Mathematical thermodynamics of fluids

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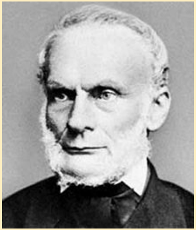
Complete fluid systems



Rudolph Clausius
[1822-1888]

Die Energie der Welt ist constant;
Die Entropie der Welt strebt einem Maximum zu

All pictures in the text thanks to wikipedia

Fluids at equilibrium

Thermodynamic state variables

mass density .... $\varrho = \varrho(t, x)$

absolute temperature .... $\vartheta = \vartheta(t, x)$

Thermodynamic functions

pressure .... $p = p(\varrho, \vartheta)$

internal energy .... $e = e(\varrho, \vartheta)$

entropy .... $s = s(\varrho, \vartheta)$

Gibbs' relation

$$\vartheta D s(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Dynamics, diffusion, transport

Macroscopic velocity

$$\mathbf{u} = \mathbf{u}(t,x), \ \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t) = \mathbf{u}(t, \mathbf{X}(t)), \ \mathbf{X}(0) = x$$

Viscosity - Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I},$$

Heat conductivity - Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Energetically insulated system

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0 \text{ or } (\mathbb{S} \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

Conservation (balance) laws

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}$$

Thermal energy vs. entropy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho \mathbf{u} e) + \mathrm{div}_x \mathbf{q} \boxed{\geq} \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho \mathbf{u} s) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \boxed{\geq} \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x = 0$$

Well posedness, classical way

Jacques Hadamard
[1865 - 1963]

- **Existence.** Given problem is solvable for any choice of (admissible) data
- **Uniqueness.** Solutions are uniquely determined by the data
- **Stability.** Solutions depend continuously on the data

Dissipation

Dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta)\right]\ \mathrm{d}x$$

$$+\int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x \leq 0$$

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta)$$

Relative energy

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\ \middle|\ r,\Theta,\mathbf{U}\right)$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho - r) - H_\Theta(r,\Theta)\right]\ \mathrm{d}x$$

Dissipative solutions

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \, \mathrm{d}t$$

Test functions

$$r > 0, \ \ \Theta > 0$$

$\mathbf{U}$ satisfying the relevant natural boundary conditions

Remainder

Remainder

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \left(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \right) \, \mathrm{d}x$$

$$+ \int_\Omega \left[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \, \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta \right.$$

$$\left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \, \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \, \mathrm{d}x$$

Weak vs. dissipative solutions

Classical (strong) solutions

Equations satisfied in the classical sense

Weak solutions

- continuity and momentum equations in the sense of distributions
- entropy (internal energy) inequality in the sense of distributions
- total energy balance

Dissipative solutions

Relative energy inequality for any trio r, Θ, $\mathbf{U}$

Properties of weak solutions

Compatibility

weak + smooth $\Rightarrow$ strong

Weak solutions with entropy inequality are dissipative

weak $\Rightarrow$ dissipative

Weak strong uniquness

Dissipative (weak) and strong solution emanating from the same initial data coincide as long as the latter exists

Global-in-time weak solutions, entropy

Pressure - density, temperature state equation

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\vartheta^4$$

$$\varrho e(\varrho, \vartheta) = \frac{3}{2}\vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0$$

Transport coefficients

$$\underline{\mu}(1 + \vartheta^\alpha) \leq \mu(\vartheta) \leq \overline{\mu}(1 + \vartheta^\alpha), \ \ \eta(\vartheta) \leq \overline{\eta}(1 + \vartheta^\alpha), \ \alpha \in (2/5, 1],$$

$$\underline{\kappa}(1 + \vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^3)$$

Global-in-time weak solutions, internal energy

Pressure - density, temperature state equation

$$e(\varrho, \vartheta) = c_v \vartheta + H(\varrho)$$

$$p(\varrho, \vartheta) = \varrho^\gamma + \varrho\vartheta, \ \gamma > 3$$

Transport coefficients

$$\mu > 0, \ \eta \geq 0 \text{ constant}$$

$$\underline{\mu}(1 + \vartheta^2) \leq \kappa(\vartheta) \leq \overline{\mu}(1 + \vartheta^2)$$

Regularity criteria

Weak solutions with entropy inequality

The weak solution emanating from smooth initial data remains smooth as soon as

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} \leq c$$

Weak solutions with internal energy inequality

The weak solution emanating from smooth initial data remains smooth as soon as

$$\|\mathbf{u}\|_{L^\infty((0,T)\times\Omega;R^3)} + \|\mathrm{div}_x \mathbf{u}\|_{L^1(0,T;L^\infty(\Omega))} \leq c, \ \ \vartheta \leq \overline{\vartheta}$$

Existence theory - *a priori* bounds

Integral bounds - conservation laws

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3))$$

$$\varrho \in L^\infty(0,T;L^{5/3}(\Omega))$$

$$\vartheta \in L^\infty(0,T;L^4(\Omega))$$

Gradient bounds - energy dissipation

$$\nabla_x \mathbf{u} \in L^2(0,T;L^q(\Omega;R^{3\times 3})),\ q = \frac{8}{5-\alpha}$$

$$\nabla_x \vartheta \in L^2(0,T;L^2(\Omega;R^3))$$

$$\nabla_x \log(\vartheta) \in L^2(0,T;L^2(\Omega;R^3))$$

Pressure bounds

$$p(\varrho,\vartheta)\varrho^\beta \in L^1((0,T)\times\Omega) \text{ for a certain } \beta > 0$$

