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Eduard Feireisl
Yongzhong Sun

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Eduard Feireisl *   Yongzhong Sun †

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic

Department of Mathematics, Nanjing University
Nanjing, Jiangsu, 210093, P. R. China

Abstract

We adapt the relative energy functional associated to the compressible Navier-Stokes system to show stability of solutions emanating from 1-D initial data with respect to multidimensional $N = 2, 3$ perturbations. Besides the application of the relative energy inequality as a suitable "distance" between two solutions, refined regularity estimates in L^p based Sobolev spaces are used.

Key words: Compressible Navier-Stokes equations, 1-D compressible fluid flow, relative energy

1 Introduction

The results obtained in the present paper may be viewed as a "compressible" counterpart of the recent paper of Bardos et al. [2], where the authors consider similar problems in the context of incompressible fluid flows governed by the standard Navier-Stokes system. More specifically, we study

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the evolution of a viscous, compressible, isentropic fluid described in terms of the mass density $\varrho = \varrho(t,x)$ and the macroscopic velocity $\mathbf{u} = \mathbf{u}(t,x)$ satisfying the *compressible Navier-Stokes system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

supplemented with the constitutive relations for the pressure

$$p(\varrho) = a\varrho^\gamma,\ a > 0,\ \gamma > 1, \tag{1.3}$$

and the viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I},\ \mu > 0,\ \eta \geq 0. \tag{1.4}$$

We introduce a one-dimensional counterpart of the system (1.1–1.4), namely

$$\partial_t R + \partial_y (RV) = 0, \tag{1.5}$$

$$\partial_t(RV) + \partial_y(RV^2) + \partial_y p(R) = \left[2\mu \left(1 - \frac{1}{N} \right) + \eta \right] \partial^2_{y,y} V. \tag{1.6}$$

With the obvious identification $x_1 = y$, $\varrho(x) = R(x_1)$, $\mathbf{u}(x) = [V(x_1), 0, \dots, 0]$, any solution of problem (1.5), (1.6) obviously satisfies also the extended system (1.1–1.4).

The one-dimensional fluid motion governed by the system of equations (1.5), (1.6) is nowadays well-understood, see the monograph by Antontsev, Kazhikhov and Monakhov [1]. In particular, problem (1.5), (1.6) considered in the slab $(0,1)$, and supplemented with the impermeability boundary conditions

$$V(t,0) = V(t,1) = 0,\ t \in (0,T), \tag{1.7}$$

and the initial conditions

$$R(0,\cdot) = R_0 > 0,\ V(0,\cdot) = V_0 \tag{1.8}$$

admits a (unique) weak solution for a fairly vast class of initial data, see Amosov and Zlotnik [19]. Moreover, the solutions are regular provided the initial data are smooth enough, see Kazhikhov [15].

Our goal in the present paper is to examine stability of the one-dimensional solution $[r, V]$ under multidimensional perturbations of the initial data. In particular, we show that the one-dimensional solutions are uniquely determined in the framework of multidimensional *weak* solutions to problem (1.1–1.4) supplemented with suitable boundary conditions provided a suitable form of *energy inequality* is satisfied. This fact is in sharp contrast to similar problems related to *inviscid* fluids, for which Chiodaroli, DeLellis and Kreml [3] obtained infinitely many weak solutions for the compressible Euler system emanating from Lipschitz initial data. Moreover, these solutions, constructed by the method of convex integration, start from one-dimensional initial data but become truly multidimensional after a critical time. Thus, in particular, our results imply that the solutions constructed in [3] *are not* viscosity solutions, meaning they cannot be obtained as an inviscid limit of solutions of the Navier-Stokes system with the same initial data, see Section 5 for details.

Our approach is based on the relative energy inequality for the compressible Navier-Stokes system identified in [10] combined with regularity properties of solutions to the one-dimensional problem (1.5–1.8) in the L^p Sobolev norms. The paper is organized as follows. In Section 2, we collect the necessary preliminary material and formulate our main results. The relative energy inequality is introduced in Section 3. In Section 4, we show the L^p-estimates for the solutions of the one-dimensional problem and complete the proof of their stability in the multidimensional setting. The paper is concluded by a short discussion in Section 5.

2 Preliminaries, main result

We consider a domain $\Omega \subset R^N$, $N = 2, 3$,

$$\Omega = (0,1) \times \mathcal{T}^{N-1}, \text{ where } \mathcal{T}^{N-1} \equiv \Big((0,1)|_{\{0,1\}} \Big)^{N-1} \text{ is the torus in } R^{N-1},$$

specifically all functions defined in Ω are 1-periodic with respect to the variables $x_j,\ j > 1$. Accordingly, any solution r, V of problem (1.5–1.7) can be extended to be constant in $x_j,\ j > 1$.

