ON SOLVABILITY SETS OF BOUNDARY VALUE PROBLEMS FOR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Consider boundary value problems for a functional differential equation

$$\begin{cases} x^{(n)}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [a,b], \\ lx = c, \end{cases}$$

where $T^+, T^-\colon \mathbf{C}[a,b] \to \mathbf{L}[a,b]$ are positive linear operators; $l\colon \mathbf{AC}^{n-1}[a,b] \to \mathbb{R}^n$ is a linear bounded vector-functional, $f \in \mathbf{L}[a,b]$, $c \in \mathbb{R}^n$, $n \geqslant 2$.

Let the solvability set be the set of all points $(\mathcal{T}^+, \mathcal{T}^-) \in \mathbb{R}_2^+$ such that for all operators T^+, T^- with $\|T^\pm\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^\pm$ the problems have a unique solution for every f and c. A method of finding the solvability sets are proposed. Some new properties of these sets are obtained in various cases.

We continue the investigations of the solvability sets started in R. Hakl, A. Lomtatidze, J. Šremr: Some boundary value problems for first order scalar functional differential equations. Folia Mathematica 10, Brno, 2002.

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1. Introduction

Consider the boundary value problem for a scalar n-th order functional differential equation on the finite interval $[a,b]$:

$$\begin{cases} x^{(n)}(t) = (Tx)(t) + f(t), \quad t \in [a,b], \\ lx = c, \end{cases} \tag{1.1}$$

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where $T = T^+ - T^-$, T^+ and T^- are positive linear operators from the space of continuous functions $\mathbf{C} \equiv \mathbf{C}[a,b]$ to the space of summable functions $\mathbf{L} \equiv \mathbf{L}[a,b]$; $l = (l_1, \dots, l_n)$ is a linear n-dimensional vector-functional bounded on the space $\mathbf{AC}^{n-1} \equiv \mathbf{AC}^{n-1}[a,b]$ of functions with absolutely continuous derivatives up to the $(n-1)$-th order; $f \in \mathbf{L}$, $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. Solutions of (1.1) belong to $\mathbf{AC}^{n-1}$. Here positive operators map every non-negative function from $\mathbf{C}$ into a nonnegative almost everywhere function from $\mathbf{L}$.

We continue the investigation of the solvability of boundary value problems for functional differential equations started in works by N. V. Azbelev and its pupils [1]. The method is a development of the method applied by R. Hakl, S. Mukhigulashvili, A. Lomtatidze, B. Půža, J. Šremr [4], [5], [6]. The work continues the investigation of book [5], where the case $n = 1$ was considered in detail. We will assume that $n \geqslant 2$.

In 1995 in the notes [2], [3] S. Mukhigulashvili and A. Lomtatidze used the method of obtaining a priori estimates of the solutions to the periodic value problem for ordinary differential equations from the work [7] by A. Lasota and Z. Opial. They expanded the results to nonlinear functional differential equations with monotone operators. It turned out [3] that for the Dirichlet problem for second order functional differential equations with positive operators such a method of obtaining a priori estimates can enlarge the solvability conditions in comparison with the Banach principle.

Later this method was used to many different boundary value problems for different functional differential equations and systems: [5], [8] (first order); [4], [9], [10] (second order); [11] (third order); [12] (fourth order); [13] (systems of two equations of first order); [14], [15] (periodical problems for n-th order equations).

Here we replace obtaining an a priori estimate by an equivalent procedure that is reduced to minimizing a given function defined on a finite-dimensional set. In some cases (which will be described below) this problem can be solved exactly. It allows to obtain unimprovable in a sense conditions of solvability. These criteria will be formulated here in the form of necessary and sufficient conditia for all problems from a certain set to be uniquely solvable.

2. The Fredholm property of a boundary value problem

Define operators $\mathcal{L}\colon \mathbf{AC}^{n-1} \to \mathbf{L}$ and $B\colon \mathbf{AC}^{n-1} \to \mathbf{L} \times \mathbb{R}^n$ by the equalities $\mathcal{L}x = x^{(n)} - T^+x + T^-x$ and $Bx = \{\mathcal{L}x, lx\}$. Then the boundary value problem (1.1) can be written in one equation $Bx = \{f, c\}$, where the operator $B = [\mathcal{L}, l]$ has the Fredholm property [1] under given conditions on the parameters of the problem (it means the operator B is normally solvable, its null-space is finite-dimensional, its dimension coincides with the dimension of the null-space of the conjugate operator).

