Equations and Various Concepts of Solutions in Thermodynamics of Compressible Viscous Fluids

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Motto: *Everything Should Be Made as Simple as Possible, But Not Simpler*, Albert Einstein [1879–1955]

1 Fluids in motion

The objective of this series of lectures is to present the recent development of the *mathematical theory* of complete fluids. Here, *complete* means capable to incorporate the basic physical principles, in particular the First, Second (and Third) laws of thermodynamics, in a correct and integral way into the mathematical model. We remain at the platform of classical *continuum mechanics*, where the fluid motion is described in term of observable macroscopic quantities: the mass density, the (absolute) temperature, and the (bulk) velocity.

The adequate mathematical model(s) is typically represented by a system of partial differential equations of evolutionary type, meaning there is a distinguished variable called *time*, and denoted t henceforth. The state-of-the art highlighted in these lecture notes can be characterized as follows:

*The research of E.F. leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

- Most of the systems arising in continuum fluid mechanics are non-linear and as such are either not known to posses or even fail to posses classical solutions, at least if the time interval is large and/or the data are not a small perturbation of an equilibrium state.

- It is possible to develop a mathematical theory of *weak* or variational solutions that are defined globally in time and for any physically relevant data. These weak solutions, however, are in general not unique in their class; several *admissible criteria* based on the underlying physical principles have been proposed to remedy this drawback.

- In general, the problems of *stability* and *convergence* of numerical schemes are easier to handle in the framework of weak solutions living in the natural function spaces.

- There is a lot of interesting mathematical issues the understanding of which may ba effectively used in the real world applications.

1.1 Introduction

Continuum mechanics describes a fluid in motion in terms of numerical values of macroscopic quantities - fields or *state variables*- depending on the time t and the spatial position x. Here we adopt the *Eulerian description*, where the coordinate frame is attached to the physical domain Ω occupied by the fluid. The fields are interrelated through a system of *field equations* - balance laws - reflecting the underlying physical principles of conservation or balance of mass, momentum, energy as well as other quantities as the case may be. The material properties of a specific fluid are characterized by *constitutive relations*. The interaction of the fluid with the outer world is specified through *boundary conditions*.

1.2 State variables

We suppose that the state of a fluid at any instant t is characterized by its *mass density* $\varrho = \varrho(t,x)$ and the *absolute temperature* $\vartheta = \vartheta(t,x)$. The motion is described by means of the velocity field $\mathbf{u} = \mathbf{u}(t,x)$. Accordingly, the fluid moves along streamlines - the spatial curves $\mathbf{X} = \mathbf{X}(t)$ solving

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X} = \mathbf{u}(t,\mathbf{X}).$$

As is well known, the velocity field $\mathbf{u}$ must enjoy certain regularity properties for this non-linear system to be well-posed. Typically, one requires $\mathbf{u}$ to be Lipschitz with respect to the spatial variable $\mathbf{X}$, otherwise the streamlines may not be uniquely determined by their initial position and the original idea of *continuous* motion breaks down. Unfortunately, however, a rigorous verification of this property for most of the fluid systems we shall discuss remains a largely open problem.

1.3 Conservation/balance laws

The conservation/balance laws in continuum mechanics are usually written in a general differential form

$$\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x) = s(t,x). \tag{1}$$

A conservation/balance law reflects the underlying physical principle relating the changes of a volume density of a physical quantity d to its flux $\mathbf{F}$ and a possible source term s as the case may be. In the Eulerian coordinate system, the flux $\mathbf{F}$ consists of a convective (conservative) component $d\mathbf{u}$, and, at least for certain physical quantities, a diffusive (dissipative) part proportional to spatial derivatives of d.

It is useful to pause here to see how (1) can be derived from certain elementary observations under the hypothetical assumption of *smoothness* of all quantities in question. Comparing the total amount of the physical quantity d in a spatial volume B evaluated at two times $t_1 < t_2$ we obtain

$$\int_B \Big[d(t_2,x) - d(t_1,x) \Big] \ \mathrm{d}x = - \int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t,x) \cdot \mathbf{n} \ \mathrm{dS}_x \ \mathrm{d}t + \int_{t_1}^{t_2} \int_B s(t,x) \ \mathrm{d}x \ \mathrm{d}t, \tag{2}$$

where $\mathbf{n}$ is the outer normal vector to the boundary ∂B. Letting $t_1 \to t_2$ and applying Gauss-Green theorem we get (1).

As a matter of fact, it is rather (2) than (1) that reflects the underlying physical principles. We may also write an "approximation" of (2) in the form

$$\int_B \frac{d(t+\Delta t) - d(t)}{\Delta t} \ \mathrm{d}x \approx - \int_{\partial B} \mathbf{F}(t) \cdot \mathbf{n} \ \mathrm{dS}_x + \int_B s(t) \ \mathrm{d}x$$

that should be satisfied for any "small" Δt, B that is reminiscent of certain numerical schemes. Formula (2) is also a suitable starting point to build up the theory of weak solutions based on the concept of distributional derivatives.

1.3.1 Equation of continuity, mass conservation

A mathematical formulation of the physical principle of mass conservation reads

$$\partial_t \varrho(t,x) + \mathrm{div}_x \left(\varrho(t,x)\mathbf{u}(t,x)\right) = 0. \tag{3}$$

The mass flux is purely convective and the source term is absent in (3).

1.3.2 Momentum equation, Newton's second law

The time evolution of the momentum $\varrho\mathbf{u}$ is governed by the system of equations

$$\partial_t(\varrho(t,x)\mathbf{u}(t,x)) + \mathrm{div}_x \left(\varrho(t,x)\mathbf{u}(t,x) \otimes \mathbf{u}(t,x)\right) = \mathrm{div}_x \mathbb{T}(t,x) + \varrho(t,x)\mathbf{f}(t,x), \tag{4}$$

where $\mathbb{T}$ denotes the Cauchy stress to be determined below, and $\mathbf{f}$ is the volume density of the external forces acting on the fluid.

1.3.3 Energy balance, First law of thermodynamics

Taking the scalar product of (4) with $\mathbf{u}$ and using (3) we easily deduce the kinetic energy balance

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2(t,x)\right) + \mathrm{div}_x \left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}(t,x)\right)$$
$$= \mathrm{div}_x(\mathbb{T}\cdot\mathbf{u}(t,x)) - \mathbb{T}(t,x) : \nabla_x\mathbf{u}(t,x) + \varrho\mathbf{f}\cdot\mathbf{u}(t,x).$$

Even in the absence of the external forces, the right-hand side does not vanish, and, accordingly, the kinetic energy is not conserved, unless $\mathbb{T} \equiv 0$.