Convergence, sequential stability

Div-Curl lemma [F.Murat, L.Tartar, 1975]

Let

$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } L^p,$$

$$\mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } L^q,$$

with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

Let, moreover,

$$\mathrm{div}[\mathbf{v}_\varepsilon],\ \mathrm{curl}[\mathbf{w}_\varepsilon] \text{ be precompact in } W^{-1,s}$$

Then

$$\mathbf{v}_\varepsilon \cdot \mathbf{w}_\varepsilon \to \mathbf{v} \cdot \mathbf{w} \text{ weakly in } L^r.$$

Weak sequential stability of convective terms

Ansatz for Div-Curl lemma

$$\mathbf{v}_\varepsilon = [\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon], \ \mathbf{w}_\varepsilon = [u^i_\varepsilon, 0, 0, 0], \ i = 1, 2, 3$$

Aubin-Lions argument (Div-Curl lemma)

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup \varrho \mathbf{u}$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \rightharpoonup \varrho \mathbf{u} \otimes \mathbf{u}$$

$$\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \vartheta_\varepsilon \rightharpoonup \varrho s(\varrho, \vartheta) \vartheta$$

Pointwise convergence of temperature, I

GOAL: Use monotonicity of $s(\varrho, \vartheta)$ in ϑ to show

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \Big) (\vartheta_\varepsilon - \vartheta) \ {\rm d}x \ {\rm d}t \to 0$$

$$\Rightarrow$$

$$\| \vartheta_\varepsilon - \vartheta \|_{L^3} \to 0$$

STEP 1: Aubin-Lions argument (Div-Curl lemma)

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) (\vartheta_\varepsilon - \vartheta) \ {\rm d}x \ {\rm d}t \to 0$$

Pointwise convergence of temperature, II

STEP 2: Renormalized equation of continuity [DiPerna and P.-L. Lions, 1989]

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

STEP 3: Aubin-Lions argument (Div-Curl lemma)

$$\overline{b(\varrho) g(\vartheta)} = \overline{b(\varrho)}\ \overline{g(\vartheta)}$$

Pointwise convergence of temperature, II

Fundamental theorem on Young measures, [J.M Ball 1989, P.Pedregal 1997]

Let $\mathbf{v}_\varepsilon : Q \subset R^N \to R^M$ be a sequence of vector fields bounded in $L^1(Q; R^M)$.
Then there exists a subsequence (not relabeled) and a family of probability measures $\{\nu_y\}_{y \in Q}$ on R^M such that:
For any Carathéodory function $\Phi = \Phi(y, Z)$, $yinQ$, $Z \in R^M$ such that

$$\Phi(\cdot, \mathbf{v}_\varepsilon) \to \overline{\Phi} \text{ weakly in } L^1(Q)$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y, Z) \, \mathrm{d}\nu_y(Z) \text{ for a.a. } y \in Q.$$

Pointwise convergence of temperature, III

STEP 4:

Since we already know from STEP 3 that

$$\nu[\varrho_\varepsilon \vartheta_\varepsilon] = \nu[\varrho_\varepsilon] \otimes \nu[\vartheta_\varepsilon],$$

Fundamental theorem yields the desired conclusion

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta)(\vartheta_\varepsilon - \vartheta) \, dx \, dt \to 0$$

Conclusion

$$\vartheta_\varepsilon \to \vartheta \text{ a.a. on } (0, T) \times \Omega$$

Pointwise convergence of density,I

STEP 1: Renormalized equation of continuity

$$\partial_t(\varrho \log(\varrho)) + \mathrm{div}_x(\varrho \log(\varrho)\mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t(\overline{\varrho \log(\varrho)}) + \mathrm{div}_x(\overline{\varrho \log(\varrho)}\mathbf{u}) + \overline{\varrho \mathrm{div}_x \mathbf{u}} = 0$$

Propagation of density oscillations

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right) \mathrm{d}x = - \int_\Omega \left(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u} \right) \mathrm{d}x$$

Pointwise convergence of density,II

STEP 2: Effective viscous pressure [P.-L.Lions, 1998]

$$\overline{p(\varrho,\vartheta)b(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{b(\varrho)} = \overline{[\mathcal{R}:\mathbb{S}]b(\varrho)} - [\mathcal{R}:\mathbb{S}]\overline{b(\varrho)}$$

where

$$\mathcal{R}_{i,j} \equiv \partial_{x_i}\Delta^{-1}\partial_{x_j}$$

Commutator

$$\mathcal{R}:\mathbb{S} = \boxed{\mathcal{R}:\mathbb{S} - \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}} + \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right)\mathrm{div}_x\mathbf{u}$$

Commutator estimates

Commutator lemma [in the spirit of Coifman and Meyer]

Let $w \in W^{1,r}(R^N)$, $\mathbf{V} \in L^p(R^N; R^N)$ be given, where

$$1 < r < N,\ 1 < p < \infty,\ \frac{1}{r} + \frac{1}{p} - \frac{1}{N} < 1.$$

The for any s satisfying

$$\frac{1}{r} + \frac{1}{p} - \frac{1}{N} < \frac{1}{s} < 1$$

there exists $\beta > 0$ such that

$$\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \|_{W^{\beta,s}(R^N,R^N)} \leq c\|w\|_{W^{1,r}} \|\mathbf{V}\|_{L^p}.$$

