ON THE CONGRUENCE LATTICE OF AN ABELIAN LATTICE ORDERED GROUP

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Abstract. In the present note we characterize finite lattices which are isomorphic to the congruence lattice of an abelian lattice ordered group.

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1. Introduction

Congruence lattices of lattices have been studied by a large number of authors (for references until 1990 we recall Grätzer (cf. [4]); for the more recent papers cf., e.g., Ploščica, Tůma and Wehrung [8] and the references quoted there).

The famous Dilworth's Theorem on the congruence lattice of lattices reads as follows:

(A) Every finite distributive lattice D can be represented as the lattice of congruence relations of a finite lattice L.

Grätzer [4] remarks that Dilworth never published this result and that (A) appeared as an exercise (marked as difficult) in Birkhoff [1]; the first published proof is in Grätzer and Schmidt [5].

In the present note we investigate the following analogous question: which finite lattices are isomorphic to the congruence lattice of an abelian lattice ordered group?

We also deal with the case when lattices of finite breadth are considered instead of finite lattices.

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Let X and Y be partially ordered sets, $X \cap Y = \emptyset$. Put $Z = X \cup Y$ and on the set Z define a partial order $\leqslant$ as follows:

1) on the set X or Y, the original partial order remains valid;
2) for $x \in X$ and $y \in Y$ we put $x < y$.

We denote $(Z, \leqslant) = X \oplus Y$.

Let $\mathcal{A}$ be a nonempty class of partially ordered sets which is closed with respect to isomorphisms. We denote by $F_1(\mathcal{A})$ the class of all partially ordered sets X having the property that there exist $n \in N$ and $X_1, X_2, \ldots, X_n \in \mathcal{A}$ with

$$X \simeq X_1 \times X_2 \times \ldots \times X_n.$$

Further, we denote by $F_2(\mathcal{A})$ the class of all partially ordered sets Z such that either

(i) $Z \in \mathcal{A}$,

or

(ii) there exist $X \in \mathcal{A}$ and a finite chain Y such that $Z \simeq X \oplus Y$.

We put $F_3(\mathcal{A}) = F_2(F_1(\mathcal{A}))$ and

$$F_0(\mathcal{A}) = \bigcup F_3^n(\mathcal{A}) \quad (n = 1, 2, 3, \ldots).$$

It is obvious that in the definition of $F_0(\mathcal{A})$ the condition (ii) above can be replaced by the condition

(ii′) there exist $X \in \mathcal{A}$ and a one-element chain Y such that $Z \simeq X \oplus Y$.

Let $\mathcal{A}_0$ be the class of all finite chains. We prove

(B) Let D be a finite lattice. Then the following conditions are equivalent:
   (i) There exists an abelian lattice ordered group G such that D is isomorphic to the congruence lattice of G.
   (ii) D belongs to the class $F_0(\mathcal{A}_0)$.

E x a m p l e s. The lattice in Fig. 1 satisfies the condition (ii) above; the lattice in Fig. 2 does not satisfy this condition.

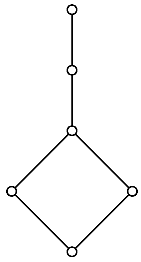

Fig. 1

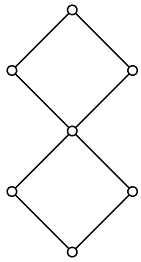

Fig. 2

3. The implication (ii) $\Rightarrow$ (i)

For lattice ordered groups we apply the notation as in Birkhoff [1] and Conrad [3].

If A is a linearly ordered group and B is a lattice ordered group, then we denote by $A \circ B$ the corresponding lexicographic product.

It is well-known that congruences on a lattice ordered group G are in a one-to-one correspondence with the ℓ-ideals of G. Moreover, if G is abelian, then the notion of an ℓ-ideal of G coincides with the notion of a convex ℓ-subgroup of G.

Let $C(G)$ be the system of all convex ℓ-subgroups of G; this system is partially ordered by the set-theoretical inclusion.

The following two assertions are easy to verify.

3.1. Lemma. *Let A and B be lattice ordered groups, $G = A \times B$. Then $C(G) \simeq C(A) \times C(B)$.*

3.2. Lemma. *Let A be a linearly ordered group with $A \neq \{0\}$; further, let B be a lattice ordered group. Let $C_0(A)$ be the system of all nonzero convex ℓ-subgroups of A. Then*

$$C(A \circ B) \simeq C(B) \oplus C_0(A).$$

We denote by Z the additive group of all integers with the natural linear order. For a positive integer n let us denote by $\overline{n}$ a chain having n elements. Then 3.2 yields

3.3. Lemma. *Let $G = G_1 \circ G_2 \circ \ldots \circ G_n$, where $G_1 = Z$ for $i = 1, 2, \ldots, n$. Then $C(G) \simeq \overline{n+1}$.*

3.4. Lemma. *Let D be a finite lattice and let* (i), (ii) *be as in* (B). *Then the implication* (ii)$\Rightarrow$(i) *is valid.*

Proof. Assume that (ii) is satisfied. We proceed by induction with respect to the number of elements of the lattice D. Put $\operatorname{card} D = k$.

