Uniqueness of rarefaction waves in compressible Euler systems

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Setting of the problem

Let us first consider the compressible isentropic Euler system in the whole 2D space

$$
\left\{
\begin{array}{rcl}
\partial_t \rho + \operatorname{div}_x(\rho v) & = & 0 \\
\partial_t (\rho v) + \operatorname{div}_x (\rho v \otimes v) + \nabla_x [p(\rho)] & = & 0 \\
\rho(\cdot, 0) & = & \rho^0 \\
v(\cdot, 0) & = & v^0 \, .
\end{array}
\right.
\tag{1}
$$

The pressure $p(\rho)$ is given. Let $\varepsilon(\rho)$ be such that $p(\rho) = \rho^2 \varepsilon'(\rho)$.

Entropy inequality:

$$
\partial_t \left(\rho \varepsilon(\rho) + \frac{\rho |v|^2}{2} \right) + \operatorname{div}_x \left(\left(\rho \varepsilon(\rho) + \frac{\rho |v|^2}{2} + p(\rho) \right) v \right) \; \leq \; 0
\tag{2}
$$

Riemann problem

Denote $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \tag{3}$$

where $\rho_\pm, v_\pm$ are constants.
In particular the initial data are "1D" and there is a classical theory about self-similar solutions to the Riemann problem in 1D (they are unique in the class of BV functions).
In the case of system (1), the initial singularity can resolve to at most 3 structures (rarefaction wave, admissible shock or contact discontinuity) connected by constant states.

First observation

If $v_{-1} = v_{+1}$, then any self-similar solution to (1), (3) has to satisfy $v_1(t,x) = v_{-1} = v_{+1}$ and in particular there is no contact discontinuity in the self-similar solution.
The initial singularity then resolves into at most 2 structures (rarefaction waves or admissible shocks) connected by constant states.

Classification of self-similar solutions I

1) If

$$v_{+2} - v_{-2} \geq \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and a 3−rarefaction wave. The intermediate state is vacuum, i.e. $\rho_m = 0$.

2) If

$$\left| \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau \right| < v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and a 3−rarefaction wave. The intermediate state has $\rho_m > 0$.

Classification of self-similar solutions II

3) If $\rho_- > \rho_+$ and

$$-\sqrt{\frac{(\rho_- - \rho_+)(p(\rho_-) - p(\rho_+))}{\rho_-\rho_+}} < v_{+2} - v_{-2} < \int_{\rho_+}^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of a 1−rarefaction wave and an admissible 3−shock.

4) If $\rho_- < \rho_+$ and

$$-\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}} < v_{+2} - v_{-2} < \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau} \mathrm{d}\tau,$$

then the self-similar solution consists of an admissible 1−shock and a 3−rarefaction wave.

5) If

$$v_{+2} - v_{-2} < -\sqrt{\frac{(\rho_+ - \rho_-)(p(\rho_+) - p(\rho_-))}{\rho_+\rho_-}}$$

then the self-similar solution consists of an admissible $1-$shock and an admissible $3-$shock.

Nonuniqueness results

Theorem 1 (Chiodaroli, De Lellis, K.)

There exist Riemann initial data of the case 3) for which there exist infinitely many admissible weak solutions of the isentropic Euler system. Moreover such Riemann data are generated by a compression wave (backwards rarefaction wave), in particular this implies nonuniqueness for Lipschitz initial data.

Theorem 2 (Chiodaroli, K.)

For any Riemann initial data of the case 5) there exist infinitely many admissible weak solutions of the isentropic Euler system.

Specification of the problem I

First we define the appropriate class of weak solutions. Consider

$$\Omega = \mathcal{T}^1 \times (-a, a),$$

with $a > 0$ sufficiently large and $\mathcal{T}^1$ is a 1D torus. We will consider weak solutions periodic in x_1 and having the same boundary fluxes on $x_2 = \pm a$ as has the self-similar solution.
We consider Riemann data satisfying $v_{\pm 1} = 0$.

Specification of the problem II

More specifically we work with weak solutions satisfying:

$$\rho v_2(t, x_1, -a) = \rho_- v_{-2}, \;\; \rho v_2(t, x_1, a) = \rho_+ v_{+2};$$

$$\left(\rho v_j v_2 + p(\rho)\right)(t, x_1, -a) = \left(\rho_- v_{-j} v_{-2} + p(\rho_-)\right)$$

$$\left(\frac{1}{2}\rho|v|^2 + \rho\varepsilon(\rho) + p(\rho)\right) v_2(t, x_1, -a) =$$
$$\left(\frac{1}{2}\rho_-|v_-|^2 + \rho_-\varepsilon(\rho_-) + p(\rho_-)\right) v_{-2}$$

and similarly for $x_2 = a$.

