DYNAMICS OF DIANALYTIC TRANSFORMATIONS OF KLEIN SURFACES

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Abstract. This paper is an introduction to dynamics of dianalytic self-maps of nonorientable Klein surfaces. The main theorem asserts that dianalytic dynamics on Klein surfaces can be canonically reduced to dynamics of some classes of analytic self-maps on their orientable double covers. A complete list of those maps is given in the case where the respective Klein surfaces are the real projective plane, the pointed real projective plane and the Klein bottle.

Keywords: nonorientable Klein surface, dianalytic self-map, Julia set, Fatou set, dianalytic dynamics

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1. Klein surfaces

Let X be a Riemann surface. A symmetry (in the sense of Klein) of X is any fixed point free antianalytic involution h of X. Although some other types of symmetries play an important role in the theory of Riemann surfaces (see [1]), we will deal in this paper only with symmetries in the sense of Klein. The couple (X, h) is called a symmetric Riemann surface.

Let S be a surface. For $p \in S$, a chart at p is is a couple (U, φ) consisting of a neighborhood U of p and a homeomorphism

$$\varphi\colon\ U \to V \subseteq \mathbb{C}^+ := \{z \in \mathbb{C};\ \operatorname{Im} z \geqslant 0\}.$$

A dianalytic atlas on S is a family of charts $\Upsilon = \{(U_\alpha, \varphi_\alpha), \alpha \in I\}$ such that $\bigcup\limits_{\alpha\in I} U_\alpha = S$ and for every couple $(U_\alpha, \varphi_\alpha)$, $(U_\beta, \varphi_\beta) \in \Upsilon$ the transfer function $\varphi_\beta \circ \varphi_\alpha^{-1}$ is either conformal, or the complex conjugate of a conformal mapping on each connected component of its domain. In the last case, it will be called an anticonformal mapping of the respective component.

Two dianalytic atlases Υ_1 and Υ_2 of S are called dianalytically compatible if $\Upsilon_1 \cup \Upsilon_2$ is a dianalytic atlas of S. We call a maximal dianalytic atlas Υ a dianalytic structure on S and the couple (S, Υ) is called a Klein surface. Klein surfaces can be orientable or nonorientable, bordered or border free surfaces. The category of Klein surfaces contains as a subcategory that of Riemann surfaces, considered here as bordered or border free. See [2] and [3] for the morphisms of Klein surfaces and [7] for their groups of automorphisms.

The following theorem (see [5]) has its origins in Klein's work.

Theorem 1.1. *If (X, h) is a symmetric Riemann surface and $\langle h \rangle$ is the two element group generated by h, then the canonical projection $\pi\colon X \to X/\langle h \rangle$ induces on $X/\langle h \rangle$ a structure of nonorientable Klein surface, with respect to which π is a morphism of Klein surfaces.*

Conversely, if S is a nonorientable Klein surface, there is a symmetric Riemann surface (X, h) such that S is dianalytically equivalent to $X/\langle h \rangle$.

X is called the *orientable double cover* of S and is uniquely determined up to a conformal mapping.

Theorem 1.2. *If (X, h) is the orientable double cover of a nonorientable Klein surface S, then any dianalytic mapping $f\colon S \to S$ can be lifted to a unique analytic mapping $F\colon X \to X$ that commutes with h and satisfies the equality $\pi \circ F = f \circ \pi$.*

P r o o f. Let (D, π_1) be the universal covering of S and let G be the covering group of D over S. It is known (see [5]) that G is a disjoint union $G = G_1 \cup gG_1$, where G_1 is the subgroup in G of all conformal transformations and $g \in G \setminus G_1$ is anticonformal. Moreover, G_1 is the covering group of D over X and $h = h_g\colon X \to X$ is the fixed point free antianalytic involution of X defined by $h(\hat{z}) = \widehat{g(z)}$, where $\hat{z}$ is the orbit of a point $z \in D$ under the action of G_1 and X has been identified with D/G_1. If we denote by π_2 the canonical projection of D onto X, it is obvious that $\pi_1 = \pi \circ \pi_2$.

