Continuum Mechanics, part 1

Background, notation, tensors

The Concept of Continuum

The molecular nature of the structure of matter is well established.

In many cases, however, the individual molecule is of no concern.

Observed macroscopic behavior is based on assumption that the material is continuously distributed throughout its volume and completely fills the space.

The continuum concept of matter is the fundamental postulate of continuum mechanics.

Adoption of continuum concept means that field quantities as stress and displacements are expressed as piecewise continuous functions of space coordinates and time.

CONTINUUM MECHANICS

- generally, it is a non-linear matter,
- theoretical foundations are known for more than two centuries – Cauchy, Euler, St. Venant, ...,
- the non-linear mechanics develops quickly during the last decades and it is substantially influenced by the availability of high-performance computers and by the progress in numerical and programming methods,

- it was the computer and modern mathematical methods which allowed to solve the difficult theoretical and engineering problems taking into account the material and geometrical non-linearities and transient phenomena,

- still, the most difficult task is the determination of validity range of the used mathematical, physical and computational models.

- The mathematical description of non-linear phenomena is difficult – for the efficient development of formulas it is suitable to use the tensor notation.
- The tensor notation can be considered as a direct hint for algorithmic evaluation of formulas, however, for the practical numerical computation the matrix notation is preferred.

- Note: To a certain extent Maple and Matlab and old Reduce could handle symbolic manipulation in a tensorial notation.

Notation

- That's why we will talk not only about the tensor notation, which is very efficient for deriving the fundamental formulas, but also about the equivalent matrix notation, which is preferable for the computer implementation.

- Besides, we will also mention a so called 'vector' notation, which is currently being used in the engineering theory of strength of material.

Example

Strain tensor in indicial notation is

$$\varepsilon_{ij}$$

Its matrix representation is

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}.$$

Due to the strain tensor symmetry a more compact 'vector' notation is often being employed in engineering, i.e.

$$\{\varepsilon\} = \{\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad \varepsilon_{12} \quad \varepsilon_{23} \quad \varepsilon_{31}\}^{\mathrm{T}}.$$

The engineering strain is

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{21} \\ 2\varepsilon_{21} \end{Bmatrix}$$

The reason for the appearance of a ‘strange’ multiplication factor of 2 will be explained later.

You should carefully distinguish between constants in

$$\sigma_{ij} = C_{ijkl}\ \varepsilon_{kl} \quad \text{and} \quad \{\sigma\} = [C]\{\varepsilon\}.$$

Continuum mechanics in Solids ...

scope of the presentation

Tensors and notation
Kinematics
Finite deformation and strain tensors
Deformation gradient
Displacement gradient
Left Cauchy deformation gradient
Green-Lagrange strain tensor
Almansi (Euler) strain tensor
Infinitesimal (Cauchy) strain tensor
Infinitesimal rotation
Stretch
Polar decomposition

Continuum mechanics in Solids ... cont.

- Rigid body motion
- Motion and flow
- Stress tensors
- Incremental quantities
- Energy principles
- Total and updated Lagrangian approach
- Numerical approaches

Tensors, Notation, Background

Continuum mechanics deals with physical quantities, which are independent of any particular coordinate system.

At the same time these quantities are often specified by referring to an appropriate system of coordinates.

Such quantities are advantageously represented by tensors. The physical laws of continuum mechanics are expressed by tensor equations.

The invariance of tensor quantities under a coordinate transformation is one of principal reasons for the usefulness of tensor calculus in continuum mechanics.

Notation being used is not unified.

Deformation and motion of a considered body could be observed from the configuration

at time 0 to that at time t,
at time t to that at time $t + \Delta t$.

Notation

symbolic	$\mathbf{A}$, $\mathbf{B}$, $\mathbf{c}$, $\vec{x}$
indicial	A_{ij}, B_{ij}, c_i, x_i
matrix	$[A]$, $[B]$, $\{c\}$, $\{x\}$

Tensors, vectors, scalars

General tensors .. transformation in curvilinear systems
Cartesian tensors .. transformation in Cartesian systems

Tensors are classified by the rank or order according to the particular form of the transformation law they obey.