We connect the boundary value problem (1.1) for all pairs $f \in \mathbf{L}$ and $c \in \mathbb{R}^n$ with the operator B. We will say that the problem (1.1) has the Fredholm property provided the appropriate operator B has the Fredholm property.

If B has the Fredholm property, then B is invertible if and only if the equation $Bx = 0$ has only the trivial solution. Therefore problem (1.1) has a unique solution for all pairs $f \in \mathbf{L}$ and $c \in \mathbb{R}^n$ if and only if the homogeneous problem

$$\begin{cases} x^{(n)}(t) = (T^+x)(t) - (T^-x)(t), \quad t \in [a,b], \\ lx = 0 \end{cases} \tag{2.1}$$

has only the trivial solution.

Hence, either for every pair f and c problem (1.1) cannot have a unique solution, or this problem for every pair f and c has only one solution. In the latter case problem (1.1) is called uniquely solvable.

3. Solvability sets

We will say that a point $(\mathcal{T}^+, \mathcal{T}^-) \in \mathbb{R}^2_+ \equiv [0,+\infty) \times [0,+\infty)$ is a point of unique solvability of problem (1.1) if this problem is uniquely solvable for all positive operators T^+ and T^- such that

$$\|T^+\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^+, \quad \|T^-\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^-.$$

Note that for any positive operator $T^\pm\colon \mathbf{C} \to \mathbf{L}$ the norm is defined by the equality

$$\|T^\pm\|_{\mathbf{C}\to\mathbf{L}} = \int_a^b (T^\pm 1)(s)\,\mathrm{d}s.$$

The set of all points of unique solvability is called the solvability set of problem (1.1) and is denoted by $\mathcal{R}$. It is clear that the set $\mathcal{R} = \mathcal{R}(l)$ for given n and $[a,b]$ depends on the vector-functional l only. Generally speaking the set $\mathcal{R}(l)$ is not open. So, the property of a point from $\mathbb{R}^2_+$ to be a point of unique solvability of problem (1.1) is not a stable property.

Our main intention is to construct or to estimate the solvability set $\mathcal{R}(l)$. It will be shown that the problem of building $\mathcal{R}$ is reduced to minimizing some known function defined on a finite-dimensional set. This approach can be easily expanded to systems of functional differential equations (in this case the solvability set will be a subset of the k-dimensional space $\mathbb{R}^k_+$ for some $k \geqslant 1$).

Now we will describe the dependence of $\mathcal{R}$ on properties of l. Denote by $N = N(l)$ the dimension of the linear space of solutions to the homogeneous problem

$$\begin{cases} x^{(n)}(t) = 0, \quad t \in [a,b], \\ lx = 0. \end{cases} \tag{3.1}$$

It turns out that the set $\mathcal{R}(l)$ depends essentially on the number $N(l)$.

Theorem 3.1. *If $N(l) \geqslant 2$, then $\mathcal{R}(l)$ is the empty set.*

P r o o f. Under the conditions of the theorem, problem (3.1) has a nontrivial solution u such that $u(\tau) = 0$ for some $\tau \in [a,b]$. Hence problem (3.1) is not uniquely solvable if $T^{\pm}x = p^{\pm}x(\tau)$ for any $p^{\pm} \in \mathbf{L}$. □

So, we have to consider only two cases: $N = 0$ and $N = 1$.

4. The problem is uniquely solvable for $T^+ = T^- = 0$

Let $N(l) = 0$. Then the problem

$$\begin{cases} x^{(n)}(t) = f(t), \quad t \in [a,b], \\ lx = 0 \end{cases}$$

is uniquely solvable, its solution has the integral representation [1]

$$x(t) = \int_a^b G(t,s)f(s)\,\mathrm{d}s, \quad t \in [a,b],$$

where the Green function $G(t,s)$ is such that $G(t,\cdot) \in \mathbf{L}_\infty$ for all $t \in [a,b]$, and there exist finite $\inf\limits_{t\in[a,b]} \operatorname*{vrai\,sup}\limits_{s\in[a,b]} G(t,s)$, $\sup\limits_{t\in[a,b]} \operatorname*{vrai\,inf}\limits_{s\in[a,b]} G(t,s)$.

