FURTHER NEW GENERALIZED TOPOLOGIES VIA MIXED CONSTRUCTIONS DUE TO CSÁSZÁR

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Abstract. The theory of generalized topologies was introduced by Á. Császár (2002). In the literature, some authors have introduced and studied generalized topologies and some generalized topologies via generalized topological spaces due to Á. Császár. Also, the notions of mixed constructions based on two generalized topologies were introduced and investigated by Á. Császár (2009). The main aim of this paper is to introduce and study further new generalized topologies called μ_{12}^C via mixed constructions based on two generalized topologies μ_1 and μ_2 on a nonempty set X and also generalized topologies called μ_C and μ_*^C for a generalized topological space (X, μ).

Keywords: mixed construction; generalized topology; generalized topological space; weak generalized topology; countable subcover; μ_{12}^C-open set; μ_C-open set; μ_*^C-open set; countable set

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1. Introduction

The theory of generalized topologies was introduced by Császár [1]. One of the generalizations of topologies introduced in [1] is known as generalized topology due to Császár. Also, some authors have introduced and studied generalized topologies and some generalized topologies via generalized topological spaces due to Császár [5], [7], [8]. Moreover, Császár introduced and investigated the notions of mixed constructions based on two generalized topologies [4]. The main goal of the present paper is to introduce and study further new generalized topologies called μ_{12}^C via mixed constructions based on two generalized topologies μ_1 and μ_2 on a nonempty set X and also generalized topologies called μ_C and μ_*^C for a generalized topological space (X, μ).

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2. Preliminaries

We recall basic notations. Let X be a nonempty set and $\mu \subset \exp X$ where $\exp X$ is the power set of X. Then μ is said to be a generalized topology [1] (briefly GT) if $\emptyset \in \mu$ and an arbitrary union of elements of μ belongs to μ. A set X with a generalized topology μ on X is called a generalized topological space (briefly GTS) and is denoted by (X, μ) [1]. The elements of μ are called μ-open sets for a generalized topological space (X, μ) and the complements of μ-open sets are called μ-closed sets [1].

Let (X, μ) be a GTS and $S \subset X$. The intersection of all μ-closed sets containing S, i.e., the smallest μ-closed set containing S is denoted by $c_\mu(S)$ [2], [3]. The union of all μ-open sets contained in S, i.e., the largest μ-open set contained in S is denoted by $i_\mu(S)$ [2], [3]. An operation $\beta\colon \exp X \to \exp X$ is called idempotent if $\beta(\beta(S)) = \beta(S)$ for $S \subset X$ and monotonic when $S \subset T \subset X$ implies $\beta(S) \subset \beta(T)$ [1], [3]. It is known that i_μ and c_μ are idempotent and monotonic [2], [3]. Let (X, μ) be a GTS, $S \subset X$ and $x \in X$. Then $x \in c_\mu(S)$ if and only if $R \cap S \neq \emptyset$ for $x \in R \in \mu$ [1], [3]. Also, $c_\mu(X \setminus S) = X \setminus i_\mu(S)$ [1], [3].

3. The GT μ_{12}^C

Definition 1. Let μ_1 and μ_2 be two GTs on a set X and $S \subset X$. Then S is said to be μ_{12}^C-open if for every $x \in S$ there exists a μ_1-open set R in X containing x such that $R \setminus i_{\mu_2}(S)$ is countable. The complement of a μ_{12}^C-open set in X is said to be μ_{12}^C-closed.

The family of all μ_{12}^C-open sets in X is denoted by μ_{12}^C.

