Maximal dissipation and well posedness for models of inviscid fluids

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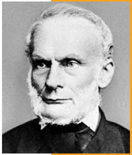
Thermodynamics



Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Well posedness - classical sense

Jacques Hadamard,
[1865 - 1963]

Existence

Given problem is solvable for any choice of (admissible) data

Uniqueness

Solutions are uniquely determined by the data

Stability

Solutions depend continuously on the data

Well posedness - modern way

Jacques-Louis Lions,
[1928 - 2001]

Approximations

Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically

Uniform bounds

Approximate solutions possess uniform bounds depending solely on the data

Stability

The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

Abstract conservation laws

System of equations (conservation laws)

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F}(\mathbf{U}) = 0,$$

$\mathbf{U}$ .... state variable
$\mathbb{F}$ .... flux

"Entropies"

$$\partial_t E_i(\mathbf{U}) + \mathrm{div}_x \mathbf{F}_{E_i}(\mathbf{U}) = \boxed{(\leq)} 0, \ i = 1, 2, \dots$$

E_i .... entropy
$\mathbf{F}_i$ .... entropy flux

***A priori* bounds**

$$\int E_i(\mathbf{U}) \, \mathrm{d}x \text{ bounded in terms of the initial data}, \ i = 1, 2, \dots$$

Weak vs. strong solutions

Lack of regularity

- bounds available only in L^p (L^∞)
- presence of oscillations
- discontinuities (shocks) appearing in finite time even for initial states

Weak solutions

$$\int_\Omega \mathbf{U} \cdot \varphi(\tau_2, \cdot) - \mathbf{U}\varphi(\tau_1, \cdot) \ \mathrm{d}x$$

$$= \int_{\tau_1}^{\tau_2} \int_\Omega \left[\mathbf{U} \cdot \partial_t \varphi + \mathbb{F}(\mathbf{U}) : \nabla_x \varphi \right] \ \mathrm{d}x \ \mathrm{d}t$$

Weak continuity

$t \mapsto \mathbf{U}(t, \cdot)$ weakly continuous

Compensated compactness - DiPerna, Tartar

Linear field equations

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0$$

$$\partial_t E_i + \mathrm{div}_x \mathbf{F}_i \leq 0,\ i = 1, 2, \dots$$

Nonlinear constitutive equations

$$\mathbb{F} = \mathbb{F}(\mathbf{U}),\ E_i = E_i(\mathbf{U}),\ \mathbf{F}_i = \mathbf{F}_i(\mathbf{U}),\ i = 1, 2, \dots$$

Compensated compactness

- linear field equations yield constraints on possible oscillations described by Young measure
- nonlinear constrained imposed by constitutive equations reduce the Young measures to Dirac masses (no oscillations)

Basic ideas of Young measures

Oscillatory sequence - convergence in the sense of averages

$$U_n \to U \text{ weakly-(*)} \Leftrightarrow \int_B U_n \to \int_B U \text{ for any } B$$

Young measure associated to $\{U_n\}$

$$< \sigma_y(U), f >= \lim_{r \to 0} \frac{1}{|B_r(y)|} \left[\lim_{n \to \infty} \int_{B_r} f(U_n) \right]$$

Strong (pointwise a.a.) convergence

$$U_n \to U \text{ for a.a.} \Leftrightarrow \sigma_y(U) = \delta_{U(y)} \text{ for a.a. } y$$

Convex integration

Linear field equations

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0$$

Replacing constitutive equation

$$\mathbb{F} = \mathbb{F}(\mathbf{U}) \Leftrightarrow \Lambda(\mathbf{U}, \mathbb{F}) = E(\mathbf{U}) \ \boxed{\text{“implicit”}}$$

$$\Lambda(\mathbf{U}, \mathbb{F}) \text{ convex}, \ \Lambda(\mathbf{U}, \mathbb{F}) \geq E(\mathbf{U})$$

Relaxation of constitutive equation

$$E(\mathbf{U}) \leq \Lambda(\mathbf{U}, \mathbb{F}) \leq e, \ e \text{ given “energy”}$$

$$\mathbb{F} = \mathbb{F}(\mathbf{U}) \Leftrightarrow \Lambda(\mathbf{U}, \mathbb{F}) = E(\mathbf{U}) \Leftarrow E(\mathbf{U}) = e$$

Convex integration - DeLellis - Székelyhidi

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} \right) + \nabla_x \left(\frac{1}{3} |\mathbf{v}|^2 + \Pi \right) = 0$$

$$\mathbb{F} = \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} \right) - \text{ symmetric traceless}$$

Relaxation

$$\Lambda(\mathbf{v}, \mathbb{F}) = \frac{3}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{F} \right]$$

$$\Lambda(\mathbf{v}, \mathbb{F}) \geq \frac{1}{2} |\mathbf{v}|^2$$

$$\Lambda(\mathbf{v}, \mathbb{F}) = \frac{1}{2} |\mathbf{v}|^2 \Leftrightarrow \mathbb{F} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I}$$

Oscillatory lemma

Subsolutions

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = 0, \ E(\mathbf{U}) \leq \Lambda(\mathbf{U}, \mathbb{F}) \boxed{<} e$$