To enforce the First law of thermodynamics, we write the volume density of the total energy of the fluid in the form

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e$$

that consists of the kinetic component $\frac{1}{2}\varrho|\mathbf{u}|^2$ and the internal energy ϱe. In accordance with our choice of the state variables, the (specific) internal energy $e = e(\varrho, \vartheta)$ is a function of the density ϱ and the temperature ϑ.

A mathematical formulation of the First law of thermodynamics reads:

$$\partial_t \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right](t,x) + \mathrm{div}_x \left(\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right](t,x)\mathbf{u}(t,x)\right) \tag{5}$$

$$+\mathrm{div}_x \mathbf{q}(t,x) - \mathrm{div}_x(\mathbb{T}(t,x)\cdot \mathbf{u}(t,x))$$
$$= \varrho(t,x)\mathbf{f}(t,x)\cdot \mathbf{u}(t,x) + \varrho(t,x)\mathcal{Q}(t,x),$$

where $\mathbf{q}$ denotes the diffusive part of the internal energy flux and $\mathcal{Q}$ the volume density of the external heat sources.

1.4 Constitutive relations for fluids

Fluids are characterized by Stokes' law

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}, \tag{6}$$

where $\mathbb{S}$ is the viscous stress tensor and p is a scalar quantity termed pressure. Similarly to the specific energy e, the pressure $p = p(\varrho, \vartheta)$ is a function of the state variables ϱ, ϑ.

1.4.1 Entropy, Second law of thermodynamics

The Second law of thermodynamics postulates the existence of another thermodynamic function - entropy. We suppose that the specific entropy $s = s(\varrho, \vartheta)$ is interrelated with the internal energy e and the pressure p by means of Gibbs' equation

$$\vartheta D s(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right), \tag{7}$$

where D stands for the differential with respect to ϱ and ϑ.

Internal energy equation In view of Stokes' relation (6), the total energy balance may be rewritten as the internal energy equation

$$\partial_t(\varrho e(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}, \tag{8}$$

or, equivalently, in the form of thermal energy balance

$$\varrho c_v(\varrho,\vartheta)\Big(\partial_t\vartheta + \mathbf{u}\cdot\nabla_x\vartheta\Big) + \mathrm{div}_x\mathbf{q} = \mathbb{S} : \nabla_x\mathbf{u} - \varrho\frac{\partial p(\varrho,\vartheta)}{\partial\vartheta}\mathrm{div}_x\mathbf{u}, \tag{9}$$

where we have introduced the specific heat at constant volume

$$c_v(\varrho,\vartheta) = \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta}.$$

Note that the passage from (8) to (9) uses the equation of continuity (3).

We say that $p = (\varrho, \vartheta)$, $e(\varrho, \vartheta)$ comply with *thermodynamic stability*, if

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ c_v(\varrho, \vartheta) = \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \tag{10}$$

for all $\varrho > 0$, $\vartheta > 0$, see [Bechtel et al(2005)Bechtel, Rooney, and Forest].

Entropy production Dividing the internal energy balance (8) on ϑ, we may use the equation of continuity (3) and Gibbs' relation (7) to obtain the entropy equation

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x \left(\varrho s(\varrho, \vartheta)\mathbf{u}\right) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) + \frac{\varrho}{\vartheta}\mathcal{Q}. \tag{11}$$

The quantity

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) \geq 0 \tag{12}$$

is termed entropy production rate and, in accordance with the Second law of thermodynamics, is always non-negative. This may (and will) imply some structural restrictions to be satisfied by the constitutive equations for $\mathbb{S}$ and $\mathbf{q}$ discussed below.

2 Basic equations of fluid dynamics

In order to close the system of fluid dynamic equations, we need constitutive equations for the viscous stress $\mathbb{S}$ and the internal energy flux $\mathbf{q}$.

2.1 Euler system, ideal fluids

Ideal fluids are those for which $\mathbb{S} = 0$, $\mathbf{q} = 0$. The associated system of equations is usually called *Euler system* see e.g. [de Groot and Mazur(1984)]:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{13}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0, \tag{14}$$

$$\partial_t \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right] + \mathrm{div}_x \left(\varrho \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right] \mathbf{u}\right) + \mathrm{div}_x(p(\varrho,\vartheta)\mathbf{u}) = 0, \tag{15}$$

where we have omitted, for the sake of simplicity, the effect of external sources in (14), (15).

2.2 Navier-Stokes-Fourier system, viscous and heat conducting fluids

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Ideal fluids introduced in the previous section may and should be seen as a hypothetical limit state of real fluids that are both viscous and heat conducting. In such a case, the viscous stress $\mathbb{S}$ as well as the internal energy flux $\mathbf{q}$ depend effectively on the velocity gradient $\nabla_x \mathbf{u}$ and the temperature gradient $\nabla_x \vartheta$, respectively.

2.2.1 Newtonian fluids

For Newtonian or linearly viscous fluids, the viscous stress tensor is a linear function of the velocity gradient.

Newton's law The viscous stress tensor $\mathbb{S}$ for a Newtonian fluid is given by *Newton's rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{16}$$

where the scalar quantities μ and η are termed the shear and bulk viscosity coefficient, respectively. In accordance with the Second law of thermodynamics enforced through (12), μ and η are non-negative and may depend on the state variables ϱ, ϑ as the case may be.

Fourier's law Similarly to (16), the internal energy flux $\mathbf{q}$ of a linearly viscous fluid is a linear function of $\nabla_x \vartheta$ determined by *Fourier's law*

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \tag{17}$$

with the heat conductivity coefficient $\kappa \geq 0$ that may depend on ϱ and ϑ.

2.2.2 Navier-Stokes-Fourier system

In accordance with the previous discussion, the time evolution of a Newtonian heat conducting fluid is determined by the *Navier-Stokes-Fourier system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{18}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) \tag{19}$$
$$= \mathrm{div}_x \left(\mu \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right] + \eta \mathrm{div}_x \mathbf{u} \mathbb{I} \right),$$

$$\varrho c_v(\varrho, \vartheta) \Big(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta \Big) - \mathrm{div}_x \left(\kappa \nabla_x \vartheta \right) \tag{20}$$
$$= \left(\mu \left[\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right] + \eta \mathrm{div}_x \mathbf{u} \mathbb{I} \right) : \nabla_x \mathbf{u} - \varrho \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \mathrm{div}_x \mathbf{u},$$

where, similarly to the Euler system (13 - 15), the effect of the external sources has been omitted. We point out that equation (20) is *formally* equivalent to the total energy balance (5), the internal energy balance (8) and even to the entropy balance (11).

3 Boundary conditions

Fluids are usually confined to a bounded spatial domain Ω, the unbounded domains considered in certain mathematical models should be seen as an idealization of large fluid domains in the real world. There is a large variety of boundary behavior of both Eulerian and Navier-Stokes fluid determined by its interaction with the real world. For definiteness, we consider a very simple situation, where the kinematic boundary $\partial\Omega$ is at rest and impermeable, meaning

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{21}$$

where the symbol $\mathbf{n}$ denotes the outer normal vector to $\partial\Omega$.

