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ON THE DYNAMICS OF VISCOUS MASONRY BEAMS

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Abstract. In this paper, we consider the longitudinal and transversal vibrations of the masonry beams and arches. The basic motivation is the seismic vulnerability analysis of masonry structures that can be modeled as monodimensional elements. The Euler–Bernoulli hypothesis is employed for the system of forces in the beam. The axial force and the bending moment are assumed to consist of the elastic and viscous parts. The elastic part is described by the no–tension material, i.e., the material with no resistance to tension and which accounts for the cases of limitless, as well as bounded compressive strength. The adaptation of this material to beams has been developed in [9, 11]. The viscous part amounts to the Kelvin–Voigt damping depending linearly on the time derivatives of the linearized strain and curvature. The dynamical equations are formulated and a mathematical analysis of them is presented. Specifically, following [3], the theorems of existence, uniqueness and regularity of the solution of the dynamical equations are recapitulated and specialized for our purposes, to support the numerical analysis applied previously in [5]. As usual, for that the Galerkin method has been used. As an illustration, two numerical examples (slender masonry tower and masonry arch) are presented in this paper with the applied forces corresponding to the acceleration in the earthquake in Emilia Romagna in May 29, 2012.

Key Words Non-linear dynamics; no-tension material; masonry slender towers and arches; coupling phenomena; Galerkin method

MSC 2010 74B99

Contents

1 Introduction

Recent years have seen increasing interest in conducting vulnerability analyses of masonry buildings, due to the need to preserve the historical heritage. Despite the great research efforts, accurate modeling of the dynamic behavior of masonry structures still represents an open problem. The difficulties are due mainly to the heterogeneities of the masonry and the characteristics of the constituent materials with their differing mechanical responses under tension and compression. Particularly interesting results have been obtained by using the so-called no-tension or masonry-like material which, with the proper numerical techniques, has been successfully applied to the study of the static of several masonry buildings [4]. However, due to the complexity of the dynamic analyses of continuous masonry-like bodies, a constitutive equation for slender masonry structures that can be represented by one-dimensional elements has been formulated [11]. This non-linear elastic constitutive equation expresses the internal forces (normal force and bending moment) as functions of the generalized strains (linear extension and change of curvature of the beams' axis), under the assumption that the material has no resistance to tension in the longitudinal direction and it accounts for the cases of limitless, as well as bounded compressive strength. The model, firstly developed for rectangular cross-section beams, has been generalized to the case of hollow, rectangular cross-sections. Moreover, applied at first to static problems [10], it has revealed to be suitable for conducting non-linear dynamic analyses of slender masonry structures with simple geometry and flexural behavior like towers, bell towers and arches [5]. Coupling phenomena between transverse and axial vibrations, which are recognized as an important factor in the seismic behavior of slender structures are also taken into account, as they are embedded in the constitutive equation. To include the damping of the structure, a linear viscous term is introduced in the equation of motion. Moreover, since the model accounts for the material's non-linear behavior in all sections of the structure, it can be useful to obtain measures of local and global damage. Such measures can be meaningful, even if the assumed constitutive equation of the material does not allow a complete account of the irreversibility of the damage process.

In this paper, in order to support the use of the numerical analysis, we prove, under suitable hypotheses, the existence, uniqueness, regularity and continuous dependence on data of the solution of the equation of motion for our viscous masonry beam. We also state general result on the convergence of the Galerkin method. In Section 2 and in the Appendix some properties of the constitutive equation of the masonry beam are described. In Section 3 we introduce the space functions to be employed in Section 4, where, following [3], the theorems of existence, uniqueness and regularity of the solution of the equation of motion are recapitulated and specialized for our purpose. In Section 5 the abstract results are applied to the study of viscous masonry beams. Finally, a brief account of the numerical method is given in Section 6 where we present two application examples. As usual, for that the Galerkin method has been used.

2 Viscous masonry beams

We consider a general (possibly) curved beam with the initial curvature χ. We parametrize the material points of the beam by the natural (arc length) parameter $s \in I$, where $I \subset \mathbb{R}$ is a closed interval, the reference configuration of the beam. Thus χ is a function of s, and we assume that χ is continuously differentiable. We express general time-dependent fields over the beam as functions of s and of time t. We denote the differentiation with respect to s by an attached prime and the differentiation with respect to time by a superimposed dot. We denote by $S = [0, T]$ the interval of time, where $T > 0$ is arbitrary, and consider the motion of the beam for times from S. The tangential and radial displacements of the beam are denoted by w and v, respectively, so that the displacement of the point $s \in I$ at time $t \in S$ is $(w(t,s), v(t,s))$. The pair $u = (w(\cdot,\cdot), v(\cdot,\cdot))$ is called the displacement function of the beam. The generalized strain $\hat{e}(u)$ of a displacement u is

$$\hat{e}(u)(t,x) = (\varepsilon(t,x), \kappa(t,x))$$

with*

$$\varepsilon = w' - \chi v, \quad \kappa = -(v'' - (\chi w)'). \tag{2.1}$$

These are the (linearized) strain and the change of curvature of the axis of the beam, respectively, that are heuristically deduced in [1].

Internal forces consist of the axial force N and the bending moment M. These are the sums of the corresponding viscous and elastic parts N_v, N_e M_v M_e,

$$N = N_v + N_e, \quad M = M_v + M_e.$$

For the viscous axial force and viscous bending moment we postulate the constitutive equations

$$N_v = N_v(\dot{\varepsilon}, \dot{\kappa}), \quad M_v = M_v(\dot{\varepsilon}, \dot{\kappa}),$$

with the linear dependencies on $\dot{\varepsilon}$, $\dot{\kappa}$. The coefficients of proportionality of the linear maps N_v and M_v are the generalized viscosity coefficients of the masonry body. We shall assume that these viscosities are positive definite in the sense that

$$N_v(\dot{\varepsilon}, \dot{\kappa})\dot{\varepsilon} + M_v(\dot{\varepsilon}, \dot{\kappa})\dot{\kappa} \ge c|(\dot{\varepsilon}, \dot{\kappa})|^2 \tag{2.2}$$

for some $c > 0$ and all $(\dot{\varepsilon}, \dot{\kappa}) \in \mathbb{R}^2$ where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^2$. For some parts of the regularity theory we need also the generalized symmetry of the coefficients of viscosity in the sense that

$$N_v(\dot{\varepsilon}_1, \dot{\kappa}_1)\dot{\varepsilon}_2 + M_v(\dot{\varepsilon}_1, \dot{\kappa}_1)\dot{\kappa}_2 = N_v(\dot{\varepsilon}_2, \dot{\kappa}_2)\dot{\varepsilon}_1 + M_v(\dot{\varepsilon}_2, \dot{\kappa}_2)\dot{\kappa}_1 \tag{2.3}$$

for every $(\dot{\varepsilon}_1, \dot{\kappa}_1) \in \mathbb{R}^2$, $(\dot{\varepsilon}_2, \dot{\kappa}_2) \in \mathbb{R}^2$. The simplest example of the maps N_v and M_v meeting the requirements (2.2) and (2.3) are

$$N_v(\dot{\varepsilon}, \dot{\kappa}) = \alpha\dot{\varepsilon}, \quad M_v(\dot{\varepsilon}, \dot{\kappa}) = \beta\dot{\kappa}$$

for all $(\dot{\varepsilon}, \dot{\kappa}) \in \mathbb{R}^2$ where α and β are positive constants.