In a three-dimensional space ($n = 3$) the number of components of a tensor is n^N, where N is the rank (order) of that tensor.

Tensors of the order zero are called scalars.

In any coordinate system a scalar is specified by one component. Scalars are physical quantities uniquely specified by magnitude.

Tensors of the order one are called vectors.

In physical space they have three components. Vectors are physical quantities possessing both magnitude and direction.

Scalars ... magnitude only (mass, temperature, energy), will be denoted by Latin or Greek letters in italics as

$$a, \alpha, E$$

Vectors ... magnitude and direction (velocity, acceleration), may be represented by directed line segments and denoted by

$$\mathbf{x}, \left\{x\right\}$$

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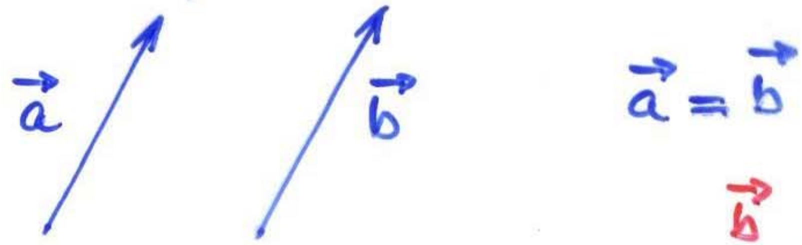
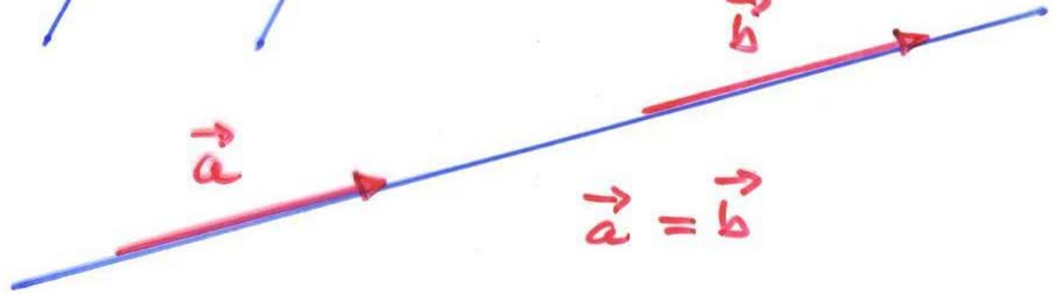
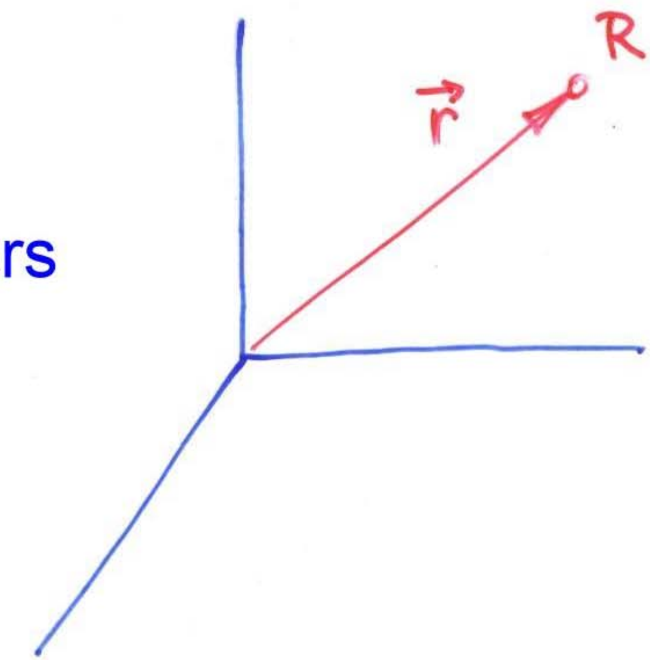
Free vectors