3.1 Slip vs. stick

While the impermeability condition (21) is sufficient for the description on an inviscid fluid governed by the Euler system (13-15), an extra piece of information is needed if the fluid is viscous.

3.1.1 No slip boundary conditions

A commonly accepted hypothesis asserts that a viscous fluid adheres completely to the boundary, meaning, in addition to (21), also the tangential component of the velocity vanishes on $\partial\Omega$. This can be written concisely in the form of *no-slip* boundary condition

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{22}$$

3.1.2 No stick, complete slip boundary conditions

In certain situations, e.g. for nanofluids, it was observed that the no-slip condition (22) is no longer a relevant description of the fluid behavior. Instead, one may postulate the *no-stick or complete slip* condition

$$(\mathbb{S} \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0. \tag{23}$$

In other words, the tangential component of the normal (viscous) stress vanishes on $\partial\Omega$.

3.1.3 Navier's slip

A compromise between (22) and (23) is *Navier's slip* condition

$$[\mathbb{S} \cdot \mathbf{n}]_{\tan} + \beta \varrho \mathbf{u}|_{\partial\Omega} = 0, \tag{24}$$

where β plays a role of a friction coefficient.

3.2 Boundary behavior of the temperature, heat flux

For heat conducting fluids, the boundary behavior must be specified. In energetically insulated system, the heat flux vanishes in the normal direction on $\partial\Omega$,

$$-\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = \kappa \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{25}$$

Alternatively, we may prescribe the distribution of the temperature on the boundary,

$$\vartheta|_{\partial\Omega} = \vartheta_b. \tag{26}$$

Of course, there are many other possibilities of the boundary behavior of ϑ including a combination of (25), (26) imposed on disjoint parts of $\partial\Omega$.

4 Well posedness, classical solutions

A system of *evolutionary* partial differential equations, supplemented with suitable boundary conditions, is *well posed* provided it admits a unique solution for any admissible initial state. The initial state for the Navier-Stokes-Fourier or complete Euler system is given by specifying the initial distribution of the density, velocity, and temperature:

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 \text{ in } \Omega. \tag{27}$$

Alternatively, the initial momentum $(\varrho\mathbf{u})_0$, initial internal energy e_0 and/or entropy s_0 can be prescribed. In view of the physical background, the initial data should obey certain admissibility conditions, in particular, the density and (absolute) temperature should be strictly positive, the total initial energy finite, among others.

4.1 Classical solvability

Given smooth and physically admissible initial data, the problems in fluid dynamics are supposed to admit a unique classical (smooth) solutions. This is true, however, only on a possibly short time interval $[0, T_{\max})$. If $T_{\max} = \infty$ is in general an open question. Solutions of the (inviscid) Euler system (13 - 15) may develop discontinuities (shock waves) in a finite time no matter how smooth and even small the initial data are, see [Smoller(1967), Chapter 15]. Regularity of solutions to the Navier-Stokes-Fourier system (18 - 20) in the long run is a famous open problem, see [Fefferman(2006)], [Tao(2013)] for a thorough discussion in the context of *incompressible fluids*.

4.2 Local in time existence

There are many results concerning local-in-time existence of smooth solutions for both the Euler and the Navier-Stokes-Fourier system, for different choices of spatial geometries, boundary conditions, classes of initial data etc.

4.2.1 Euler system - classical solutions

We state the result in the physically relevant domain - the whole Eucleidean space R^3 - to avoid technicalities connected with the boundary behavior of solutions, see [Benzoni-Gavage and Serre(2007), Chapter 13, Theorem 13.1.].

Theorem 4.1 *Let $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ be given. Suppose that the pressure $p = p(\varrho, \vartheta)$, $e(\varrho, \vartheta)$ are twice continuously differentiable functions satisfying Gibbs' relation (7) and the thermodynamic stability condition (10) in an open set $\mathcal{U} \subset (0, \infty)^2$ containing $[\overline{\varrho}, \overline{\vartheta}]$. Let the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ be given such that*

$$[\varrho_0(x), \mathbf{u}_0(x)] \text{ belong to a compact subset of } \mathcal{U} \text{ for all } x \in R^3,$$

$$\varrho_0 - \overline{\varrho}, \vartheta_0 - \overline{\vartheta} \in W^{k,2}(R^3),\ \mathbf{u}_0 \in W^{k,2}(R^3; R^3) \text{ for some } k > \frac{5}{2}.$$

Then there exists a positive time $T > 0$ such that the Euler system (13 - 15) admits a solution ϱ, ϑ, $\mathbf{u}$ unique in the class

$$\varrho - \overline{\varrho}, \vartheta - \overline{\vartheta} \in C([0,T]; W^{k,2}(R^3)) \cap C^1([0,T]; W^{k-1,2}(R^3)),$$

$$\mathbf{u} \in C([0,T]; W^{k,2}(R^3; R^3)) \cap C^1([0,T]; W^{k-1,2}(R^3, R^3)).$$

Remark 4.1 *The symbol $W^{k,2}(R^3)$ denotes the* Sobolev space *of functions having (generalized) derivatives up to order k square integrable in R^3.*

4.2.2 Navier-Stokes-Fourier system - classical solutions

A short-time existence result for the Navier-Stokes-Fourier system (13 - 15), endowed, for definiteness, with the boundary conditions (22), (25) may be stated as follows, see [Valli(1982), Theorem A and Remark 3.3].

Theorem 4.2 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, $\nu > 0$. Let the initial data $\varrho_0, \vartheta_0 \in W^{3,2}(\Omega)$, $\mathbf{u}_0 \in W^{3,2}(\Omega; R^3)$ be given such that $[\varrho_0(x), \mathbf{u}_0(x)]$ belong to a compact subset of an open set $\mathcal{U} \subset (0, \infty)^2$, and satisfying the* compatibility conditions

$$\mathbf{u}_0|_{\partial\Omega} = 0,\ \nabla_x \vartheta_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

$$\nabla_x p(\varrho_0, \vartheta_0)\Big|_{\partial\Omega}$$

$$= \mathrm{div}_x \left(\mu(\varrho_0, \vartheta_0) \left[\nabla_x \mathbf{u}_0 + \nabla_x^t \mathbf{u}_0 - \frac{2}{3} \mathrm{div}_x \mathbf{u}_0 \mathbb{I} \right] + \eta(\varrho_0, \vartheta_0) \mathrm{div}_x \mathbf{u}_0 \mathbb{I} \right) \Big|_{\partial\Omega}.$$

Suppose that the pressure $p = p(\varrho, \vartheta)$, the specific heat at constant volume $c_v = c_v(\varrho, \vartheta)$, as well as the transport coefficients $\mu = \mu(\varrho, \vartheta)$, $\eta = \eta(\varrho, \vartheta)$, and $\kappa = \kappa(\varrho, \vartheta)$ are three-times continuously differentiable in $\mathcal{U}$ and satisfy

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0,\ c_v(\varrho, \vartheta) > 0,\ \mu(\varrho, \vartheta) > 0,\ \eta(\varrho, \vartheta) \geq 0,\ \kappa(\varrho, \vartheta) > 0$$

for all $[\varrho, \vartheta] \in \mathcal{U}$.