Pointwise convergence of density,III

STEP 3: Effective viscous pressure revisited

$$0 \leq \overline{p(\varrho,\vartheta)\varrho} - \overline{p(\varrho,\vartheta)}\varrho = (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\Big(\overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}\Big)$$

yielding

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho)$$

Conclusion

$$\varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T) \times \Omega$$

Pointwise convergence of density - general case, I

STEP 1: Renormalized equation of continuity:

$$\partial_t(\varrho L_k(\varrho)) + \mathrm{div}_x(\varrho L_k(\varrho)\mathbf{u}) + T_k(\varrho)\mathrm{div}_x\mathbf{u} = 0$$

$$\partial_t(\overline{\varrho L_k(\varrho)}) + \mathrm{div}_x(\overline{\varrho L_k(\varrho)}\mathbf{u}) + \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} = 0$$

Cut-off functions

$$T_k(\varrho) = \min\{\varrho, k\}$$

$$L_k(\varrho) = \log(\varrho),\ \ \varrho \leq k$$

Pointwise convergence of density - general case, II

Density oscillations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \right) \ \mathrm{d}x = \int_\Omega \left(T_k(\varrho)\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right) \ \mathrm{d}x$$

$$+ \int_\Omega \left(\overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} \right) \ \mathrm{d}x$$

STEP 2: Effective viscous flux revisited

$$\overline{p(\varrho,\vartheta) T_k(\varrho)} - \overline{p(\varrho,\vartheta)} \ \overline{T_k(\varrho)}$$

$$= (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta))\left(\overline{T_k(\varrho)\mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x \mathbf{u} \right)$$

Oscillations defect measure

Oscillations description

$$\sup_{k \geq 1} \left[\limsup_{\varepsilon \to 0} \int_0^T \int_\Omega |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q \ \mathrm{d}x \ \mathrm{d}t \right] < \infty$$

$$q = 5/3 + 1 = 8/3$$

Convergence of the density - general case,III

STEP 3: Boundedness of oscillation deffect measure

- The limit functions ϱ, $\mathbf{u}$ satisfy the renormalized equation of continuity
- $$\int_\Omega \left(T_k(\varrho) \mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \right) \, \mathrm{d}x \to 0 \text{ for } k \to \infty$$

Pointwise convergence of density

$$\overline{\varrho \log(\varrho)} = \lim_{k \to \infty} \overline{\varrho L_k(\varrho)} = \lim_{k \to \infty} \varrho L_k(\varrho) = \varrho \log(\varrho)$$

$$\varrho_\varepsilon \to \varrho \text{ a.a. on } (0, T) \times \Omega$$

Well-posedness of inviscid fluids

Compressible Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Energy (entropy) inequality

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) \mathbf{u} + p(\varrho)\mathbf{u} \right] \leq 0$$

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \ \mathrm{d}z$$

Result of Chiodaroli, DeLellis, Kreml [2013]

There exist Lipschitz initial data such that the compressible Euler system admits infinitely many admissible (entropy) weak solutions.

Riemann problem

Riemann initial data

$$\varrho(0, x_1, \dots, x_N) = \begin{cases} \varrho_L \text{ if } x_1 \leq 0 \\ \varrho_R \text{ if } x_1 > 0 \end{cases}$$

$$u^1(0, x_1, \dots, x_N) = \begin{cases} u^1_L \text{ if } x_1 \leq 0 \\ u^1_R \text{ if } x_1 > 0 \end{cases}$$

$$u^k(0, x_1, \dots, x_N) = 0, \ \ k = 2, \dots, N$$

Wild solutions

The wild solutions emanate from the 1D Riemann data but the velocity admits non-zero second component

Shock free solutions

Geometry, pressure

$$\Omega = (-a, a) \times \mathcal{T}^1 \text{ (periodic in } x_2\text{)}$$

$$p(0) = 0, \ p'(r) > 0 \text{ for } r > 0, \ p \text{ convex}$$

Theorem EF, O.Kreml [2014]

Let $\tilde{\varrho} = \tilde{\varrho}(x_1/t)$, $\tilde{\mathbf{u}} = [\tilde{u}^1(x_1/t), 0]$ be the self-similar solution to the Riemann problem consisting of rarefaction waves (locally Lipschitz for $t > 0$) and such that

$$\mathrm{ess}\inf_{(0,t)\times R} \tilde{\varrho} > 0.$$

Let $[\varrho, \mathbf{u}]$ be a bounded admissible weak solution such that

$$\varrho \geq 0 \text{ a.a. in } (0, T) \times \Omega.$$

Then

$$\varrho \equiv \tilde{\varrho}, \ \mathbf{u} \equiv \tilde{\mathbf{u}} \text{ in } (0, T) \times \Omega.$$

Method of relative energy

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_\Omega \left(\frac{1}{2}\varrho |\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right) \, \mathrm{d}x$$

Relative energy inequality

$$\left[\mathcal{E}\left(\varrho, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\mathbf{u}}\right) \right]_{t=0}^{t=\tau}$$

$$\leq - \int_0^\tau \int_\Omega \left[\varrho |u^1 - \tilde{u}^1|^2 + p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho}) \right] \partial_{x_1} \tilde{u}^1 \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \text{"other terms"}$$

Full Euler system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \varrho \vartheta = 0$$

Energy balance

$$\partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta \right] + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + c_v \varrho \vartheta + \varrho \vartheta \right) \mathbf{u} \right] = 0$$

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) \geq 0, \ \ s = s(\varrho, \vartheta) \equiv \log \left(\frac{\vartheta^{c_v}}{\varrho} \right)$$

Riemann problem

Geometry

$$\Omega = R^1 \times \mathcal{T}^1, \text{ where } \mathcal{T}^1 \equiv [0,1]_{\{0,1\}} \text{ is the ``flat'' sphere}$$