Remark 1. Let μ_1 and μ_2 be two GTs on a set X. Suppose that $\mu_1 = \mu_2$. Then for every $S \subset X$, the following implication holds:

$$S \text{ is } \mu_1\text{-open (respectively, } \mu_2\text{-open)} \Rightarrow S \text{ is } \mu_{12}^C\text{-open.}$$

This implication is not reversible as shown in the following example:

Example 1. Let $\mathbb{R}$ be the set of real numbers with the GTs $\mu_1 = \mu_2 = \{\emptyset, A, B, C, \mathbb{R}, \mathbb{Q}^*, \mathbb{Q}^* \cup A, \mathbb{Q}^* \cup B, \mathbb{Q}^* \cup C\}$ where $\mathbb{Q}^*$ is the set of irrational numbers, $A = [1, 3]$, $B = [2, 4]$ and $C = [1, 4]$. Take $S = \mathbb{Q}^* \cup \{-2, -1, 0\}$. Then S is μ_{12}^C-open but it is not μ_1-open and it is not μ_2-open.

Remark 2. Let μ_1 and μ_2 be two GTs on a set X.

(1) Suppose that $\mu_1 \subsetneq \mu_2$. Then for every $S \subset X$, the following implication holds:

$$S \text{ is } \mu_1\text{-open} \Rightarrow S \text{ is } \mu_{12}^C\text{-open.}$$

(2) Suppose that $\mu_2 \subsetneq \mu_1$. Then for every $S \subset X$, the following implication holds:

$$S \text{ is } \mu_2\text{-open} \Rightarrow S \text{ is } \mu_{12}^C\text{-open}.$$

These implications are not reversible as shown in the following examples:

Example 2. Let $\mathbb{R}$ be the set of real numbers with the GT $\mu_1 = \{\emptyset, A, \mathbb{N}, \mathbb{Q}^*, \mathbb{Q}^* \cup A, \mathbb{N} \cup A, \mathbb{N} \cup Q^*, \mathbb{N} \cup Q^* \cup A\}$ and the GT $\mu_2 = \{\emptyset, A, \mathbb{Q}^*, \mathbb{Q}^* \cup A\}$ where $\mathbb{Q}^*$ is the set of irrational numbers, $\mathbb{N}$ is the set of natural numbers and $A = [0, 1]$. Take $S = \{1, 2, 3\}$. Then S is μ_{12}^C-open but it is not μ_2-open.

Example 3. Let $\mathbb{R}$ be the set of real numbers with the GT $\mu_1 = \{\emptyset, \mathbb{R}, \mathbb{Q}^*, A, \mathbb{Q}^* \cup A\}$ and the GT $\mu_2 = \{\emptyset, A, B, C, \mathbb{R}, \mathbb{Q}^*, \mathbb{Q}^* \cup A, \mathbb{Q}^* \cup B, \mathbb{Q}^* \cup C\}$ where $\mathbb{Q}^*$ is the set of irrational numbers, $A = [1, 3]$, $B = [2, 4]$ and $C = [1, 4]$. Take $S = \mathbb{Q}^* \cup \{0, 1\}$. Then S is μ_{12}^C-open but it is not μ_1-open.

Theorem 1. *For two GTs μ_1 and μ_2 on a set X, (X, μ_{12}^C) is a GTS.*

Proof. It is obvious that $\emptyset \in \mu_{12}^C$. Suppose that $\{S_i \colon i \in I\}$ is a family of μ_{12}^C-open sets in X and $x \in \bigcup_{i \in I} S_i$. We have $x \in S_{i_0}$ for some $i_0 \in I$. By Definition 1 exists a μ_1-open set R in X containing x and also $R \setminus i_{\mu_2}(S_{i_0})$ is countable. Since

$$R \setminus i_{\mu_2}\left(\bigcup_{i \in I} S_i\right) \subset R \setminus \bigcup_{i \in I} i_{\mu_2}(S_i) \subset R \setminus i_{\mu_2}(S_{i_0}),$$

$R \setminus i_{\mu_2}\left(\bigcup_{i \in I} S_i\right)$ is countable. Consequently, we have $\bigcup_{i \in I} S_i \in \mu_{12}^C$. □

