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Cluster Algebras Based on Vertex Operator Algebras

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Abstract. Starting from Zhu recursion formulas for correlation functions for vertex operator algebras with formal parameters associated to local coordinates around marked points on a Riemann surfaces, we introduce a cluster algebra structure over a non-commutative set of variables. Cluster elements and mutation rules are explicitly defined. In particular, we propose an elliptic version of vertex operator cluster algebras arising from correlation functions and Zhu reduction procedure for vertex operators on the torus.

INTRODUCTION

Since cluster algebras have been introduced in [15], they found numerous reincarnations in Mathematics [6, 7, 8, 9, 21, 22, 14, 29] and Mathematical Physics [42, 43, 3, 20]. The theory of cluster algebras is connected to many different areas of mathematics, e.g., the representation theory of finite dimensional algebras, Lie theory, Poisson geometry and Teichmüller theory [21]–[23]. Among those topics are dilogarithm identities for conformal field theories [42, 43], quantum algebras [19, 25], quivers [27, 28, 41]. The list of references above is far from being complete.

Several application of cluster algebras in Conformal Field Theory are known [3, 42, 43]. Thus it would not be a surprise if we could establish a direct algebraic connections between them. In particular, one could ask for a version of cluster algebras over non-commutative set of variables. In this note we would like to sketch a way to relate cluster algebras [15, 16, 17, 44] with vertex operator algebras [1, 12, 26]. Base on axioms of vertex algebras and their theory of correlation functions on Riemann surfaces, we formulate definition of a vertex operator cluster algebra. These algebras possess a structure similar to ordinary cluster algebras. At the same time, seeds are defined over non-commutative variables (modes of vertex operators), coordinates around marked points, and matrix elements of a number of vertex operators with formal parameters on a Riemann surface. Rich structure of a vertex operator algebra enables us to enlarge the cluster algebra setup by introducing a number of non-commutative parameters.

Since the definition of a vertex operator cluster algebra includes correlation functions depending on local coordinate on Riemann surfaces it would be interesting to understand possible relations of this construction with the quantum dilogarithm identities as well as with the origin of ordinary cluster algebras arising [14] on Riemann surfaces.

CLUSTER ALGEBRAS

Let us first recall the notion of a cluster algebra [15, 16, 17] following notations of [44]. We consider commutative cluster algebras of rank n . The set of all cluster variables is constructed recursively from an initial set of n cluster variables using mutations. Every mutation defines a new cluster variable as a rational function of the cluster variables constructed previously. Thus recursively, every cluster variable is a certain rational function in the initial n cluster variables. These rational functions are Laurent polynomials [15].

A cluster algebra is determined by its initial seed which consists of a cluster $\mathbf{x} = \{x_1, \dots, x_n\}$, of algebraically independent set of generators, a coefficient tuple $\mathbf{y} = \{y_1, \dots, y_n\}$, and a skew-symmetrizable $n \times n$ integer matrix (the exchange matrix) $B = (b_{ij})$, i.e., $b_{i,j} = -b_{j,i}$. The coefficients $\{y_1, \dots, y_n\}$ are taken in a torsion free abelian group \mathbb{P} .

The mutation in direction k defines a new cluster

$$x'_k x_k = y^+ \prod_{b_{k,i} > 0} x_i^{b_{k,i}} + y^- \prod_{b_{k,i} < 0} x_i^{-b_{k,i}}, \quad (1)$$

where y^\pm are certain monomials in the y_1, \dots, y_n . Mutations also transform the coefficient tuple \mathbf{y} and the matrix B . If u is any cluster variable, thus u is obtained from the initial cluster $\{x_1, \dots, x_n\}$ by a sequence of mutations, then, by [15], u can be written as a Laurent polynomial in the variables x_1, \dots, x_n , that is,

$$u = \frac{f(x_1, \dots, x_n)}{\prod_{i=1}^n x_i^{d_i}}, \quad (2)$$

for some d_i , where $f(x_1, \dots, x_n)$ is a polynomial with coefficients in the group ring $\mathbb{Z}\mathbb{P}$ of the coefficient group \mathbb{P} . A cluster algebra is said to be of finite type if it has only a finite number of seeds. In [16] it was shown that cluster algebras of finite type can be classified in terms of the Dynkin diagrams of finite-dimensional simple Lie algebras. For formal definition of a cluster algebra see subsection a).