Initial data

$$\varrho(0, x_1, x_2) = R_0(x_1), \ \ R_0 = \begin{cases} R_L \text{ for } x_1 \leq 0 \\ R_R \text{ for } x_1 > 0 \end{cases}$$

$$\vartheta(0, x_1, x_2) = \Theta_0(x_1), \ \ \Theta_0 = \begin{cases} \Theta_L \text{ for } x_1 \leq 0 \\ \Theta_R \text{ for } x_1 > 0 \end{cases}$$

$$u^1(0, x_1, x_2) = U_0(x_1), \ \ U_0 = \begin{cases} U_L \text{ for } x_1 \leq 0, \\ U_R \text{ for } x_1 > 0 \end{cases} \qquad u^2(0, x_1, x_2) = 0.$$

Shock free Riemann solutions

Solution class

$$0 < \varrho \leq \overline{\varrho},\ 0 < \vartheta \leq \overline{\vartheta},\ |s(\varrho, \vartheta)| < \overline{s},\ |\mathbf{u}| < \overline{u}$$

Isentropic solutions

- the entropy S is *constant* in $[0, T] \times \Omega$
- $\Theta = R^{\frac{1}{c_v}} \exp\left(\frac{1}{c_v} S\right)$
- $R = R(t, x_1)$ and $U = U(t, x_1)$ represent a rarefaction wave solution of the 1-D *isentropic* system

$$\partial_t R + \partial_{x_1}(RU) = 0,\ R\left[\partial_t U + U \partial_{x_1} U\right] + \exp\left(\frac{1}{c_v} S\right) \partial_{x_1} R^{\frac{c_v+1}{c_v}} = 0$$

Uniqueness

Theorem, EF, O.Kreml, A.Vasseur [2014]

Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Euler system in $(0, T) \times \Omega$ originating from the Riemann data. Suppose in addition that the Riemann data give rise to the shock-free solution $[R, \Theta, U]$ of the 1-D Riemann problem.

Then

$$\varrho = R, \ \vartheta = \Theta, \ \mathbf{u} = [U, 0] \text{ a.a. in } (0, T) \times \Omega$$

Relative energy

Relative energy (entropy) functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \tilde{\mathbf{u}}|^2 + H_{\tilde{\vartheta}}(\varrho, \vartheta) - \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho}(\varrho - \tilde{\varrho}) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) \right] \ \mathrm{d}x$$

Ballistic free energy

$$H_{\tilde{\vartheta}}(\varrho, \vartheta) = \varrho \left(c_v \vartheta - \tilde{\vartheta} s(\varrho, \vartheta) \right).$$

$$\varrho \mapsto H_{\tilde{\vartheta}}(\varrho, \tilde{\vartheta}) \ \text{convex}$$

$$\vartheta \mapsto H_{\tilde{\vartheta}}(\varrho, \vartheta) \begin{cases} \text{decreasing for } \vartheta < \tilde{\vartheta} \\ \text{increasing for } \vartheta > \tilde{\vartheta} \end{cases}$$

Relative energy inequality, dissipative solutions

Relative energy inequality

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\middle|\tilde\varrho,\tilde\vartheta,\tilde{\mathbf{u}}\right)\right]_{t=0}^{t=\tau} \leq \int_0^\tau \mathcal{R}\left(\varrho,\vartheta,\mathbf{u},\tilde\varrho,\tilde\vartheta,\tilde{\mathbf{u}}\right)\ \mathrm{d}t$$

Test functions

$$\tilde\varrho > 0,\ \tilde\vartheta > 0,\ \left\{ \begin{array}{c} \tilde\varrho = R_L, \tilde\vartheta = \Theta_L,\ \tilde u^1 = U_L,\ \tilde u^2 = 0 \text{ if } x_1 < -A, \\ \tilde\varrho = R_R, \tilde\vartheta = \Theta_R,\ \tilde u^1 = U_R,\ \tilde u^2 = 0 \text{ if } x_1 > A \end{array} \right\}$$

Remainder

Remainder in the relative energy inequality

$$\mathcal{R}\left(\varrho, \vartheta, \mathbf{u}, \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right)$$

$$= \int_\Omega \Big[\varrho(\tilde{\mathbf{u}} - \mathbf{u}) \cdot \partial_t \tilde{\mathbf{u}} + \varrho(\tilde{\mathbf{u}} - \mathbf{u}) \otimes \mathbf{u} : \nabla_x \tilde{\mathbf{u}} + (\tilde{\varrho}\tilde{\vartheta} - \varrho\vartheta) \mathrm{div}_x \tilde{\mathbf{u}} \Big] \ \mathrm{d}x$$

$$- \int_\Omega \Big[\varrho \Big(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \Big) \partial_t \tilde{\vartheta} + \varrho \Big(s(\varrho, \vartheta) - s(\tilde{\varrho}, \tilde{\vartheta}) \Big) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} \Big] \ \mathrm{d}x$$

$$+ \int_\Omega \left[\left(1 - \frac{\varrho}{\tilde{\varrho}} \right) \partial_t (\tilde{\varrho}\tilde{\vartheta}) + \left(\tilde{u} - \frac{\varrho}{\tilde{\varrho}} \right) \mathbf{u} \cdot \nabla_x (\tilde{\varrho}\tilde{\vartheta}) \right] \ \mathrm{d}x$$

Robustness of 1D viscosity solutions

Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}),$$

Pressure, viscous stress

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1,$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0.$$