Corollary 1. *Let μ_1 and μ_2 be two GTs on a set X.*

(1) *Suppose that $\mu_1 = \mu_2$. Then (X, μ_{12}^C) is a GTS such that*

$$\mu_1 \subset \mu_{12}^C \quad (\textit{resp. } \mu_2 \subset \mu_{12}^C).$$

(2) *Suppose that $\mu_1 \subsetneq \mu_2$. Then (X, μ_{12}^C) is a GTS such that $\mu_1 \subset \mu_{12}^C$.*

(3) *Suppose that $\mu_2 \subsetneq \mu_1$. Then (X, μ_{12}^C) is a GTS such that $\mu_2 \subset \mu_{12}^C$.*

Proof. It follows from Remark 1 and Remark 2. □

Theorem 2. *Let μ_1 and μ_2 be two GTs on a set X and $S \subset X$. Then S is μ_{12}^C-open if and only if for every $x \in S$ there exists a μ_1-open set P in X containing x and a countable set R such that $P \setminus R \subset i_{\mu_2}(S)$.*

P r o o f. Let $S \in \mu_{12}^C$ and $x \in S$. Then there exists a μ_1-open set P in X containing x such that $P \setminus i_{\mu_2}(S)$ is countable. Put

$$R = P \setminus i_{\mu_2}(S) = P \cap (X \setminus i_{\mu_2}(S)).$$

Consequently, we have $P \setminus R \subset i_{\mu_2}(S)$.

Conversely, let $x \in S$. There exists a μ_1-open set P in X containing x and a countable subset R such that $P \setminus R \subset i_{\mu_2}(S)$. Thus, $P \setminus i_{\mu_2}(S)$ is countable and hence S is μ_{12}^C-open. □

Theorem 3. *Let μ_1 and μ_2 be two GTs on a set X. Suppose that S is a μ_{12}^C-closed set in X. Then $c_{\mu_2}(S) \subset P \cup R$ for a μ_1-closed set P in X and a countable set R in X.*

P r o o f. Let S be a μ_{12}^C-closed set in X. Since S is μ_{12}^C-closed, then $X \setminus S$ is μ_{12}^C-open. From Theorem 2 implies that there exist a μ_1-open set T in X containing x and a countable set R in X such that

$$T \setminus R \subset i_{\mu_2}(X \setminus S) = X \setminus c_{\mu_2}(S)$$

for each $x \in X \setminus S$. Then we have

$$c_{\mu_2}(S) \subset X \setminus (T \setminus R) \subset X \setminus (T \cap (X \setminus R)) = X \cap ((X \setminus T) \cup R) = (X \setminus T) \cup R.$$

Put $P = X \setminus T$. Since T is μ_1-open, then set P is μ_1-closed and also $c_{\mu_2}(S) \subset P \cup R$. □

Definition 2. Let (X, μ) be a GTS. Then X is said to be μ-locally countable if every $x \in X$ has a countable μ-neighborhood.

Theorem 4. *Let μ_1 and μ_2 be two GTs on a set X. If X is a μ_1-locally countable GTS, then S is μ_{12}^C-open for every $S \subset X$.*

P r o o f. Suppose that X is a μ_1-locally countable GTS. Let $S \subset X$ and $x \in S$. It follows that there exist a countable μ_1-neighborhood P of x and a μ_1-open set R in X containing x such that $R \subset P$. We have

$$R \setminus i_{\mu_2}(S) \subset P \setminus i_{\mu_2}(S) \subset P.$$

Hence, $R \setminus i_{\mu_2}(S)$ is countable and S is μ_{12}^C-open. Consequently, S is μ_{12}^C-open for every $S \subset X$. □

Definition 3. Let (X, μ) be a GTS. Then a set S in X is said to be μ-Lindelöf if every μ-open cover of S in (X, μ) has a countable subcover. The GTS (X, μ) is said to be μ-Lindelöf if every μ-open cover of X has a countable subcover.