Then there exists $T > 0$ *such that the Navier-Stokes-Fourier system (13 - 15), supplemented with the boundary conditions (22), (25) admits a unique solution in the class*

$$\varrho,\ \vartheta \in C([0,T]; W^{3,2}(\Omega)) \cap C^1([0,T]; W^{2,2}(\Omega)),$$

$$\mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3)) \cap C^1([0,T]; W^{2,2}(\Omega; R^3)).$$

Remark 4.2 *It can be shown that any solution belonging to the class specified in Theorem 4.2 possess all the necessary derivatives and is therefore a classical solution in the* open set $(0,T) \times \Omega$.

4.3 Classical solvability - conclusion

The systems of equations considered in mathematical fluid dynamics are nonlinear and as such susceptible to develop singularities, either in the form of steep gradients (shock waves) or concentrations (mass collapse). Such phenomena have been rigorously verified for the inviscid Euler system. A mathematical theory based on global-in-time solutions is beyond the reach of the available mathematical methods and up-to-date knowledge, even for the Navier-Stokes-Fourier system. On the other hand, these problems are being solved numerically with continuously improving capacity of modern computers. *Some* concept of solutions is therefore needed to perform a rigorous analysis of convergence of the numerical methods. The weak solutions discussed in the next part offer such alternative.

5 Weak solutions

The idea of weak solutions is based on the concept of *generalized derivatives* or distributions. Classical functions are replaced by their *internal averages* or, more precisely

$$f : Q \mapsto R \ \approx \int_Q f\varphi,\ \varphi \in C_c^\infty(Q)$$

where the symbol $C_c^\infty(Q)$ denotes the set of infinitely differentiable functions with compact support in Q. Differential operators D can be conveniently expresses by means of a formal by-parts integration:

$$Df \approx - \int_Q f D\varphi, \ \varphi \in C_c^\infty(Q).$$

Accordingly, any (locally) integrable function possesses derivatives of arbitrary order! The Sobolev spaces $W^{k,2}$ used in the previous part are based on distributional derivatives.

5.1 Euler system - weak solutions

We say that $[\varrho, \vartheta, \mathbf{u}]$ is a *weak solution* of the Euler system (13 - 15) in the set $(0,T) \times \Omega$ if:

$$\int_0^T \int_\Omega \left(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{28}$$

for any $\varphi \in C_c^\infty((0,T) \times \Omega)$;

$$\int_0^T \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{29}$$

for any $\varphi \in C_c^\infty((0,T) \times \Omega; R^3)$;

$$\int_0^T \int_\Omega \left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \partial_t \varphi \right. \tag{30}$$

$$\left. + \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + p(\varrho, \vartheta) \right] \mathbf{u} \cdot \nabla_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t = 0$$

for any $\varphi \in C_c^\infty((0,T) \times \Omega)$.

Note that the integral identities (28-30) are well defined as soon as all the compositions of ϱ, ϑ, $\mathbf{u}$ with all nonlinearities are at least locally integrable.

5.1.1 Weak continuity, initial and/or boundary conditions

Functions that are merely (locally) integrable do not posses traces on lower-dimensional structures in Ω, in particular, it is not clear how to define the initial and/or boundary conditions in the class of weak solutions. Fortunately, the necessary piece of information is already encoded in the weak formulation. For example, if ϱ is a weak solution of (28), we may take a special test function $\varphi(t,x) = \psi(t)\phi(x)$, $\psi \in C_c^\infty(0,T)$, $\phi \in C_c^\infty(\Omega)$ to obtain

$$\int_0^T \psi'(t) \int_\Omega \varrho(t,\cdot)\phi \ \mathrm{d}x \ \mathrm{d}t = - \int_0^T \psi(t) \int_\Omega \varrho \mathbf{u}(t,\cdot) \cdot \nabla_x \phi \ \mathrm{d}x \ \mathrm{d}t,$$

from which we may deduce that the function

$$t \mapsto \int_\Omega \varrho(t,\cdot)\phi \ \mathrm{d}x \text{ admits an integrable generalized derivate in } (0,T)$$

and as such can be represented, upon modification on a set of zero measure, by an absolutely continuous function. Thus the initial conditions can be interpreted in the sense of integral averages:

$$\varrho(0,\cdot) = \varrho_0 \ \approx \int_\Omega \varrho(t,\cdot)\phi \ \mathrm{d}x \to \int_\Omega \varrho_0 \phi \ \mathrm{d}x \text{ as } t \to 0+ \ \text{ for any } \phi \in C_c^\infty(\Omega).$$

The anticipated weak continuity in time enables us to incorporate the initial conditions in the weak formulation, replacing (28-29) by

$$\int_0^T \int_\Omega \left(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x \tag{31}$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega)$;

$$\int_0^T \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho,\vartheta) \mathrm{div}_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t \tag{32}$$

$$= - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0,\cdot) \ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$;

$$\int_0^T \int_\Omega \left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right] \partial_t \varphi \right. \tag{33}$$

$$\left. + \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + p(\varrho,\vartheta) \right] \mathbf{u} \cdot \nabla_x \varphi \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$= - \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) \right] \varphi(0,\cdot) \ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \Omega)$.

Boundary conditions, or at least the normal traces of the fluxes can be interpreted in a similar way. We will discuss this issue in the context of the Navier-Stokes-Fourier system.

Remark 5.1 *As a matter of fact, the weak formulation can be derived directly (without passing from classical to generalized derivatives) from the underlying physical principles written in their natural* integral form*, see [Feireisl and Novotný(2009), Chapter 1].*

5.2 Navier-Stokes-Fourier system - weak solutions

In order to introduce a weak formulation of the Navier-Stokes-Fourier system, we first rewrite the energy equation (20) in the conservative form

$$\partial_t(\varrho e(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e(\varrho,\vartheta)\mathbf{u}) - \mathrm{div}_x\,(\kappa\nabla_x\vartheta)$$
$$= \left(\mu\left[\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right] + \eta\mathrm{div}_x\mathbf{u}\mathbb{I}\right) : \nabla_x\mathbf{u} - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}.$$

Note that this is possible as long as p, e and $c_v = \partial_\vartheta e$ are interrelated through (7).