Bounded vectors



Positional (reference) vectors
are completely fixed



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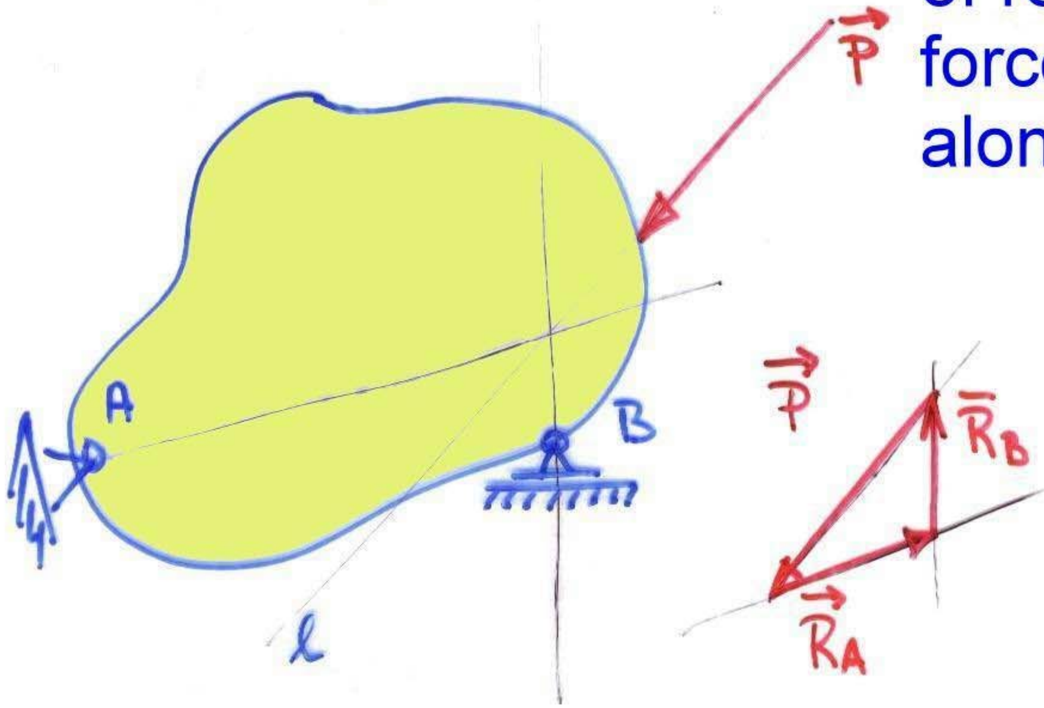
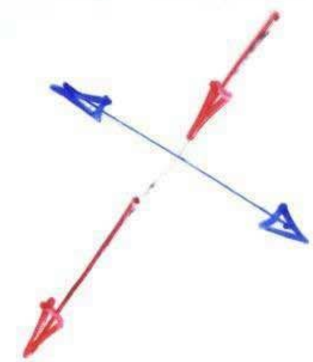
Rigid body mechanics

For the correct calculation of reactions the loading force can be freely moved along the line ℓ



It cannot, however, be shifted lateraly



A model ... no deformation due to an external loading

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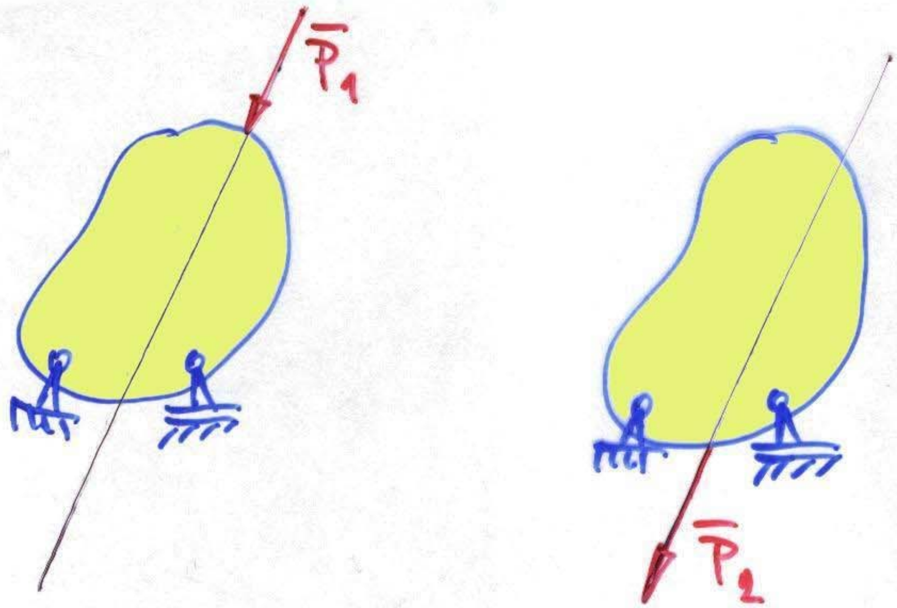
Mechanics of deformable bodies