For given $\mathcal{T}^+$ and $\mathcal{T}^-$, for all non-positive values $\mathcal{T}_1^+$, $\mathcal{T}_2^+$, $\mathcal{T}_1^-$, $\mathcal{T}_2^-$ such that $\mathcal{T}_1^+ + \mathcal{T}_2^+ = \mathcal{T}^+$, $\mathcal{T}_1^- + \mathcal{T}_2^- = \mathcal{T}^-$, and for all points $\tau_1, \tau_2, c^+, c^-, d^+, d^- \in [a,b]$ define a function $\Delta_{\mathcal{T}^+,\mathcal{T}^-}$ by the equality

$$\begin{aligned} &\Delta_{\mathcal{T}^+,\mathcal{T}^-}(\tau_1,\tau_2,c^+,c^-,d^+,d^-,\mathcal{T}_1^+,\mathcal{T}_2^+,\mathcal{T}_1^-,\mathcal{T}_2^-) \\ &\quad\equiv 1 - \mathcal{T}_2^+ g_1(d^+) - \mathcal{T}_1^+ g_2(c^+) + \mathcal{T}_1^- g_1(c^-) + \mathcal{T}_2^- g_2(d^-) \\ &\qquad + \mathcal{T}_2^+ g_1(d^+)(\mathcal{T}_1^+ g_2(c^+) - \mathcal{T}_2^- g_2(d^-)) + \mathcal{T}_2^+ g_2(d^+)(\mathcal{T}_2^- g_1(d^-) - \mathcal{T}_1^+ g_1(c^+)) \\ &\qquad + \mathcal{T}_1^- g_1(c^-)(\mathcal{T}_2^- g_2(d^-) - \mathcal{T}_1^+ g_2(c^+)) + \mathcal{T}_1^- g_2(c^-)(\mathcal{T}_1^+ g_1(c^+) - \mathcal{T}_2^- g_1(d^-)), \end{aligned} \tag{4.1}$$

where $g_1(s) = G(\tau_1,s)$, $g_2(s) = G(\tau_2,s)$.

We will say that for given $\mathcal{T}^+$ and $\mathcal{T}^-$ the condition A is fulfilled if there exist no sets E_{c^+}, E_{c^-}, E_{d^+}, $E_{d^-} \subset [a,b]$ with positive measure such that for almost all points $c^+ \in E_{c^+}$, $c^- \in E_{c^-}$, $d^+ \in E_{d^+}$, $d^- \in E_{d^-}$ the equality $\Delta_{\mathcal{T}^+,\mathcal{T}^-} = 0$ is fulfilled for some fixed other arguments.

Theorem 4.1. *If $N(l) = 0$, then*

1) *the set $\mathcal{R}(l)$ is not empty and contains a neighborhood of zero in $\mathbb{R}^2_+$;*
2) *$\mathcal{R}(l)$ has at most two connected components; every component is bounded; the component that contains the origin is always present, it belongs to the intersection of all pairs $(\mathcal{T}^+, \mathcal{T}^-)$ satisfying the inequality*

$$\operatorname*{vrai\,sup}_{s\in[a,b]} G(t,s)\mathcal{T}^+ - \operatorname*{vrai\,inf}_{s\in[a,b]} G(t,s)\mathcal{T}^- \leqslant 1 \quad \text{for all } t \in [a,b]; \tag{4.2}$$

the other connected component can exist if the Green function $G(t,s)$ is essentially separated from zero; in this case for all points $(\mathcal{T}^+, \mathcal{T}^-)$ of the second connected component, the inequality

$$\operatorname*{vrai\,inf}_{s\in[a,b]} G(t,s)\mathcal{T}^+ - \operatorname*{vrai\,sup}_{s\in[a,b]} G(t,s)\mathcal{T}^- \geqslant 1 \quad \text{for all } t \in [a,b] \tag{4.3}$$

is fulfilled;

3) *a point $(\mathcal{T}^+, \mathcal{T}^-) \in \mathbb{R}^2_+$ belongs to the solvability set $\mathcal{R}(l)$ if and only if the condition A is fulfilled and either (4.2) is fulfilled and $\Delta_{\mathcal{T}^+,\mathcal{T}^-} \geqslant 0$ almost everywhere, or (4.3) is fulfilled and $\Delta_{\mathcal{T}^+,\mathcal{T}^-} \leqslant 0$ almost everywhere;*
4) *if the equality $lx = 0$ implies that the function x has a zero on $[a,b]$, then the solvability set $\mathcal{R}(l)$ has only one connected component; moreover, if $(\mathcal{T}^+, \mathcal{T}^-) \in \mathcal{R}(l)$, then $(\widetilde{\mathcal{T}}^+, \widetilde{\mathcal{T}}^-) \in \mathcal{R}(l)$ for all $\widetilde{\mathcal{T}}^+ \in [0, \mathcal{T}^+]$, $\widetilde{\mathcal{T}}^- \in [0, \mathcal{T}^-]$.*

P r o o f. The item 1) can be proved by the Banach principle.