The elastic part of the response is assumed to be of the format

$$N_e = N_e(\varepsilon, \kappa), \quad M_e = M_e(\varepsilon, \kappa)$$

* Equations (2.1) correct the corresponding equations in [7].

with specified response functions N_e and M_e. We assume that the beam is made of a masonry material. Several constitutive models are available for this situation. If the beam is made of a no-tension material with infinite compressive strength, then the response functions take the form adopted in [9], [7; Eq. (17)]. If the beam is rectangular with solid cross-section and if it is modelled as a no-tension material with limited compressive strength, then the constitutive equation is more complicated; it is described in [11; Section 2] and recapitulated here in the Appendix. Even more complicated is the response for hollow, rectangular cross-section beams made of a no-tension material with limited compressive strength. It is described in [5; Section 2]. Our treatment of the existence and regularity theory in Section 4 covers all these cases, as will be explained later.

If q and p denote the longitudinal and transversal forces distributed along the beam, the equations of motion for axial and transverse displacements read

$$\left.\begin{aligned} m\ddot{w} &= N' - \chi M' + q, \\ m\ddot{v} &= M'' + \chi N + p, \end{aligned}\right\} \tag{2.4}$$

where m is the mass per unit length of beam. In the treatment of the existence and regularity theory below we shall formulate the weak version of these equations. These are based on the Virtual Power Principle. It takes the following form in the static case:

$$\int_0^l (N\varepsilon + M\kappa)\, ds = \int_0^l (qw + pv)\, ds \tag{2.5}$$

for every sufficiently smooth displacement field (w, v) which satisfies the boundary conditions. Let us now outline the derivation of the equilibrium equations corresponding to the dynamic equations (2.4) from this Virtual Power Principle.

From (2.1) and (2.5) by integrating by parts we deduce

$$\int_0^l (qw + pv)\, ds = \int_0^l \left(N(w' - \chi v) + M(-v'' - (\chi w)')\right)\, ds =$$

$$[Nw]_0^l - \int_0^l N'w ds - \int_0^l N\chi v\, ds - [Mv']_0^l + \int_0^l M'v'\, ds - [M\chi w]_0^l + \int_0^l M'\chi w\, ds =$$

$$[Nw]_0^l - [M(v' + \chi w)]_0^l + [M'v]_0^l - \int_0^l N'w\, ds - \int_0^l N\chi v\, ds - \int_0^l M''v\, ds + \int_0^l M'\chi w\, ds.$$

In view of the boundary conditions (and because we assume that there are no concentrated forces and couples) we obtain

$$[Nw]_0^l = 0, \quad [M'v]_0^l = 0, \quad [M(v' + \chi w)]_0^l = 0, \tag{2.6}$$

and then

$$\int_0^l (qw + pv)\, ds = \int_0^l (-N' + \chi M')w\, ds + \int_0^l -(N\chi + M'')v\, ds.$$

From the arbitrariness of w and v we obtain the equilibrium equations

$$\left.\begin{aligned} N' - \chi M' + q &= 0, \\ M'' + \chi N + p &= 0. \end{aligned}\right\} \tag{2.7}$$

These are the static analogs of (2.4).

2.1 Remark. In [1] the equilibrium equation are given in the form

$$\left.\begin{aligned} N' - \chi T + q &= 0, \\ T' + \chi N + p &= 0, \\ M' - T + m &= 0 \end{aligned}\right\} \tag{2.8}$$

where T is the shear force and m are the distribuited couples along the axis of the beam. For $m = 0$ we have $T = M'$ by $(2.8)_3$ and then equations (2.7) and (2.8) are equivalent. Moreover, the boundary condition $(2.6)_2$ and $(2.6)_3$ can be written as

$$[Tv]_0^l = 0$$

$$[M\phi]_0^l = 0,$$

respectively, where

$$\phi = -(v' + \chi w)$$

is the rotation of the axis of the beam [1].

We now return to the general dynamic case.

In view of the occurence of many possibilities of the boundary conditions, we interpret them abstractly as a given linear subspace V of the space Z of all possible (w, v) pairs of displacements in the same way as in [7]. Anticipating, in the treatment of the existence theory we take for Z the product $W^{1,2}(I) \times W^{2,2}(I)$ of Sobolev spaces and so $V \subset W^{1,2}(I) \times W^{2,2}(I)$. In the example of a beam clamped at the bottom we take

$$V = \{(w, v) \in Z : w(0) = 0, v(0) = v'(0) = 0\}.$$

We interpret the loads applied to the beam as a time-dependent linear functional $\boldsymbol{l} = \boldsymbol{l}(t)$ on the space Z of pairs of displacements.

3 Spaces of time–dependent functions

Let V be a separable Hilbert space, continuously and densely contained in a Hilbert space H and let V^* be its dual, i.e., the space of all continuous linear functionals on V. Identifying the dual of H with H itself, we have the inclusions

$$V \subset H \subset V^*. \tag{3.1}$$

We call every triple as in (3.1) the evolutionary triple.

Let Y be a Banach space with norm $\|\cdot\|_Y$, let $T > 0$ and put $S = [0, T]$. By $L^2(S, Y)$ we denote the space of all (classes of equivalence of) Bochner integrable maps $u : S \to Y$ such that

$$\|u\|^2 := \int_S \|u(t)\|_Y^2 \, dt < \infty.$$

If $f \in L^2(S, Y^*)$ then the duality pairing $\langle u, f\rangle \equiv \langle f, u\rangle$ is defined by

$$\langle u, f\rangle = \int_S \langle u(t), f(t)\rangle_{Y \times Y^*} \, dt$$

where $\langle \cdot, \cdot \rangle_{Y \times Y^*}$ is the duality pairing between Y and Y^*, i.e., for a given $t \in S$, the value $\langle u(t), f(t) \rangle_{Y \times Y^*}$ is the value of the linear functional $f(t)$ on the element $u(t)$.

We denote by $C(S, Y)$ the set of all norm continuous maps $u : S \to Y$.

A map $u : S \to Y$ is said to be strongly differentiable at $t \in S$ if there exists an element $\dot{u}(t) \in Y$ such that

$$\lim_{\substack{h \to 0 \\ t+h \in S}} \|((u(t+h) - u(t))/h - \dot{u}(t)\|_Y = 0.$$

The element $\dot{u}(t)$ is called the strong derivative of u at t. The function $u : S \to Y$ is called strongly differentiable if it is strongly differentiable at every point of the interval S. We denote by $C^1(S, Y)$ the space of all maps $u : S \to Y$ that are strongly differentiable and the derivative $t \mapsto \dot{u}(t)$ is norm continuous on S. Proceeding inductively, we define the strong derivatives of order m where m is a positive integer, and we further denote by $C^m(S, Y)$ the set of all $u : S \to Y$ which have strong derivatives of all orders $\leq m$ and these are norm continuous functions on S.

A map $u : S \to Y$ is said to be weakly differentiable at the point $t \in S$, if there exists an element $\dot{u}(t) \in Y$ satisfying the condition

$$\lim_{\substack{h \to 0 \\ t+h \in S}} \langle (u(t+h) - u(t))/h - \dot{u}(t), f \rangle_{Y \times Y^*} = 0 \quad \text{for each} \quad f \in Y^*.$$

The element $\dot{u}(t)$ is called the weak derivative of u. The function $u : S \to Y$ is called weakly differentiable if it is weakly differentiable at every point of the interval S. By induction we define the weak derivative of order m of $u \in S \to Y$. We denote by $C_w^m(S, X)$ the set of all functions $u : S \to Y$ possessing demicontinuous *weak* derivatives of order $\leq m$. Here a demicontinuous function $u : S \to Y$ is a function such that the function $t \mapsto \langle u(t), f \rangle$ is continuous for each $f \in Y^*$.