Even if the reactions are the same, the stress distribution is different

It should be reminded that in linear mechanics the stress is calculated from the undeformed configuration of the body

$$\sigma_{\text{engineering}} = \frac{F_{\text{current}}}{A_{\text{initial}}}$$

A vector may be defined with respect to a particular coordinate system by specifying the components of the vector in that system.

The choice of coordinate system is arbitrary, but in certain situations a particular choice may be advantageous.

The Cartesian rectangular system is represented by mutually perpendicular axes. Any vector may be expressed as a linear combination of three, arbitrary, nonzero, noncomplanar vectors, which are called

base vectors.

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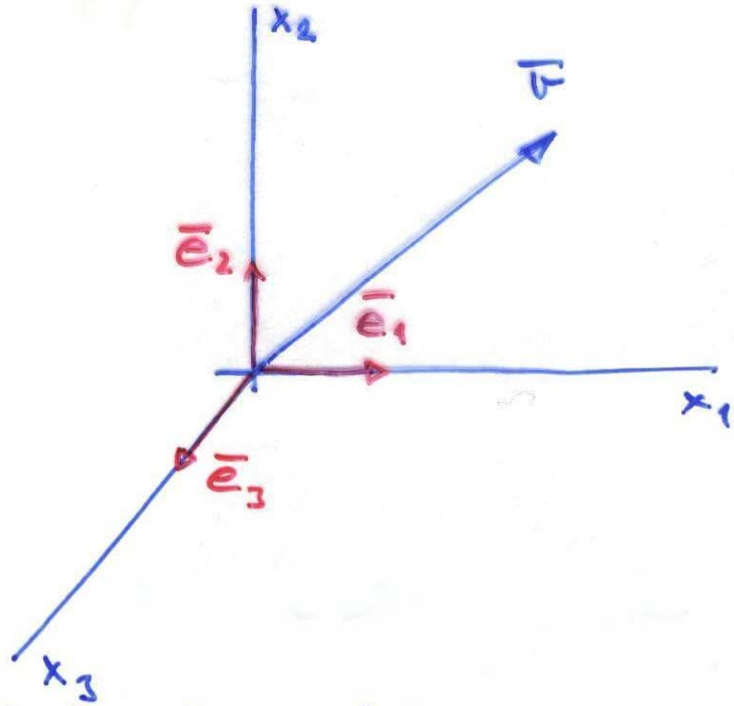
The most frequent choice of base vectors for the rectangular Cartesian system is the set of unit vectors along the coordinate axis {x} that are often denoted {e}



Notation used

$$\bar{v} = v_1\bar{e}_1 + v_2\bar{e}_2 + v_3\bar{e}_3 =$$

$$= \sum_{i=1}^{3} v_i\bar{e}_i = v_i\bar{e}_i$$

Notice the summation sign being dropped.
Einstein or summation convention - dummy index.

Summation rule

When an index appears twice in a term, that index is understood to take on all values of its range, and the resulting terms summed.

$$c = a_i\, b_i = \sum_{i=1}^{3} a_i\, b_i$$

So the repeated indices are often referred to as dummy indices, since their replacement by any other letter, not appearing as a free index, does not change the meaning of the term in which they occur.