Let z_0 be an arbitrary point of D. For every $w \in D$ over $f(\pi_1(z_0))$ we define a lift f_w of f in the following way (compare [12], p. 145). If γ is a path in D from z_0 to z, then $f_w(z)$ is by definition the end-point of the path γ_w obtained by lifting $f \circ \pi_1 \circ \gamma$ from w. If γ' is another path in D from z_0 to z then, since D is simply connected, γ and γ' are homotopic, hence $f \circ \pi_1 \circ \gamma$ and $f \circ \pi_1 \circ \gamma'$ are homotopic, and by the Monodromy Theorem γ_w and γ'_w are homotopic. Consequently, f_w is well defined and we have $\pi_1 \circ f_w = f \circ \pi_1$. Since π and f are dianalytic functions and D

is an orientable surface, by the Uniqueness Theorem of analytic functions, f_w must be either an analytic or an antianalytic mapping. Moreover, if $w' \equiv w''$ (mod G_1), then $\pi_1(f_{w'}(z)) = \pi_1(f_{w''}(z))$ and therefore we can define $F\colon X \to X$ by the formula $F([z]_1) = \pi_1 \circ f_w(z)$, where $[z]_1$ is the equivalence class of z modulo G_1 and X is identified with D/G_1. It can be easily seen that $\pi \circ F = f \circ \pi$ and therefore F is a lift of f over X. Since $\pi \circ h = \pi$, we also have $\pi \circ h \circ F = f \circ \pi$, therefore $h \circ F$ is another lift of f over X. The mappings π, h and f being dianalytic, so should be F and $h \circ F$, and since X is orientable, each one of them must be either analytic, or antianalytic. Obviously, if F is analytic, then $h \circ F$ is antianalytic and vice-versa. Supposing that F is the analytic one, we need only to show that F commutes with h. This will be obvious if we can show that $h \circ F$ and $F \circ h$ are both lifts of f. Indeed, since X is the double cover of S, and there is an analytic lift of f over X, there cannot be two antianalytic lifts of f over X. and therefore we must have $h \circ F = F \circ h$. The equalities

$$\pi \circ F = f \circ \pi \quad \text{and} \quad \pi \circ h = \pi$$

show that

$$\pi \circ (F \circ h) = (\pi \circ F) \circ h = (f \circ \pi) \circ h = f \circ (\pi \circ h) = f \circ \pi$$

and

$$\pi \circ (h \circ F) = (\pi \circ h) \circ F = \pi \circ F = f \circ \pi,$$

which implies that indeed both $h \circ F$ and $F \circ h$ are lifts of f to X. □

Theorem 1.3. *If (X, h) is the orientable double cover of a nonorientable Klein surface S, and $F\colon X \to X$ is a continuous mapping that commutes with h, then there is a unique continuous mapping $f\colon S \to S$ such that $\pi \circ F = f \circ \pi$. If F is an analytic mapping, then f is dianalytic.*

P r o o f. Let us define $f\colon S \to S$ by $f(\tilde{z}) = \tilde{\zeta}$, where $\tilde{z} = \{z, h(z)\}$, $\tilde{\zeta} = \{\zeta, h(\zeta)\}$ and $\zeta = F(z)$. Then

$$\pi(F(z)) = \pi(\zeta) = \tilde{\zeta} = f(\tilde{z}) = f(\pi(z)).$$

The uniqueness and the continuity of f follow from the fact that for every set V where π is injective $f(\tilde{z}) = \pi_{|V} \circ F \circ \pi_{|V}^{-1}(\tilde{z})$ and all the functions on the right hand side of this equality are continuous. If F is an analytic mapping, then f is dianalytic in every set $f(V)$ such that π is injective in V. By consequence f is dianalytic in S. □

Theorems 1.2 and 1.3 show that the equality $\pi \circ F = f \circ \pi$ establishes a one to one correspondence between dianalytic mappings $f\colon S \to S$ and analytic mappings $F\colon X \to X$ that commute with h. When studying dynamics on Riemann surfaces particular classes of functions F are considered. The question arises of what would be the corresponding classes of functions f. We will give in this paper a partial answer to this question.

