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The Convexity of $\mathbf{C} \mapsto \mathbf{h}(\det \mathbf{C})$

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A simple proof is given of the characterization of the convexity of the function $\mathbf{C} \mapsto \mathbf{h}(\det \mathbf{C})$ on positive definite symmetric matrices due to Lehmich et al. (2014). The proof uses the classical characterization of convex functions depending on a symmetric matrix through its eigenvalues due to Davis (1957).

Introduction and proof

Motivated by the study of the neo-Hookean model energy function, Lehmich et al. (2014) examine the convexity of the stored energy function expressed as a function of the Cauchy–Green deformation tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. Specifically, they prove the following result.

Theorem 1 *Let $n \geq 2$ and let $h : (0, \infty) \to \mathrm{R}$ be a twice continuously differentiable function. Then the function*

$$\mathbf{C} \mapsto h(\det \mathbf{C}) \tag{1}$$

is convex on the set SymP of positive definite symmetric n by n matrices if and only if

$$nsh''(s) + (n-1)h'(s) \geq 0 \quad \textit{and} \quad h'(s) \leq 0 \quad \textit{for every} \quad s > 0. \tag{2}$$

This was established in Lehmich et al. (2014) by a prevalently computational proof. Another proof was given recently by Spector (2015). Both the two proofs are rather involved compared with the simplicity of the context and result. Here I shall give a short proof based on the following result of (Davis, 1957, Corollary 1):

Theorem 2 *Let $\phi : \Delta \to \mathrm{R}$ be a symmetric (under permutations of the components of $x \in \Delta$) function on an open symmetric domain $\Delta \subset \mathrm{R}^n$. Let $D = \{\mathbf{C} : (\lambda_1, \ldots, \lambda_n) \in \Delta\}$ and let $f : D \to \mathrm{R}$ be defined by $f(\mathbf{C}) = \phi(\lambda_1, \ldots, \lambda_n)$, $\mathbf{C} \in D$ where $(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $\mathbf{C}$ respecting the multiplicities but otherwise arranged in arbitrary order. Then D is convex if and only if Δ is convex and f is convex if and only if ϕ is convex.*

It is well known that the class of functions f admitting a representation in terms of ϕ as above is exactly the class of isotropic functions, i.e., those satisfying

$$f(\mathbf{Q}\mathbf{C}\mathbf{Q}^{\mathrm{T}}) = f(\mathbf{C})$$

for all $\mathbf{C} \in D$ and all orthogonal matrices $\mathbf{Q}$ with $\det \mathbf{Q} = 1$.

Proof of Theorem 1 The function (1) is convex on SymP if and only if the function $\phi(\lambda_1, \ldots, \lambda_n) = h(\lambda_1 \cdots \lambda_n)$ is convex on $(0, \infty)^n$. If $(\lambda_1, \ldots, \lambda_n) \in (0, \infty)^n$, $s := (\lambda_1 \cdots \lambda_n)$, and $x = (x_1, \ldots, x_n) \in \mathrm{R}^n$ then the hessian of ϕ is found to be

$$H(x, x) := \sum_{i,j=1}^{n} \mathrm{D}^2_{ij}\phi(\lambda_1, \ldots, \lambda_n)x_i x_j = (s^2h''(s) + sh'(s))\alpha^2 - sh'(s)\beta$$

where

$$\alpha = y_1 + \ldots + y_n, \quad \beta = y_1^2 + \ldots + y_n^2, \quad y_i = x_i/\lambda_i.$$

Let us show that $H(x, x) \geq 0$ for every $x \in \mathrm{R}^n$ if and only if (2) hold at s. To see the necessity, take $y = (1, \ldots, 1)$ and $y = (1, -1, 0, \ldots, 0)$, respectively. To prove the sufficiency, note that the numbers α, β satisfy $\alpha^2/n \leq \beta$. Indeed, using $y_i y_j \leq \frac{1}{2}(y_i^2 + y_j^2)$, we find

$$\alpha^2 = \sum_{i,\, j=1}^{n} y_i y_j \leq \sum_{i,\, j=1}^{n} \frac{1}{2}(y_i^2 + y_j^2) = n\beta.$$

Thus if (2) hold, we have

$$H(x, x) = (s^2 h''(s) + sh'(s))\alpha^2 - sh'(s)\beta \geq (s^2 h''(s) + sh'(s))\alpha^2 - sh'(s)\alpha^2/n \geq 0.$$

This completes the proof.

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