Accordingly, the weak formulation of the Navier-Stokes-Fourier system (18 - 20) reads as follows:

$$\int_0^T\int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega \varrho_0\varphi(0,\cdot)\ \mathrm{d}x \tag{34}$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega)$;

$$\int_0^T\int_\Omega (\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho,\vartheta)\mathrm{div}_x\varphi)\ \mathrm{d}x\ \mathrm{d}t \tag{35}$$
$$\int_0^T\int_\Omega \mu\left[\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right] : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t + \int_0^T\int_\Omega \eta\mathrm{div}_x\mathbf{u}\mathrm{div}_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$
$$= -\int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega;R^3)$;

$$\int_0^T\int_\Omega (\varrho e(\varrho,\vartheta)\partial_t\varphi + \varrho e(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi)\ \mathrm{d}x\ \mathrm{d}t - \int_0^T\int_\Omega \kappa\nabla_x\vartheta\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{36}$$
$$= -\int_0^T\int_\Omega \left(\mu\left[\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right] + \eta\mathrm{div}_x\mathbf{u}\mathbb{I}\right) : \nabla_x\mathbf{u}\varphi\ \mathrm{d}x\ \mathrm{d}t$$
$$-\int_0^T\int_\Omega p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega \varrho_0 e(\varrho_0,\vartheta_0)\varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega)$. Similarly to the previous part, the weak formulation already includes the satisfaction of the initial conditions.

5.2.1 Boundary conditions

The reader will have noticed that in contrast with the Euler system, the weak formulation of the Navier-Stokes-Fourier system included *first* derivatives of the velocity $\mathbf{u}$ as well as the temperature ϑ. Anticipating that the first derivatives are integrable functions, the fields $\mathbf{u}$ and ϑ have well defined *traces* on the boundary $\partial\Omega$, see e.g. [Ziemer(1989), Chapter 3]. Thus we may incorporate the *Dirichlet type* boundary conditions (22), (26) in the definition of the function spaces the solution belong to. In particular, the no-slip condition (22) corresponds to the Sobolev space $W_0^{1,p}(\Omega)$ od functions with integrable first order derivatives in power p and vanishing on the boundary.

The boundary conditions of *Neumann type* like can be accommodated in the weak formulation by extending the class of admissible test functions. Thus for instance the no-flux condition (25) is enforced by postulating (36) for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. The complete slip (21), (23) requires $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ and (35) to be satisfied for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, etc.

5.3 A disturbing example

The class of weak solutions to a given problem is apparently much larger than required by the classical theory. In other words, it might be easier to establish *existence* but definitely more delicate to show *uniqueness* among all possible weak solutions emanating form the same initial data. Indeed there exist weak solutions to the (incompressible) variant of the Euler system that can be obtained in a completely non-constructive way by a method recently developed in [De Lellis and Székelyhidi(2010)]. Adapting this technique, we may show a rather illustrative but at the same time disturbing example of non-uniqueness in the context of fluid thermodynamics. To this end, consider the so-called Euler-Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{37}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho\vartheta) = 0, \tag{38}$$

$$\frac{3}{2}\left[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\right] - \Delta\vartheta = -\varrho\vartheta\mathrm{div}_x\mathbf{u}. \tag{39}$$

The system (37-39) is a special case of the Navier-Stokes-Fourier system with $p = \varrho\vartheta$, $c_v = \frac{3}{2}$, $\mu = \eta = 0$, $\kappa = 1$. Although a correct physical justification of an inviscid heat conducting fluid my be dubious, the system has been used as a suitable approximation in certain models, see [Wilcox(1984)].

For the sake of simplicity, we consider the spatially periodic boundary conditions, meaning the underlying spatial domain

$$\Omega = \mathcal{T}^3 = \Big([-1,1] \,|_{\{-1;1\}} \Big)^3$$

is the "flat" torus. We report the following result,
see [Chiodaroli et al(2014)Chiodaroli, Feireisl, and Kreml, Theorem 3.1].

Theorem 5.1 *Let $T > 0$ be given. Let the initial data satisfy*

$$\varrho_0, \vartheta_0 \in C^3(\mathcal{T}^3),\ \mathbf{u} \in C^3(\mathcal{T}^3; R^3)),\ \varrho_0 > 0,\ \vartheta_0 > 0 \ \mathit{in}\ \mathcal{T}^3.$$

The the initial-value problem for the Euler-Fourier system (37-39) admits infinitely many *weak solutions in $(0,T) \times \Omega$ belonging to the class*

$$\varrho \in C^2([0,T] \times \Omega),\ \partial_t \vartheta \in L^p(0,T; L^p(\Omega)),\ \nabla_x^2 \vartheta \in L^p(0,T; L^p(\Omega; R^{3\times 3}))$$

for any $1 \le p < \infty$,

$$\mathbf{u} \in C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^3)) \cap L^\infty((0,T) \times \Omega; R^3),\ \mathrm{div}_x \mathbf{u} \in C^2([0,T] \times \Omega).$$

The conclusion of Theorem 5.1 reveals the main drawback of the mathematical theory based on the concept of weak solutions, namely, the restrictions imposed by the problem upon the class of possible solutions are too weak to ensure uniqueness. Apparently, the weak formulation must be augmented by certain *admissibility conditions* dictated by physics to pick up the relevant solution. On the other hand, the extra conditions should not be too strong to prevent global-in-time existence. We will address this and related issues in the remaining part of this chapter devoted to the mathematical theory of the complete Navier-Stokes-Fourier system.

6 Mathematical theory of compressible, viscous and heat conducting fluids

We start by an alternative weak formulation of the Navier-Stokes-Fourier system based on the Second law of thermodynamics. The theory accommodates, in particular, the *energetically closed systems*, mechanically and thermally insulated from the outer world. Accordingly, we focus on the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{40}$$

in particular the total energy E is a constant of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x = 0. \tag{41}$$

We use (41), together with

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{42}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \tag{43}$$

and the entropy *inequality*

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \tag{44}$$

as a basis of a new weak formulation of the Navier-Stokes-Fourier system. Similarly to the above, we take

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mathbf{q} = -\kappa \nabla_x \vartheta. \tag{45}$$

6.1 Finite energy weak solutions to the Navier-Stokes-Fourier system

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We shall say that a trio of functions ϱ, ϑ, $\mathbf{u}$ is a *finite energy weak solution* to the Navier-Stokes-Fourier system (41-45), supplemented with the boundary conditions (40) if:

- $$\varrho \in L^\infty(0,T; L^\gamma(\Omega)), \ \vartheta \in L^\infty(0,T; L^q(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$$

  for certain $\gamma > 1, q > 1$,

  $$\varrho \geq 0, \ \vartheta > 0 \text{ a.a. in } (0,T) \times \Omega,$$

  $$\mathbf{u} \in L^2(0,T; W_0^{1,r}(\Omega; R^3)) \text{ for a certain } r > 1;$$

- $$\int_0^T \int_\Omega (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi) \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x \tag{46}$$
  for any $\varphi \in C_c^\infty([0,T) \times \Omega)$;

- $$\int_0^T \int_\Omega (\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho, \vartheta) \mathrm{div}_x \varphi) \ \mathrm{d}x \ \mathrm{d}t \tag{47}$$
  $$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \ \mathrm{d}x$$
  for any $\varphi \in C_c^\infty([0,T) \times \Omega; R^3)$;

- $$\int_0^T \int_\Omega \left(\varrho s(\varrho, \vartheta) \partial_t \varphi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q} \cdot \nabla_x \varphi}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t \tag{48}$$
  $$+ \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \varphi \ \mathrm{d}x \ \mathrm{d}t$$
  $$\leq - \int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) \varphi(0, \cdot) \ \mathrm{d}x$$
  for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$.

