Mathematical properties of certain models of two-phase compressible fluids

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Model by Lowengrub-Truskinovski [1998]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \boxed{\mathrm{div}_x \mathbb{S}(c, \nabla_x \mathbf{u})}$$

$$+\mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard system

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$

Model by Anderson-McFadden-Wheeler [1998]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, c) = \mathrm{div}_x \mathbb{S}(c, \nabla_x \mathbf{u})$$

$$+\mathrm{div}_x \left(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard system

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$

Constitutive equations

General constitutive equation

$$p(\varrho, c) = \varrho^2 \frac{\partial f(\varrho, c)}{\partial \varrho}, \ \mu(\varrho, c) = \frac{\partial f(\varrho, c)}{\partial c}$$

Free energy

$$f(\varrho, c) = f_{\mathrm{e}}(\varrho) + f_{\mathrm{mix}}(\varrho, c), \ f_{\mathrm{mix}}(\varrho, c) = H(c)\log(\varrho) + G(c)$$

Pressure

$$p(\varrho, c) = p_{\mathrm{e}}(\varrho) + \varrho H(c), \ f_{\mathrm{e}}(\varrho) = \int_1^{\varrho} \frac{p_{\mathrm{e}}(z)}{z^2} \, \mathrm{d}z$$

More hypotheses

Growth hypotheses

$$p_{\mathrm{e}}(0) = 0, \;\; p_1 \varrho^{\gamma-1} - p_2 \leq p_{\mathrm{e}}'(\varrho) \leq p_3 \varrho^{\gamma-1} + p_4$$

$$-H_1 \leq H'(c), H(c) \leq H_2, \;\; , \;\; G_1 c - G_2 \leq G'(c) \leq G_3 c + G_4$$

Viscosity - Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu(c) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(c) \mathrm{div}_x \mathbf{u} \mathbb{I},$$

Periodic boundary conditions

$$\Omega = \left([0,1] \Big|_{\{0,1\}} \right)^3 .$$

Global existence - viscous case

Weak solutions - H.Abels, EF, Indiana Univ. Math. J. 2008

Let $\gamma > \frac{3}{2}$. Then the Anderson-McFadden-Wheeler model admits a global-in-time weak solution for any finite energy initial data.

Global existence - inviscid case

Weak solutions, EF 2014

Let

$$f(\varrho, c) = H(c) + \log(\varrho)\left(\alpha_1 \frac{1-c}{2} + \alpha_2 \frac{1+c}{2}\right),$$

$$H \in C^2(R),\ |H''(c)| \leq \overline{H} \text{ for all } c \in R^1$$

and let the initial data be given such that

$$\varrho(0,\cdot) = \varrho_0 \in C^3(\Omega),\ \inf_{\Omega} \varrho_0 > 0,$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \in C^3(\Omega; R^3),\ c(0,\cdot) = c_0 \in C^2(\Omega).$$

Then the Lowengrub-Truskinovski *inviscid* model admits infinitely many global-in-time weak solutions.

Energy

Total energy functional

$$\mathcal{E}_{\rm tot}(\varrho, \mathbf{u}, c) \equiv \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}\varrho|\nabla_x c|^2 + \varrho f_0(\varrho, c) \right) \ {\rm d}x$$

Initial energy jump

$$\liminf_{t\to 0+} \mathcal{E}_{\rm tot}(\varrho, \mathbf{u}, c)(t) > \mathcal{E}_{\rm tot}(\varrho_0, \mathbf{u}_0, c_0).$$

Dissipative solutions

Global-in-time admissible weak solutions, EF 2015

For any given $T > 0$ and the initial data

$$\varrho(0,\cdot) = \varrho_0 \in C^3(\Omega),\ \inf_{\Omega} \varrho_0 > 0,$$

there exists a set of initial concentrations $c_0 \in C^3(\Omega)$ dense in $C(\Omega)$ such that for any $[\varrho_0, c_0]$ there exists $\mathbf{u}_0 \in L^\infty(\Omega; R^3)$ such that the inviscid Lowengrub-Truskinovski system with

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0$$

admits infinitely many weak solutions in $(0,T) \times \Omega$ satisfying the energy inequality

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}\varrho|\nabla_x c|^2 + \varrho f_0(\varrho, c) \right)(t,\cdot)\ \mathrm{d}x$$

$$\leq \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{2}\varrho_0|\nabla_x c_0|^2 + \varrho_0 f_0(\varrho_0, c_0) \right)\ \mathrm{d}x$$

Convex integration ansatz

Helmholtz decomposition

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Phi, \ \mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \Phi \ \mathrm{d}x = 0$$

Modified system

$$\partial_t \varrho + \Delta \Phi = 0,$$

$$\partial_t (\mathbf{v} + \nabla_x \Phi) + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} \right) + \nabla_x p_0(\varrho, c)$$

$$= \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right),$$

$$\varrho \partial_t c + \mathbf{v} \cdot \nabla_x c + \nabla_x \Phi \cdot \nabla_x c$$

$$= -\Delta \left(\frac{1}{\varrho} \mathrm{div}_x (\varrho \nabla_x c) \right) + \Delta \left(H'(c) + \frac{\alpha_2 - \alpha_1}{2} \log(\varrho) \right).$$

Cahn-Hilliard system

Solution mapping

$$\mathbf{v} \mapsto c[\mathbf{v}], \ \mathbf{v} \in L^\infty((0,T)\times\Omega; R^3) \cap C^1((0,T)\times\Omega; R^3), \ \mathrm{div}_x \mathbf{v} = 0$$

Maximal regularity - Denk, Hieber, Pruess [2007]

$$\partial_t c + \frac{1}{\varrho}\Delta\left(\frac{1}{\varrho}\mathrm{div}_x\left(\varrho\nabla_x c\right)\right) = h,$$

Reformulation via convex integration theory

Abstract Euler system

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{3} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} \mathbb{I} \right)$$

$$= \mathrm{div}_x \left(\varrho \left(\nabla_x c[\mathbf{v}] \otimes \nabla_x c[\mathbf{v}] - \frac{1}{3} |\nabla_x c[\mathbf{v}]|^2 \right) \right)$$

Energy

$$\overline{E}[\mathbf{v}] = \frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho}$$

$$= \Lambda(t) - \frac{3}{2} \left(\frac{1}{6} |\nabla_x c[\mathbf{v}]|^2 + p_0(\varrho, c[\mathbf{v}]) + \partial_t \nabla_x \Phi \right)$$