2. Dianalytic dynamics

Let $f\colon S \to S$ be a dianalytic self-map of a Klein surface, and let $f^{\circ n}$ be its n-th iterate. The orbit of a point $\tilde{z} = \{z, h(z)\} \in S$ is the set $\{f^{\circ n}(\tilde{z})\}_{n=0}^{\infty}$, where $f^{\circ 0}(\tilde{z}) = \tilde{z}$, and for $n \geqslant 1$, $f^{\circ n}(\tilde{z}) = f \circ f^{\circ(n-1)}(\tilde{z})$. The limit set of the orbit of $\tilde{z}$ is denoted by $\omega_f(\tilde{z})$. For the corresponding analytic self-map $F\colon X \to X$ of the orientable double cover of X, we have the orbit sets of z and $h(z)$, namely $\{F^{\circ n}(z)\}_{n=0}^{\infty}$ and, $\{F^{\circ n}(h(z))\}_{n=0}^{\infty}$ respectively, and their limit sets $\omega_F(z)$ and $\omega_F(h(z))$, respectively. The following propositions have elementary proofs that will be omitted.

Proposition 2.1. *If $F\colon X \to X$ commutes with h, then for every $n \geqslant 1$, $F^{\circ n}$ commutes with h. Moreover, if the dianalytic self-map f of S corresponds to the analytic map $F\colon X \to X$, then $f^{\circ n}$ corresponds to $F^{\circ n}$, i.e. $\pi \circ F^{\circ n} = f^{\circ n} \circ \pi$.*

Proposition 2.2. *If $\tilde{z}_0 = (z_0, h(z_0))$ is a fixed point of f, then z_0 and $h(z_0)$ are fixed points for F.*

Proposition 2.3. *If F commutes with h and z_0 is a fixed point of F, then $h(z_0)$ is also a fixed point of F and $\widetilde{z}_0 = \pi(z_0) = \pi(h(z_0))$ is a fixed point of f.*

Proposition 2.4. *If $\tilde{z} = \{z, h(z)\}$, then $\pi(\omega_F(z)) = \pi(\omega_F(h(z)) = \omega_f(\tilde{z})$.*

A point $\tilde{z}_0 \in S$ is said to be periodic of order p if $f^{\circ p}(\tilde{z}_0) = \tilde{z}_0$ and $f^{\circ j}(\tilde{z}_0) \neq \tilde{z}_0$ for $j < p$. The set $\{\tilde{z}_n = f_n(\tilde{z}_0)\}_{n=0}^{p-1}$ is called a cycle.

The following proposition gives a correspondence between cycles on S relative to f and cycles on X relative to F.

Proposition 2.5. *If $\tilde{z}_0$ is a periodic point of order p for f, then z_0 and $h(z_0)$ are periodic of order p for F. Vice-versa, if z_0 is periodic of order p for F, then $h(z_0)$ is also periodic of order p for F and $\tilde{z}_0 = \{\tilde{z}_0,\ h(\tilde{z}_0\}$ is periodic of order p for f.*

The concept of multiplier plays an important role in the study of dynamics on Riemann surfaces. Namely, for a cycle $\{z_n = F^{\circ n}(\tilde{z}_0)\}_{n=0}^{p-1}$ the multiplier is the complex number

$$\lambda = F'(z_0)\dots F'(z_{p-1}) = (F^{\circ p})'(z_0),$$

where the derivatives are taken by using local coordinates at z_n. Obviously, λ has an invariant meaning, that is it does not depend on a particular choice of local coordinates. Since we are dealing here with dianalytic maps, it would make sense to replace the operator $\mathrm{d}/\mathrm{d}z$ appearing in the definition of the multiplier by $\partial/\partial z + \partial/\partial\bar{z} = \partial/\partial x$, which reduces to $\partial/\partial z = \mathrm{d}/\mathrm{d}z$ in the analytic case, and to $\partial/\partial\bar{z}$ in the antianalytic case. Then the multiplier of a cycle $\{\tilde{z}_n = f^{\circ n}(\tilde{z}_0)\}_{n=0}^{p-1}$ would be:

$$\lambda = \frac{\partial f}{\partial x}(\tilde{z}_0)\frac{\partial f}{\partial x}(\tilde{z}_1)\dots\frac{\partial f}{\partial x}(\tilde{z}_{p-1}).$$