Formal definition

Let \mathbb{P} be an abelian group with binary operation \oplus . Let $\mathbb{Z}\mathbb{P}$ be the group ring of \mathbb{P} and let $\mathbb{Q}\mathbb{P}(x_1, \dots, x_n)$ be the field of rational functions in n variables with coefficients in $\mathbb{Q}\mathbb{P}$. A seed is a triple $(\mathbf{x}, \mathbf{y}, B)$, where $\mathbf{x} = \{x_1, \dots, x_n\}$ is a basis of $\mathbb{Q}\mathbb{P}(x_1, \dots, x_n)$, $\mathbf{y} = \{y_1, \dots, y_n\}$, is an n -tuple of elements $y_i \in \mathbb{P}$, and B is a skew-symmetrizable matrix.

Given a seed $(\mathbf{x}, \mathbf{y}, B)$ its mutation $\mu_k(\mathbf{x}, \mathbf{y}, B)$ in direction k is a new seed $(\mathbf{x}', \mathbf{y}', B')$ defined as follows. Let $[x]_+ = \max(x, 0)$. Then we have $B' = (b'_{ij})$ with

$$b'_{ij} = \begin{cases} b_{ij} & \text{for } i = k \text{ or } j = k, \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik} [b_{kj}]_+, & \text{otherwise.} \end{cases} \quad (3)$$

For new coefficients $\mathbf{y}' = (y'_1, \dots, y'_n)$, with

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k, \end{cases} \quad (4)$$

and $\mathbf{x} = \{x_1, \dots, x_n\}$, where

$$x'_k = \frac{y_k \prod_{i=1}^n x_i^{[b_{ik}]_+} + \prod_{i=1}^n x_i^{[-b_{ik}]_+}}{(y_k \oplus 1) x_k}. \quad (5)$$

Mutations are involutions, i.e., $\mu_k \mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}, \mathbf{y}, B)$.

VERTEX OPERATOR ALGEBRAS

A vertex operator algebra (VOA) [1, 5, 10, 12, 26, 31, 34] is determined by a quadruple $(V, Y, \mathbf{1}, \omega)$, where V is a linear space endowed with a \mathbb{Z} -grading with $V = \bigoplus_{r \in \mathbb{Z}} V_r$ with $\dim V_r < \infty$. The state $\mathbf{1} \in V_0$, $\mathbf{1} \neq 0$, is the vacuum vector and $\omega \in V_2$ is the conformal vector with properties described below. The vertex operator Y is a linear map $Y : V \rightarrow \text{End}(V)[[z, z^{-1}]]$ for formal variable z so that for any vector $u \in V$ we have a vertex operator

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}. \quad (6)$$

The linear operators (modes) $u(n) : V \rightarrow V$ satisfy creativity

$$Y(u, z)\mathbf{1} = u + O(z), \quad (7)$$

and lower truncation

$$u(n)v = 0, \quad (8)$$

conditions for each $u, v \in V$ and $n \gg 0$. For the conformal vector ω one has

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad (9)$$

where $L(n)$ satisfies the Virasoro algebra for some central charge C

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{C}{12}(m^3 - m)\delta_{m,-n}\text{Id}_V, \quad (10)$$

where Id_V is identity operator on V . Each vertex operator satisfies the translation property

$$Y(L(-1)u, z) = \partial_z Y(u, z). \quad (11)$$

The Virasoro operator $L(0)$ provides the \mathbb{Z} -grading with $L(0)u = ru$ for $u \in V_r$, $r \in \mathbb{Z}$. Finally, the vertex operators satisfy the Jacobi identity

$$z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) Y(v, z_2) - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2) Y(u, z_1) = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \quad (12)$$