If $u \in L^2(S, Y)$, we say that $\dot{u} \in L^2(S, Y)$ is a derivative of u in the sense of distributions if we have

$$\int_S u(t)\dot{\phi}(t)\, dt = -\int_S \dot{u}(t)\phi(t)\, dt$$

for every $\phi : \mathbb{R} \to \mathbb{R}$ that is infinitely many times continuously differentiable and has a support contained in $(0, T)$. The derivative in the sense of distributions, if it exists, is uniquely determined as an element of $L^2(S, Y)$, i.e., up to the set of vanishing Lebesgue measure. More generally, if m is a positive integer we define the derivative of order m in the sense of distributions $u^{(m)} \in L^2(S, Y)$ of $u \in L^2(S, Y)$ by

$$\int_S u(t)\phi^{(m)}(t)\, dt = (-1)^m \int_S u^{(m)}(t)\phi(t)\, dt$$

for every $\phi : \mathbb{R} \to \mathbb{R}$ that is infinitely many times continuously differentiable and has a support contained in $(0, T)$.

4 Operator differential equations of the second order

In this section we recapitulate and specialize the results of [3] relevant to our purpose.

Consider an evolutionary triple (3.1). We shall deal with the operator equations

$$\left.\begin{aligned} \ddot{u}(t)+A\dot{u}(t)+Bu(t)=f(t) \quad &\text{for almost every } t\in S,\\ u(0)=a_0, \quad \dot{u}(0)=a_1, \quad u\in C(S,V), \quad &\dot{u}\in V, \end{aligned}\right\} \quad (4.1)$$

where A and B are possibly nonlinear operators mapping V into V^*, $f\in L^2(S,V^*)$, $a_0\in V$ and $a_1\in H$. Here $\ddot{u}\in L^2(S,V^*)$ is the second derivative in the sense of distributions of u, considered as a map from $S\to V^*$, while $\dot{u}$ is the first derivative in the sense of distributions of u, considered as a map from S to $V\subset V^*$. Thus $\ddot{u}:S\to V^*$ satisfies

$$\int_S u(t)\ddot{\phi}(t)\,dt=\int_S \ddot{u}(t)\phi(t)\,dt$$

for every $\phi:\mathbb{R}\to\mathbb{R}$ that is infinitely many times continuously differentiable and has a support contained in $(0,T)$ while $\dot{u}:S\to V$ satisfies

$$\int_S u(t)\dot{\phi}(t)\,dt=-\int_S \dot{u}(t)\phi(t)\,dt$$

for every $\phi:\mathbb{R}\to\mathbb{R}$ that is infinitely many times continuously differentiable and has a support contained in $(0,T)$. Both these definitions are particular cases of the general definition in Section 3, the first one with the choice $Y=V^*$ and the second with $Y=V$. We note that we have $\dot{u}\in W$ where

$$W:=\{w\in L^2(S,V),\dot{w}\in L^2(S,V^*))\}\subset C(S,H)$$

where the last inclusion follows from [3; Chapter IV, Theorem 1.17]. Thus the initial condition $\dot{u}(0)=a_1\in H$ is meaningful. We note that (4.1) is a special case of the second order operator differential equations considered in [3; Chapter VII] where Volterra operators, i.e., those depending on the history of u up to t are admitted and where the exponents $p\geq 2$ are admitted in some results, but not in the results pertinent to our development.

An operator A from V to V^* is said to be strongly monotone if

$$\langle Au-Av,u-v\rangle_{V\times V^*}\geq c\|u-v\|_V^2$$

for all $u,v\in V$ and some $c>0$. An operator A from V to V^* is said to be radially continuous if $A(u+hv)\to A(u)$ whenever $h\in\mathbb{R}$ satisfies $h\to 0$ and the assertion holds for every $u,v\in V$. An operator B from V to V^* is said to be Lipschitz continuous if

$$\|Bu-Bv\|_{V^*}\leq c\|u-v\|_V$$

for all $u,v\in V$ and some $c>0$.

Specializing [3; Theorem 1.2 and Remark 1.3, Chapter VII] to our situation we obtain the following existence result.

4.1 Theorem. *Let $A:V\to V^*$ be a radially continuous strongly monotone operator and let $B:V\to V^*$ be Lipschitz continuous operator. Then for arbitrary $a_0\in V$, $a_1\in H$ and $f\in L^2(S,V^*)$ the problem* (4.1) *has exactly one solution. For this solution we have $\dot{u}\in W$. Moreover, the solution depends continuously on the data in the sense that the map $(a_0,a_1)\mapsto(u,\dot{u})$ is a continuous map from $V\times H$ to $C(S,V)\times C(S,H)$.*

The existence of the solution is obtained via the application of the Galerkin method that we now briefly outline.

Let $\{h_1, h_2, \dots\}$ denote a linearly independent infinite sequence such that its span is dense in V. The span of $\{h_1, h_2, \dots\}$ is also dense in H as a consequence. Let H_n denote the span of $\{h_1, \dots, h_n\}$ with the scalar product induced by the scalar product of H. We assume that the dual space H_n^* is identified with H_n. For each operator $D : V \to V^*$ there exists an operator $D_n : H_n \to H_n$ satisfying

$$\langle D_n u, v\rangle_{V\times V^*} = \langle Du, v\rangle_{V\times V^*} \qquad \text{for each } u, v \in H_n.$$

We shall use this possibility in particular for the operators A and B.

Let $\{a_{0n}\}$ be an arbitrary sequence of elements of H_n converging in V to a_0. Let, further, $\{a_{1n}\}$ be any sequence of elements of H_n converging to a_1 in H. Let, finally, $f_n \in L^2(S, H_n)$ be defined by the relation

$$\langle f_n, v\rangle = \langle f, v\rangle \qquad \text{for each } v \in L^2(S, H_n).$$

We consider the following Galerkin's equations

$$\left.\begin{aligned} \ddot{u}_n + A_n\dot{u}_n + B_n u_n = f_n,\\ u_n(0) = a_{0n}, \quad \dot{u}_n(0) = a_{1n},\\ u_n \in C(S; H_n), \quad \dot{u}_n \in L^2(S, H_n). \end{aligned}\right\} \tag{4.2}$$

Specializing [3; Lemma 2.2 and Theorem 2.2, Chapter VII] to our situation, we obtain the following two assertions.

4.2 Lemma. *Under the assumptions of Theorem* 4.1 *for each positive integer n the problem* (4.2) *has exactly one solution* $u_n \in C^1(S, H_n)$*, having the property that* $\ddot{u}_n \in L^2(S, H_n)$*. The sequence* $\{u_n\}$ *is bounded in* $C(S, V)$ *and the sequence* $\{\dot{u}_n\}$ *is bounded in* $C(S, H)$ *and in* $L^2(S, V)$*; the sequence* $A\dot{u}_n$ *is bounded in* $L^2(S, V^*)$*.*

4.3 Theorem. *Assume that the hypotheses of Theorem* 4.1 *hold. Denote by* u_n *the solutions of the Galerkin's equations* (4.2) *and by u the solution of the problem* (4.1)*. Then for* $n \to \infty$ *we have*

a) $u_n \to u$ *in* $C(S, V)$*,*

b) $\dot{u}_n \to \dot{u}$ *in* $C(S, H)$ *and in* $L^2(S, V)$;

c) $A\dot{u}_n \rightharpoonup A\dot{u}$ *in* $L^2(S, V^*)$*.*

Next we are going to discuss the regularity of the dependence of the solution on t. [3] gives two kinds of such results, the first type giving the regularity on S under a special choice of the initial conditions and the second type for general initial conditions but on intervals of the type $[\delta, T]$ for any δ satisfying $0 < \delta < T$.

From [3; Theorem 3.2, Chapter VII] we derive the following result.