Vectors will be denoted in a following way

Symbolic or Gibbs notation $\vec{a}, \bar{a}, \mathbf{a}$

Indicial notation; a component or all of them a_i

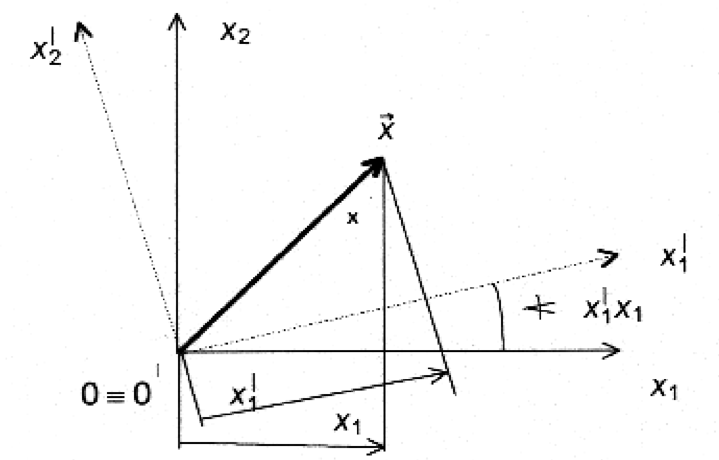
Matrix algebra notation $\{a\}$

Note

Tensor indicial notation does not distinguish between row and column vectors

$$\{a\} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ \vdots \\ a_n \end{Bmatrix} \quad \{a\}^{\mathrm{T}} = \{a_1 \quad a_2 \quad a_3 \quad \dots \quad \dots \quad a_n\}; \; s = \sqrt{\{a\}^{\mathrm{T}}\{a\}} = (a_i a_i)^{\frac{1}{2}}$$

Orthogonal transformation $\mathbf{x}' \leftarrow \mathbf{x}$



Direction cosines

$$a_{ij} = \cos\left(x_i' x_j\right)$$

9 quantities, 6 of them independent

are stored in 3 by 3 matrix

$\mathbf{A} = a_{ij}$

Transformation law is

$x_i' = a_{ij}\, x_j$ or $\{x'\} = [A]\{x\}$ or $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$

for the first order Cartesian tensors.

Inverse transformation

$$\mathbf{x} \leftarrow \mathbf{x}'$$

$$x_i = a_{ji}\ x'_j \qquad \{x\} = [A]^{\mathrm{T}} \{x'\}$$

Combining forward and inverse transformations for an arbitrary vector

$$x'_i = a_{ij}\, x_j \qquad x_j = a_{kj}\ x'_k \qquad \{x'\} = [A]\{x\} \qquad \{x\} = [A]^{\mathrm{T}}\{x'\}$$

$$x'_i = a_{ij}\ a_{kj}\ x'_k \qquad \{x'\} = [A][A]^{\mathrm{T}}\ \{x'\}$$

$$x'_i = \delta_{ik}\ x'_k \qquad \{x'\} = [I]\{x'\}$$

$$x'_i = x'_i \qquad \{x'\} = \{x'\}$$

The coefficient

$$a_{ij}\ a_{kj} \quad \text{or} \quad [A]^{\mathrm{T}}[A]$$

gives the symbol or variable which is equal either to one or to zero according to whether the values *i* and *k* are the same or different. This may be simply expressed by

$$\delta_{ij} \quad \text{or} \quad [I]$$

i.e. by Kronecker delta or unit matrix

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So the Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \phantom{\text{for }} i \neq j \end{cases}$$

thus for direction cosines we can write

$$a_{ij} a_{ik} = \delta_{jk} \qquad [A]^T [A] = [I]$$

or

$$a_{ji} a_{ki} = \delta_{jk} \qquad [A][A]^T = [I]$$

this relation in expanded form consists of nine equations, known as orthogonality conditions. (six are independent)

Coordinate axis rotations and reflection of the axis in a coordinate plane – both lead to orthogonal transformation.