- $$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] (\tau, \cdot) \ \mathrm{d}x = \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right] \ \mathrm{d}x \tag{49}$$
  for a.a. $\tau \in (0, T)$.

The weak solutions satisfying (46-49) enjoy the important *compatibility property*, namely any weak solution that is sufficiently smooth satisfies the classical formulation of the Navier-Stokes-Fourier system (18-20), see [Feireisl and Novotný(2009), Chapter 2].

6.2 Global-in-time existence of finite energy weak solutions

The weak formulation of the Navier-Stokes-Fourier system based on the integral identities (inequalities) (46–49) is mathematically tractable. Under certain technical but still physically grounded restrictions imposed on the constitutive relations, the problem admits global-in-time solution for any finite energy initial data.

6.2.1 Hypotheses, constitutive relations

We shall assume that the thermodynamics functions $p = p(\varrho, \vartheta)$, $e = e(\varrho, \vartheta)$, and $s = s(\varrho, \vartheta)$ are interrelated through Gibbs' equation (7) and comply with the hypothesis of thermodynamics stability (10). In addition, we suppose that the internal energy $e = e(\varrho, \vartheta)$ and the pressure take the form

$$e(\varrho, \vartheta) = e_m(\varrho, \vartheta) + \frac{a}{\varrho}\vartheta^4, \ p(\varrho, \vartheta) = p_m(\varrho, \vartheta) + \frac{a}{4\varrho}\vartheta^4, \ a > 0, \tag{50}$$

where e_m, p_m represent molecular components augmented in (50) by radiation, see [Feireisl and Novotný(2009), Chapter 1]. Moreover, p_m and e_m satisfies the monoatomic gas equation of state

$$p_m(\varrho, \vartheta) = \frac{2}{3} e_m(\varrho, \vartheta). \tag{51}$$

The relation (51) is compatible with Gibbs' equation (7) provided

$$p_m(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right); \text{ whence } e_m(\varrho, \vartheta) = \frac{3}{2}\vartheta\frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right). \tag{52}$$

In this setting, the hypothesis of thermodynamics stability (10) gives rise to

$$P(0) = 0, \ P'(Z) > 0, \ 0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for any } Z > 0, \tag{53}$$

where, in addition, we require the specific heat at constant volume to be uniformly bounded.

Finally, by virtue of (53), the function $Z \mapsto \frac{P(Z)}{Z}$ is non-increasing, and we suppose

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{54}$$

6.2.2 Hypotheses, transport coefficients

We suppose that the transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, and $\kappa = \kappa(\vartheta)$ appearing in (45) are effective functions of the absolute temperature, specifically,

$$\underline{\mu}(1 + \vartheta^\alpha) \le \mu(\vartheta) \le \overline{\mu}(1 + \vartheta^\alpha), \ |\mu'(\vartheta)| \le c \text{ for all } \vartheta > 0, \ \frac{2}{5} < \alpha \le 1, \ \underline{\mu} > 0, \tag{55}$$

$$0 \le \mu(\vartheta) \le \overline{\eta}(1 + \vartheta^\alpha) \text{ for all } \vartheta > 0, \tag{56}$$

and

$$\underline{\kappa}(1 + \vartheta^3) \le \kappa(\vartheta) \le \overline{\kappa}(1 + \vartheta^3) \text{ for all } \vartheta > 0, \ \underline{\kappa} > 0. \tag{57}$$

6.2.3 Existence of finite energy weak solutions

We report the following result, see [Feireisl and Novotný(2009), Chapter 3, Theorem 3.1]:

Theorem 6.1 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$. Suppose that the pressure p and the internal energy e are interrelated through (50–52), where $P \in C[0,\infty) \cap C^3(0,\infty)$ satisfies the structural hypotheses (53), (54). Let the transport coefficients μ, η, κ be continuously differentiable functions of the temperature ϑ satisfying (55–57). Finally, let the initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ be given such that*

$$\varrho_0,\ \vartheta_0 \in L^\infty(\Omega),\ \varrho_0 > 0,\ \vartheta_0 > 0 \ a.a.\ in\ \Omega, \mathbf{u}_0 \in L^2(\Omega; R^3). \tag{58}$$

Then the Navier-Stokes-Fourier system (41–45), supplemented with the boundary conditions (40) possesses a finite energy weak solution $\varrho, \vartheta, \mathbf{u}$ in $(0,T) \times \Omega$ in the sense specified in (46–49). The weak solution belongs to the class:

$$\begin{gathered} \varrho \geq 0 \ a.a.\ in\ (0,T) \times \Omega, \\ \varrho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^{5/3}(\Omega)) \cap L^\beta((0,T) \times \Omega) \end{gathered} \tag{59}$$

for a certain $\beta > \frac{5}{3}$;

$$\vartheta > 0 \ a.a.\ in\ (0,T) \times \Omega,\ \vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \tag{60}$$

$$\vartheta^3, \log(\vartheta) \in L^2(0,T; W^{1,2}(\Omega)); \tag{61}$$

$$\mathbf{u} \in L^2(0,T; W_0^\Lambda(\Omega; R^3)),\ \Lambda = \frac{8}{5-\alpha},\ \varrho\mathbf{u} \in C_{\text{weak}}(0,T; L^{5/4}(\Omega; R^3)). \tag{62}$$

In the remaining part of this text, we will discuss various properties of the finite energy weak solutions, the existence of which is guaranteed by Theorem 6.1. An alternative approach based on the internal energy formulation (34–36) was proposed in [Feireisl(2004)]. Although mathematically less sophisticated and physically limited by more restrictive constitutive relations than in Theorem 6.1, the approach [Feireisl(2004)] proved to be convenient when studying stability and convergence properties of certain numerical methods, [Feireisl et al(2014a)Feireisl, Karper, and Novotný].