4.4 Theorem. *Let* $A : V \to V^*$ *be radially continuous and strongly monotone and let the operator* $B : V \to V^*$ *be Lipschitz continuous and let* $f \in L^2(S, V^*)$ *be Lipschitz continuous, i.e.,*

$$\|f(s) - f(t)\|_{V^*} \le c|s - t| \tag{4.3}$$

for every $s, t \in S$ *and some* $c > 0$*. Then for every* $a_0, a_1 \in V$ *with* $Aa_1 + Ba_0 + f(0) \in H$ *the problem*

$$\left.\begin{aligned} \ddot{u}(t) + A\dot{u}(t) + Bu(t) = f(t) \quad \textit{for each} \quad t \in S,\\ u(0) = a_0, \quad \dot{u}(0) = a_1, \quad u \in C^1(S, V) \cap C_w^2(S, H) \end{aligned}\right\} \tag{4.4}$$

has exactly one solution.

We here note that we apply [3; Theorem 3.2, Chapter VII] with the family $\{A(t)\}, t \in S$, of that theorem equal to our time independent operator A and with the family $\{B(t)\}, t \in S$, given by

$$B(t)w = Bw - f(t)$$

for each $t \in S$ and $w \in V$. The function ρ_A of [3; Theorem 3.2, Chapter VII] may be set equal to 0 and the function ρ_B may be given by $\rho_B(\tau) = c$ for all $\tau \geq 0$ where the constant c is as in (4.3). Then the hypotheses of [3; Theorem 3.2, Chapter VII] are satisfied and the conclusion gives Theorem 4.4.

We say that $A : V \to V^*$ is a potential operator if there exists a function $F : V \to \mathbb{R}$ such that A is the Gateaux derivative of F.

4.5 Theorem. *Let the assumptions of Theorem* 4.4 *be satisfied and assume, in addition, that A is a potential operator. Then for every $a_0 \in V$, $a_1 \in H$ the problem*

$$\left.\begin{aligned} \ddot{u}(t) + A\dot{u}(t) + Bu(t) = f(t), \quad & 0 < t \leq T, \\ u(0) = a_0, \quad \dot{u}(0) = a_1, \quad & u \in C(S, V), \\ \dot{u} \in C^1_w([\delta, T], H) \cap C([\delta, T]; V) \cap W \quad & \textit{for each} \quad \delta \in (0, T] \end{aligned}\right\} \quad (4.5)$$

has exactly one solution.

This result follows from [3; Theorem 3.4, Chapter VII], where we identify the operator $D : V \to V^*$ of that theorem with our A and where we put the operator C of [3; Theorem 3.4, Chapter VII] to be equal identicaly to 0.

5 Application of the abstract results to viscous masonry beams

We take for V a closed subspace of $W^{1,2}(I) \times W^{2,2}(I)$, which is the same subspace as equally denoted subspace in [7; Section 2]. This represents the boundary conditions. We furthermore take the time-dependent loads $\boldsymbol{l} \in L^2(S, V^*)$. We define H to be the closure in $L^2(I, \mathbb{R}^2)$ of the space V with the obvious inclusion of V in H. Then

$$V \subset H \subset V^*$$

forms an evolutionary triplet.

We define the operators $A, B : V \to V^*$ by

$$\langle Aa, \alpha \rangle = \int_I (N_v(\hat{e}(a)), M_v(\hat{e}(a)) \cdot \hat{e}(\alpha)\, ds$$

and

$$\langle Ba, \alpha \rangle = \int_I (N_e(\hat{e}(a)), M_e(\hat{e}(a))) \cdot \hat{e}(\alpha)\, ds$$

for any $a = (b, c) \in V$ and $\alpha = (\beta, \gamma) \in V$. We denote the arguments of the operators A and B by the neutral symbol a although in the application of A and B to our differential equation the arguments will be $a = \dot{u} = (\dot{w}, \dot{v})$ and $a = u = (w, v)$, respectively, where w and v are the longitudinal and transversal displacements. Since we assume that the viscous response functions N_v and M_v are linear, we see that A is a linear transformation from V to V^*; clearly, it is bounded. Finally, $f \in L^2(S, V^*)$ is defined by

$$f(t) = \boldsymbol{l}(t), \quad t \in S.$$

To apply Theorem 4.1, we only need to assume the positive definite property (2.2), which guarantees that A is a strongly monotone operator. That B is Lipschitz continuous follows from the Lipschitz continuity of the maps N_e, M_e. This is proved in [7] for the case of a beam with infinite compressive strength. In the cases of materials with finite compressive strength considered in [11; Section 2] and [5; Section 2] this is verified by a detailed analysis of the response functions described in the last two references. Thus under (2.2), we have a well-defined global evolution of the viscous masonry beam. The same hypothesis also suffices for the convergence of the Galerkin approximations in Theorem 4.3.

The regularity Theorem 4.4 for the special choice of the initial conditions requires, besides (2.2), also the Lipschitz continuity of the dependence of the loads on time.

Finally, the regularity Theorem 4.5 requires (2.2), the Lipschitz continuity of the loads on time, and the symmetry property (2.3), which guarantees that A is a potential operator with a quadratic potential.

6 Numerical examples

In this section, firstly we briefly describe the numerical method that has been used to integrate the motion equations and then we present two numerical examples. In the first example we study a slender masonry tower while in the second one we study a masonry arch. In both the cases, as dynamic action, we consider a horizontal acceleration recorded during the recent Emilia Romagna earthquake which occurred in May 29, 2012, having the magnitude 6.0, PGA= 2.89 m/s^2, and duration of 35.3 s. The computations have been performed trough the code MADY which implements the described numerical method [6].

Because for the discretization of the structures we use rectilinear beam elements, we rewrite equations (2.5) with $\chi = 0$,

$$\ddot{v} + M'' + p = 0, \quad \ddot{u} - N' - q = 0. \tag{6.1}$$

If we multiply $(6.1)_1$ and $(6.1)_2$ by the test functions u_f and u_a, respectively, which satisfy the appropriate boundary conditions, and integrate by parts, we obtain

$$\int_0^l u_f'' M \, ds - \int_0^l u_f m \ddot{v} \, ds + \int_0^l u_f q \, ds = 0, \tag{6.2}$$

$$\int_0^l -u_a' N \, ds - \int_0^l u_a m \ddot{w} \, ds + \int_0^l u_a p \, ds = 0. \tag{6.3}$$

In discretizing the structure into finite elements, conforming elements and Hermite shape functions have been selected in order to guarantee continuity of both the transverse beam-axis displacement and its rotation, while linear shape functions have been used for the axial displacement [8]. Therefore, for each node there are three degrees of freedom: the axial and transverse displacement plus the rotation.

Let us now denote the selected functions defined over the master element $\hat{\Omega}$ (i.e., for $-1 \le \xi \le 1$) as

$$\hat{\Psi}_{01}(\xi) = \frac{1}{4}(\xi - 1)^2 (\xi + 2), \quad \hat{\Psi}_{02}(\xi) = \frac{1}{4}(\xi + 1)^2 (2 - \xi), \tag{6.4}$$

$$\hat{\Psi}_{11}(\xi) = \frac{1}{4}(\xi - 1)^2 (\xi + 1), \quad \hat{\Psi}_{12}(\xi) = \frac{1}{4}(\xi + 1)^2 (\xi - 1), \tag{6.5}$$

$$\hat{\Phi}_1(\xi) = \frac{1}{2}(1 - \xi), \quad \hat{\Phi}_2(\xi) = \frac{1}{2}(1 + \xi) \tag{6.6}$$

and let $\hat{\mathbf{\Psi}}(\xi)$ and $\hat{\mathbf{\Phi}}(\xi)$ be the vectors

$$\hat{\mathbf{\Psi}}^T = \left\{ 0 \ \hat{\Psi}_{01} \ \hat{\Psi}_{11} \ 0 \ \hat{\Psi}_{02} \ \hat{\Psi}_{12} \right\}, \quad \hat{\mathbf{\Phi}}^T = \left\{ \hat{\Phi}_1 \ 0 \ 0 \ \hat{\Phi}_2 \ 0 \ 0 \right\}.$$