Theorem 5. *Let μ_1 and μ_2 be two GTs on a set X. If (X, μ) is a μ_1-Lindelöf GTS, then $S \setminus i_{\mu_2}(S)$ is countable for every μ_1-closed set $S \in \mu_{12}^C$.*

P r o o f. Suppose that (X, μ) is a μ_1-Lindelöf GTS. Let $S \in \mu_{12}^C$ be a μ_1-closed set in X. Then for every $x \in S$ there exists a μ_1-open set R_x containing x such that $R_x \setminus i_{\mu_2}(S)$ is countable. Since $x \in R_x$ for every $x \in S$, then $\{R_x \colon x \in S\}$ is a μ_1-open cover for the set S. Since (X, μ) is a μ_1-Lindelöf GTS and S is μ_1-closed, then S is μ_1-Lindelöf. Since S is μ_1-Lindelöf, S has a countable subcover $\{R_{x_n} \colon n \in \mathbb{N}\}$ of $\{R_x \colon x \in S\}$. We have

$$S \setminus i_{\mu_2}(S) \subset \left(\bigcup_{n \in \mathbb{N}} R_{x_n} \right) \setminus i_{\mu_2}(S) \subset \bigcup_{n \in \mathbb{N}} (R_{x_n} \setminus i_{\mu_2}(S)).$$

Thus, $S \setminus i_{\mu_2}(S)$ is countable. $\square$

Definition 4. Let (X, μ) be a GTS. Then X is said to be μ-anti-locally countable if all nonempty μ-open sets in X are uncountable.

Theorem 6. *Let μ_1 and μ_2 be two GTs on a set X. Suppose that X is a μ_1-anti-locally countable GTS. Then (X, μ_{12}^C) is a μ_{12}^C-anti-locally countable GTS.*

P r o o f. Suppose that X is a μ_1-anti-locally countable GTS. Let $S \in \mu_{12}^C$ and $x \in S$. By Theorem 2 exist a μ_1-open set P in X containing x and a countable set R in X such that $P \setminus R \subset i_{\mu_2}(S)$. It follows that $i_{\mu_2}(S)$ is not countable. Thus, S is not countable. Hence, (X, μ_{12}^C) is a μ_{12}^C-anti-locally countable GTS. $\square$

Definition 5 ([6])**.** Let (X, μ) and (Y, λ) be two GTSs. Then a function $f \colon (X, \mu) \to (Y, \lambda)$ is said to be (μ, λ)-open if $f(S) \in \lambda$ for every $S \in \mu$.

Theorem 7. *Suppose that μ_1 and μ_2 are two GTs on a set X and λ_1 and λ_2 are two GTs on a set Y. Let $f \colon X \to Y$ be a (μ_1, λ_1)-open and (μ_2, λ_2)-open function. Then $f(S)$ is λ_{12}^C-open for every μ_{12}^C-open set S in X.*

P r o o f. Let S be a μ_{12}^C-open set in X and $x \in S$. Take $y = f(x) \in f(S)$. By Definition 1 exists a μ_1-open set R in X containing x such that $R \setminus i_{\mu_2}(S)$ is countable. Since f is a (μ_1, λ_1)-open function, then $f(R)$ is a λ_1-open set in Y. On the other hand, we have $i_{\mu_2}(S) \subset S$ and then $f(i_{\mu_2}(S)) \subset f(S)$. Since f is a (μ_2, λ_2)-open function, $i_{\lambda_2}(f(i_{\mu_2}(S))) = f(i_{\mu_2}(S)) \subset i_{\lambda_2}(f(S))$. We have $y = f(x) \in f(R)$

and

$$f(R) \setminus i_{\lambda_2}(f(S)) \subset f(R) \setminus f(i_{\mu_2}(S)) \subset f(R \setminus i_{\mu_2}(S))$$

is countable. Thus, $f(S)$ is a λ_{12}^C-open set in Y. $\square$

Corollary 2. *Suppose that $\mu_1 = \mu_2$ $(= \mu)$ and $\lambda_1 = \lambda_2$ $(= \lambda)$ are two GTs on sets X and Y, respectively. Let $f\colon X \to Y$ be a (μ, λ)-open function. Then $f(S)$ is λ_{12}^C-open for every μ_{12}^C-open set S in X.*

Proof. It follows from Theorem 7. $\square$

4. The GTs μ_C and μ_*^C

Definition 6. Let (X, μ) be a GTS and $S \subset X$. Then S is said to be μ_C-closed if $c_\mu(R) \subset S$ for every countable subset $R \neq \emptyset$ of S. Also, S is said to be μ_C-open if the complement of S is a μ_C-closed set.