Sometimes, in order to study dianalytic dynamics on a nonorientable Klein surface, the use of its orientable double cover and of analytic dynamics on it might be more economical. This last surface can be spherical, Euclidean, or hyperbolic, according as its universal covering surface is $\overline{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{C}^+$. A specific metric has been associated with each one of these cases, that allows one to define the normality in the sense of Montel of the families of iterates of analytic self-maps and to describe the corresponding Fatou and Julia sets. Only the spherical metric is h-invariant, so that it can be projected on the corresponding nonorientable Klein surface. In this case the canonical projection is an isometry with respect to the spherical metric and its projection, and the normality of any family of iterates is preserved by projection. Therefore the Fatou and the Julia sets of f on S are projections of the Fatou and Julia sets of F on X, respectively. On the other hand, it is well known that this is the most interesting case (see [13], [14]), since it offers an unlimited variety of situations. In the other two cases there is not an obvious relationship between these sets. Indeed, the Euclidean metric and its symmetric component are not equivalent metrics (see [6]) and consequently we might expect that, when related to these metrics, some normal families of functions on X do not project into normal families of functions on S. We do not know for the moment if this is true or false, but we are convinced of the fact that a parallel study of dynamics of f on S and that of dynamics of F on X is worthwhile.

3. Dianalytic self-maps of the real projective plane

The real projective plane P^2 can be realized factorizing the Riemann sphere $\overline{\mathbb{C}}$ by the group $\langle h\rangle$ generated by the fixed point free antianalytic involution $h\colon\ z \to -1/\bar{z}$. Here $h(0) = \infty$ and $h(\infty) = 0$. There is a unique dianalytic structure on P^2 making the canonical projection $\pi\colon\ \overline{\mathbb{C}} \to P^2$ a dianalytic function. Thus, P^2 endowed with that dianalytic structure becomes a nonorientable Klein surface.

We are interested in studying iterations of dianalytic transformations $f\colon P^2 \to P^2$. As seen in the previous sections, there is a one to one correspondence between these transformations and the analytic self-maps of the Riemann sphere (= rational functions) which commute with h. Therefore the dianalytic dynamics of the real projective plane can be completely described in terms of the analytic dynamics of these rational functions.

Theorem 3.1. *Dianalytic transformations $f\colon P^2 \to P^2$ are of the form:*

$$f(\tilde{z}) = \widetilde{F(z)} \text{ or } f(\tilde{z}) = \widetilde{F(\bar{z})} \text{ where } \tilde{z} = \{z, h(z)\}$$

and

$$F(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{a_0 z^{2n+1} + a_1 z^{2n} + \ldots + a_{2n+1}}{-\bar{a}_{2n+1} z^{2n+1} + \bar{a}_{2n} z^{2n} - \ldots + \bar{a}_0}, \quad |a_0| + |a_{2n+1}| \neq 0.$$