Moreover, denoting as $\mathbf{\Psi}(\zeta)$ and $\mathbf{\Phi}(\zeta)$ the vectors

$$\mathbf{\Psi}^T(\zeta) = \left\{ 0 \ \Psi_{01} \ \frac{l_e}{2}\Psi_{11} \ 0 \ \Psi_{02} \ \frac{l_e}{2}\Psi_{12} \right\}, \quad \mathbf{\Phi}^T(\zeta) = \left\{ \Phi_1 \ 0 \ 0 \ \Phi_2 \ 0 \right\},$$

where ζ is the local coordinate along each mesh element, with origin at the first node, and l_e is the element's length. Lastly, denoting $\mathbf{q}$ as the vector of nodal displacements of each element,

$$\mathbf{q}^T = \left\{ w_1 \ v_1 \ \varphi_1 \ w_2 \ v_2 \ \varphi_2 \right\},$$

the functions $v(s,t)$ and $w(s,t)$, as well as the test functions, can be approximated over each element at each time t as

$$v(s) = \mathbf{\Psi}^T(s)\mathbf{q}(t), \quad w(s) = \mathbf{\Phi}^T(s)\mathbf{q}(t).$$

Therefore, by substituting into equation (6.2) and (6.3), for the i-th element E, we obtain the discretized equations

$$\mathbf{M}^E \ddot{q}^E + \boldsymbol{f}_i^E - \boldsymbol{f}_e^E = 0 \tag{6.7}$$

where the total matrix of mass M^E, the vector of the external forces $\boldsymbol{f}_e^E$ and the vector of internal forces $\boldsymbol{f}_i^E$ are given by

$$\mathbf{M}^E = \int_{-1}^{1} m\mathbf{\Psi}^T\mathbf{\Psi}\, d\xi + \int_{-1}^{1} m\mathbf{\Phi}^T\mathbf{\Phi}\, d\xi,$$

$$\boldsymbol{f}_e^E = \int_{-1}^{1} p\mathbf{\Psi}\, d\xi + \int_{-1}^{1} q\mathbf{\Phi} d\xi,$$

$$\boldsymbol{f}_i^E = -\int_{-1}^{1} \mathbf{\Psi}'' M\, d\xi + \int_{-1}^{1} \mathbf{\Phi}' N\, d\xi.$$

By suitably assembling equation (6.7), we have determined the motion equations for the entire structure. The Newmark method is then used to integrate the ordinary, non-linear differential equation over time, and the resulting non-linear algebraic system can be solved via the Newton-Raphson iterative method. As most of these techniques are standard, a detailed explanation is omitted. Nevertheless, it should be noted that defining the stiffness matrix $\tilde{\mathbf{K}}^E$ requires calculating the derivatives of the generalized stress with respect to the generalized strain in each of the domain regions, since $\tilde{\mathbf{K}}^E$ is given by

$$\tilde{\mathbf{K}}_s^E = \frac{\partial}{\partial \mathbf{q}}\Big(-\int_{-1}^{1} \mathbf{\Psi}'' M \, d\xi + \int_{-1}^{1} \mathbf{\Phi}' N \, d\xi\Big).$$

Regarding the effects of viscous damping, these have been accounted for by including in the motion equation a constant symmetric viscous damping matrix, $\mathbf{C}$ which is obtained as a linear combination of $\mathbf{M}$ and the initial elastic stiffness matrix $\mathbf{K}$, as per the Rayleigh assumption. Then the positive definite property (2.2) and the symmetry property (2.3) are satisfied.

6.1 Masonry tower Let us consider a free-standing masonry tower with height $H = 45\,m$, having a square section with external side $b = 5.5\,m$ and thickness $t = 1.5\,m$. As the constitutive equation we consider the relations set forth in [5; Section 2] for hollow rectangular cross section, under the assumption that the material has no resistance to tension and limited compressive strength, with the following values of parameters: Young modulus $E = 3\,10^3\,MPa$, compressive strength $\sigma_o = -3.0\,MPa$ and density $\gamma = 1800\,kg/m^3$. The assumed damping is 2%.

Figure 6.1 shows the behavior of the displacement of the top of the tower as a function of time; the maximum value of the displacement is about .3 meters and it is reached at $t = \bar{t} = 7.895\,s$. For $t = \bar{t}$, the behavior of the displacement, velocity and acceleration along the height of the tower are shown in Figure 6.2, while Figure 6.3 shows the behavior of the bending moment, eccentricity $e = M/N$ and shear force T. Finally Figure 6.4 shows, for $t = \bar{t}$ and for each cross section, the percentage of cracked and crusched areas that gives an estimation of the damage of the structure.

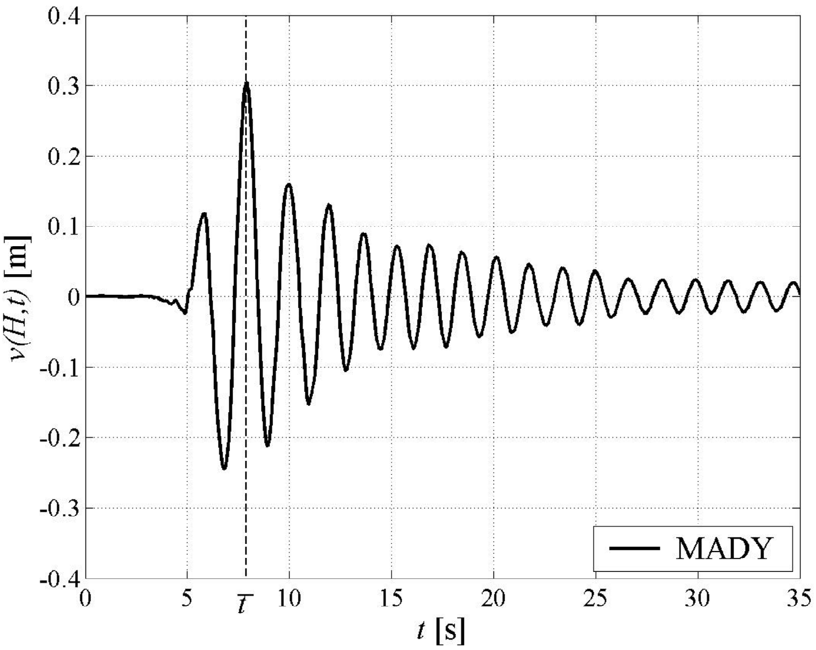

Fig. 6.1. Transversal displacement $v(H,t)$ at the top of the tower as a function of time.

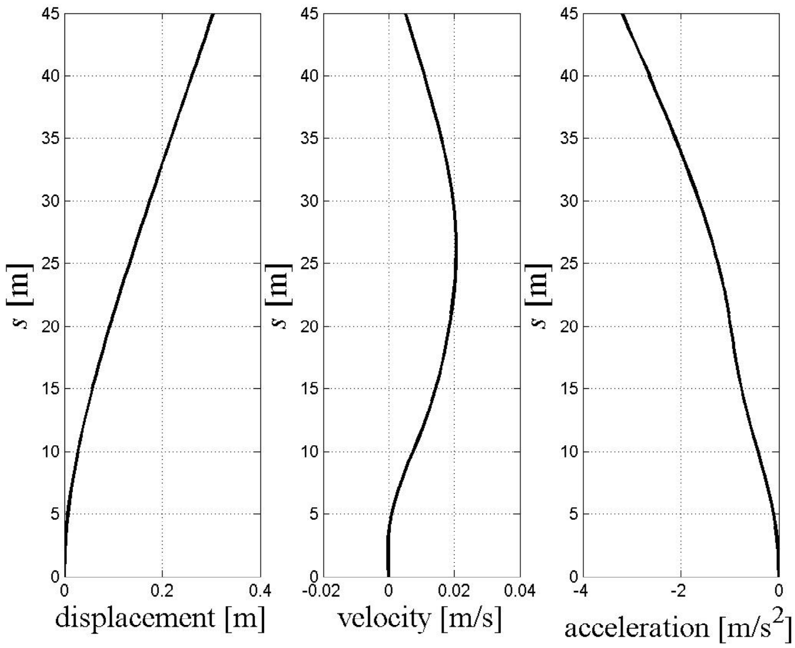

Fig. 6.2. Transversal displacement $v(s,\bar{t})$, velocity $\dot{v}(s,\bar{t})$ and acceleration $\ddot{v}(s,\bar{t})$ as a function of s, at $\bar{t} = 7.895\,s$.