These axioms imply locality, skew-symmetry, associativity and commutativity conditions:

$$(z_1 - z_2)^N Y(u, z_1) Y(v, z_2) = (z_1 - z_2)^N Y(v, z_2) Y(u, z_1), \quad (13)$$

$$Y(u, z)v = e^{zL(-1)} Y(v, -z)u,$$

$$(z_0 + z_2)^N Y(u, z_0 + z_2) Y(v, z_2)w = (z_0 + z_2)^N Y(Y(u, z_0)v, z_2)w,$$

$$u(k)Y(v, z) - Y(v, z)u(k) = \sum_{j \geq 0} \binom{k}{j} Y(u(j)v, z) z^{k-j}, \quad (14)$$

for $u, v, w \in V$ and integers $N \gg 0$. For $v = \mathbf{1}$ one has

$$Y(\mathbf{1}, z) = \text{Id}_V. \quad (15)$$

Note also that modes of homogeneous states are graded operators on V , i.e., for $v \in V_k$,

$$v(n) : V_m \rightarrow V_{m+k-n-1}.$$

In particular, let us define the zero mode $o(v)$ of a state of weight $wt(v) = k$, i.e., $v \in V_k$, as

$$o(v) = v(wt(v) - 1), \quad (16)$$

extending to V additively.

MATRIX ELEMENTS ON THE SPHERE

VOAs via rational matrix elements

There is a number of equivalent sets of axioms for vertex operator algebra theory [36]. In [12, 10] it was proven that one can describe a vertex operator algebra by the set of all its correlation functions. For our purposes here we require one of these equivalent approaches wherein the properties of a vertex operator algebra are expressed in terms of properties of matrix elements which turn out to be rational functions of the formal vertex operator parameters. In this approach vertex operator algebra formal parameters can be taken to be complex numbers with the matrix elements considered as rational functions on the Riemann sphere, i.e., this corresponds to the genus one Riemann surface case.

Let us first define matrix elements. Assume that our VOA is of CFT-type, i.e.,

$$V = \mathbb{C}\mathbf{1} \oplus V_1 \oplus \dots$$

We define the restricted dual space of V by [10]

$$V' = \bigoplus_{n \geq 0} V_n^* \tag{17}$$

where V_n^* is the dual space of linear functionals on the finite dimensional space V_n . Let $\langle \cdot, \cdot \rangle$ denote the canonical pairing between V' and V . In what follows prime marks states that belong to the dual space. Define matrix elements for $v' \in V'$, $v \in V$ and n vertex operators $Y(v_1, z_1), \dots, Y(v_n, z_n)$ by

$$\langle v', Y(v_1, z_1) \dots Y(v_n, z_n)v \rangle \tag{18}$$

In particular, choosing $v = \mathbf{1}$ and $v' = \mathbf{1}'$ we obtain the n -point correlation function on the sphere:

$$F_V^{(0)}(v_1, z_1; \dots; v_n, z_n) = \langle \mathbf{1}', Y(v_1, z_1) \dots Y(v_n, z_n)\mathbf{1} \rangle \tag{19}$$

Here the upper index of $F^{(0)}$ stands for the genus. One can show in general that every matrix element is a homogeneous rational function of z_1, \dots, z_n , [10], [4]. Thus the formal parameters of VOA theory can be replaced by complex parameters on (appropriate subdomains of) the genus zero Riemann sphere \mathbb{CP}^1 .

We illustrate [36] this by considering matrix elements containing one or two vertex operators. Recall from (16) that for $u \in V_n$,

$$u(k) : V_m \rightarrow V_{m+n-k-1} \tag{20}$$

Hence it follows that for $v' \in V'_{m'}$, $v \in V_m$ and $u \in V_n$ we obtain a monomial

$$\langle v', Y(u, z)v \rangle = C_{v'v}^u z^{m'-m-n} \tag{21}$$

where

$$C_{v'v}^u = \langle v', u(m+n-m'-1)v \rangle \tag{22}$$

Recall now the following formal expansion: for variable x, y we adopt the convention that

$$(z_1 + z_2)^m = \sum_{n \geq 0} \binom{m}{n} z_1^{m-n} z_2^n \tag{23}$$

i.e., for $m < 0$ we formally expand in the second parameter z_2 . Next consider the matrix element of two vertex operators. We then find