Let $k = 1$. We set $G = \{0\}$. Hence $K(G) \simeq D$, whence (i) holds.

Suppose that $k > 1$ and that the assertion is valid for lattices whose numbers of elements are less than k.

Since D belongs to $F_0(\mathcal{A}_0)$, by applying the relation between (ii) and (ii′) we conclude that either

a) there exist lattices $X_1, X_2, \ldots, X_n$ belonging to $F_0(\mathcal{A}_0)$ such that

$$D \simeq X_1 \times X_2 \times \ldots \times X_n$$

and that $\operatorname{card} X_i < k$ for $i = 1, 2, \ldots, n$,

or

b) there exists a lattice D_1 and a one-element chain B such that $D \simeq D_1 \oplus B$.

Assume that a) holds. In view of the induction hypothesis there are abelian lattice ordered groups G_i ($i = 1, 2, \ldots, n$) such that $C(G_i) \simeq X_i$. Put $G = G_1 \times G_2 \times \ldots \times G_n$. In view of 3.1 we have $C(G) \simeq D$.

Next, let us suppose that b) holds. In view of the induction hypothesis there is an abelian lattice ordered group G_1 with $C(G_1) \simeq D_1$. Put $G_2 = Z$ and denote

$$G = G_2 \circ G_1.$$

According to 3.2 we have

$$C(G) \simeq D_1 \oplus B,$$

whence (i) is satisfied. □

We remark that without using the equivalence between (ii) and (ii′) when defining $F_0(\mathcal{A}_0)$ the above proof must be slightly modfied and instead of 3.2 we have to apply 3.3.

4. The implication (i) ⇒ (ii)

A lattice ordered group H is said to be a lexico extension of its convex ℓ-subgroup H_1 if, whenever $0 < x \in H \setminus H_1$, then $x > h_1$ for each $h_1 \in H_1$. (Cf. [3].) We express this situation by writing $H = \langle H_1 \rangle$. If, moreover, $H \neq H_1$, then H is called a proper lexico extension of H_1.

We recall the following well-known result.

4.1. Lemma. *Let $H = \langle H_1 \rangle$. Then H_1 is an ℓ-ideal of H and the factor ℓ-group H/H_1 is linearly ordered.*

4.2. Lemma. *Let H be a proper lexico extension of H_1; put $H_2 = H/H_1$. Let B be the system of all nonzero convex ℓ-subgroups of H_2. Then*

$$C(H) \simeq C(H_1) \oplus B.$$

The proof is simple, it will be omitted.

From 4.1 we conclude that the present lemma is, in fact, a generalization of 3.2.

Let $\mathcal{A}$ be a nonempty class of lattice ordered groups. Suppose that $\mathcal{A}$ is closed with respect to isomorphisms. We denote by $\varphi(\mathcal{A})$ the class of all lattice ordered groups G such that either $G \in \mathcal{A}$ or there are $G_1, G_2, \ldots, G_n \in \mathcal{A}$ such that $n > 1$ and

$$G = \langle G_1 \times G_2 \times \ldots \times G_n \rangle.$$

Put

$$\varphi_0(\mathcal{A}) = \bigcup \varphi^n(\mathcal{A}) \quad (n = 1, 2, 3, \ldots).$$

Let G be a lattice ordered group. A subset X of G^+ is called disjoint if $x_1 \wedge x_2 = 0$ whenever x_1 and x_2 are distinct elements of G.

From the result of Conrad [2] (a slight sharpening of this result is given in the author's paper [7]) we obtain

4.3. Lemma. *The following conditions for a lattice ordered group G are equivalent:*

(i) *Each disjoint subset of G is finite.*

(ii) *There exists a finite system $\mathcal{A} \neq \emptyset$ of linearly ordered groups such that $G \in \varphi_0(\mathcal{A})$.*

Let $0 \leqslant x \in G$. The convex ℓ-subgroup of G generated by x is the set

$$\bigcup [-nx, nx] \quad (n = 1, 2, 3, \ldots);$$

we denote this set by $G[x]$.

If x_1 and x_2 are disjoint elements of G (i.e., if $x_1 \wedge x_2 = 0$), then

$$G[x_1] \cap G[x_2] = \{0\}.$$

Hence we have

4.4. Lemma. *Let G be a lattice ordered group such that the set $C(G)$ is finite. Then each disjoint subset of G is finite.*

Let $\mathcal{A}_1 \neq \emptyset$ be a finite system of finite chains. Put

$$T_1 = \mathcal{A}_1, \quad T_n = \varphi^{n-1}(\mathcal{A}_1) \quad \text{for } n = 2, 3, \ldots$$

In view of the results of Section 3 (namely, cf. 3.3), for each $X \in \mathcal{A}_1$ there exists an abelian linearly ordered group G_X such that $C(G_X) \simeq X$. Denote

$$\overline{\mathcal{A}_1} = \{G_X \colon X \in \mathcal{A}_1\}.$$

4.5. Lemma. *Let $\mathcal{A}_1$ and $\overline{\mathcal{A}_1}$ be as above. Let G be a lattice ordered group such that $G \in \varphi_0(\overline{\mathcal{A}_1})$. Put $C(G) = D$. Then D belongs to $F_0(\mathcal{A}_1)$.*

P r o o f. There exists the least positive integer n with $G \in T_n$. We proceed by induction with respect to n. If $n = 1$, then the assertion is a consequence of the definition of $F_0(\mathcal{A}_1)$.