P r o o f. The equality $\pi \circ F = f \circ \pi$ implies indeed that for $\tilde{z} = \{z, h(z)\}$, we have $f(\tilde{z}) = \widetilde{F(z)}$, or $f(\tilde{z}) = \widetilde{F(\bar{z})}$, where F is an analytic transformation of $\overline{\mathbb{C}}$. Consequently

$$F(z) = \frac{a_0 z^p + a_1 z^{p-1} + \ldots + a_p}{b_0 \; z^q + b_1 z^{q-1} + \ldots + b_q}, \text{ where } a_0 \neq 0, \; b_0 \neq 0$$

and F commutes with h, i.e. $F(-1/\bar{z}) = -1/\overline{F(z)}$. This last identity implies that

$$-\frac{\bar{b}_0 \bar{z}^q + \bar{b}_1 \bar{z}^{q-1} + \ldots + \bar{b}_q}{\bar{a}_0 \bar{z}^p + \bar{a}_1 \bar{z}^{p-1} + \ldots + \bar{a}_p} = \begin{cases} \dfrac{a_p \bar{z}^q - a_{p-1} \bar{z}^{q-1} + \ldots + (-1)^p a_0 \bar{z}^{q-p}}{b_q \bar{z}^q - b_{q-1} \bar{z}^{q-1} + \ldots + (-1)^q b_0}, & \text{if } q \geqslant p, \\ \dfrac{a_p \bar{z}^p - a_{p-1} \bar{z}^{p-1} + \ldots + (-1)^p a_0}{b_q \bar{z}^p - b_{q-1} \bar{z}^{p-1} + \ldots + (-1)^q b_0 \bar{z}^{p-q}}, & \text{if } q < p. \end{cases}$$

Since $a_0 \neq 0$ and $b_0 \neq 0$, we have necessarily

$$b_q = b_{q-1} = \ldots = b_{p+1} = 0, \quad b_p \neq 0, \quad \text{when } q \geqslant p$$

and

$$a_p = a_{p-1} = \ldots = a_{q+1} = 0, \quad a_q \neq 0, \quad \text{when } q < p.$$

Moreover, if $q \geqslant p$, there is a constant k such that, for $j = 1, 2, \ldots, p$

$$(-1)^j a_{p-j} = -k \bar{b}_j \text{ and } (-1)^{q+j} b_j = k \bar{a}_{p-j}.$$

These two equalities imply that $|k|^2 = (-1)^{q+1}$, and therefore q must be an odd number and $k = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$.

Analogously, if $q < p$, then there should exist a constant k such that

$$(-1)^{p+j} a_j = -k\bar{b}_{q-j} \quad \text{and} \quad (-1)^j b^j_{q-j} = k\bar{a}_j$$

for $j = 1, 2, \ldots, q$.

These equalities imply this time that p is an odd number and $k = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$. Putting together these results we obtain that indeed F should be a rational function of odd degree, whose coefficients of the numerator and denominator satisfy the indicated relationships.

We notice that separately a_0 and a_{2n+1} can cancel, and therefore the two polynomials producing F might not have necessarily the same degree. However, if we want the denominator to be the constant $\bar{a}_0$, then the numerator takes the particular form $\mathrm{e}^{\mathrm{i}\theta} z^{2n+1}$ and therefore $F(z) = \mathrm{e}^{\mathrm{i}\theta} z^{2n+1}$, for a real θ. This implies that only special facts related to the dynamics of polynomials in $\overline{\mathbb{C}}$ might admit extensions to P^2. □

Corollary 3.2. *Dianalytic automorphisms of P^2 are of the form:*

$$g(\tilde{z}) = \widetilde{G(z)}, \quad \text{or} \quad g(\tilde{z}) = \widetilde{G(\bar{z})}$$

where:

$$G(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{a\ z\ +\ b}{-\ \bar{b}\ z\ +\ \bar{a}}, \quad \text{where } |a| + |b| \neq 0.$$

Indeed, G should be a Möbius transformation, whose coefficients satisfy this obvious relationship.

4. Dianalytic self-maps of the pointed real projective plane P^2_*

According to Theorem 1.2, every dianalytic self-mapping $f\colon\ P^2_* \to P^2_*$ has exactly two lifts F and $F \circ h$ to its orientable double cover $\mathbb{C}^*$, one analytic and the other one antianalytic. Vice-versa, every analytic or antianalytic self-map F of $\mathbb{C}^*$, that commutes with h, generates by projection such a dianalytic f. For dymamics of F see, for example, [8]. These mappings satisfy:

$$F\Big(-\frac{1}{\bar{z}}\Big) = -\frac{1}{\overline{F(z)}} \quad \text{and} \quad \pi \circ F = f \circ \pi, \tag{1}$$

where $\pi\colon\ C^* \to P_2^*$ is the canonical projection $z \to \tilde{z} = \{z;\ -\frac{1}{\bar{z}}\}$.