Oscillatory increments

$$\partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbb{G}_\varepsilon = 0, \ \Lambda(\mathbf{U} + \mathbf{w}_\varepsilon, \mathbb{F} + \mathbb{G}_\varepsilon) < e$$

$$\mathbf{w}_\varepsilon, \ \mathbb{G}_\varepsilon \text{ compactly supported in } Q, \ \mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(Q)$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{w}_\varepsilon|^2 \ \mathrm{d}x \geq C \int_B \Big(e - E(\mathbf{U}) \Big)^\alpha \ \mathrm{d}x$$

$$\Rightarrow$$

$$\liminf_{\varepsilon \to 0} \int_B E(\mathbf{U} + \mathbf{w}_\varepsilon) \ \mathrm{d}x \geq \int_B E(\mathbf{U}) \ \mathrm{d}x + C \int_B \Big(e - E(\mathbf{U}) \Big)^\alpha \ \mathrm{d}x$$

Admissibility criteria

Entropy admissibility criterion - Second law

$$\partial_t E(\mathbf{U}) + \mathrm{div}_x \mathbf{F}(\mathbf{U}) \leq 0$$

Entropy rate admissibility criterion - Dafermos [1973]

$\mathbf{U}$ maximal with respect to the relation $\succ$

$$\mathbf{U} \succ \mathbf{V}$$

$$\Leftrightarrow$$

$$\mathbf{U}(t,\cdot) = \mathbf{V}(t,\cdot) \text{ for } t \leq \tau$$

$$\int_\Omega E(\mathbf{U}(t,\cdot))\ \mathrm{d}x \leq \int_\Omega E(\mathbf{V})(t,\cdot)\ \mathrm{d}x \text{ for a.a. } t \in (\tau, \tau + \delta)$$

for some $\delta > 0$

Pointwise maximal entropy rate criterion

Maximal dissipation admissibility criterion I

$$\mathbf{U} \succ_{\max} \mathbf{V}$$

$$\Leftrightarrow$$

$$\mathbf{U}(t,\cdot) = \mathbf{V}(t,\cdot) \text{ for } t \leq \tau$$

$$E(\mathbf{U}(t,\cdot)) \leq E(\mathbf{V})(t,\cdot) \text{ for a.a. } t \in (\tau, \tau+\delta)$$

$$\text{for some } \delta > 0$$

$$\mathbf{U} \succ_{\max-\mathrm{sharp}} \mathbf{V}$$

$$\Leftrightarrow$$

$$\mathbf{U}(t,\cdot) = \mathbf{V}(t,\cdot) \text{ for } t \leq \tau$$

$$E(\mathbf{U}(t,\cdot)) < E(\mathbf{V})(t,\cdot) \text{ for a.a. } t \in (\tau, \tau+\delta)$$

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = \boxed{0} \ \text{(inviscid)}$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta \mathbf{u})\Big] - \boxed{\Delta \vartheta} = -\varrho\vartheta \mathrm{div}_x \mathbf{u} \ \text{(heat conductive)}$$

Existence of weak solutions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence [Chiodaroli, F.,Kreml [2013]]

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta\ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0,T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Application of the convex integration method, I

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \Big(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \Big) - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

"Energy"

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}]$$

Application of the convex integration method, II

Construction of solutions

1. Fix ϱ and compute the acoustic potential Ψ

$$-\Delta \Psi = \partial_t \varrho$$

2. Compute $\vartheta = \vartheta[\mathbf{v}]$ for $\mathbf{v} \in L^\infty$

$$\frac{3}{2}\left(\partial_t(\varrho\vartheta) + \mathrm{div}_x \Big(\vartheta(\mathbf{v} + \nabla_x \Psi) \Big) \right) - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

3. Observe that $0 < \vartheta < \overline{\vartheta}$, $\overline{\vartheta}$ independent of $\mathbf{v}$
4. Take

$$e = \chi(t) - \frac{3}{2}\varrho\vartheta[\mathbf{v}]$$

and use a non-local variant of the results od DeLellis and Székelyhidi for the *incompressible* Euler system to find $\mathbf{v}$

Conservative solutions to the Euler-Fourier system

Total energy conservation

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{3}{2} \varrho \vartheta \right) (\tau, \cdot) \ \mathrm{d}x = \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \frac{3}{2} \varrho_0 \vartheta_0 \right) \ \mathrm{d}x$$

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

Conservative weak solutions [Chiodaroli, F., Kreml [2013]]

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many conservative weak solutions in $(0, T)$

Weak formulation revisited

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Entropy production

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma \boxed{\geq} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta^2}$$

Total energy conservation

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{3}{2}\varrho\vartheta\right)(\tau, \cdot)\ \mathrm{d}x = \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{3}{2}\varrho_0\vartheta_0\right)\ \mathrm{d}x$$

Maximal dissipation criterion?

Entropy production rate

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

$$s(\varrho,\vartheta) = -\log\left(\frac{\varrho}{\vartheta^{3/2}}\right)$$

$$\sigma \boxed{\geq} -\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta^2}$$

Maximal dissipation

- Maximize the entropy production rate σ
- Maximize the total entropy $\int_\Omega \varrho s(\varrho,\vartheta)\ \mathrm{d}x$
- Maximize the entropy $\varrho s(\varrho,\vartheta)$