Specification of the problem III

This means in particular that in the weak formulation of the Euler system appear additional boundary integrals on $x_2 = \pm a$, for example the equation of continuity in the weak formulation looks as follows:

$$
\begin{aligned}
&\int_\Omega [\rho(\tau, x)\varphi(\tau, x) - \rho_0(x)\varphi(0, x)]\,\mathrm{d}x \\
&+ \int_0^\tau \int_{\mathcal{T}^1} \rho_+ v_{+2}\varphi(t, x_1, a)\ \mathrm{d}x_1\mathrm{d}t \\
&- \int_0^\tau \int_{\mathcal{T}^1} \rho_- v_{-2}\varphi(t, x_1, -a)\ \mathrm{d}x_2\mathrm{d}t \\
&= \int_0^\tau \int_\Omega [\rho(t, x)\partial_t\varphi(t, x) + \rho v(t, x) \cdot \nabla\varphi(t, x)]\,\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Main Theorem in the isentropic case

Theorem 3 (Feireisl, K.)

Let $p(\rho) = \rho^\gamma$, $\gamma > 1$. *Let* $\tilde{\rho}(t,x) = R(x_2/t)$, $\tilde{v}(t,x) = (0, V(x_2/t))$ *be the self-similar solution to the Riemann problem consisting of rarefaction waves (locally Lipschitz for* $t > 0$*) and such that*

$$\operatorname{ess\,inf}_{(0,t)\times\Omega} \tilde{\rho} > 0. \tag{4}$$

Let (ρ, v) *be a bounded admissible weak solution such that*

$$\rho \geq 0 \text{ a.a. in } (0,T)\times\Omega.$$

Then

$$\rho \equiv \tilde{\rho}, \;\; v \equiv \tilde{v} \text{ in } (0,T)\times\Omega.$$

Note that according to our earlier study the self-similar solution to the Riemann problem consists only of rarefaction waves and satisfies (4) if and only if the initial Riemann data satisfy

$$\left|\int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau}\mathrm{d}\tau\right| \leq v_{+2} - v_{-2} < \int_0^{\rho_-} \frac{\sqrt{p'(\tau)}}{\tau}\mathrm{d}\tau + \int_0^{\rho_+} \frac{\sqrt{p'(\tau)}}{\tau}\mathrm{d}\tau.$$

Relative entropy inequality

The proof is based on the relative entropy inequality. Define the relative entropy functional

$$\mathcal{E}\left(\rho, v \middle| r, V\right) = \frac{1}{2}\rho|v - V|^2 + \left(H(\rho) - H'(r)(\rho - r) - H(r)\right),$$

where $H(s) = s\varepsilon(s)$.
Concept of relative entropies goes back to DiPerna and Dafermos. Similarly as in papers by Feireisl, Novotný and others in the case of Navier-Stokes equations we first prove that any bounded admissible weak solution satisfies the relative entropy inequality with any couple of functions (r, V) such that

$$r \in C^1([0,T] \times \overline{\Omega}),\ \ V \in C^1([0,T] \times \overline{\Omega}),\ \ r > 0.$$

Relative entropy inequality II

$$
\begin{aligned}
&\int_\Omega \mathcal{E}\left(\rho, v \middle| r, V\right)(\tau, x)\mathrm{d}x - \int_\Omega \mathcal{E}\left(\rho_0, v_0 \middle| r(0,x), V(0,x)\right)\mathrm{d}x \\
&+ \text{boundary terms} \leq \\
&\int_0^\tau \int_\Omega \left[\rho\left(\partial_t V + v \cdot \nabla V\right) \cdot (V - v) + \left(p(r) - p(\rho)\right) \operatorname{div} V\right] \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[(r - \rho)\partial_t H'(r) + (rV - \rho v) \cdot \nabla H'(r)\right](t, x)\mathrm{d}x\mathrm{d}t
\end{aligned}
$$

Relative entropy inequality III

Observe that the rarefaction wave solution $(\tilde{\rho}, \tilde{v})$ may be taken as the test couple (r, V) in the relative entropy inequality as

- $\rho, \tilde{\rho}, v, \tilde{V}$ bounded,
- $\partial_t \tilde{\rho}, \partial_t \tilde{v}_2, \partial_{x_2} \tilde{\rho}, \partial_{x_2} \tilde{v}_2 \in L^\infty(0, T; L^1(\Omega))$

and such step thus can be justified by a density argument and Lebesgue dominated convergence theorem.
Therefore the initial term and the boundary terms in the relative entropy inequality vanish.