The weak formulation of the Navier-Stokes-Fourier system based on the complete energy balance has also been studied in the framework of weak solutions. [Hoff and Jenssen(2004)] established global existence for radially symmetric data in R^3. They also identified one of the main stumbling blocks in the analysis of the Navier-Stokes-Fourier system, namely the (hypothetical) appearance of *vacuum zones*, where the density vanishes and the classical understanding of the equations breaks down. More recently, [Bresch and Desjardins(2006)], [Bresch and Desjardins(2007)] discovered a new *a priori* bound on the density gradient leading to global-in-time existence in the truly $3D-$setting conditioned, unfortunately, by a very specific relation satisfied by the density dependent viscosity coefficients and a rather unrealistic formula for the pressure that has to be infinite (negative) for $\varrho \to 0$.

The constraint represented by (46–49) may seem too weak to ensure, at least formally, the *well-posedness* of the problem, meaning uniqueness and possibly stability of solutions with respect to the initial data. Note, however, that this issue remains largely open even for the seemingly simpler *incomressible* Navier-Stokes system despite a concerted effort of generations of excellent mathematicians, see [Fefferman(2006)]. Bellow, we provide an answer to a less ambitious but still interesting question, namely *the weak-strong uniquness* principle. This principle asserts that weak and strong solutions emanating from the *same* initial data coincide as long as the latter exists. To attack the problem, more thermodynamics is need encoded in the so-called relative energy inequality and the resulting concept of *dissipative solution* discussed in the next section.

7 Dissipative solutions

Motivated by the work of [Dafermos(1979)], we introduce a *relative energy functional* associated to the Navier-Stokes-Fourier system. Here again, the Second law of thermodynamics, enforced through Gibbs' equation (9) and the hypothesis of thermodynamics stability (10), will play a crucial role.

7.1 Ballistic free energy

Following [Ericksen(1998)] we consider the so-called *ballistic free energy* functional in the form

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta).$$

The thermodynamic stability relation (10) gives rise to the following two properties of the functions H_Θ:

$$\varrho \mapsto H_\Theta(\varrho, \Theta) \text{ is strictly convex,} \tag{63}$$

and

$$\vartheta \mapsto H_\Theta(\varrho, \vartheta) \text{ attains its global minimum at } \vartheta = \tilde{\vartheta}. \tag{64}$$

As observed by [Bechtel et al(2005)Bechtel, Rooney, and Forest], the above properties are intimately related to *stability* of the equilibrium solutions to the Navier-Stokes-Fourier system. As we shall see, (63), (64) contain the necessary piece of information that will be used later in the proof of weak-strong uniqueness.

7.2 Relative energy

The relative energy is defined as

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2} \varrho \left| \mathbf{u} - \mathbf{U} \right|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x,$$

where $\varrho, \vartheta, \mathbf{u}$ is a weak solution to the Navier-Stokes-Fourier system, and $r, \Theta, \mathbf{U}$ is an arbitrary trio of functions satisfying the relevant compatibility conditions. More precisely, we need

$$r > 0, \ \Theta > 0 \text{ and } \mathbf{U}|_{\partial\Omega} = 0 \tag{65}$$

as soon as the no-slip conditions (40) for the velocity are imposed.

Given the coercivity properties of the ballistic free energy stated in (63), (64), it is easy to see that $\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right)$ plays a role of "distance" between $[\varrho, \vartheta, \mathbf{u}]$ and $[r, \Theta, \mathbf{U}]$, meaning $\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U}\right) \geq 0$ vanishing only if $[\varrho, \vartheta, \mathbf{u}] = [r, \Theta, \mathbf{U}]$.

7.3 Relative energy inequality, dissipative solutions

The strength of the mathematical theory based on the weak solutions in the setting (46–49) consists in the fact that it is possible to derive a functional relation for $\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right)$ reminiscent of the Gronwall inequality. Specifically, we report the following result:

$$
\begin{aligned}
&\left[\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U}\right)\right]_{t=0}^{t=\tau} \qquad (66)\\
&+\int_0^\tau \int_\Omega \frac{\Theta}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)\cdot \nabla_x \vartheta}{\vartheta}\right)\ \mathrm{d}x\ \mathrm{d}t\\
&\leq \int_0^\tau \int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot \nabla_x \mathbf{U} \cdot (\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau \int_\Omega \varrho\Big(s(\varrho,\vartheta) - s(r,\Theta)\Big)\Big(\mathbf{U}-\mathbf{u}\Big)\cdot \nabla_x \Theta\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau \int_\Omega \varrho\Big(\partial_t \mathbf{U} + \mathbf{U}\cdot \nabla_x \mathbf{U}\Big)\cdot (\mathbf{U}-\mathbf{u})\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau \int_\Omega \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - p(\varrho,\vartheta)\mathrm{div}_x \mathbf{U}\right)\ \mathrm{d}x\ \mathrm{d}t\\
&-\int_0^\tau \int_\Omega \left(\varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\partial_t\Theta + \varrho\Big(s(\varrho,\vartheta)-s(r,\Theta)\Big)\mathbf{U}\cdot\nabla_x\Theta\right)\ \mathrm{d}x\ \mathrm{d}t\\
&-\int_0^\tau \int_\Omega \frac{\mathbf{q}(\vartheta, \nabla_x\vartheta)}{\vartheta}\cdot \nabla_x \Theta\ \mathrm{d}x\ \mathrm{d}t\\
&+\int_0^\tau \int_\Omega \left(\left(1-\frac{\varrho}{r}\right)\partial_t p(r,\Theta) - \frac{\varrho}{r}\mathbf{u}\cdot\nabla_x p(r,\Theta)\right)\ \mathrm{d}x\ \mathrm{d}t
\end{aligned}
$$

for any finite energy weak solution of the Navier-Stokes-Fourier system (40–45) and any trio of (smooth) test functions satisfying the compatibility conditions (65), see [Feireisl and Novotný(2012), Section 3]. Motivated by [Lions(1996)], where a similar definition is proposed for the incompressible Euler system, we say that $\varrho, \vartheta, \mathbf{u}$ is a *dissipative solution* to the Navier-Stokes-Fourier system (40–45) if (i) $\varrho, \vartheta, \mathbf{u}$ belong to the regularity class specified in Theorem 6.1, (ii) $\varrho, \vartheta, \mathbf{u}$ satisfies the relative energy inequality (66) for any trio $r, \Theta, \mathbf{U}$ of sufficiently smooth (for all integrals in (66) to be well defined) test functions satisfying the compatibility conditions (65). As observed in [Feireisl and Novotný(2012), Section 3], any finite energy weak solution of the Navier-Stokes-Fourier system is a dissipative solution. The revers implication is an interesting open problem. The concept as well as a relevant existence theory in the framework of dissipative solutions can be extended to a vast class of physical spaces, including unbounded domains in R^3, see [Jesslé et al(2013)Jesslé, Jin, and Novotný].