Here we can only give a plan of the proof for items 2) and 3). Problem (1.1) is not uniquely solvable if and only if there exists a solution to the homogeneous problem (2.1). This solution is a solution of problem (2.1) with a special operator $\widetilde{T}x = p_1 x(\tau_1) + p_2 x(\tau_2)$, where τ_1, $\tau_2 \in [a,b]$, $p_1, p_2 \in \mathbf{L}$, $p_1 + p_2 = T1$, and $-T^-1 \leqslant p_1, p_2 \leqslant T^+1$. The necessary and sufficient conditions of the unique solvability of all these problem with $\|T^\pm\| = \mathcal{T}^\pm$ are the conditions 2) and 3) of the theorem.

Under the conditions of item 4), if $(\mathcal{T}^+, \mathcal{T}^-) \notin \mathcal{R}(l)$, then for some positive operators T^+, T^- with $\|T^\pm\| = \mathcal{T}^\pm$ there exists a solution to the problem (2.1) that has a zero at some point $\tau \in [a,b]$. So, for the operator $Tx \equiv T^+x + p^+x(\tau) - T^-x - p^-x(\tau)$, where functions p^+, p^- are positive and summable, problem (1.1) is not uniquely solvable. Therefore, for all $\widetilde{\mathcal{T}}^+ \geqslant \mathcal{T}^+$ and $\widetilde{\mathcal{T}}^- \geqslant \mathcal{T}^-$ we have $(\widetilde{\mathcal{T}}^+, \widetilde{\mathcal{T}}^-) \notin \mathcal{R}(l)$. $\square$

E x a m p l e 4.2. Consider the boundary value problem

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [0,1], \\ \dot{x}(0) = 0, \quad \dot{x}(1) = x(0), \end{cases} \tag{4.4}$$

which can be used for investigation of economic processes.

A point $(\mathcal{T}^+, \mathcal{T}^-)$ belongs to the solvability set $\mathcal{R}_{4.4}$ of problem (4.4) if and only if at least one of the following conditions is fulfilled:

$$\mathcal{T}^+ \in \Big[0, \frac{1}{2}\Big], \quad 0 \leqslant \mathcal{T}^- \leqslant 2 + \sqrt{8}\sqrt{1 - \mathcal{T}^+}, \tag{4.5}$$

$$\mathcal{T}^+ \in \Big(\frac{1}{2}, \frac{\sqrt{5}+1}{4}\Big], \quad \frac{2\mathcal{T}^+ - 1}{1 - \mathcal{T}^+} \leqslant \mathcal{T}^- \leqslant 2 + \sqrt{8}\sqrt{1 - \mathcal{T}^+}, \tag{4.6}$$

$$\mathcal{T}^- \in \Big[0, \frac{1}{7}\Big], \quad \frac{2\mathcal{T}^- + 1}{1 - \frac{\mathcal{T}^-}{4}} \leqslant \mathcal{T}^+ \leqslant 3 + 2\sqrt{1 - \mathcal{T}^-}, \tag{4.7}$$

$$\mathcal{T}^- \in \Big(\frac{1}{7}, \frac{\sqrt{5}-1}{2}\Big], \quad \frac{\mathcal{T}^- + 1}{1 - \mathcal{T}^-} \leqslant \mathcal{T}^+ \leqslant 3 + 2\sqrt{1 - \mathcal{T}^-}. \tag{4.8}$$

The direct estimate of the norm of the operator T gives only the conclusion that $(\mathcal{T}^+, \mathcal{T}^-) \in \mathcal{R}_{4.4}$ if $\mathcal{T}^+ + \mathcal{T}^- \leqslant 1/2$. However, we see from (4.7) that $(5,0) \in \mathcal{R}_{4.4}$.

E x a m p l e 4.3 (see also [4], [9]). For the Dirichlet problem

$$\begin{cases} \ddot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [0,1], \\ x(0) = 0, \quad x(1) = 0, \end{cases} \tag{4.9}$$

the solvability set is defined by the equality

$$\mathcal{R}_{4.9} = \big\{(\mathcal{T}^+, \mathcal{T}^-)\colon\ \mathcal{T}^- \in [0,4],\ \mathcal{T}^+ \in \big[0, 8\big(1 + \sqrt{1 - \mathcal{T}^-/4}\big)\big]\big\}.$$

The direct estimate of the norm of the operator T gives only the conclusion that $(\mathcal{T}^+, \mathcal{T}^-) \in \mathcal{R}_{4.9}$ if $\mathcal{T}^+ + \mathcal{T}^- \leqslant 4$.

