} V \right] \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \Big[\rho\Big(s(\rho, \theta) - S\Big)(\partial_t \Theta + V \partial_{x_2}\Theta) \\
&\qquad + \rho\Big(s(\rho, \theta) - S\Big)(v_2 - V)\partial_{x_2}\Theta \Big] \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\rho}{R}\right) (\partial_t(R\Theta) + V\partial_{x_2}(R\Theta)) \right] \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Slope of the rarefaction waves

Recall that we already know that

$$\partial_{x_2} V \geq 0.$$

Analyzing the rarefaction waves more carefully and taking into account the condition $S = \mathrm{const}$ we deduce the following property which will be important later:

$$\left| \frac{\partial_{x_2} \Theta}{\partial_{x_2} V} \right|^2 = \frac{1}{c_v(c_v + 1)} \Theta.$$

Some calculations

Using the Young inequality we obtain

$$
\begin{aligned}
&\rho\Big(s(\rho,\theta) - S\Big)(v_2 - V)\partial_{x_2}\Theta \\
\leq&\rho\frac{1}{4}(s-S)^2\frac{|\partial_{x_2}\Theta|^2}{\partial_{x_2}V} + \rho|v_2 - V|^2\partial_{x_2}V.
\end{aligned}
$$

Moreover using the fact that $[R,\Theta,(0,V)]$ solve the equations we can rewrite

$$
\begin{aligned}
&(R\Theta - \rho\theta)\partial_{x_2}V - \rho\,(s-S)\,(\partial_t\Theta + V\partial_{x_2}\Theta) \\
&+\frac{R-\rho}{R}\,(\partial_t(R\Theta) + V\partial_{x_2}(R\Theta)) \\
=&\rho\left[\Theta - \theta - \frac{1}{c_v}\left(\frac{R}{\rho} - 1\right)\Theta + \frac{1}{c_v}(s-S)\Theta\right]\partial_{x_2}V
\end{aligned}
$$

Finally we arrive to the following form of the relative entropy inequality

$$
\begin{aligned}
&\left[\mathcal{E}\left(\rho, \theta, v \middle| R, \Theta, (0, V)\right)\right]_{t=0}^{t=\tau} \\
\leq &\rho \frac{1}{4}(s - S)^2 \frac{|\partial_{x_2}\Theta|^2}{\partial_{x_2} V} \\
+ &\rho \left[\Theta - \theta - \frac{1}{c_v}\left(\frac{R}{\rho} - 1\right)\Theta + \frac{1}{c_v}(s - S)\Theta\right]\partial_{x_2} V
\end{aligned}
$$

Negative definite form

Note that for $\Theta = \Theta(\overline{V}, S)$ expressed as a function of the specific volume $\overline{V} = \frac{1}{R}$ and the entropy S, we get

$$\Theta(\overline{V}, S) = \exp\left(\frac{1}{c_v} S\right) \overline{V}^{-\frac{1}{c_v}},$$

$$\partial_{\overline{V}} \Theta(\overline{V}, S) = -\frac{1}{c_v} \Theta R, \ \partial_S \Theta(\overline{V}, S) = \frac{1}{c_v} \Theta;$$

whence

$$\Theta - \theta - \frac{1}{c_v}\left(\frac{R}{\rho} - 1\right)\Theta + \frac{1}{c_v}(s - S)\Theta$$

is a *negative-definite* quadratic form in the variables $(\frac{1}{\rho} - \frac{1}{R})$, $s - S$.

Final step

Now it is easy to see that it is enough to show that the function

$$F_{R,\Theta}(\rho, s) : [\rho, s] \mapsto$$
$$\left[\Theta - \theta(\rho, s) - \frac{1}{c_v}\left(\frac{R}{\rho} - 1\right)\Theta + \frac{1}{c_v}(s - S)\Theta\right] + \frac{\Theta}{4c_v(c_v + 1)}(s - S)^2$$

is non-positive for any choice $\rho > 0$, $s \geq S$.
Note that $s \geq S$ a.e. in $(0, T) \times \Omega$ for any weak solution. This is a consequence of the entropy inequality and the choice $s^0 = S$.

Final step II

Plug in

$$\theta(\rho, s) = \exp\left(\frac{1}{c_v} s\right) \rho^{\frac{1}{c_v}}, \;\; \Theta = \exp\left(\frac{1}{c_v} S\right) R^{\frac{1}{c_v}}$$

and introduce new variables

$$z = \frac{1}{c_v}(s - S) \geq 0, \;\; y = \left(\frac{\rho}{R}\right)^{\frac{1}{c_v}} > 0.$$

This way we obtain

$$G(y, z) = \left[1 - \exp(z)\, y - \frac{1}{c_v}\left(\frac{1}{y^{c_v}} - 1\right) + z + \frac{c_v}{4(c_v + 1)} z^2\right]$$

and we want to show that G is non-positive

Final step III

- Observation 1:

  $y \mapsto G(y,0) \leq 0$ attaining strong global maximum $G(1,0)=0$.

- Observation 2: There are no critical points of G in the open set $z>0$, $y>0$.

Indeed, compute

$$\partial_y G(y,z) = -\exp(z) + \frac{1}{y^{c_v+1}}, \; \partial_z G(y,z) = -\exp(z)y + 1 + \frac{c_v}{2(c_v+1)}z.$$

and assuming both partial derivatives equal zero yields

$$\exp\left(\frac{c_v}{c_v+1}z\right) = 1 + \frac{c_v}{2(c_v+1)}z,$$

which holds only for $z=0$.

Final step IV

- Observation 3:

$$G(y,z) \to -\infty \text{ as } y \to 0,\ \ y \to \infty \text{ for any fixed } z \geq 0.$$

- Observation 4: For $y \geq 1$ simply use $exp(z) \leq 1 + z + \frac{z^2}{2}$ to conclude that in this case $G(y,z) \leq 0$

Consequently, it remains to control G for $y \in (0,1)$ and large $z > 0$.

Final step V

Finally, fix $z > 0$ and examine the function

$$y \mapsto G(y,z), \ \ y \in (0,1].$$

We already know that $G(1,z) < 0$ and $\lim_{y\to 0} G(y,z) = -\infty$. There is exactly one critical point, namely

$$y = \exp\left(-\frac{1}{c_v+1}z\right),$$

with the corresponding critical value

$$\frac{c_v+1}{c_v}\left(1-\exp\left(\frac{c_v}{c_v+1}z\right)\right)+z+\frac{c_v}{4(c_v+1)}z^2$$
$$\leq -z-\frac{c_v}{2(c_v+1)}z^2+z+\frac{c_v}{4(c_v+1)}z^2 \leq 0.$$

The theorem is proved.

Thank you

Thank you for your attention.