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the Kronecker delta is sometimes called the substitution operator for which

$$\delta_{ik} x_k = x_i$$

since

$$\delta_{ik} x_k = \delta_{i1} x_1 + \delta_{i2} x_2 + \delta_{i3} x_3 = \begin{cases} x_1 & i = 1 \\ x_2 \text{ for } & i = 2 \\ x_3 & i = 3 \end{cases} \rightarrow x_i$$

Second order tensors, definition and properties

Let $\bar{R} \equiv \bar{R}'$ and $\bar{S} \equiv \bar{S}'$ are two vectors expressed in unprimed and primed coordinate systems. Applying the orthogonal transformation we get

cm0014b

$R'_i = a_{ik} R_k$ $\qquad\qquad$ $\{R'\} = [A]\{R\}$

$S'_j = a_{jl} S_l$ $\qquad\qquad$ $\{S'\} = [A]\{S\}$

For all possible products of vector components we can write

$R'_i S'_j = a_{ik} a_{jl} R_k S_l$ $\qquad\qquad$ $\{R'\}\{S'\}^T = [A]\{R\}\{S\}^T[A]^T$

the second-order tensor is defined as

$T'_{ij} = R'_i S'_j$ $\qquad\qquad$ $[T'] = \{R'\}\{S'\}^T$

$T_{kl} = R_k S_l$ $\qquad\qquad$ $[T] = \{R\}\{S\}^T$

Second order tensor transformation

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$$T'_{ij} = a_{ik}\, a_{jl}\, T_{kl} \qquad [T'] = [A][T][A]^T$$

With the help of orthogonality conditions it is easy to invert the previous relation, thereby giving the transformation rule from primed to unprimed components in the form

$$T_{ij} = a_{ki}\, a_{lj}\, T'_{kl} \qquad [T] = [A]^T[T'][A] .$$

indicial notation

matrix algebra notation

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Unrepeated indeces are known as free indices
Number of unrepeated indices is equal to tensorial rank

So for a range of three on both indices i, j
the symbol A_{ij} represents in three-dimensional
space nine components that may be arranged
into the form of 3 by 3 square matrix

A_{ij} – indicial notation

$\mathbf{A}$ – symbolic (bold faced) notation

$[A]$ – matrix algebra notation

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$ – tensor presented explicitly by giving all components arranged in a square array

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the higher-order tensors are defined similarly by means of the transformation law

$$T'_{ijkl} = a_{ir}\, a_{js}\, a_{kt}\, a_{lu}\, T_{rstu}$$

the precise meaning of this tensorial equation can easily be clarified by the following segment of the basic program

Fourth order tensor transformation

$$T'_{ijkl} = a_{ir} a_{js} a_{kt} a_{lu} \, T_{rstu}$$

```
d = 3; DIM T’(d,d,d,d),T(d,d,d,d),a(d,d)
for i = 1 to d
    for j = 1 to d
        for k = 1 to d
            for l = 1 to d
                T’(i,j,k,l) = 0;
                for r = 1 to d
                    for s = 1 to d
                        for t = 1 to d
                            for u = 1 to d
T’(i,j,k,l) = T’(i,j,k,l) + a(i,r)*a(j,s)*a(k,t)*a(l,u)*T(r,s,t,u);
                            next u
                        next t
                    next s
                next r
            next l
        next k
    next j
next i
```

The inverse transformation law is

$$T_{ijkl} = a_{ri}\, a_{sj}\, a_{tk}\, a_{ul}\, T'_{rstu}$$

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Addition and substraction of Cartesian tensors:

$$T_{ijk} = A_{ijk} \pm B_{ijk}$$

Multiplication by a scalar

$$b_i = \lambda a_i \qquad \{b\} = \lambda \{a\}$$

$$B_{ij} = \alpha A_{ij} \qquad [B] = \alpha [A]$$

cm0017b

Contraction of a tensor

with respect to two free indices is the process of assigning to both indices the same letter subscript — changing thereby these indices to dummy indices.
Contraction produces a tensor having an order two less than the original.

$T_{ij} \rightarrow T_{ii} \Rightarrow s = T_{ii}$

```
s := 0;
for i:=1 to 3 do s:=s + t[i,i];
```

$R_{ijk} \rightarrow R_{ijj} \Rightarrow v_i = R_{ijj}$

```
for i:=1 to 3 do
  begin
    v[i]:=0;
    for j:=1 to 3 do
      v[i]:=v[i]+t[i,j,j];
  end;
```

Tensor multiplication

A) Outer product of two tensors of arbitrary order is the tensor whose components are formed by multiplying each component of one tensor by every component of the other

tensors of the first order (dyadic product)