It is known (see, for example [11]) that the analytic transformations $F\colon\ \mathbb{C}^* \to \mathbb{C}^*$ are those of the form

$$F(z) = z^n \exp\Big[\varphi(z) + \psi\Big(\frac{1}{z}\Big)\Big], \tag{2}$$

where φ and ψ are entire functions that fix zero and n is an integer. Using this result we can prove:

Theorem 4.1. *Dianalytic transformations $f\colon\ P^2_* \to P^2_*$ are of the form*

$$f(\tilde{z}) = \tilde{\chi} \quad \textit{with} \quad \chi = F(z) \quad \textit{or} \quad F(\bar{z}) \tag{3}$$

where

$$F(z) = \mathrm{e}^{\mathrm{i}\theta} z^{2k+1}, \quad \textit{with} \quad \theta \in \mathbb{R} \quad \textit{and} \quad k \in Z, \tag{4}$$

or

$$F(z) = \frac{1}{z} \exp\Big[\sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\bar{a}_n}{z^n}\Big]. \tag{5}$$

Here the series $\sum\limits_{n=1}^{\infty} a_n z^n$ *converges in* $\mathbb{C}$ *and for at least one* n *we have* $a_n \neq 0$.

P r o o f. The relationship (1) implies (3), therefore we only have to check (4) and (5).

If $\varphi(z) + \psi(1/z) = \text{const.}$, then (2) implies $F(z) = az^n$ with $a \in \mathbb{C}$ and $n \in \mathbb{Z}$. In this case (1) means

$$(-1)^n \frac{a}{\bar{z}^n} = -\frac{1}{\bar{a}\bar{z}^n},$$

hence $|a| = (-1)^{n+1}$, which implies that $a = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$, and $n = 2k+1$, $k \in \mathbb{Z}$. Consequently, in this case F is of the form (4).

If $\varphi(z) + \psi(1/z)$ is not a constant, let us denote

$$G(z) = \varphi(z) + \psi\Big(\frac{1}{z}\Big).$$

Then

$$F(z) = z^n e^{G(z)} \quad \text{and} \quad F\Big(-\frac{1}{\bar{z}}\Big) = (-1)^n \bar{z}^{-n} \mathrm{e}^{G(-\frac{1}{\bar{z}})}.$$

Therefore

$$(-1)^n \bar{z}^{-n} \mathrm{e}^{G(-\frac{1}{\bar{z}})} = -\frac{1}{\bar{z}^n} \mathrm{e}^{-\overline{G(z)}},$$

which implies that

$$\exp\Big[\overline{G(z)} + G\Big(-\frac{1}{\bar{z}}\Big)\Big] = (-1)^{n+1}.$$

Thus

$$\operatorname{Re}\Big[\overline{G(z)} + G\Big(-\frac{1}{\bar{z}}\Big)\Big] = 0 \quad \text{and} \quad \operatorname{Im}\Big[\overline{G(z)} + G\Big(-\frac{1}{\bar{z}}\Big)\Big] = (n+1)\pi.$$

Consequently

$$G(z)+G\Big(-\frac{1}{\bar z}\Big)=(n+1)\pi\mathrm{i},$$

meaning that

$$\overline{\varphi(z)}+\overline{\psi\Big(\frac{1}{z}\Big)}+\varphi\Big(-\frac{1}{\bar z}\Big)+\psi(-\bar z)=(n+1)\pi\mathrm{i}.$$