2.1 Finite energy weak solutions to the multidimensional system

We say that the functions $[\varrho, \mathbf{u}]$ represent a *finite energy weak solution* to the Navier-Stokes system in the space-time cylinder $(0,T) \times \Omega$, supplemented with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{2.1}$$

and the initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \tag{2.2}$$

if:

The density ϱ is a non-negative function, $\varrho \in C_{\rm weak}([0,T]; L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0,T; W^{1,2}_0(\Omega; R^N))$, $\varrho\mathbf{u} \in C_{\rm weak}([0,T]; L^{2\gamma/(\gamma-1)}(\Omega))$, and

$$\left[\int_\Omega \varrho\varphi \ {\rm d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi \right] \ {\rm d}x{\rm d}t \tag{2.3}$$

for any $0 \leq \tau_1 \leq \tau_2 \leq T$ and any $\varphi \in C_c^\infty([0,T] \times \overline{\Omega})$;

$$\left[\int_\Omega \varrho \mathbf{u} \cdot \varphi \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t \tag{2.4}$$

for any $0 \leq \tau_1 \leq \tau_2 \leq T$ and any $\varphi \in C_c^\infty([0,T] \times \Omega; R^N)$.

In addition, the *energy inequality*

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ \mathrm{d}x, \text{ with } P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma, \tag{2.5}$$

holds for a.a. $\tau \in (0,T)$.

Remark 2.1 *Obviously, we set $\varrho \mathbf{u}(0,\cdot) = \varrho_0 \mathbf{u}_0$ in (2.3). In the weak formulation, however, it is more natural to prescribe the initial distribution of the momentum $\varrho \mathbf{u}(0,\cdot) = (\varrho \mathbf{u})_0$ stipulating finiteness of the kinetic energy*

$$\int_\Omega \frac{1}{2} \frac{|(\varrho \mathbf{u})_0|^2}{\varrho_0} \ \mathrm{d}x < \infty,$$

in particular $(\varrho \mathbf{u})_0$ vanishes a.a. on the vacuum set $\{\varrho_0 = 0\}$. Such a generality is not needed here as we always consider the initial data with strictly positive density.

Remark 2.2 *It is possible to replace (2.5) by a stronger assumption*

$$\psi(\tau) \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] \ \mathrm{d}x - \int_0^\tau \partial_t \psi \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] \ \mathrm{d}x \ \mathrm{d}t \leq \psi(0) \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ \mathrm{d}x \tag{2.6}$$

for a.a. $\tau \in (0,T)$ and any $\psi \in C_c^\infty[0,T]$, $\psi \geq 0$. Weak solutions satisfying (2.6) were constructed by Kukučka [16] for a fairly general class of spatial domains including those with non-Lipschitz boundary.

Finite energy weak solutions to problem (1.1–1.4), (2.1), (2.2) are known to exist for any finite energy initial data whenever $\gamma > \frac{N}{2}$, see Lions [17] and [9].

2.2 Solutions to the one-dimensional system

Solvability of the one-dimensional problem was established by Kazhikhov [15]. He showed that system (1.5), (1.6) supplemented with the conditions (1.7), (1.8) admits a (strong) solution $[R, V]$, unique in the class

$$R \in L^\infty(0,T; W^{1,2}(0,1)), \ V \in L^\infty(0,T; W_0^{1,2}(0,1)) \cap L^2(0,T; W^{2,2}(0,1)), \tag{2.7}$$

as soon as

$$R(0,\cdot) = r_0 \in W^{1,2}(0,1),\ R_0 > 0 \text{ in } [0,1],\ V(0,\cdot) = V_0 \in W_0^{1,2}(0,1).$$

Moreover, a vacuum cannot appear in a finite time,

$$0 < \underline{R}(t) \le R(t,\cdot) \le \overline{R}(t) \text{ for any } t \ge 0. \tag{2.8}$$

Remark 2.3 *The absence of vacuum in one-dimensional flow was extended to a fairly general class of weak solutions by Hoff and Smoller [14].*

The well-posedness theory for problem (1.5–1.8) was later extended by Amosov and Zlotnik [19] to a general class of initial data

$$R_0, \frac{1}{R_0} \in L^\infty(0,1),\ V_0 \in L^2(0,1). \tag{2.9}$$

On the other hand, if the initial data are smooth, specifically,

$$\left\{ \begin{array}{c} R_0 \in C^{1+\beta}[0,1],\ R_0 V_0 \in C^{2+\beta}[0,1],\ \beta > 0, \\ \\ \text{with the compatibility conditions } V_0|_{y=0,1} = \partial^2_{y,y} V_0|_{y=0,1} = \partial_y R_0|_{0,1} = 0, \end{array} \right\} \tag{2.10}$$

than the solution $[R,V]$ is classical (smooth), see Kazhikhov [15].