1D Navier-Stokes system

$$\partial_t R + \partial_y (RV) = 0,$$

$$\partial_t (RV) + \partial_y (RV^2) + \partial_y p(R) = \left[2\mu \left(1 - \frac{1}{N} \right) + \eta \right] \partial^2_{y,y} V.$$

Stability of 1D solutions - hypotheses

Theorem EF, Y.Sun [2015]

$$\gamma > \frac{N}{2},\ q > \max\{2, \gamma'\},\ \frac{1}{\gamma} + \frac{1}{\gamma'} = 1 \text{ if } N = 2$$

$$q > \max\left\{3, \frac{6\gamma}{5\gamma - 6}\right\} \text{ if } N = 3$$

Let $[R, V]$ be a (strong) solution of the one-dimensional problem, with the initial data belonging to the class

$$R_0 \in W^{1,q}(0,1),\ R_0 > 0,\ V_0 \in W_0^{1,q}(0,1)$$

Let $[\varrho, \mathbf{u}]$ be a finite energy weak solution to the Navier-Stokes system in

$$(0,T) \times \Omega,\ \Omega = (0,1) \times \mathcal{T}^{N-1},$$

with the initial data

$$\varrho_0 \in L^\infty(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in L^2(\Omega; R^3).$$

Stability of 1D solutions - conclusion

Conclusion

Then

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{V}|^2 + P(\varrho) - P'(R)(\varrho - R) - P(R) \right] (\tau, \cdot) \ \mathrm{d}x$$

$$\leq c(T) \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{V}_0|^2 + P(\varrho_0) - P'(R_0)(\varrho_0 - R_0) - P(R_0) \right] \ \mathrm{d}x$$

for a.a. $\tau \in (0, T)$,

$$P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma.$$

Full Navier-Stokes-Fourier system

Navier-Stokes-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) - \lambda \mathbf{u}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right),$$

Slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Constitutive relations - scaling

Pressure

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\varrho,\vartheta),\ p_M = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right),\ p_R(\varrho,\vartheta) = \frac{\boxed{a}}{3}\vartheta^4$$

Viscous stress

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \boxed{\nu}\left[\mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}\right]$$

Heat flux

$$\mathbf{q} = -\boxed{\omega}\kappa(\vartheta)\nabla_x \vartheta$$

Brinkman type “damping”

$$D = -\boxed{\lambda}\mathbf{u}$$

Target system

Full Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_M(\varrho, \vartheta) = 0$$

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right)$$

$$+\mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e_M(\varrho, \vartheta) \right) \mathbf{u} + p_M(\varrho, \vartheta) \mathbf{u} \right] = 0$$

Slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Dissipative solutions

Relative energy

$$\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right)$$

$$=\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+H_\Theta(\varrho,\vartheta)-\frac{\partial H_\Theta(r,\Theta)}{\partial\varrho}(\varrho-r)-H_\Theta(r,\Theta)\right]\ \mathrm{d}x$$

Relative energy inequality

$$\left[\mathcal{E}\left(\varrho,\vartheta,\mathbf{u}\Big|r,\Theta,\mathbf{U}\right)\right]_{t=0}^{t=\tau}$$

$$+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x\ \mathrm{d}t$$

$$+\lambda\int_0^\tau\int_\Omega|\mathbf{u}|^2\ \mathrm{d}x\ \mathrm{d}t\le\int_0^\tau\mathcal{R}(\varrho,\vartheta,\mathbf{u},r,\Theta,\mathbf{U})\mathrm{d}t$$

$$r,\Theta>0,\ \ \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega}=0$$

Dissipative solutions - remainder

Remainder

$$\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) = \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \left[\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta + \lambda \mathbf{u} \cdot \mathbf{U} \right] \, \mathrm{d}x$$

$$+ \int_\Omega \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \Big(\mathbf{U} - \mathbf{u} \Big) \cdot \nabla_x \Theta \, \mathrm{d}x$$

$$+ \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x - \int_\Omega p(\varrho, \vartheta) \mathrm{div}_x \mathbf{U} \, \mathrm{d}x$$

$$- \int_\Omega \left(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{U} \cdot \nabla_x \Theta \right) \, \mathrm{d}x$$

$$+ \int_\Omega \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_x p(r, \Theta) \right) \, \mathrm{d}x$$

Vanishing dissipation limit

Theorem EF [2015]

Let $[\varrho_E, \vartheta_E, \mathbf{u}_E]$ be the classical solution of the Euler system in a time interval $(0, T)$, with the initial data $[\varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}]$. Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak (dissipative) solution of the Navier-Stokes-Fourier system, with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$.
Then

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| \varrho_E, \vartheta_E, \mathbf{u}_E\right)(\tau)$$

$$\leq c_1(T, \mathrm{data})\mathcal{E}\left(\varrho_0, \vartheta_0, \mathbf{u}_0 \Big| \varrho_{0,E}, \vartheta_{0,E}, \mathbf{u}_{0,E}\right)$$

$$+c_2(T, \mathrm{data}) \max\left\{a, \nu, \omega, \lambda, \frac{\nu}{\sqrt{a}}, \frac{\omega}{a}, \left(\frac{a}{\sqrt{\nu^3 \lambda}}\right)^{1/3}\right\}$$

for a.a. $\tau \in (0, T)$.