NOTATION

- indicial $c_{ij} = a_i b_j$
- symbolic $\mathbb{C} = \vec{a} \otimes \vec{b}$
- matrix algebra $[c] = \{a\}\{b\}^T$

the exact meaning is clarified by

```
for i:=1 to n do
    for j:= 1 to n do c[i,j]:= a[i] * b[j];
```

cm0018b

tensors of the second order (tensor product)

NOTATION

- indicial $\qquad C_{ijkl} = A_{ij} B_{kl}$
- symbolic $\qquad \mathbb{C} = \mathbb{A} \otimes \mathbb{B}$
- matrix $\qquad$ —

the exact meaning is clarified by

```
for i:=1 to n do
   for j:=1 to n do
      for k:= 1 to n do
         for l:=1 to n do c[i,j,k,l]:=a[i,j] * b[k,l];
```

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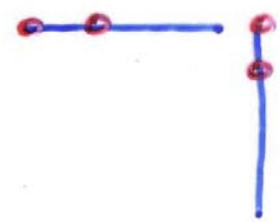
Tensor multiplication

1) Inner products

Tensors of the first order (scalar or dot product)

NOTATION

- indicial $\quad s = a_i\, b_i$
- symbolic $\quad s = \vec{a} \cdot \vec{b}$
- matrix algebra $\quad s = \{a\}^T \{b\}$



Meaning

```
s := 0;
for i := 1 to n do s := s + a[i] * b[i];
```

An interlude (vector or cross product)

NOTATION

– indicial

$$c_i = \varepsilon_{ijk}\, a_j b_k ,$$

where

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{even permutation } 1,2,3 - 2,3,1 - 3,1,2 \\ 0 & \text{repeated indizes as } 1,1,2 - \text{etc.} \\ -1 & \text{odd permutation } 3,2,1 - 2,1,3 - 1,3,2 \end{cases}$$

is so called permutating, alternating or Levi-Civit symbol

– symbolic

$$\bar{c} = \bar{a} \times \bar{b} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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Inner products – cont.

Tensors of the second order (tensor dot product or matrix product)

NOTATION

- indicial $\quad c_{ij} = a_{ik}\, b_{kj}$
- symbolic $\quad \mathbb{C} = \mathbb{A} \cdot \mathbb{B}$
- matrix algebra $\quad [C] = [A][B]$

Meaning

```
for i:=1 to n do
  for j:=1 to n do
    begin
      c[i,j]:=0;
      for k:=1 to n do c[i,j]:=c[i,j]+a[i,k]*b[k,j];
    end;
```

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Other possibilities

$d_{ij} = a_{ki}\, b_{kj}$ $\qquad [D] = [A]^T [B]$

$e_{ij} = a_{ik}\, b_{jk}$ $\qquad [E] = [A][B]^T$

$f_{ij} = a_{ki}\, b_{jk} = b_{jk}\, a_{ki}$ $\qquad [F]^T = [B][A]$

$b_i = a_{ij}\, c_j$ $\qquad \{b\} = [A]\{c\}$

$d_i = a_{ji}\, c_j$ $\qquad \{d\} = [A]^T\{c\}$

$e_j = c_k\, a_{kj}$ $\qquad \{e\}^T = \{c\}^T[A]$

cannot be distinguished by tensor calc

Inner products – cont.

Double dot product of two second order tensors

NOTATION

- indicial $\qquad s = A_{ij} B_{ij}$
- symbolic $\qquad s = \mathbb{A} : \mathbb{B}$
- matrix $\qquad \longrightarrow s = [A] : [B]$

Meaning

$$
\begin{aligned}
s = \; & A_{11} B_{11} + A_{12} B_{12} + A_{13} B_{13} + \\
& A_{21} B_{21} + A_{22} B_{22} + A_{23} B_{23} + \\
& A_{31} B_{31} + A_{32} B_{32} + A_{33} B_{33}
\end{aligned}
$$

PASCAL

```
s := 0 ;
for i := 1 to n do
    for j := 1 to n do  s := s + a[i,j] * b[i,j] ;
```