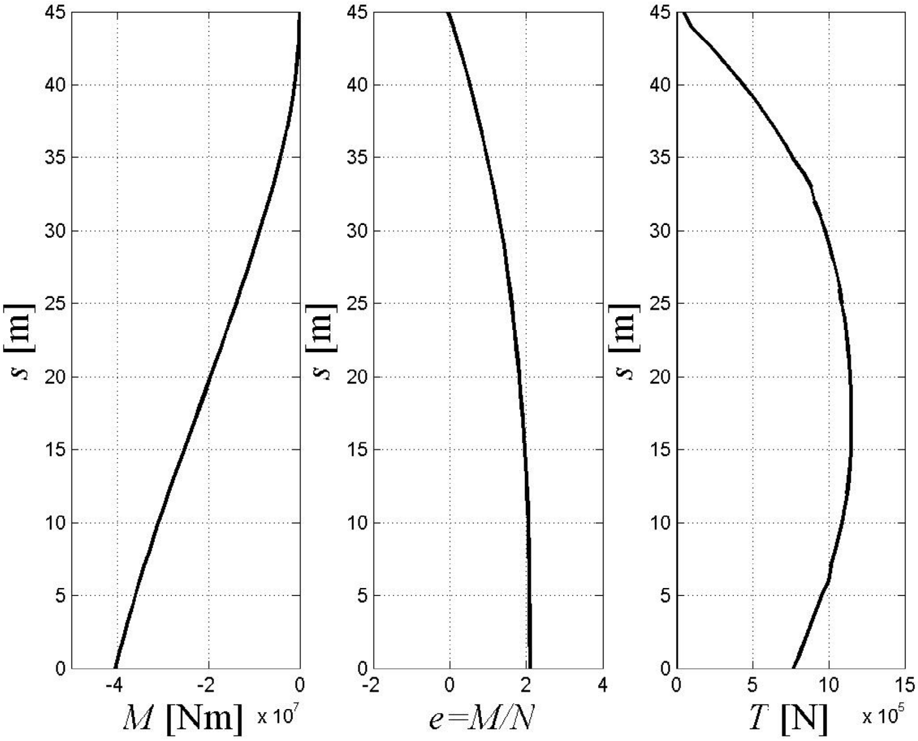

Fig. 6.3. Bending moment $M(s, \bar{t})$, eccentricity $e(s, \bar{t})$ and shear force $T(s, \bar{t})$ as a function of s, at $\bar{t} = 7.895\,s$.

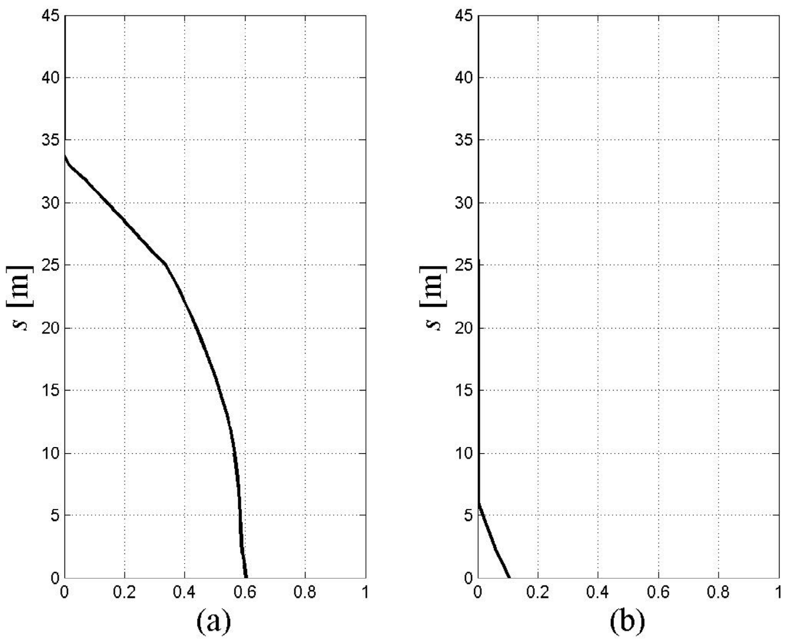

Fig. 6.4. Percentage of (a) cracked and (b) crushed area as a function of s, at $\bar{t} = 7.895\,s$.

6.2 Masonry arch Let us consider a masonry circular arch with span $l = 15\,m$, internal radius $R = 7.98\,m$, width $d = 1\,m$, thickness $2h = 0.75\,m$ and springing angle of $20°$ [2]. As the constitutive equation we consider the relations set forth in [11; Section 2] for solid rectangular cross section, under the assumption that the material has no resistance to tension and limited compressive strength (see the Appendix), with the following values of parameters: Young modulus $E = 1.5\,10^4\,MPa$, compressive strength $\sigma_0 = -5.0\,MPa$, density $\gamma = 2200\,Kg/m^3$ and damping 4%.

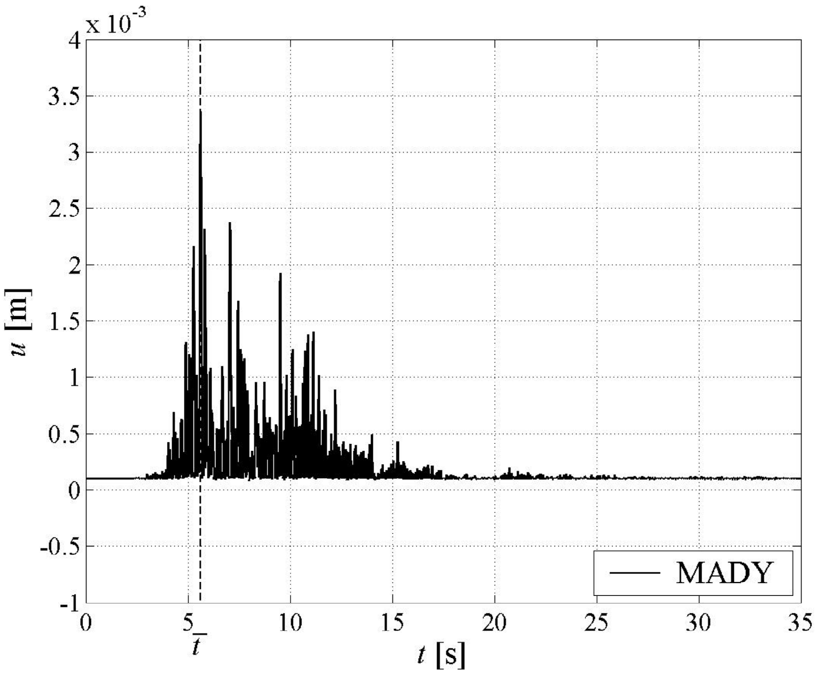

Fig. 6.5. Total displacement $u = \sqrt{w^2 + v^2}$ at the point P.

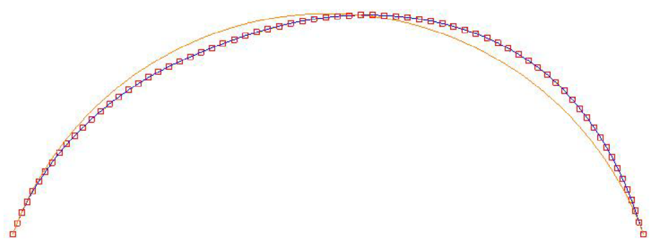

Fig. 6.6. Deformation of the central line of the arch at $\bar{t} = 5.615s$.