Navier-Stokes-Fourier system - numerics

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \equiv \mu \Delta \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u}, \ \ \mu > 0, \ \ \lambda \geq 0$$

Internal energy equation

$$c_v \left[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\vartheta \mathbf{u}) \right] - \mathrm{div}_x \left(\kappa(\vartheta) \nabla_x \vartheta \right)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p_\vartheta(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}, \ \ \kappa(\vartheta) > 0$$

Initial conditions and boundary conditions

$$\varrho(0, \cdot) = \varrho_0 > 0, \ \ \vartheta(0, \cdot) = \vartheta_0 > 0, \ \ \mathbf{u}(0, \cdot) = \mathbf{u}_0$$

$$\mathbf{u}|_{\partial\Omega} = 0, \ \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Total energy balance, weak formulation

Pressure

$$p(\varrho, \vartheta) = a\varrho^\gamma + b\varrho + \varrho\vartheta, \ \gamma > 3, \ a, b > 0$$

Total energy balance

$$E(t) = \int \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v\varrho\vartheta + \frac{a}{\gamma - 1}\varrho^\gamma + b\varrho\log(\varrho) \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0, \ \frac{\mathrm{d}}{\mathrm{d}t}E(t) \boxed{\leq} 0$$

Internal energy inequality

$$c_v \left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\vartheta\mathbf{u}) \right] - \mathrm{div}_x \left(\kappa(\vartheta)\nabla_x\vartheta \right)$$

$$\boxed{\geq} \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - p_\vartheta(\varrho, \vartheta)\mathrm{div}_x\mathbf{u}, \ \kappa(\vartheta) > 0$$

Analytical approximation

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = \varepsilon \boxed{\Delta_x \varrho}$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$+\varepsilon \boxed{\Delta_x(\varrho \mathbf{u})}$$

Internal energy equation

$$c_v \left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\vartheta \mathbf{u})\right] - \mathrm{div}_x \left(\kappa(\vartheta)\nabla_x \vartheta\right)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p_\vartheta(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}, \ \kappa(\vartheta) > 0$$

Numerical solution

FV framework

regular tetrahedral mesh

$$\Omega \subset \Omega_h \text{ -- polygonal domain}$$

$$\Omega_h \to \Omega \text{ in the sense of compacts}$$

$$Q_h = \{ v \mid v = \text{piece-wise constant} \}$$

FE framework - Crouzeix - Raviart

$$V_h = \left\{ v \;\middle|\; v = \text{piece-wise affine, } \tilde{v}_\Gamma \text{ continuous on face } \Gamma \right\}$$

$$\tilde{v}_\Gamma \equiv \frac{1}{|\Gamma|} \int_\Gamma v \, \mathrm{dS}_x$$

Hyperbolic part - finite volumes

Time discretization

$$D_t r_h^k = \frac{r_h^k - r_h^{k-1}}{\Delta t}$$

Upwind

$$r\mathbf{u} \cdot \nabla_x \phi \approx \mathrm{Up}[r, \mathbf{u}][[\phi]] \text{ on a face } \Gamma$$

$$[[\phi]] = \phi^{\mathrm{out}} - \phi^{\mathrm{in}}$$

Standard upwind

$$\mathrm{Up}[r, \mathbf{u}] = \mathrm{Up}[r, \mathbf{u}] = \frac{r^{\mathrm{in}}}{2}\left([<\mathbf{u}>_\Gamma \cdot \mathbf{n}]^+ + [<\mathbf{u}>_\Gamma \cdot \mathbf{n}]^+\right)$$

$$+\frac{r^{\mathrm{out}}}{2}\left([<\mathbf{u}>_\Gamma \cdot \mathbf{n}]^- + [<\mathbf{u}>_\Gamma \cdot \mathbf{n}]^-\right)$$

"Dissipative upwind"

$$\mathrm{Up}[r,\mathbf{u}] = \frac{r^{\mathrm{in}}}{2}\left([< \mathbf{u} >_\Gamma \cdot\mathbf{n} + h^\alpha]^+ + [< \mathbf{u} >_\Gamma \cdot\mathbf{n} - h^\alpha]^+\right)$$
$$+\frac{r^{\mathrm{out}}}{2}\left([< \mathbf{u} >_\Gamma \cdot\mathbf{n} + h^\alpha]^- + [< \mathbf{u} >_\Gamma \cdot\mathbf{n} - h^\alpha]^-\right)$$

Hyperbolic part - upwind

Artificial dissipation

$$\mathrm{Up}[r, \mathbf{u}] = \underbrace{r^{\mathrm{in}}[< \mathbf{u} >_\Gamma \cdot \mathbf{n}]^+ + r^{\mathrm{out}}[< \mathbf{u} >_\Gamma \cdot \mathbf{n}]^-}_{\text{conventional upwind}} - \underbrace{[[r]]_\Gamma h^\alpha \chi\left(\frac{< \mathbf{u} >_\Gamma \cdot \mathbf{n}}{h^\alpha}\right)}_{\text{dissipative component}},$$

$$\chi(z) = \begin{cases} 0 \text{ for } z < -1, \\ \frac{1}{2}(z+1) \text{ if } -1 \leq z \leq 0, \\ -\frac{1}{2}(z-1) \text{ if } 0 < z \leq 1, \\ 0 \text{ for } z > 1. \end{cases}$$