The equality

$$\lim_{z\to 0}[\overline{\varphi(z)}+\psi(-\bar z)]=0$$

and the previous equation implies

$$\lim_{u\to\infty}[\varphi(-u)+\overline{\psi(\bar u)}]=(n+1)\pi\mathrm{i}.$$

Thus the analytic function

$$z\to\varphi(-z)+\overline{\psi(\bar z)}$$

is bounded in $\mathbb{C}$ and according to Liouville's Theorem, it is a constant. Since $\varphi(0)=\psi(0)=0$, we must have $n=-1$ and consequently $\varphi(z)=-\overline{\psi(-\bar z)}$ for every $z\in\mathbb{C}$. If

$$\varphi(z)=\sum_{n=1}^{\infty}a_nz^n \text{ and } \psi(z)=\sum_{n=1}^{\infty}b_nz^n,$$

then this equation implies that

$$\sum_{n=1}^{\infty}a_nz^n=\sum_{n=1}^{\infty}(-1)^{n+1}\overline{b_n}z^n.$$

Therefore $b_n=(-1)^{n+1}\bar a_n$, which implies (5), and the theorem is completely proved.

Example. For $F\colon\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ defined by $F(z)=\mathrm{e}^{\mathrm{i}\theta}z^{2k+1}$, $\theta\in\mathbb{R}$, $k\in\mathbb{Z}-\{0\}$, the Julia set is the unit circle, as can be seen by an easy computation. Correspondingly, for $f\colon P^2\to P^2$ defined by $f(\tilde z)=\widetilde{F(z)}$, the Julia set is $\{\tilde z|;\ |z|=1\}$.

5. The case of a Klein bottle

The list of dianalytic self-maps of a Klein bottle can be obtained by a technique similar to that used in the previous sections, here the double orientable surface being a torus. Namely, every Klein bottle endowed with a dianalytic structure is (dianalytically isomorphic with) an orbit space $\mathbb{C}/G$, where $G = \{S; V\}$ is the group of analytic and antianalytic transformations of $\mathbb{C}$ generated by $S(z) = \bar{z} + 1/2$ and $V(z) = z + \mathrm{i}\beta$, $\beta \geqslant 1$.

The group $G_1 = \{S^2; V\}$, where $S^2(z) = z+1$ represents the subgroup of conformal elements of G. The orbit of zero with respect to G_1 is the lattice $\Sigma = Z \oplus (\mathrm{i}\beta)Z$. It is more convenient to use the standard notation $\mathbb{C}/\Sigma$ for the torus $\mathbb{C}/G_1$. If $z \in \mathbb{C}$, we will denote by $\hat{z}$ the G_1-orbit of z and by $\tilde{z}$ the G-orbit of z. Thus:

$$\hat{z} = z + \Sigma := \{z + \zeta\colon\ \zeta \in \Sigma\} \quad \text{and} \quad \tilde{z} = \hat{z} \cup \widehat{S(z)} = \{\hat{z}; \widehat{S(z)}\}.$$

Let us denote, as in the previous sections, by $\pi_2\colon\ \mathbb{C} \to \mathbb{C}/G$ the universal covering of the Klein bottle $\mathbb{C}/G$, by $\pi_1\colon\ \mathbb{C} \to \mathbb{C}/\Sigma$ the universal covering of its orientable double cover, and by $\pi\colon\ \mathbb{C}/\Sigma \to \mathbb{C}/G$ the canonical projection of the Riemann surface $\mathbb{C}/\Sigma$ onto the nonorientable Klein surface $\mathbb{C}/G$. Obviously, $\pi_1 = \pi \circ \pi_2$. The antianalytic involution $h\colon\ \mathbb{C}/\Sigma \to \mathbb{C}/\Sigma$ (Compare Theorem 1.1 and [4]) is given by

$$h(\hat{z}) = \widehat{S(z)} = \widehat{\bar{z} + 1/2}.$$

In a more general setting, it is known (see [10]) that every compact Riemann surface of genus one is analytically equivalent to a torus $\mathbb{C}/\Sigma_\tau$, where $\operatorname{Im}\tau > 0$ and $\Sigma_\tau = Z \oplus \tau Z$. Two tori $\mathbb{C}/\Sigma_\tau$ and $\mathbb{C}/\Sigma_\mu$ are analytically equivalent if and only if τ and μ are equivalent with respect to the modular group