The family of all μ_C-open sets in a GTS (X, μ) is denoted by μ_C.

Remark 3. Let (X, μ) be a GTS. Then for every $S \subset X$, the following implication holds:

$$S \text{ is } \mu\text{-open} \Rightarrow S \text{ is } \mu_C\text{-open}.$$

This implication is not reversible as shown in the following example:

Example 4. Let $\mathbb{R}$ be the set of real numbers with the GT $\mu = \{\emptyset, A \subset \mathbb{R}\colon \mathbb{R} \setminus A$ is countable and $A \neq \mathbb{R}\}$. Take $S = (1, 2)$. Then S is μ_C-open but it is not μ-open.

Theorem 8. *Let (X, μ) be a GTS and $S \subset X$. Then S is μ_C-open if and only if $S \subset i_\mu(X \setminus R)$ for every countable set $R \neq \emptyset$ in X such that $S \subset X \setminus R$.*

Proof. Let S be a μ_C-open set and $R \neq \emptyset$ be a countable set in X such that $S \subset X \setminus R$. By Definition 6 the set $X \setminus S$ is μ_C-closed. Since $R \subset X \setminus S$ and R is countable, we have $c_\mu(R) \subset X \setminus S$. Consequently, we have $S \subset X \setminus c_\mu(R) = i_\mu(X \setminus R)$.

Conversely, suppose that $S \subset i_\mu(X \setminus R)$ for every countable set $R \neq \emptyset$ in X such that $S \subset X \setminus R$. Let $T \neq \emptyset$ be a countable subset of $X \setminus S$. This implies $S \subset X \setminus T$ and

$$S \subset i_\mu(X \setminus T) = X \setminus c_\mu(T).$$

We have $c_\mu(T) \subset X \setminus S$. It follows that $X \setminus S$ is μ_C-closed. Thus, S is μ_C-open in X. $\square$

Theorem 9. *For a GTS (X, μ), (X, μ_C) is a GTS.*

P r o o f. It is obvious that $\emptyset \in \mu_C$. Let $\{S_i\}_{i \in I}$ be a family of μ_C-open sets in (X, μ_C). By Definition 6 the set $\{X \setminus S_i\}_{i \in I}$ is a family of μ_C-closed sets in (X, μ_C). Take a subset $R \neq \emptyset$ of $\bigcap_{i \in I}(X \setminus S_i)$ and assume that R is a countable set in X. We have $R \subset X \setminus S_i$ for each $i \in I$. Since $X \setminus S_i$ is a μ_C-closed set in X for each $i \in I$, we have $c_\mu(R) \subset X \setminus S_i$ for each $i \in I$. It follows that

$$c_\mu(R) \subset \bigcap_{i \in I}(X \setminus S_i).$$

By Definition 6 the set $\bigcap_{i \in I}(X \setminus S_i)$ is a μ_C-closed set in X. Consequently, $\bigcup_{i \in I} S_i$ is a μ_C-open set in X. □

Corollary 3. *Let (X, μ) be a GTS. Then (X, μ_C) is a GTS such that $\mu \subset \mu_C$.*

P r o o f. It follows from Remark 3 and Theorem 9. □

Definition 7. Let (X, μ) be a GTS and $S \subset X$. The union of all μ_C-open sets in X contained in S is said to be the μ_C-interior of S and is denoted by $i_{\mu_C}(S)$.

R e m a r k 4. Let (X, μ) be a GTS and $S \subset X$. If S is μ-open set in X, then $i_{\mu_C}(S) = i_\mu(S)$.

The following example shows that the converse of this implication is not true in general.