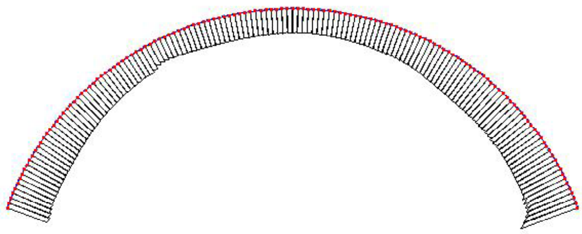

Fig. 6.7. Axial force N at $\bar{t} = 5.615s$. (Maximum value $2.879\,10^5$ N)

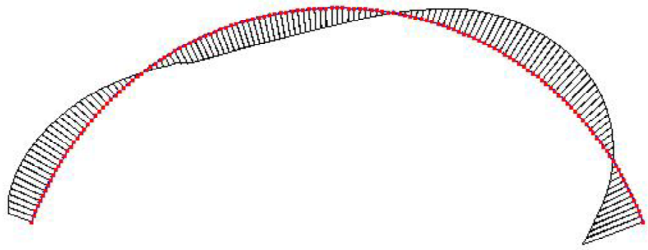

Fig. 6.8. Bending moment M at $\bar{t} = 5.615s$. (Maximum value $9.305\,10^4$ Nm)

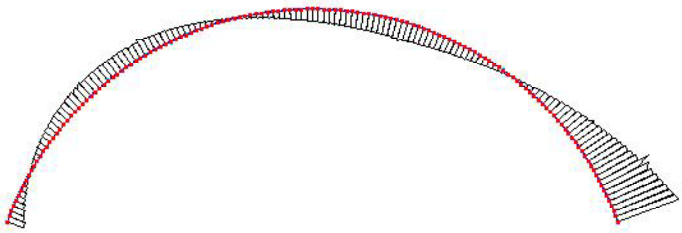

Fig. 6.9. Shear force T at $\bar{t} = 5.615s$. (Maximum value $7.15\,10^4$ N)

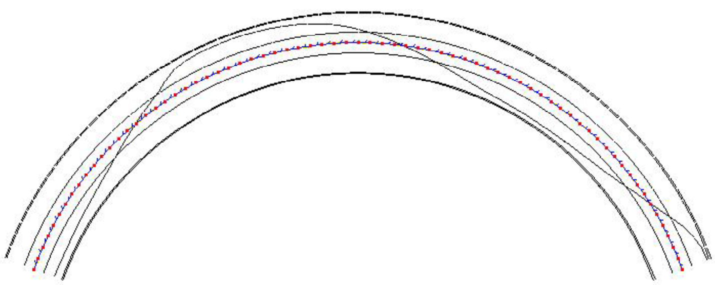

Fig. 6.10. Line of trust at $\bar{t} = 5.615s$.

Figure 6.5 shows the total displacement $u = \sqrt{w^2 + v^2}$ at point the P at an angular distance of $34°$ from the right springing, as a function of time. The maximum value of u is about $3.4\ 10^{-3}\ m$ and it is reached at $t = \bar{t} = 5.615s$. For $t = \bar{t}$, Figure 6.6 shows the strained center line and Figures 6.7, 6.8, 6.7 show the generalized stresses, axial force N, bending moment M and shear force T, respectively. The maximum values of the generalized stresses are reached at the right springing and they are $N_{\max} = 2.879\,10^5 N$, $M_{\max} = 9.305\,10^4 Nm$ and $T_{\max} = 7.15\,10^4 N$. Finally, Figure 6.10 shows the line of trust which is wholly contained in the "reduced arch", i.e. an arch with thickness equal to $h(1 - N/2\sigma_0 hd)$, according to (7.1) (below).

7 Appendix: The stored energy of a no-tension beam with bounded compressive strength

In this Appendix we consider beams having rectangular cross section of height $2h$, made of no-tension material with compressive Young modulus E and compressive strenght $\sigma_0 < 0$ [9, 11]. In the following, in order to use the dimensionless quantities, we will write κ for $h\kappa$, ϕ for ϕ/EA, N for N/EA and M for M/EAh, where A is the area of the section. Moreover we put $\varepsilon_0 = \sigma_0/E$.

Let us consider the partition of the generalized strains $\mathrm{I\!R}^2 = \Sigma_1 \cup \Sigma_2^{\pm} \cup \Sigma_3^{\pm} \cup \Sigma_4^{\pm} \cup \Sigma_5 \cup \Sigma_6$ (Fig. 7.1), where

$$\begin{cases} \Sigma_1 = \{\boldsymbol{e} = (\varepsilon, \kappa) \in \mathrm{I\!R}^2; |\kappa| \le -\varepsilon, |\kappa| \le \varepsilon - \varepsilon_0\}, \\ \Sigma_2^{\pm} = \{\boldsymbol{e} = (\varepsilon, \kappa) \in \mathrm{I\!R}^2; \pm\kappa < |\varepsilon|, \pm\kappa > \varepsilon_0 - \varepsilon\}, \\ \Sigma_3^{\pm} = \{\boldsymbol{e} = (\varepsilon, \kappa) \in \mathrm{I\!R}^2; \pm\kappa < |\varepsilon - \varepsilon_0|, \pm\kappa > \varepsilon\}, \\ \Sigma_4^{\pm} = \{\boldsymbol{e} = (\varepsilon, \kappa) \in \mathrm{I\!R}^2; \pm\kappa < \varepsilon, \pm\kappa > \varepsilon_0 - \varepsilon\}, \\ \Sigma_5 = \{\boldsymbol{e} = (\varepsilon, \kappa) \in \mathrm{I\!R}^2; \pm|\kappa| \le \varepsilon\}, \\ \Sigma_6 = \{\boldsymbol{e} = (\varepsilon, \kappa) \in \mathrm{I\!R}^2; \pm|\kappa| \ge \varepsilon - \varepsilon_0\}, \end{cases}$$

and the subset Ω of the generalized stress $\mathrm{I\!R}^2$ (Fig. 7.2), $\Omega = \Omega_1 \cup \Omega_2^{\pm} \cup \Omega_3^{\pm} \cup \Omega_4^{\pm} \cup \Omega_5 \cup \Omega_6$ where

$$\begin{cases} \Omega_1 = \{\boldsymbol{t} = (N, M) \in \Omega; N \le 0, |M| \le -N/3, |M| \le (N - \varepsilon_0)/3\}, \\ \Omega_2^{\pm} = \{\boldsymbol{t} = (N, M) \in \Omega; \pm M < N/3, \pm M \ge -4N^2/3\varepsilon_0 + N\}, \\ \Omega_3^{\pm} = \{\boldsymbol{t} = (N, M) \in \Omega; \pm M < (\varepsilon_0 - N)/3, \pm M \ge 4N^2/3\varepsilon_0 + 5N/3 - \varepsilon_0/3\}, \\ \Omega_4^{\pm} = \{\boldsymbol{t} = (N, M) \in \Omega; \pm M < -4N^2/3\varepsilon_0 + 5N/3 - \varepsilon_0/3, \\ \qquad \pm M < -4N^2/3\varepsilon_0 + N, \pm M \ge -N^2/\varepsilon_0 + N\}, \\ \Omega_5 = \mathbf{0}, \\ \Omega_6 = \{\varepsilon_0, 0\}. \end{cases}$$