2.3 Main result

As already pointed out in the introduction, the solution $[R,V]$ of the one-dimensional problem can be viewed as a particular solution of the multidimensional system (1.1–1.4), (2.1), (2.2) after the natural extension

$$R(t,x_1,x_2,x_3) = R(t,x_1), \mathbf{V}(t,x_1,x_2,x_3) = (V(t,x_1),0,0).$$

We are ready to state our main result:

Theorem 2.1 *Let*

$$\gamma > \frac{N}{2},\ q > \max\{2,\gamma'\},\ \frac{1}{\gamma} + \frac{1}{\gamma'} = 1 \textit{ if } N = 2,\ q > \max\left\{3, \frac{6\gamma}{5\gamma - 6}\right\} \textit{ if } N = 3. \tag{2.11}$$

Let $[R,V]$ be a (strong) solution of the one-dimensional problem (1.5–1.8), with the initial data belonging to the class

$$R_0 \in W^{1,q}(0,1),\ R_0 > 0 \textit{ in } [0,1],\ V_0 \in W_0^{1,q}(0,1). \tag{2.12}$$

Let $[\varrho, \mathbf{u}]$ be a finite energy weak solution to the Navier-Stokes system (1.1–1.4) in $(0,T)\times\Omega$, supplemented with the conditions

$$\varrho_0 \in L^\infty(\Omega),\ \ \varrho_0 > 0 \ \textit{a.a. in}\ \Omega;\ \ \mathbf{u}_0 \in L^2(\Omega; R^N). \tag{2.13}$$

Then

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{V}|^2 + P(\varrho) - P'(R)(\varrho - R) - P(R) \right](\tau,\cdot)\ \mathrm{d}x \tag{2.14}$$

$$\leq c(T)\int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{V}_0|^2 + P(\varrho_0) - P'(R_0)(\varrho_0 - R_0) - P(R_0) \right]\ \mathrm{d}x$$

for a.a. $\tau \in (0,T)$.

Seeing that $\varrho \mapsto P(\varrho) \equiv \frac{a}{\gamma-1}\varrho^\gamma$ is a strictly convex function, relation (2.13) implies that $\varrho = R$, $\mathbf{u} = \mathbf{V}$ whenever $\varrho_0 = R_0$, $\mathbf{u}_0 = \mathbf{V}_0 = (V_0, 0, 0)$.

The next two sections are devoted to the proof of Theorem 2.1.

3 Relative energy

Following Dafermos [4], Germain [11], Mellet and Vasseur [18], among others, we introduce the *relative energy functional*

$$\mathcal{E}\left([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right]\ \mathrm{d}x.$$

Under the hypotheses of Theorem 2.1, any finite energy weak solution $[\varrho, \mathbf{u}]$ of the compressible Navier-Stokes system satisfies the *relative energy inequality*:

$$\mathcal{E}\left([\varrho,\mathbf{u}]\Big|[r,\mathbf{U}]\right)(\tau) + \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \Big)\ \mathrm{d}x\ \mathrm{d}t \tag{3.1}$$

$$\leq \mathcal{E}\left([\varrho_0,\mathbf{u}_0]\Big|[r,\mathbf{U}](0,\cdot)\right) + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})\ \mathrm{d}t,$$

for all "test" functions

$$r \in C^\infty_c([0,T]\times\overline{\Omega}),\ \ r>0,\ \ \mathbf{U} \in C^\infty_c([0,T]\times\overline{\Omega}; R^N),\ \ \mathbf{U}|_{\partial\Omega} = 0, \tag{3.2}$$

where the remained term reads

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \equiv \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{u}\nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u})\ \mathrm{d}x \tag{3.3}$$

$$+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x$$

$$+ \int_\Omega \left((r - \varrho)\partial_t P'(r) + \nabla_x P'(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) \right) \ \mathrm{d}x - \int_\Omega \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p(r) \Big) \ \mathrm{d}x,$$

see [8, Section 3.2.1].