Relative entropy inequality IV

Thus we get

$$\int_\Omega \mathcal{E}\left(\rho, v \middle| \tilde{\rho}, \tilde{v}\right)(\tau, x)\mathrm{d}x \leq$$

$$\int_0^\tau \int_\Omega \left[\rho\left(\partial_t \tilde{v}_2 + v_2 \partial_{x_2} \tilde{v}_2\right)\left(\tilde{v}_2 - v_2\right) + \Big(p(\tilde{\rho}) - p(\rho)\Big)\partial_{x_2}\tilde{v}_2\right](t, x)\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \Big[(\tilde{\rho} - \rho)\partial_t H'(\tilde{\rho}) + (\tilde{\rho}\tilde{v}_2 - \rho v_2)\partial_{x_2} H'(\tilde{\rho})\Big](t, x)\mathrm{d}x\mathrm{d}t$$

Some calculations

We rewrite the terms as follows. First:

$$
\begin{aligned}
&\rho\left(\partial_t \tilde{v}_2 + v_2 \partial_{x_2} \tilde{v}_2\right)\left(\tilde{v}_2 - v_2\right) = \\
&\rho(\partial_t \tilde{v}_2 + \tilde{v}_2 \partial_{x_2} \tilde{v}_2)(\tilde{v}_2 - v_2) - \rho \partial_{x_2} \tilde{v}_2 (\tilde{v}_2 - v_2)^2 = \\
&-(\rho/\tilde{\rho})\partial_{x_2} p(\tilde{\rho})(\tilde{v}_2 - v_2) - \rho \partial_{x_2} \tilde{v}_2 (\tilde{v}_2 - v_2)^2.
\end{aligned}
$$

Next:

$$
\begin{aligned}
&\Big(p(\tilde{\rho}) - p(\rho)\Big)\partial_{x_2} \tilde{v}_2 = \\
&-\Big(p(\rho) - p'(\tilde{\rho})(\rho - \tilde{\rho}) - p(\tilde{\rho})\Big)\partial_{x_2} \tilde{v}_2 - p'(\tilde{\rho})(\rho - \tilde{\rho})\partial_{x_2} \tilde{v}_2
\end{aligned}
$$

Finally just using the property $\partial_z H'(\tilde{\rho}) = (p'(\tilde{\rho})/\tilde{\rho})\partial_z \tilde{\rho}$ we have

$$
\begin{aligned}
&(\tilde{\rho} - \rho)\partial_t H'(\tilde{\rho}) + (\tilde{\rho}\tilde{v}_2 - \rho v_2)\partial_{x_2} H'(\tilde{\rho}) = \\
&\frac{\tilde{\rho} - \rho}{\tilde{\rho}} p'(\tilde{\rho})\partial_t \tilde{\rho} + \frac{\tilde{\rho}\tilde{v}_2 - \rho v_2}{\tilde{\rho}} p'(\tilde{\rho})\partial_{x_2} \tilde{\rho}
\end{aligned}
$$

Final step

Summing up all the terms and using again the fact that $\tilde{\rho}, \tilde{v}_2$ solve the continuity equation we end up with

$$\int_\Omega \mathcal{E}\left(\rho, v \middle| \tilde{\rho}, \tilde{v}\right)(\tau, x)\mathrm{d}x \leq$$
$$-\int_0^\tau \int_\Omega \left[\rho\left(\tilde{v}_2 - v_2\right)^2 + \Big(p(\rho) - p'(\tilde{\rho})(\rho - \tilde{\rho}) - p(\tilde{\rho})\Big)\right] \partial_{x_2}\tilde{v}_2(t, x)\mathrm{d}x\mathrm{d}t$$

Since $p(\rho)$ is convex the theorem follows from the fact that $\partial_{x_2}\tilde{v}_2(t, x) \geq 0$ which is a consequence of the classical theory of the self-similar solutions in the case of rarefaction waves.