$$M = \Big\{z \to \frac{az+b}{cz+d};\ a, b, c, d \in Z, ad - bc = 1\Big\}.$$

The set

$$A = \Big\{\tau;\ -\frac{1}{2} < \operatorname{Re}\tau \leqslant -\frac{1}{2}, \operatorname{Im}\tau > 0, |\tau| > 1\Big\} \cup \Big\{\mathrm{e}^{\mathrm{i}\theta};\ -\frac{\pi}{3} \leqslant \theta \leqslant -\frac{\pi}{2}\Big\}$$

is a fundamental set for M. Thus A gives a parametrization of the entire family of analytically non-isomorphic tori. Among these tori there are very "few" which are symmetric tori, i.e. which are double covers of Klein bottles, namely only those of the form $\mathbb{C}/\Sigma_\tau$ with $\tau = \mathrm{i}\beta$, $\beta \geqslant 1$. We shall give next the complete list of dianalytic transformations of Klein bottles (compare [4]).

Theorem 6.1. *The dianalytic self-maps of the Klein bottle $\mathbb{C}/G$ are of the form:*

$$f(\tilde{z}) = \tilde{\varsigma} \quad \textit{with} \quad \varsigma = \alpha z + a \quad \textit{or} \quad \alpha\bar{z} + a, \tag{6}$$

where

$$\alpha \in \mathbb{R},\ a \in \mathbb{C},\ \alpha\Sigma \subseteq \Sigma \quad \textit{and} \quad \tfrac{1}{2}\alpha + 2\mathrm{i}\,\mathrm{Im}(a) \in \Sigma + \tfrac{1}{2}.$$

The map f is injective if and only if $\alpha\Sigma = \Sigma$.

Proof. If the lift G to $\mathbb{C}$ of $f \circ \pi_1$, such that $G(0) = a$, is analytic, then $G(z) = \alpha z + a$, where α and a satisfy (6). (Compare [4], Theorem 4 and [9] p. 26.) Then, for $G_1 = G \circ S$ we have:

$$G_1(z) = \alpha\bar{z} + \tfrac{1}{2}\alpha + a$$

for every $z \in \mathbb{C}$ and this represents the unique antianalytic lift of $f \circ \pi_1$, that satisfies $G_1(0) = \frac{1}{2}\alpha + a$, with α and a satisfying (6).

The maps $F_k\colon \mathbb{C}/\Sigma \to \mathbb{C}/\Sigma$ given by $F_1(\hat{z}) := \hat{\eta}$ with $\eta = \alpha z + a$ and $F_2(\hat{z}) := \hat{\xi}$ with $\xi = \alpha\bar{z} + S(a)$ are well defined ($\overline{\Sigma} = \Sigma$!) and F_1 is analytic, while F_2 is antianalytic. Both of them are liftings of $f \circ \pi$, with $f(\tilde{0}) = \tilde{a}$. Moreover,

$$(F_2 \circ h)(\hat{z}) = F_2(\widehat{S(z)}) = \hat{\eta},$$

since $\alpha\overline{S(z)} + S(a) = \alpha z + a + (\frac{1}{2}\alpha + S(a) - a)$ and $\frac{1}{2}\alpha + S(a) - a \in \Sigma$. Thus $F_2 \circ h = F_1$. Similar arguments can be used in the case where the lift G of $f \circ \pi_1$ is antianalytic. Since $\pi^{-1}(\tilde{a}) = \{\hat{a}; \widehat{S(a)}\}$, F_1 and F_2 are the only lifts to $\mathbb{C}/\Sigma$ of $f \circ \pi$, and the theorem is completely proved.

Now the dynamics of dianalytic self-maps of the Klein bottle $\mathbb{C}/G$ are canonically reduced to the dynamics of the analytic/antianalytic self-maps of the tori $\mathbb{C}/\Sigma$ (compare [13], Section 6). $\square$

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