5. The space of solutions of the problem for $T^+ = T^- = 0$ is one-dimensional

Consider the last case, when $N = 1$. Obviously, the point $(0, 0)$ does not belong to the solvability set. We cannot even assert that the solvability set is not empty.

If $N = 1$, there exists a linear bounded vector-functional $\tilde{l}\colon \mathbf{AC}^{n-1} \to \mathbb{R}^n$ such that $\ker \tilde{l} = \ker l$, and the n-th component $\tilde{l}_n$ has the form $\tilde{l}_n x = \int_a^b \varphi_n(s) x^{(n)}(s)\, \mathrm{d}s$, where $\varphi_n \in \mathbf{L}_\infty$ (all other components have the representation $\tilde{l}_i x = \sum\limits_{j=0}^{n-1} K_{ij} x^{(j)}(a) + \int_a^b \varphi_i(s) x^{(n)}(s)\, \mathrm{d}s$, $\varphi_i \in \mathbf{L}_\infty$, $K_{ij} \in \mathbb{R}$, $j, i = 1, \ldots, n-1$).

Further we will suppose that the vector-functional of the boundary value problem has such a form. Since the null-space is not changed, the space of solutions to the homogeneous problem (2.1) is not changed, either. So, this replacement does not affect the unique solvability of problem (1.1).

Theorem 5.1. *If $N = 1$ and $\varphi_n(t) = 0$ on a set with positive measure, then the solvability set $\mathcal{R}(l)$ is empty.*

P r o o f. Suppose $\varphi_n(t) = 0$ for all $t \in e$, where $e \subset [a, b]$ is a set with positive measure. It is sufficient to take the operator T of the form $Tx = p_1 x(\tau_1) + p_2 x(\tau_2)$, where $a \leqslant \tau_1 < \tau_2 \leqslant b$ and $p_1(t) = p_2(t) = 0$ for all $t \in [a, b] \setminus e$. Then problem (2.1) has a nontrivial solution, and (1.1) is not uniquely solvable. □

Now suppose $\varphi_n(t) > 0$ for almost all $t \in [a, b]$ (we don't consider the case when φ_n changes its sign). Let $v \in \mathbf{AC}^{n-1}$ be any nontrivial element of the one-dimensional set $\{Kv\colon\ K \in \mathbb{R}\}$ of all solutions to problem (3.1).

Theorem 5.2. *If $N = 1$ and v has a zero on $[a, b]$, then the solvability set $\mathcal{R}(l)$ is empty.*

P r o o f. Let $v(\tau) = 0$ for some $\tau \in [a, b]$. Problem (3.1) is not uniquely solvable if $T^\pm x = p^\pm x(\tau)$ for any $p^\pm \in \mathbf{L}$. □

Suppose further that the function v has no zeros (let $v(t) > 0$ for all $t \in [a, b]$). Then the problem

$$\begin{cases} x^{(n)}(t) = f(t), \quad t \in [a, b], \\ x(a) = 0, \quad l_i x = 0, \quad i = 1, \ldots, n-1 \end{cases}$$

is uniquely solvable. Denote by $G(t, s)$ its Green function.

We need some more definitions. Let $\mathcal{R}^v_{\varphi_n}$ (modified solvability set) be the set of points $(\mathcal{T}^+, \mathcal{T}^-) \in \mathbb{R}^2_+$ such that for all positive operators $T^+, T^-\colon \mathbf{C} \to \mathbf{L}$ with

$$\int_a^b \varphi_n(s)(T^+v)(s)\,\mathrm{d}s = \mathcal{T}^+, \quad \int_a^b \varphi_n(s)(T^-v)(s)\,\mathrm{d}s = \mathcal{T}^-,$$

problem (1.1) is uniquely solvable.

Obviously, for the periodic problem (for n-th order equations) and the Neumann problem (for second order equations) we have $v = 1$, $\varphi_n = 1$. Note also that $\mathcal{R}^1_1 = \mathcal{R}$.