The requirement (3.2) of smoothness of the test functions can be relaxed by means of a density argument. In particular, following [8, Section 4.1.1], we may take $r = R$, $\mathbf{U} = [V, 0, 0]$, where $[R, V]$ is a *regular* solution of the one-dimensional problem (1.5–1.8) to obtain

$$\mathcal{E}\left([\varrho, \mathbf{u}] \Big| [R, \mathbf{V}] \right)(\tau) + \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{V}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{V} \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{3.4}$$

$$\leq \mathcal{E}\left([\varrho_0, \mathbf{u}_0] \Big| [R, \mathbf{V}](0, \cdot) \right) + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, R, \mathbf{V}) \ \mathrm{d}t,$$

$$\mathcal{R}\left(\varrho, \mathbf{u}, R, \mathbf{V}\right) = -\int_\Omega \varrho (u^1 - V)^2 \partial_y V \ \mathrm{d}x - \int_\Omega \Big[p(\varrho) - p'(R)(\varrho - R) - p(R) \Big] \partial_y V \ \mathrm{d}x \tag{3.5}$$

$$+ \left[2\mu \left(1 - \frac{1}{N} \right) + \eta \right] \int_\Omega \frac{1}{R} (\varrho - R)(V - u^1) \partial^2_{y,y} V \ \mathrm{d}x.$$

Now, following step by step the arguments specified in [8, Section 4.1.1] we can establish the desired relation (2.14) using basically the relative entropy inequality (3.1), combined with (3.5) and a Gronwall type argument.

Unfortunately, the lower regularity of the one-dimensional solutions $[r, V]$ required in Theorem 2.1 *does not allow* us to derive (3.5) from (3.3) in a direct manner. Instead, having (3.5) granted for the smooth solutions in Kazhikhov's class (2.10) we derive suitable uniform bounds for solutions of the one-dimensional problem under the data regularity hypothesis (2.12). Then we close the argument by applying the density argument in the class of (strong) solutions satisfying (2.7). This will be done in the next section.

4 Uniform estimates

We start by a *formal* argument yielding the desired conclusion (2.14) from the relative energy inequality

$$\mathcal{E}\left([\varrho, \mathbf{u}] \Big| [R, \mathbf{V}] \right)(\tau) + \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{V}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{V} \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{4.1}$$

$$\leq \mathcal{E}\left([\varrho_0, \mathbf{u}_0] \Big| [R_0, \mathbf{V}_0] \right) + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, R, \mathbf{V}) \ \mathrm{d}t,$$

where $\mathcal{R}$ satisfies (3.5).

Obviously, the first two integrals on the right-hand side of (3.5) may be "absorbed" by the left-hand side via Gronwall's lemma as soon as

$$V \in L^1(0,T; W^{1,\infty}(0,1)). \tag{4.2}$$

Thanks to the embedding $W^{2,2}(0,1) \hookrightarrow W^{1,\infty}(0,1)$, the relation (4.2) holds for the (strong) solutions belonging to the class (2.7), specifically for any initial data satisfying hypothesis (2.12) with $q = 2$.

As for the last integral in (3.5), we first observe that

$$P(\varrho) - P'(R)(\varrho - R) - P(R) \geq \left\{ \begin{array}{l} c(R)(\varrho - R)^2 \text{ if } R/2 < \varrho < 2R \\ \\ c(R)|\varrho - R|^\gamma \end{array} \right.$$

with $c(R) > 0$ is uniformly bounded for R belonging to a compact set in $(0,\infty)$, where the latter condition is satisfied by solutions of the one-dimensional problem thanks to (2.8).

Next, applying Korn's inequality to the class of functions satisfying the boundary conditions (1.7), (2.1), we obtain

$$\|\mathbf{u} - \mathbf{V}\|_{W^{1,2}(\Omega;R^N)} \leq c \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{V}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{V} \Big) \ \mathrm{d}x$$

Consequently, by virtue of the Sobolev embedding

$$W^{1,2}(\Omega) \hookrightarrow L^6 \text{ for } N = 3, \ W^{1,2}(\Omega) \subset L^p \text{ for any finite } p \geq 1 \text{ if } N = 2,$$

the relation (2.14) will follow from (4.1) with (3.5) as soon as we establish the bound

$$V \in L^2(0,T; W^{2,q}(0,1)), \tag{4.3}$$

for q satisfying (2.11), see [8, Section 4.1.1] for details. Consequently, the above formal procedure can be justified by means of a density argument as soon as we show (4.3) with a bound depending only on the norm of R_0, V_0 in the Sobolev space $W^{1,q}(0,1)$.

