Recursive functions vs. classification theory

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Recursive functions and R

Robinson's theory R

NOT Robinson's arithmetic (Q), but equally illustrious

Simple presentation: language $\langle 0, S, +, \cdot, < \rangle$, axioms

$$S^n(0) + S^m(0) = S^{n+m}(0)$$
$$S^n(0) \cdot S^m(0) = S^{nm}(0)$$
$$\forall x \left(x < S^n(0) \leftrightarrow x = 0 \vee \cdots \vee x = S^{n-1}(0) \right)$$

- Axiomatizes true Σ_1 sentences
- Essentially undecidable, no r.e. completion
- Locally finitely satisfiable
- Visser '12: Strongest locally finitely satisfiable r.e. theory up to interpretation

Representability of recursive functions

Representation of a (partial) function $f\colon \mathbb{N}^k \rightharpoonup \mathbb{N}$ in T:

Formula $\varphi(x_1, \dots, x_k, y)$, constant terms $\underline{n}$ for $n \in \mathbb{N}$ s.t. T proves

- $\underline{n} \neq \underline{m}$ whenever $n \neq m$
- $\varphi(\underline{n_1}, \dots, \underline{n_k}, z) \leftrightarrow z = \underline{m}$ whenever $f(n_1, \dots, n_k) = m$

Essential undecidability of R follows from:

- Theories representing all recursive functions are essentially undecidable
- R represents all recursive functions (even partial)

Converse?

R was designed to represent recursive functions, while being as weak as possible

This suggests the following question:

Problem

If a theory represents recursive functions,
does it interpret R?

Representability revisited

Representation of $f \overset{\text{almost}}{\Longleftrightarrow}$ interpretation of a certain theory

The extra requirements are pointless $\Longrightarrow$ better definition:

Definition

A representation of $f \colon \mathbb{N}^k \rightharpoonup \mathbb{N}$ in T is an interpretation of the following theory Rep_f in T:

- Language:
  - constants $\underline{n}$ for $n \in \mathbb{N}$
  - function symbol $\underline{f}$
- Axioms:
  - $\underline{n} \neq \underline{m}$ for $n \neq m$
  - $\underline{f}(\underline{n_1}, \dots, \underline{n_k}) = \underline{m}$ for $f(n_1, \dots, n_k) = m$

New statement of the problem

Definition

$PRF = \bigcup \{ Rep_f : f \text{ partial recursive function} \}$

PRF can be equivalently expressed in a finite language:

$$0, S(x), \langle x, y \rangle, \phi_x(y)$$

Our question reduces to:

Problem

Does PRF interpret R?

Model completions

Basic idea

PRF has quantifier-free axioms
$\Longrightarrow$ shouldn't interpret much of anything

Trouble: interpretations may use formulas of arbitrary quantifier complexity $\Longrightarrow$ not easy to analyze directly

Strategy: extend *PRF* to a theory with quantifier elimination

- get a handle on possible interpretations
- embed the standard model of *PRF* in a randomly looking structure so that any combinatorial features are dissolved

Model completion

Definition

Let T be a universal theory. A theory T^* is a

- companion of T if every model of T embeds in a model of T^* and vice versa
  - equivalently: $(T^*)_\forall = T$
- model companion of T if it is a companion, and it is model-complete
  - if $M \subseteq N$ are models of T^*, then $M \preceq N$
  - equivalently: over T^*, all formulas are existential
- model completion of T if it is a (model) companion, and it has quantifier elimination

Properties of model companions

- The model companion T^* of T is unique if it exists
- Models of T^* are the existentially closed models of T
  - $M \vDash T$
  - if an existential formula holds in an extension $M \subseteq N \vDash T$, it already holds in M
- T has a model companion $\iff$ the class of e.c. models of T is elementary

T	T^*
linear orders	dense linear orders
integral domains	algebraically closed fields
Boolean algebras	atomless Boolean algebras
groups	N/A