Numerical scheme [Karlsen-Karper], I

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Continuity method

$$\int_\Omega D_t \varrho_h^k \varphi_h \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k, u_h^k] \ [[\varphi_h]]_\Gamma \ \mathrm{dS}_x = 0$$

for all $\varphi_h \in Q_h(\Omega_h)$

Numerical scheme [Karlsen-Karper], II

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Momentum method

$$\int_\Omega D_t \left(\varrho_h^k \widehat{\mathbf{u}}_h^k\right) \cdot \varphi_h \ \mathrm{d}x - \sum_{\Gamma \in \Gamma_h} \int_\Gamma \mathrm{Up}[\varrho_h^k \widehat{\mathbf{u}}_h^k, \mathbf{u}_h^k] \ \cdot [[\widehat{\varphi}_h]] \mathrm{dS}_x$$

$$= \int_\Omega p(\varrho_h^k, \vartheta_h^k) \mathrm{div}_h \varphi_h \ \mathrm{d}x - \int_\Omega \left(\mu \nabla_h \mathbf{u}_h : \nabla_h \varphi + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \varphi_h\right) \ \mathrm{d}x$$

for all $\varphi_h \in V_{h,0}(\Omega_h)$

Numerical scheme, III

Energy equation

$$c_v \left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\vartheta\mathbf{u})\right] - \mathrm{div}_x\left(\kappa(\vartheta)\nabla_x\vartheta\right)$$
$$= \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - p_\vartheta(\varrho,\vartheta)\mathrm{div}_x\mathbf{u},\ \ \kappa(\vartheta) > 0$$

Energy method

$$\int_\Omega D_t\left(\varrho_h^k\vartheta_h^k\right)\varphi_h\ \mathrm{d}x - \sum_{\Gamma\in\Gamma_h}\int_\Gamma \mathrm{Up}[\varrho_h^k\vartheta_h^k, \mathbf{u}_h^k]\ [[\varphi_h]]_\Gamma \mathrm{dS}_x$$
$$+ \sum_{\Gamma\in\Gamma_h}\int_\Gamma \frac{1}{d_h}[[K(\vartheta_h^k)]]_\Gamma\ [[\varphi_h]]_\Gamma \mathrm{dS}_x$$
$$= \int_\Omega\left(\mu|\nabla_h\mathbf{u}_h^k|^2 + \lambda|\mathrm{div}_h\mathbf{u}_h^k|^2\right)\varphi_h\ \mathrm{d}x - \int_\Omega \vartheta_h^k\partial_\vartheta p(\varrho_h^k,\vartheta_h^k)\mathrm{div}_x\mathbf{u}_h^k\varphi_h\ \mathrm{d}x$$

for all $\varphi_h \in Q_h(\Omega_h)$

Renormalization, I

Renormalized continuity equation (exact solution)

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Renormalization, II

Renormalized continuity method (numerical scheme)

$$\int_{\Omega_h} D_t b(\varrho_h^k)\phi \, \mathrm{d}x - \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}[b(\varrho_h^k), \mathbf{u}_h^k] \, [[\phi]] \, \mathrm{dS}_x$$

$$+ \int_{\Omega_h} \phi \, \left(b'(\varrho_h^k)\varrho_h^k - b(\varrho_h^k) \right) \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x$$

$$= - \int_{\Omega_h} \frac{\Delta t}{2} b''(\xi_{\varrho,h}^k) \left(\frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right)^2 \phi \, \mathrm{d}x$$

$$- h^\alpha \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \phi \; b''(\overline{\eta}_{\varrho,h}^k)[[\varrho_h^k]]^2 \, \mathrm{dS}_x$$

$$- \frac{1}{2} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \phi \; b''(\eta_{\varrho,h}^k)[[\varrho_h^k]]^2 |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| \, \mathrm{dS}_x$$

for any $\phi \in Q_h(\Omega_h)$, $b \in C^2(0, \infty)$

Renormalized energy equation (exact solution)

$$\partial_t(\varrho\chi(\vartheta)) + \mathrm{div}_x(\varrho\chi(\vartheta)\mathbf{u}) - \mathrm{div}_x(\chi'(\vartheta)\nabla_x K(\vartheta))$$

$$= \chi'(\vartheta)\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \chi''(\vartheta)K'(\vartheta)|\nabla_x\vartheta|^2 - \chi'(\vartheta)p_\vartheta(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}$$

Renormalization, IV

Renormalized energy method (numerical solution)

$$c_v \int_{\Omega_h} D_t \left(\varrho_h^k \chi(\vartheta_h^k) \right) \phi \ \mathrm{dx} - c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \mathrm{Up}(\varrho_h^k \chi(\vartheta_h^k), \mathbf{u}_h^k) \ [[\phi]] \ \mathrm{dS}_x$$

$$+ \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \frac{1}{d_\Gamma} [[K(\vartheta_h^k)]] \ [[\chi'(\vartheta_h^k)\phi]] \ \mathrm{dS}_x =$$

Renormalization, IV, continuation

$$
= \int_{\Omega_h} \left(\mu |\nabla_h \mathbf{u}_h^k|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \chi'(\vartheta_h^k) \phi \ \mathrm{d}x
$$

$$
- \int_{\Omega_h} \chi'(\vartheta_h^k) \varrho_h^k \vartheta_h^k \mathrm{div}_h \mathbf{u}_h^k \phi \ \mathrm{d}x
$$

$$
- c_v \frac{\Delta t}{2} \int_{\Omega_h} \chi''(\xi_{\vartheta,h}^k) \varrho_h^{k-1} \left(\frac{\vartheta_h^k - \vartheta_h^{k-1}}{\Delta t} \right)^2 \phi \ \mathrm{d}x
$$

$$
+ \frac{c_v}{2} \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} \phi \chi''(\eta_{\vartheta,h}^k) [[\vartheta_h^k]]^2 (\varrho_h^k)^{\mathrm{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \ \mathrm{dS}_x
$$

$$
- h^\alpha c_v \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_{\Gamma} [[\varrho_h^k]] \ [[\left(\chi(\vartheta_h^k) - \chi'(\vartheta_h^k) \vartheta_k^k \right) \phi]] \ \mathrm{dS}_x
$$

for any $\phi \in Q_h(\Omega_h)$, $\chi \in C^2(0, \infty)$

Total energy

Total energy balance (exact solution)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_h}\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + c_v\varrho\vartheta + \frac{a}{\gamma-1}(\varrho)^\gamma + b\varrho\log(\varrho)\right]\,\mathrm{d}x = 0 \; (\boxed{\leq} 0)$$