7.4 Weak-strong uniqueness

The important feature of the dissipative solutions is that they comply with the weak-strong uniqueness principle, for the proof see [Feireisl(2012), Theorem 6.2], and [Feireisl and Novotný(2012), Theorem 2.1]:

Theorem 7.1 *In addition to the hypotheses of Theorem 6.1, suppose that*

$$s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \text{ with } S(Z) \to 0 \text{ as } Z \to \infty. \tag{67}$$

Let $\varrho, \vartheta, \mathbf{u}$ be a dissipative (weak) solution to the Navier-Stokes-Fourier system in the set $(0,T) \times \Omega$. Suppose that the Navier-Stokes-Fourier system admits a strong solution $\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}$ in the time interval $(0,T)$, emanating from the same initial data and belonging to the class

$$\partial_t \tilde{\varrho},\ \partial_t \tilde{\vartheta},\ \partial_t \tilde{\mathbf{u}},\ \partial_x^m \tilde{\varrho},\ \partial_x^m \tilde{\vartheta},\ \partial_x^m \tilde{\mathbf{u}} \in L^\infty((0,T) \times \Omega),\ m = 0, 1, 2.$$

Then

$$\varrho \equiv \tilde{\varrho},\ \vartheta \equiv \tilde{\vartheta},\ \mathbf{u} \equiv \tilde{\mathbf{u}}.$$

The extra hypothesis (67) reflects the Third law of thermodynamics and can be possibly relaxed.

As we have seen in Theorem 4.2, the Navier-Stokes-Fourier system admits a local in time regular solution as soon as the initial data are regular. In view of Theorem 7.1 we know that any weak solution coincides with this strong solution as long as the latter exists. On the other hand, by virtue of Theorem 6.1, the weak solutions exist globally in time and as such provide a possible alternative of extending the local smooth solution beyond its existence interval. Whether or not strong solutions exist globally in time is an interesting open question, for small data results in this direction see [Matsumura and Nishida(1980)], [Matsumura and Nishida(1983)].

7.4.1 Back to the Euler-Fourier system

At this moment, it is useful to go back to Theorem 5.1, where we produced an example of a system (Euler-Fourier) possessing infinitely many weak solutions. We can introduce the relative energy and define the dissipative solutions for the Euler-Fourier system (37–39), exactly as for the Navier-Stokes-Fourier system. Moreover, it can be shown that the dissipative solutions of the Euler-Fourier system enjoy the property of weak-strong uniqueness similarly to the solutions of the Navier-Stokes-Fourier system, see [Chiodaroli et al(2014)Chiodaroli, Feireisl, and Kreml, Theorem 4.1]. However, one can still obtain the following result rather disturbing result, see [Chiodaroli et al(2014)Chiodaroli, Feireisl, and Kreml, Theorem 4.2]:

Theorem 7.2 *Under the hypotheses of Theorem 5.1, let $T > 0$ be given, together with the initial data*

$$\varrho_0, \vartheta_0 \in C^3(\mathcal{T}^3),\ \varrho_0 > 0,\ \vartheta_0 > 0 \ \textit{in}\ \mathcal{T}^3.$$

Then there exists the initial velocity $\mathbf{u}_0 \in L^\infty(\mathcal{T}^3, R^3)$ such that the corresponding initial-value problem for the Euler-Fourier system (37-39) admits infinitely many *weak solutions in $(0,T) \times \Omega$ belonging to the class*

$$\varrho \in C^2([0,T] \times \Omega),\ \partial_t \vartheta \in L^p(0,T; L^p(\Omega)),\ \nabla_x^2 \vartheta \in L^p(0,T; L^p(\Omega; R^{3\times 3}))$$

for any $1 \le p < \infty$,

$$\mathbf{u} \in C_{\rm weak}([0,T]; L^2(\Omega; R^3)) \cap L^\infty((0,T) \times \Omega; R^3),\ {\rm div}_x \mathbf{u} \in C^2([0,T] \times \Omega).$$

It is worth-noting that the conclusion of Theorem 7.2 does not contradict the principle of weak-strong uniqueness as $\mathbf{u}_0$ *is not* smooth. The problem of "maximal" smoothness of such data is closely related to the so-called *Onsager's conjecture* that have been intensively studied in the context of the incompressible Euler system, see [De Lellis and Székelyhidi(2013)], [De Lellis and Székelyhidi(2014)].

8 Conditional regularity

A conditional regularity criterion is a condition which, if satisfied by a weak solution to a given system, implies that the latter is regular. Similarly, such a condition may be applied to guarantee that a local (strong) solution can be extended to a given time interval. The most celebrated conditional regularity criteria in the context of the incompressible Navier-Stokes and Euler systems are due to [Prodi(1959)], [Serrin(1962)], and, more recently by [Beale et al(1984)Beale, Kato, and Majda], [Constantin and Fefferman(1993)]. Similar conditions were obtained also in the context of compressible barotropic fluids and the full Navier-Stokes-Fourier system, the reader may consult [Fan et al(2010)Fan, Jiang, and Ou], [Huang et al(2013)Huang, Li, and Wang], [Sun et al(2011)Sun, Wang, and Zhang], [Wen and Zhu(2013)], and also the references cited therein.

In view of the results of [Hoff(2002)], [Hoff and Santos(2008)], certain discontinuities imposed through the initial data in the compressible Navier-Stokes system propagate in time. In other words, unlike its incompressible counterpart, the hyperbolic-parabolic compressible Navier-Stokes system does not enjoy the smoothing property typical for purely parabolic equations. Analogously, a solution of the full Navier-Stokes-Fourier system can be regular only if regularity is enforced by a proper choice of the initial data.

8.1 Conditional regularity via the relative energy

A possible approach to conditional regularity of *weak solutions* is to show that:

- the problem admits local-in-time strong solution;
- the problem enjoys the weak-strong uniqueness property;
- show conditional regularity for the strong solution.

This procedure applied in the context of the finite-energy weak solutions to the Navier-Stokes-Fourier system gives rise to the following result, see [Feireisl et al(2014b)Feireisl, Novotný, and Sun, Theorem 2.1].

Theorem 8.1 *Under the hypotheses of Theorem 7.1, let $\varrho, \vartheta, \mathbf{u}$ be a finite energy weak solution of the Navier-Stokes-Fourier system on the time interval $(0,T)$ belonging to the regularity class specified in Theorem 6.1, emanating from (regular) initial data satisfying the hypotheses of Theorem 4.2. Suppose, in addition, that*

$$\operatorname{ess}\sup_{t\in(0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^\infty(\Omega;R^{3\times 3})} < \infty.$$

Then $\varrho, \vartheta, \mathbf{u}$ is a classical solution of the Navier-Stokes-Fourier system in $(0, T) \times \Omega$.

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