E x a m p l e 5. Let $\mathbb{R}$ be the set of real numbers with the GT $\mu = \{\emptyset, (0, 2), (1, 3), (0, 3)\}$. Then for the set of natural numbers $\mathbb{N}$ we have $i_{\mu_C}(\mathbb{N}) = i_\mu(\mathbb{N})$, but $\mathbb{N}$ is not μ-open.

Theorem 10. *Let (X, μ) be a GTS and $S \subset X$. Then S is μ-open if and only if S is μ_C-open and $i_{\mu_C}(S) = i_\mu(S)$.*

P r o o f. Let S be μ-open in X. It follows from Remark 3 and Remark 4 that S is μ_C-open and also we have $i_{\mu_C}(S) = i_\mu(S)$.

Conversely, let S be a μ_C-open set in X and $i_{\mu_C}(S) = i_\mu(S)$. By Definition 7 $S = i_{\mu_C}(S) = i_\mu(S)$. Consequently, S is a μ-open set in X. □

Definition 8. Let (X, μ) be a GTS and $S \subset X$. Then S is said to be μ_*^C-open if for every $x \in S$ there exists a μ-open set R in X containing x such that $R \setminus S$ is countable. The complement of a μ_*^C-open set in X is said to be μ_*^C-closed.

The family of all μ_*^C-open sets in a GTS (X, μ) is denoted by μ_*^C.

R e m a r k 5. Suppose that μ_1 and μ_2 are two GTs on a set X. Then for every $S \subset X$, the following implication holds:

$$S \text{ is } \mu_{12}^C\text{-open} \Rightarrow S \text{ is } (\mu_1)_*^C\text{-open}.$$

This implication is not reversible as shown in the following example:

E x a m p l e 6. Let $\mathbb{R}$ be the set of real numbers with the GT $\mu_1 = \mu_2 = \mu = \{\emptyset, \{-1\}, \mathbb{R}, \mathbb{Q}^+, \mathbb{Z}, \mathbb{Q}^+ \cup \{-1\}, \mathbb{Q}^+ \cup \mathbb{Z}\}$ where $\mathbb{Q}^+$ is the set of positive rational numbers and $\mathbb{Z}$ is the set of integer numbers. Take the set $S = \mathbb{Q}^*$ where $\mathbb{Q}^*$ is the set of irrational numbers. Then S is μ_*^C-open but it is not μ_{12}^C-open.

The notions of μ_*^C-openness and μ_C-openness are independent from each other as shown in the following examples:

E x a m p l e 7. Let $\mathbb{R}$ be the set of real numbers with the GT $\mu = \{\emptyset, \{-1, 0, 1\}, \mathbb{R}, \mathbb{Q}^+, \mathbb{Z}, \mathbb{Q}^+ \cup \{-1, 0\}, \mathbb{Q}^+ \cup \mathbb{Z}\}$ where $\mathbb{Q}^+$ is the set of positive rational numbers and $\mathbb{Z}$ is the set of integer numbers. Take the set $T = \mathbb{Q}^*$ where $\mathbb{Q}^*$ is the set of irrational numbers. Then T is μ_*^C-open but it is not μ_C-open.

E x a m p l e 8. Let $\mathbb{R}$ be the set of real numbers with the GT $\mu = \{\emptyset, A \subset \mathbb{R} \colon \mathbb{R} \setminus A$ is countable and $A \neq \mathbb{R}\}$. Take $T = \mathbb{Q}$ where $\mathbb{Q}$ is the set of rational numbers. Then T is μ_C-open but it is not μ_*^C-open.

Theorem 11. *For a GTS (X, μ), μ_*^C is a GT on X.*

P r o o f. It is analogous to that of Theorem 1. □

Corollary 4. *Suppose that μ_1 and μ_2 are two GTs on a set X. Then $(X, (\mu_1)_*^C)$ is a GTS such that*

$$\mu_{12}^C \subset (\mu_1)_*^C.$$

P r o o f. It follows from Remark 5 and Theorem 11. □

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