Intended application

If the empty L-theory $\varnothing_L$ had a model completion $\varnothing_L^*$:

- every L-structure extends to a model of $\varnothing_L^*$
- every consistent existential L-theory is consistent with $\varnothing_L^*$
- a theory interpretable in a consistent existential L-theory is weakly interpretable in $\varnothing_L^*$
- (weak) interpretations in $\varnothing_L^*$ are quantifier-free

PRF is existential, so what we'll do:

- show that indeed, $\varnothing_L$ has a model completion
- exhibit theories interpretable in R ($\sim$ locally finitely satisfiable) and not weakly interpretable in $\varnothing_L^*$

Random structure

$\varnothing_L^*$ is well known for relational languages L:
the theory of random structure(s)

- sentences that hold with asymptotic probability 1 in n-element random L-structures, $n \to \infty$
- or: the countable random L-structure
- Fraïssé limit of the class of all finite L-structures
- ω-categorical, quantifier elimination, ...
- axiomatized by extension axioms:
  - for any distinct $a_1, \dots, a_k$, there is another element b that bears any prescribed relations to $a_1, \dots, a_k$

The general case

If L includes function symbols:

- no 0–1 or limit law; no uniform distribution on ω
- 2^ω quantifier-free types $\implies$ no hope for ω-categoricity
- cannot assign values of terms willy-nilly:
  $f(a) = f(b) \to g(f(a)) = g(f(b))$

Luckily, it all works out in the end:

Theorem

For every language L, $\varnothing_L$ has a model completion $\varnothing_L^*$.

Warning: $\varnothing_L^*$ may be incomplete (quantifier-free sentences)

$\varnothing^*_L$ and existential theories

Corollary

If T is interpretable in a consistent existential theory, it is weakly quantifier-free interpretable in $\varnothing^*_L$ for some L.

A partial converse $\implies$ we are on the right track:

Proposition

Let T be an $\exists\forall$ theory in a relational language (?).
If T is weakly interpretable in some $\varnothing^*_L$,
it is interpretable in a consistent existential theory.

NB: $\varnothing^*_L$ is $\forall\exists$

Classification theory

Classification theory

- Various criteria to separate tame and wild theories ("dividing lines")
- Structure theory for models of tame theories
  - geometry of definable sets and types
  - models with special properties (prime, saturated, . . .)
  - interpretable algebraic structures (groups, . . .)
- Uncountable categoricity, stability, o-minimality, simplicity . . .
- Shelah

Why is it relevant here?

- Many dividing lines amount to weak interpretability of $\exists\forall$ locally finitely satisfiable theories!

Metaterminology

Let T be a theory:

- A formula φ has the xghg xiljxa property (XXP) in T if there are
  - a model $M \vDash T$
  - tuples $\overline{a}_i \in M$ $(i \in I)$

  such that hflijesai ff jai l jklf ajlifa $\varphi(\overline{a}_i, \overline{x})$ kah f h ahfdj k
- T has XXP if some formula has it in T
- T has the no xghg xiljxa property (NXXP) if it doesn't have XXP
- NXXP is good, XXP is bad

Main classes

picture

missing

see `http://forkinganddividing.com`

Order and independence properties

- order property (OP)
  $M \vDash T$, $\varphi(\overline{x}, \overline{y})$, $(\overline{a}_i)_{i \in \mathbb{N}}$ s.t.
  - $M \vDash \varphi(\overline{a}_i, \overline{a}_j) \iff i < j$

  NOP = stable = NIP&NSOP
- strict order property (SOP)
  - $\varphi(\overline{x}, \overline{y})$ defines a strict partial order
  - $M \vDash \varphi(\overline{a}_i, \overline{a}_j)$ for $i < j$
- k-strong order property ($\mathrm{SOP_k}$), $k \geq 3$
  - $\{\varphi(\overline{x}_1, \overline{x}_2), \varphi(\overline{x}_2, \overline{x}_3), \dots, \varphi(\overline{x}_k, \overline{x}_1)\}$ is inconsistent
  - $M \vDash \varphi(\overline{a}_i, \overline{a}_j)$ for $i < j$
- independence property (IP)
  $\varphi(\overline{x}, \overline{y})$, $(\overline{a}_i)_{i \in \mathbb{N}}$, $(\overline{b}_X)_{X \subseteq \mathbb{N}}$ s.t.
  - $M \vDash \varphi(\overline{a}_i, \overline{b}_X) \iff i \in X$