We note that from the previous relations it follows that the eccentricity $e = M/N$ has to satisfy the inequality

$$|e| \le h(1 - \frac{N}{A\sigma_0}). \tag{7.1}$$

Let $\Phi : \mathrm{I\!R}^2 \to \mathrm{I\!R}^2$ be defined by

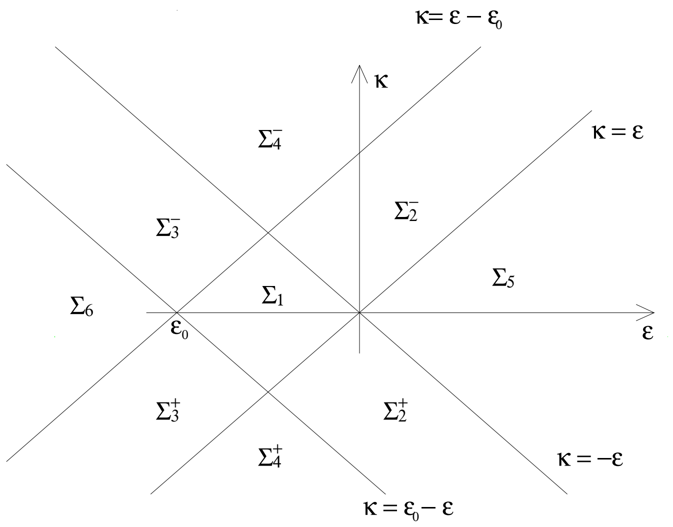

Fig. 7.1. Partition of the generalized strains.

$$\Phi(\varepsilon,\kappa)=\begin{cases}\dfrac{3\varepsilon^2+\kappa^2}{6} & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_1,\\ \dfrac{(\kappa\mp\varepsilon)^3}{6\kappa} & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_2^{\pm},\\ \pm\dfrac{\varepsilon^3+3\varepsilon^2(\pm\kappa-\varepsilon_0)+3\varepsilon(\varepsilon_0\pm\kappa)^2-\varepsilon_0^3-3\kappa^2\varepsilon_0\pm\kappa^3}{6\kappa} & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_3^{\pm},\\ \mp\dfrac{\varepsilon_0(3\varepsilon^2-3\varepsilon(\varepsilon_0\pm 2\kappa)+\varepsilon_0^2+3\kappa^2)}{6\kappa} & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_4^{\pm},\\ 0 & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_5,\\ \varepsilon\varepsilon_0 & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_6,\end{cases}\tag{7.2}$$

$(\varepsilon,\kappa)\in\mathbb{R}^2$. It is easy to see that (7.2) is the energy function corresponding to the constitutive equation given in [9, 11].

7.1 Proposition. *The function Φ is convex, continuously differentiable and nonnegative with derivative* $\mathrm{D}\,\Phi=(\hat N,\hat M)$ *given by*

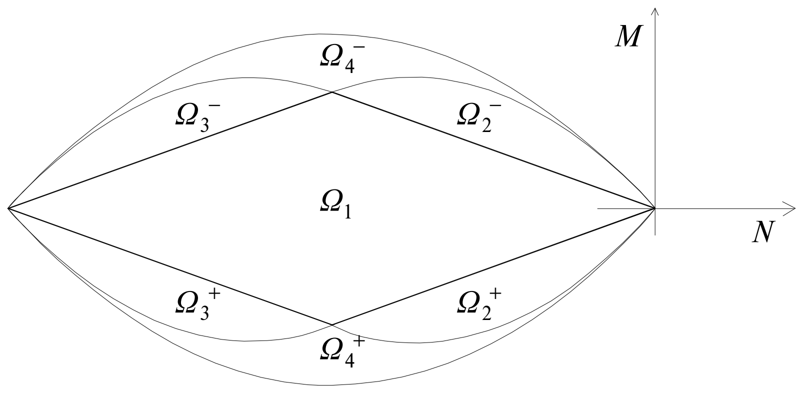

Fig. 7.2. Partition of the generalized stresses.

$$
(\hat{N},\hat{M})(\varepsilon,\kappa)=\begin{cases}
(\varepsilon,\kappa/3) & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_1,\\
\dfrac{(\varepsilon\mp\kappa)^2}{12\kappa^2}(\mp 3\kappa,\pm\varepsilon+2\kappa) & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_2^\pm,\\
\Big(\dfrac{(\varepsilon-\varepsilon_0\pm\kappa)^2\pm 4\varepsilon_0\kappa}{4\kappa},\dfrac{(\varepsilon-\varepsilon_0+\kappa)^2(\pm\varepsilon\mp\varepsilon_0-2\kappa)}{12\kappa^2}\Big) & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_3^\pm,\\
\Big(\mp\dfrac{\varepsilon_0(2\varepsilon-\varepsilon_0-2\kappa)}{4\kappa},\pm\dfrac{\varepsilon_0(3\varepsilon^2-3\varepsilon\varepsilon_0+\varepsilon_0^2-3\kappa^2)}{12\kappa^2}\Big) & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_4^\pm,\\
(0,0) & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_5,\\
(\varepsilon_0,0) & \text{if}\quad (\varepsilon,\kappa)\in\Sigma_6
\end{cases}
$$

$(\varepsilon,\kappa)\in\mathbb{R}^2$*; one has* $\mathrm{D}\,\Phi(\Sigma_1)=\Omega_1$, $\mathrm{D}\,\Phi(\Sigma_2^\pm)=\Omega_2^\pm$, $\mathrm{D}\,\Phi(\Sigma_3^\pm)=\Omega_3^\pm$, $\mathrm{D}\,\Phi(\Sigma_4^\pm)=\Omega_4^\pm$, $\mathrm{D}\,\Phi(\Sigma_5)=\mathbf{0}$, $\mathrm{D}\,\Phi(\Sigma_6)=(\varepsilon_0,0)$. *Moreover,* $\Phi(\varepsilon,\kappa)\le c|(\varepsilon,\kappa)|^2$, $|\mathrm{D}\,\Phi(\varepsilon,\kappa)|\le c|(\varepsilon,\kappa)|$ *for some* $c\in\mathbb{R}$ *and all* $(\varepsilon,\kappa)\in\mathbb{R}^2$.

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8 References

1 Baldacci, R.: *Scienza delle Costruzioni* Torino, UTET (1983)

2 Brencich, A.; Morbiducci, R.: *Masonry Arches: Historical Rules and Modern Mechanics* International Journal of Architectural Heritage: Conservation, Analysis and Restoration **1** (2007) 165–189

3 Gajewski, H.; Gröger, K.; Zacharias, K.: *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen* Berlin, Akademie–Verlag (1974)

4 Lucchesi, M.; Padovani, C.; Pasquinelli, G.; Zani, N.: *Masonry Constructions: Mechanical Models and Numerical Applications* Berlin, Springer (2008)

5 Lucchesi, M.; Pintucchi, B.: *A numerical model for non-linear dynamic analysis of slender masonry structures* European Journal of Mechanics A/Solids **26** (2007) 88–105

6 Lucchesi, M.; Pintucchi, B.; Zani, N.: *The finite elements code MADY for non-linear static and dynamic analysis of masonry structures* (2013) (In preparation)

7 Lucchesi, M.; Šilhavý, M.; Zani, N.: *Equilibrium problems and limit analysis for masonry beams* J. Elasticity **106** (2012) 165–188

8 Oden, J.T.; Carey, G.F.: *Finite Elements, Special problems in solid mechanics. Volume V* Prentice-Hall Inc. (1984)

9 Orlandi, D.: *Analisi non lineare di strutture ad arco in muratura* (1999) Ph.D. thesis, Universitá degli Studi di Firenze, Firenze

10 Pintucchi, B.; Zani, N.: *Effects of material and geometric non-linearities on the collapse of masonry arches* European J. Mechanics A/Solids **28** (2009) 45–61

11 Zani, N.: *A constitutive equation and a closed-form solution for no-tension beams with limited compressive strength* European J. Mechanics A/Solids **23** (2004) 467–484