Let

$$\begin{aligned} G_{t_1,t_2}(s) &\equiv \frac{G(t_1,s)}{v(t_1)} - \frac{G(t_2,s)}{v(t_2)}, \\ M_{t_1,t_2} &\equiv \operatorname*{vrai\,sup}_{s\in[a,b]} \frac{G_{t_1,t_2}(s)}{\varphi_n(s)}, \\ m_{t_1,t_2} &\equiv \operatorname*{vrai\,inf}_{s\in[a,b]} \frac{G_{t_1,t_2}(s)}{\varphi_n(s)}. \end{aligned}$$

For finite numbers M_{t_1,t_2}, m_{t_1,t_2} denote

$$\mathcal{M} \equiv \sup_{t_1,t_2\in[a,b]} (M_{t_1,t_2} - m_{t_1,t_2}).$$

We will say that for the system l_i, $i = 1,\dots,n$, of the functionals of boundary value problem (1.1) the condition C is fulfilled if the equality $\mathcal{M} = M_{\tilde t_1,\tilde t_2} - m_{\tilde t_1,\tilde t_2}$, where $\tilde t_1$, $\tilde t_2 \in [a,b]$, implies that the function $g_{\tilde t_1,\tilde t_2}(s) = G(\tilde t_1,s) - G(\tilde t_2,s)$, $s \in [a,b]$, takes at least one of the values $M_{\tilde t_1,\tilde t_2}$, $m_{\tilde t_1,\tilde t_2}$ on a zero-measure set only.

Remark 5.3. The condition C is fulfilled if all functions φ_i, $i = 1,\dots,n$, are polynomials. It is fulfilled for all two-point problems with the functional $l_i x = \sum_{j=0}^{n-1} (A_{ij}x^{(j)}(a) + B_{ij}x^{(j)}(b))$, A_{ij}, $B_{ij} \in \mathbb{R}$, $i,j = 1,\dots,n$.

Theorem 5.3 (see also [10], [11], [12] for periodic conditions). *Let $N = 1$, $v(t) > 0$ for all $t \in [a,b]$, $v(a) = 1$, let the function φ_n be positive almost everywhere.*

If for some $\varepsilon > 0$ for almost all $s \in [a,b]$ the inequality $\varphi_n(s) \geqslant \varepsilon$ is fulfilled or the function $G_{t_1,t_2}(s)/\varphi_n(s)$ is essentially bounded on $s \in [a,b]$ uniformly on t_1, $t_2 \in [a,b]$, then the set $\mathcal{R}^v_{\varphi_n}$ is not empty. Moreover, a pair of different nonnegative numbers $(\mathcal{T}^+, \mathcal{T}^-)$ belongs to the solvability set $\mathcal{R}^v_{\varphi_n}$ if and only if

$$\frac{Y}{1-Y} \leqslant X \leqslant 2(1+\sqrt{1-Y}) \quad \textit{and the condition C is fulfilled,}$$

or

$$\frac{Y}{1-Y} < X < 2(1+\sqrt{1-Y}) \quad \textit{and the condition } \mathrm{C} \textit{ is not fulfilled},$$

where $X = \mathcal{M}\max(\mathcal{T}^+, \mathcal{T}^-)$, $Y = \mathcal{M}\min(\mathcal{T}^+, \mathcal{T}^-)$.

P r o o f. The outline of the proof is similar to the proof for periodic problems [15]. □

E x a m p l e 5.5. If different nonnegative numbers $\mathcal{T}^+$, $\mathcal{T}^-$ are given, then

$$\begin{cases} \ddot{x}(t) = T^+x(t) - T^-x(t) + f(t), \quad t \in [0,1], \\ \dot{x}(1) = c_1, \ x(0) - x(1) = c_2 \end{cases}$$

is uniquely solvable for all positive operators $T^+, T^- \colon \mathbf{C} \to \mathbf{L}$ such that

$$\int_0^1 s(T^+1)(s)\,\mathrm{d}s = \mathcal{T}^+, \ \int_0^1 s(T^-1)(s)\,\mathrm{d}s = \mathcal{T}^-,$$

if and only if

$$\frac{\min(\mathcal{T}^+, \mathcal{T}^-)}{1-\min(\mathcal{T}^+, \mathcal{T}^-)} \leqslant \max(\mathcal{T}^+, \mathcal{T}^-) \leqslant 2\big(1+\sqrt{1-\min(\mathcal{T}^+, \mathcal{T}^-)}\big).$$

Here $v(t) = 1$, $\varphi_2(t) = t$, $\mathcal{M} = 1$.

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