Tree properties

$\mathbb{N}^{<\omega}$ = tree of finite sequences over a countable alphabet

- tree property (TP)
  M, $\varphi(\overline{x},\overline{y})$, $(\overline{a}_s)_{s\in\mathbb{N}^{<\omega}}$ s.t.
  - $\{\varphi(\overline{x},\overline{a}_{\sigma\restriction n}) : n\in\omega\}$ is consistent for each path $\sigma\in\mathbb{N}^{\omega}$
  - $\{\varphi(\overline{x},\overline{a}_{s\smile i}),\varphi(\overline{x},\overline{a}_{s\smile j})\}$ is inconsistent for $s\in\mathbb{N}^{<\omega}$, $i<j$

  NTP = simple = NTP_1&NTP_2
- TP_1 (= "SOP_2")
  - $\{\varphi(\overline{x},\overline{a}_{\sigma\restriction n}) : n\in\omega\}$ is consistent for $\sigma\in\mathbb{N}^{\omega}$
  - $\{\varphi(\overline{x},\overline{a}_s),\varphi(\overline{x},\overline{a}_t)\}$ is inconsistent for s,t incomparable
- TP_2
  M, $\varphi(\overline{x},\overline{y})$, $(\overline{a}_{n,i})_{n,i\in\omega}$
  - $\{\varphi(\overline{x},\overline{a}_{n,\sigma(n)}) : n\in\omega\}$ is consistent for $\sigma\in\mathbb{N}^{\omega}$
  - $\{\varphi(\overline{x},\overline{a}_{n,i}),\varphi(\overline{x},\overline{a}_{n,j})\}$ is inconsistent for $n\in\omega$, $i<j$

$\varnothing^*_L$ not quite domesticated

NB: random relational structures are supersimple

Observation

Any consistent extension of

- PRF, or
- $\varnothing^*_L$ if L contains a binary function

is TP_2 (hence IP and non-simple).

Proof: Take $\bar{a}_{n,i} = (\underline{n}, \underline{i})$, and

$$(x)_{y_1} = y_2$$

for the formula $\varphi(x, y_1, y_2)$

Elimination of infinity

Definition

T has elimination of infinity if for every formula $\varphi(\bar{z}, x)$, there is a bound n such that

$$|\varphi(\bar{a}, M)| > n \implies |\varphi(\bar{a}, M)| \geq \aleph_0$$

for every $M \vDash T$ and $\bar{a} \in M$

Elimination of infinity $\iff$ FO formulas are closed under $\exists^\infty$:

$$M \vDash \exists^\infty x \, \varphi(\bar{a}, x) \quad \text{iff} \quad \varphi(\bar{a}, M) \text{ is infinite}$$

Tameness of $\varnothing_L^*$

Main theorem

For any language L:

- $\varnothing_L^*$ has NSOP$_3$ (hence NSOP)
- $(\varnothing_L^*)^{\mathrm{eq}}$ eliminates infinity

Consequences

Corollary

The following theories are interpretable in R,
but not in PRF:

- (partial) orders with arbitrarily long chains
- "for each standard n, there is a set with n elements"
- directed graphs with arbitrarily long transitive chains, and no directed 3-cycle

Problems

- Does *PRF* interpret all consistent r.e. existential theories?
- Is the random graph interpretable in a consistent existential theory?
- Does $\varnothing_L^*$ have NTP_1, or even NSOP_1?
- Does $\varnothing_L^*$ have weak elimination of imaginaries?

Thank you for attention!

References

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