An example

$$s = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$$

strain

stress

energy or work

$$s = \{\sigma\}^T\{\varepsilon\}\,\frac{1}{2} \qquad s = [\sigma]:[\varepsilon]\,\frac{1}{2}$$

cm0022a

A NOTE ON VECTOR AND TENSOR INVARIANCE

Let $\{e\}$ and $\{e'\}$ are unit vectors in two coordinate systems $\{x\}$ and $\{x'\}$ respectively. The coordinate systems are related by orthogonal transformation defined by direction cosines a_{ij} so that

$$x_i = a_{ji}\, x_j' \qquad \{x\} = [A]^T \{x'\}$$

$$e_i = a_{ji}\, e_j' \qquad \{e\} = [A]^T \{e'\}$$

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If $\vec{z}$ is an arbitrary vector, then for its components we can write

$$z_i = a_{ji} \, z'_j \qquad\qquad \{z\} = [A]^T \{z'\}$$

Which gives a conclusion which is almost trivial

$$\vec{z} = z_i \vec{e}_i = z'_i \vec{e}'_i = \vec{z}' \quad \text{but} \quad \{z\} \neq \{z'\}.$$

The same vector does not have the same components in different coordinate systems.

There is another form which is used as a notation for a second order tensor.

$$\mathbb{B} = B_{ij} \left(\bar{e}_i \circledast \bar{e}_j \right) = B_{11}\, e_1 \circledast e_1 + B_{12}\, \bar{e}_1 \circledast \bar{e}_2 + B_{13}\, \bar{e}_1 \circledast \bar{e}_3 +$$

$$+ B_{21}\, \bar{e}_2 \circledast \bar{e}_1 + B_{22}\, \bar{e}_2 \circledast \bar{e}_2 + B_{23}\, \bar{e}_2 \circledast \bar{e}_3 +$$

$$+ B_{31}\, \bar{e}_3 \circledast \bar{e}_1 + B_{32}\, \bar{e}_2 \circledast \bar{e}_3 + B_{33}\, \bar{e}_3 \circledast \bar{e}_3$$

This notation expresses the fact that tensor components can be specified only after a coordinate system has been introduced. It carries information about the coordinate system.
The tensor $\mathbb{B}$ however is a quantity which is independent of the chosen system of coordinates

So

$$\mathbb{B} = B_{ij}\, \bar{e}_i \circledast \bar{e}_j = \underbrace{a_{pi}\, B_{ij}\, a_{qj}}_{B'_{pq}}\, \bar{e}'_p \circledast \bar{e}'_q =$$

$a_{qj}\, \bar{e}'_q$

$a_{pi}\, \bar{e}'_p$

$$= B'_{pq}\, \bar{e}'_p \circledast \bar{e}'_q = \mathbb{B}'.$$

So again we can state that

$$\mathbb{B} = \mathbb{B}' \qquad\qquad [B_{ij}] \neq [B'_{ij}]$$

THE CONTINUUM CONCEPT

The molecular nature of the structure of matter is well established.

In many cases, however, the individual molecule is of no concern

Observed macroscopic behaviour is based on assumption that the material is continuously distributed throughout its volume and completely fill the space it occupies.

This continuum concept of matter is the fundamental postulate of continuum mechanics.

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Within the limitation for which the continuum assumption is valid this concept provides a framework for studying the behaviour of solids, liquids and gases alike.

Addoption of continuum viewpoint means that field quantities as stress and displacement are expressed as piecewice continuous functions of space coordinates and time.

homogeneity – identical properties at all points

isotropy – with respect to some property if that property is the same in all directions at a point.

Terminology

Měli bychom být schopni jasně rozlišovat mezi veličinami stavu (deformace, napjatost) a mírami, které tyto veličiny kvantifikují (posuny a přetvoření jsou mírou deformace, napětí je mírou napjatosti)

Stav

Č	A	R	F
napjatost	state of stress	напряжённое состояние	état de tension
deformace	deformation	деформация	déformation

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Míra stavu

Č	A	R	F
napětí σ_{ij}	stress	напряжение	tension
1) posun u_i	displacement	перемещение	déplacement
2) přetvoření ε_{ij}	strain	деформация	déformation

neurčí ani ČSN 01 1302 z 1976

ε_{ij} ... poměrné prodloužení či zkrácení

Původní vydání definovalo

.... poměrná deformace