Mathematical models of viscous heat conducting fluids

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Equadiff 2015, Lyon, 6 July - 10 July 2015

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Navier-Stokes-Fourier system

Equation of contintuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \ \mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}), \ \mathbf{u}|_{\partial \Omega} = 0$$

Second law, entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \sigma, \ \mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

First law, total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \ \mathrm{d}x = 0$$

Constitutive relations

Gibbs' law, thermodynamics stability

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right), \ \frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$

Viscosity, Newton's law

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}$$

Heat conductivity, Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta$$

Second law, entropy production

$$\sigma \boxed{\geq} \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)\cdot \nabla_x \vartheta}{\vartheta}\right)$$

Relative energy for NSF system

Ballistic free energy [Ericksen]

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta \varrho s(\varrho,\vartheta)$$

Relative NSF energy

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \middle|\ r,\Theta,\mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta) \right) \ \mathrm{d}x$$

Relative entropy vs. relative energy

Dafermos [1979] - relative entropy for the full Euler system

factor $\frac{1}{\Theta}$

Relative energy inequality

Relative entropy inequality

$$
\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}
$$

$$
+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t
$$

$$
\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t
$$

Test functions

$$
r > 0, \ \ \Theta > 0
$$

$\mathbf{U}$ satisfying the relevant natural boundary conditions

Remainder

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \ \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Global-in-time weak solutions, hypotheses

Pressure - density, temperature state equation

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\vartheta^4$$

$$\varrho e(\varrho, \vartheta) = \frac{3}{2}\vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0$$

Transport coefficients

$$\underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\alpha), \quad \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^\alpha), \ \alpha \in (2/5, 1],$$

$$\underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3)$$

Results

Existence of weak solutions [Indiana Univ. Math. J. 2004]

Weak solutions exist (under certain constitutive restrictions) for any finite energy initial data and on an arbitrary time interval. Smooth weak solutions are strong solutions

Weak-strong uniqueness [with A.Novotný ARMA 2012]

Weak and strong solutions to the Navier-Stokes-Fourier system emanating from the same initial data coincide as long as the latter exists. Strong solutions are unique in the class of weak solutions

Conditional regularity [with A.Novotný, Y. Sun ARMA 2014]

A weak solution emanating from smooth initial data is smooth as soon as

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} < \infty$$

Inviscid fluids?

Euler-Fourier system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x (\varrho \vartheta) = 0$$

$$\frac{3}{2} \left[\partial_t (\varrho \vartheta) + \mathrm{div}_x (\varrho \vartheta \mathbf{u}) \right] - \Delta \vartheta = - \varrho \vartheta \mathrm{div}_x \mathbf{u}$$

Existence of weak solutions [with E.Chiodaroli, O.Kreml AIHP 2014]

The Euler-Fourier system admits infinitely many global-in-time weak solutions for any smooth initial data ϱ_0, ϑ_0, $\mathbf{u}_0$. Moreover, for any ϱ_0, ϑ_0 there exists a velocity field $\mathbf{u}_0 \in L^\infty$ such that the problem admits infinitely many global *admissible* weak solutions.

Solutions via convex integration

Step 1: density and acoustic potential

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0, \ \partial_t \varrho + \Delta \Psi = 0$$

Step 2: temperature as a function of v

$$\frac{3}{2} \left(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \right) - \Delta \vartheta = - \varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

Step 3: Euler system with nonconstant coefficients

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta [\mathbf{v}]) = 0$$

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}]$$

Vanishing dissipation limit

Pressure

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\varrho,\vartheta),\ p_M = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right),\ p_R(\varrho,\vartheta) = \frac{\boxed{a}}{3}\vartheta^4$$

Viscous stress

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \boxed{\nu}\left[\mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}\right]$$

Heat flux

$$\mathbf{q} = -\boxed{\omega}\kappa(\vartheta)\nabla_x \vartheta$$

Brinkman type “damping”

$$D = -\boxed{\lambda}\mathbf{u}$$

Target system

Full Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_M(\varrho, \vartheta) = 0$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right)$$

$$+ \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right) \mathbf{u} + p_M(\varrho, \vartheta) \mathbf{u} \right] = 0$$

Slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Vanishing dissipation limit

Theorem EF [2015]

Let $[\varrho_E, \vartheta_E, \mathbf{u}_E]$ be the classical solution of the Euler system in a time interval $(0, T)$, with the initial data $[\varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}]$. Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak (dissipative) solution of the Navier-Stokes-Fourier system, with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$.
Then

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| \varrho_E, \vartheta_E, \mathbf{u}_E\right)(\tau)$$

$$\leq c_1(T, \mathrm{data})\mathcal{E}\left(\varrho_0, \vartheta_0, \mathbf{u}_0 \middle| \varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}\right)$$

$$+c_2(T, \mathrm{data}) \max\left\{ a, \nu, \omega, \lambda, \frac{\nu}{\sqrt{a}}, \frac{\omega}{a}, \left(\frac{a}{\sqrt{\nu^3\lambda}}\right)^{1/3} \right\}$$

for a.a. $\tau \in (0, T)$.

Numerics

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Thermal energy balance

$$c_v \left(\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u}) \right) + \mathrm{div}_x \mathbf{q} \boxed{\geq} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p_\vartheta \mathrm{div}_x \mathbf{u}$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) \ \mathrm{d}x \boxed{\leq} 0$$

Numerical analysis

Numerical analysis with T.Karper, A.Novotný, R.Hošek, M.Michálek [2014]

- A mixed finite-volume finite element implicit scheme converges to a weak solution
- Convergence is unconditional provided the numerical solutions $\varrho_h, \vartheta_h, \mathbf{u}_h$ and $\mathrm{div}_x \mathbf{u}_h$ remain uniformly bounded

Blow-up criterion

Blow-up of smooth solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth ($W^{2,3}$). Then the Navier-Stokes-Fourier system admits a strong solution defined on a (possibly short) time interval $(0, T)$.
If

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} \right] < \infty,$$

then the solution can be extended beyond T.

Regularity criterion

Regularity for weak solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth ($W^{2,3}$). Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system such that

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is regular.

Synergy analysis - numerics, assumptions

Numerical solutions with regular initial data

Suppose that $[\varrho_h, \vartheta_h, \mathbf{u}_h]$ is a sequence of numerical solutions for regular initial data

Boundedness

Suppose that

$$\varrho_h^k,\ \vartheta_h^k,\ \mathbf{u}_h^k,\ \mathrm{div}_h \mathbf{u}_h^k$$

are bounded independently of the order of discretization h.

Synergy analysis - numerics, conclusion

Conclusion

The numerical solutions converge to a weak solution with

$$\sup_{t\in(0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Consequently:

- the limit solution is smooth
- the limit solution is unique
- the numerical scheme converges unconditionally
- error estimates (?)