Full Euler system

Now we consider the full compressible Euler system in the whole 2D space

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x(\rho\theta) \;=\; 0 \\ \partial_t(\frac{1}{2}\rho\,|v|^2 + c_v\rho\theta) + \operatorname{div}_x((\frac{1}{2}\rho\,|v|^2 + c_v\rho\theta + \rho\theta)v) \;=\; 0 \\ \rho(\cdot,0) \;=\; \rho^0 \\ v(\cdot,0) \;=\; v^0 \\ \theta(\cdot,0) \;=\; \theta^0\,. \end{cases} \tag{5}$$

Here θ is the temperature of the gas and $c_v > 0$ is constant called specific heat at constant volume

Entropy inequality

The associated entropy inequality to the system reads as follows

$$\partial_t(\rho s) + \mathrm{div}_x(\rho s v) \ \geq \ 0,$$

where

$$s(\rho, \theta) = \log\left(\frac{\theta^{c_v}}{\rho}\right)$$

Domain and weak solutions

Similarly as in the isentropic case we consider the domain

$$\Omega = \mathcal{T}^1 \times \mathbb{R}^1, \text{ where } \mathcal{T}^1 \equiv [0,1]_{\{0,1\}} \text{ is the “flat” sphere,}$$

Again we consider solutions periodic with respect to x_1.
In the variable x_2 we will prescribe far field conditions in order to prove uniqueness of solutions to the Riemann problem.

Riemann Problem

The Riemann initial data are as follows

$$(\rho^0(x), v^0(x), \theta^0(x)) := \begin{cases} (\rho_-, v_-, \theta_-) & \text{if } x_2 < 0 \\ \\ (\rho_+, v_+, \theta_+) & \text{if } x_2 > 0, \end{cases} \tag{6}$$

and again we avoid the contact discontinuity formed by the higher dimension by assuming

$$v_\pm = (0, v_{\pm 2}).$$

Of course $\rho_\pm, \theta_\pm$ and $v_{\pm 2}$ are constants.

General form of self-similar solutions

The Riemann problem admits a solution

$$\rho(t,x) = R(t,x_2) = R(\xi),\ \theta(t,x) = \Theta(t,x_2) = \Theta(\xi),$$
$$v(t,x) = (0, V(t,x_2)) = (0, V(\xi))$$

depending solely on the self-similar variable $\xi = \frac{x_2}{t}$. Such a solution is unique in the class of BV solutions of the 1-D problem. The initial singularity resolves to at most 3 structures connected by constant states, where the first and the last structures are always either admissible shocks or rarefaction waves, whereas the middle structure is always a contact discontinuity. In special cases some of the structures can disappear.

Shock-free solutions

We consider special Riemann initial data such that the contact discontinuity does not appear and both remaining structures are rarefaction waves, more precisely:

- the entropy S is *constant* in $[0, T] \times \Omega$;
- the density R and the temperature Θ components of the Riemann solutions are interrelated through

$$\Theta = R^{\frac{1}{c_v}} \exp\left(\frac{1}{c_v} S\right);$$

- the density $R = R(t, x_2)$ and the velocity $V = V(t, x_2)$ represent a rarefaction wave solution of the 1-D *isentropic* system

$$\partial_t R + \partial_{x_2}(RV) = 0, \; R\left[\partial_t V + V \partial_{x_2} V\right] + \exp\left(\frac{1}{c_v} S\right) \partial_{x_2} R^{\frac{c_v+1}{c_v}} = 0,$$

Far field conditions I

We consider weak solutions satisfying the following far field conditions

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |\rho(t, x_1, x_2) - \rho_-| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$
$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |\rho(t, x_1, x_2) - \rho_+| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |\theta(t, x_1, x_2) - \theta_-| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$
$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |\theta(t, x_1, x_2) - \theta_+| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

Far field conditions II

and similarly

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |v_1(t, x_1, x_2)| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to -\infty} \int_0^T \int_{\mathcal{T}^1} |v_2(t, x_1, x_2) - v_{-2}| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |v_1(t, x_1, x_2)| \, \mathrm{d}x_1 \, \mathrm{d}t = 0,$$

$$\lim_{x_2 \to \infty} \int_0^T \int_{\mathcal{T}^1} |v_2(t, x_1, x_2) - v_{+2}| \, \mathrm{d}x_1 \, \mathrm{d}t = 0.$$

Main theorem

Theorem 4 (Feireisl, K., Vasseur)

Let $[\rho, \theta, v]$ be a weak solution of the Euler system in $(0, T) \times \Omega$ originating from the Riemann data and satisfying the far field conditions. Suppose in addition that the Riemann data give rise to the shock-free solution $[R, \Theta, V]$ of the 1-D Riemann problem specified above.
Then

$$\rho = R, \ \theta = \Theta, \ v = (0, V) \ \text{a.a. in} \ (0, T) \times \Omega.$$

Thank you

Thank you for your attention.