Following Danchin [5], we use the maximal regularity estimates for the velocity V adapted to the parabolic equation (1.6). As we already know that R satisfies (2.7), (2.8) we may use the Sobolev embedding $W^{1,2}(0,1) \hookrightarrow C^\alpha[0,1]$, $0 < \alpha < 1/2$ to obtain

$$R \in C([0,T]; C^\alpha[0,1]),$$

with the associated norm bounded only in terms of $\|R\|_{W^{1,2}(0,1)}$, $\|V\|_{W^{1,2}(0,1)}$. Consequently, the maximal regularity theory applies to the parabolic problem (1.6), (1.7), specifically Denk, Hieber, Prüss [7, Theorem 2.3] (cf. also Danchin [5]), yielding

$$\int_0^\tau \|V\|^2_{W^{2,q}(0,1)} \ \mathrm{d}t \leq c(T) \left(\|V_0\|^2_{W_0^{1,q}(0,1)} + \int_0^\tau \|\partial_y R\|^2_{L^q(0,1)} \right) \tag{4.4}$$

for any $0 \leq \tau \leq T$.

On the other hand, denoting $s = \partial_y R$ we deduce easily that

$$\partial_t s + V \partial_y s = -2s \partial_y V - R \partial^2_{y,y} V,$$

and, consequently,

$$\|s(\tau, \cdot)\|_{L^q(0,1)} \leq \|\partial_y V_0\|_{L^q(0,1)} + c(T) \int_0^\tau \|\partial^2_{y,y} V\|_{L^q(0,1)} \, \mathrm{d}t, \tag{4.5}$$

which, combined with (4.4) and the standard Gronwall argument, yields the desired estimates

$$\sup_{t \in (0,T)} \|\partial_y R\|_{L^q(0,1)} + \int_0^T \|V\|^2_{W^{2,q}(0,1)} \, \mathrm{d}t \leq c(T). \tag{4.6}$$

We have proved Theorem 2.1.

5 Concluding remarks

We conclude the paper by several comments concerning the implications of Theorem 2.1.

5.1 Viscosity solutions to the Euler system

Recently, DeLellis and Székelyhidi [6], Chiodaroli, DeLellis, and Kreml [3] identified a vast class of initial data for which the isentropic Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0,$$

supplemented with the entropy inequality

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{u} \right] \leq 0,$$

admits infinitely many *weak* solutions defined in $(0, T)$ provided the space dimension $N = 2, 3$. In particular, it was shown that this happens even for certain *Lipschitz* initial data ϱ_0, $\mathbf{u}_0$, see Chiodaroli, DeLellis and Kreml [3]. These solutions are "constructed" via the method of convex integration. Similarly our setting in this paper, the initial data are taken one-dimensional, the solution develops discontinuity in a finite time leading to the initial value problems with Riemann type data problem. The infinitely many solutions than emanate from the one-dimensional Riemann data but are truly multidimensional, in the sense that the second component of the velocity becomes non-trivial.

In this Theorem 2.1 shows that these solutions *are not* viscosity solutions of the Euler system, meaning they cannot be obtained as a vanishing viscosity limit of the solutions of the Navier-Stokes system (1.1–1.4), in which we let $\mu \to 0$, $\eta \to 0$.

5.2 More general initial data

As we know (see Amosov and Zlotnik [19]), the one-dimensional Navier-Stokes system (1.5), (1.6) is well-posed for a larger class of the initial data than those considered in Theorem 2.1, specifically,

$$R_0 \in L^\infty(0,1),\ R_0 > 0,\ V_0 \in L^2(0,1). \tag{5.1}$$

It seems therefore natural and also interesting in view of possible application to extend the conclusion of Theorem 2.1 to the initial data (5.1). Unfortunately, this seems not achievable by our technique based on the relative energy inequality (3.3), (3.5), where the existence of the second derivative of the velocity V is required. As observed by Hoff [12], [13], however, singularities in the pressure for the Navier-Stokes system propagate in time, and $\partial_y V$ enjoys the same regularity with $p(R)$, in particular $\partial_y V$ experiences discontinuities induced by possible jumps in R_0.

5.3 Symmetry preserving

The results of the present paper are easy to extend to the case of *radially symmetric* initial data provided the problem is considered on an *annulus*

$$\Omega = \left\{ x \in R^N \ \middle|\ 0 < \underline{r} < |x| < \overline{r} \right\}.$$

Similarly to Bardos et al. [2], we may assert that the symmetry of the data is preserved in time in the class of weak solutions to the compressible Navier-Stokes system.

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