Let $n > 1$. Hence there are $G_1, G_2, \ldots, G_m \in \varphi^{n-1}(\mathcal{A}_1)$ such that $m > 1$ and

$$G = \langle G_1 \times G_2 \times \ldots \times G_m \rangle.$$

By the induction hypothesis we conclude that all $C(G_i)$ $(i = 1, 2, \ldots, m)$ belong to $F_0(\mathcal{A}_1)$. Then in view of 3.1 we obtain

$$C(G_1 \times G_2 \times \ldots \times G_m) \in F_0(\mathcal{A}_1).$$

From this and from 4.2 we infer that the relation

$$C(\langle G_1 \times G_2 \times \ldots \times G_m \rangle) \in F_0(\mathcal{A}_1)$$

is valid. Hence $D \in F_0(\mathcal{A}_1)$. $\square$

4.6. Lemma. *Let D be a finite lattice. Let* (i) *and* (ii) *be as in* (B). *Then* (i)$\Rightarrow$(ii).

P r o o f. It suffices to apply 4.4, 4.3 and 4.5. $\square$

Now, 3.4 and 4.6 imply that (B) holds.

For a lattice L we denote by $A(L)$ the system of all antichains of L. We put

$$b(L) = \sup\{\operatorname{card} X\colon\ X \in A(L)\}.$$

The cardinal $b(L)$ will be called the breadth of L.

Let $\mathcal{B}_f$ be the class of all lattices L such that $b(L)$ is finite.

It is obvious that if H is a linearly ordered group then the lattice $C(H)$ is a chain.

Let $\mathcal{C}_0$ be the class of all chains C_1 having the property that there exists a linearly ordered group H with $C_1 \simeq C(H)$.

The chains belonging to $\mathcal{C}_0$ can be completely characterized by using merely their order properties; this characterization has been given by Iwasawa [6].

We prove

(C) Let D be a lattice of finite breadth. Then the following conditions are equivalent:

  (i_1) = the condition (i) of (B).

  (ii_1) D belongs to the class $F_0(\mathcal{C}_0)$.

5.1. Lemma. *Let G be a lattice ordered group such that $b(C(G))$ is finite. Then each disjoint subset of G is finite.*

Proof. By way of contradiction, assume that there exists an infinite disjoint subset $\{x_i\}_{i\in I}$ of G. Then without loss of generality we can suppose that $x_i > 0$ for each $i \in I$. Let us apply the notation as in Section 4. Then the system $\{G[x_i]\}_{x\in I}$ is an infinite antichain in the lattice $C(G)$, which is a contradiction. □

From this and from 4.3 we obtain

5.2. Corollary. *Let G be as in* 5.1. *Then there is a finite system $\mathcal{A} \neq \emptyset$ of linearly ordered groups such that $G \in \varphi_0(\mathcal{A})$.*

Let G and $\mathcal{A}$ be as in 5.2. Denote

$$\overline{\mathcal{A}} = \{C(H)\colon\ H \in \mathcal{A}\}.$$

Thus $\overline{\mathcal{A}} \subseteq \mathcal{C}_0$.

In view of 5.2 and 4.5 we have

5.3. Lemma. *Let G be as in* 5.1. *Then $C(G) \in F_0(\mathcal{C}_0)$.*

In view of 5.3, the implication (i_1)$\Rightarrow$(ii_1) is valid.

5.4. Lemma. *Let D be a lattice of finite breadth. Then the implication* $(\mathrm{ii}_1)\Rightarrow(\mathrm{i}_1)$ *holds.*

Proof. Assume that (ii_1) is satisfied. Denote

$$Q_1 = \mathcal{C}_0 \quad \text{and} \quad Q_n = F_3^{n-1}(Q_1) \quad \text{for} \quad n > 1.$$

Thus there is $n \in N$ such that $D \in Q_n$. We proceed by induction with respect to n.

Let $n = 1$. Then in view of the definition of $\mathcal{C}_0$ we conclude that (i_1) is valid.

Further, suppose that $n > 1$ and that the assertion is valid whenever $D_1 \in Q_m$, $m \in N$, $m < n$. We have $D \in F_2\ (F_1(Q_{n-1}))$. Hence there are $X_1, X_2, \ldots, X_k \in Q_{n-1}$ and a one-element lattice Y such that

$$D = (X_1 \times X_2 \times \ldots \times X_k) \oplus Y.$$

In view of the induction hypothesis there are abelian lattice ordered groups X_i such that $X_i = C(G_i)$ for $i = 1, 2, \ldots, k$. The remaining steps are as in the proof of 3.4. $\square$

In view of 5.3 and 5.4 we conclude that (C) is valid.

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