Total energy (numerics)

Discretized total energy

$$D_t \int_{\Omega_h} \left[\frac{1}{2} \varrho_h^k |\widehat{\mathbf{u}}_h^k|^2 + c_v \varrho_h^k \vartheta_h^k + \frac{a}{\gamma - 1} \left(\varrho_h^k \right)^\gamma + b \varrho_h^k \log(\varrho_h^k) \right] \, \mathrm{d}x$$

$$+ \frac{\Delta t}{2} \int_{\Omega_h} \left(A \left| \frac{\varrho_h^k - \varrho_h^{k-1}}{\Delta t} \right|^2 + \varrho_h^{k-1} \left| \frac{\widehat{\mathbf{u}}_h^k - \widehat{\mathbf{u}}_h^{k-1}}{\Delta t} \right|^2 \right) \, \mathrm{d}x$$

$$- \sum_{E \in E_h} \sum_{\Gamma_E \subset \partial E} \int_{\Gamma_E} (\varrho_h^k)^{\mathrm{out}} [\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}]^- \frac{\left| \widehat{\mathbf{u}}_h^k - \left(\widehat{\mathbf{u}}_h^k \right)^{\mathrm{out}} \right|^2}{2} \, \mathrm{dS}_x$$

$$+ \frac{A}{2} \sum_{\Gamma \in \Gamma_{h,\mathrm{int}}} \int_\Gamma \left(h^\alpha + |\tilde{\mathbf{u}}_h^k \cdot \mathbf{n}| \right) [[\varrho_h^k]]^2 \, \mathrm{dS}_x \leq 0$$

Stability - uniform estimates

Density

$$\varrho \in L^\infty(0,T;L^\gamma(\Omega_h))$$

Velocity, momentum

$$\sqrt{\varrho}\mathbf{u} \in L^\infty(0,T;L^2(\Omega_h;R^3))$$

$$\nabla_x \mathbf{u} \in L^2((0,T)\times\Omega_h;R^{3\times 3})$$

Temperature

$$\varrho\vartheta \in L^\infty(0,T;L^1(\Omega_h))$$

$$\nabla_x\vartheta,\ \nabla_x \log(\vartheta) \in L^2((0,T)\times\Omega_h;R^3)$$

Existence vs. convergence

Existence of weak solutions [E.F.2003]

The Navier-Stokes-Fourier system admits a global-in-time weak solution for any finite energy initial data

Convergence of the numerical scheme [E.F., R. Hošek, M. Michálek, T.Karper, A.Novotný 2014]

Consistency estimates; the numerical solutions converge, up to a subsequence, to a weak solution of the Navier-Stokes-Fourier system

Compactness - convergence

Density oscillations

$$\partial_t \overline{\varrho \log(\varrho)} + \mathrm{div}_x \left(\overline{\varrho \log(\varrho)} \right) \mathbf{u}) + \overline{\varrho \mathrm{div}_x \mathbf{u}} = 0$$

$$\partial_t (\varrho \log(\varrho)) + \mathrm{div}_x \left(\varrho \log(\varrho) \right) \mathbf{u}) + \varrho \mathrm{div}_x \mathbf{u} = 0$$

Effective viscous flux

$$0 \leq \overline{p(\varrho)\varrho} - \overline{p(\varrho)}\ \varrho = \overline{\varrho \mathrm{div}_x \mathbf{u}} - \varrho \mathrm{div}_x \mathbf{u}$$

Biting limit of the temperature

$$\lim K_\alpha(\vartheta_\varepsilon) = K_\alpha(\vartheta),\ \ K_\alpha \nearrow K$$

Blow-up criterion

Blow-up of smooth solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth ($W^{2,3}$). Then the Navier-Stokes-Fourier system admits a strong solution defined on a (possibly short) time interval $(0, T)$.
If

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} \right] < \infty,$$

then the solution can be extended beyond T.

Regularity criterion

Regularity for weak solutions [E.F., Y.Sun 2014]

Suppose that the initial data ϱ_0, ϑ_0, and $\mathbf{u}_0$ are smooth ($W^{2,3}$). Let $[\varrho, \vartheta, \mathbf{u}]$ be a weak solution of the Navier-Stokes-Fourier system such that

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Then $[\varrho, \vartheta, \mathbf{u}]$ is regular.

Synergy analysis - numerics, assumptions

Numerical solutions with regular initial data

Suppose that $[\varrho_h, \vartheta_h, \mathbf{u}_h]$ is a sequence of numerical solutions for regular initial data

Boundedness

Suppose that

$$\varrho_h^k,\ \vartheta_h^k,\ \mathbf{u}_h^k,\ \mathrm{div}_h \mathbf{u}_h^k$$

are bounded independently of the order of discretization h.

Conclusion

The numerical solutions converge to a weak solution with

$$\sup_{t \in (0,T)} \left[\|\varrho\|_{L^\infty} + \|\vartheta\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} + \|\mathrm{div}_x \mathbf{u}\|_{L^\infty} \right] < \infty.$$

Consequently:

- the limit solution is smooth
- the limit solution is unique
- the numerical scheme converges unconditionally
- error estimates (?)