Theorem 0.1 [FHL] *Let $v' \in V'_{m'}$, $v \in V_m$, $u_1 \in V_{n_1}$ and $u_2 \in V_{n_2}$. Then*

$$\langle v', Y(u_1, z_1)Y(u_2, z_2)v \rangle = \frac{f(z_1, z_2)}{z_1^{m+n_1} z_2^{m+n_2} (z_1 - z_2)^{n_1+n_2}}, \tag{24}$$

$$\langle v', Y(u_2, z_2)Y(u_1, z_1)v \rangle = \frac{f(z_1, z_2)}{z_1^{m+n_1} z_2^{m+n_2} (-z_2 + z_1)^{n_1+n_2}}, \tag{25}$$

where $f(z_1, z_2)$ is a homogeneous polynomial of degree $m + m' + n_1 + n_2$.

As we see from this theorem, the matrix elements (24), (25) are thus determined by a unique homogeneous rational function which can be evaluated on \mathbb{CP}^1 in the domains $|z_1| > |z_2|$ and $|z_2| > |z_1|$ respectively. Note that properties (24) and (25) are equivalent to locality of vertex operators $Y(u_1, z_1)$ and $Y(u_2, z_2)$ so that the axioms of a VOA can be alternatively formulated in terms of rational matrix elements [4], [10]. In [10] the canonical description of a vertex operator algebra in terms of rational functions had been given. In Appendix we recall some auxiliary notions (in particular, maps t_{i_1, \dots, i_n} , and definition of the subring $\mathbb{F}[z_1, \dots, z_n]_S$ of the field of rational functions $\mathbb{F}[z_1, \dots, z_n]$) defined in [10].

Theorem 0.1 can also be generalized for all matrix elements of several variables. Recall the proposition 3.5.1 of [10]:

Proposition 0.2 For $v_1, \dots, v_n, v \in V$, and $v' \in V'$, with any permutation of (i_1, \dots, i_n) of $(1, \dots, n)$, lies in the image of the map $\iota_{1, \dots, n}$:

$$\langle v', Y(u_{i_1}, z_{i_1}) \dots Y(u_{i_n}, z_{i_n})v \rangle = \iota_{1, \dots, n} f(z_1, \dots, z_n), \quad (26)$$

where (uniquely determined) element $f \in \mathbb{F}[z_1, \dots, z_n]_S$ is independent of the permutation and is of the form

$$f(z_1, \dots, z_n) = \frac{g(z_1, \dots, z_n)}{\prod_{i=1}^n z_i^{r_i} \prod_{j < k} (z_j - z_k)^{s_{jk}}}, \quad (27)$$

for some $g(z_1, \dots, z_n) \in \mathbb{F}[z_1, \dots, z_n]$ and $r_i, s_{jk} \in \mathbb{Z}$.

Genus zero Zhu reduction

Using the vertex commutator property (14), i.e.,

$$[u(m), Y(v, z)] = \sum_{i \geq 0} \binom{m}{i} Y(u(i)v, z) z^{m-i},$$

one can also derive [47] a recursive relationship in terms of rational functions between matrix elements for $n+1$ vertex operators and a finite sum of matrix elements for n vertex operators. In [47] we find a recurrent formula expressing an $n+1$ -point matrix element on the sphere as a finite sum of n -point matrix elements in the following

Lemma 0.3 ([Zhu], Lemma 2.2.1) For $v_1, \dots, v_n \in V$, and a homogeneous $v \in V$, we find

$$\begin{aligned} \langle v', Y(v_1, z_1) \dots Y(v_n, z_n)v \rangle &= \sum_{r=2}^n \sum_{m \geq 0} f_{wt(v_1), m}(z_1, z_r) \cdot \langle v', Y(v_2, z_2) \dots Y(v_1(m) v_r, z_r) \dots Y(v_n, z_n)v \rangle \\ &+ \langle v', o(v_1) Y(v_2, z_2) \dots Y(v_n, z_n)v \rangle, \end{aligned} \quad (28)$$

where $f_{wt(v_1), m}(z_1, z_r)$ is a rational function defined by

$$\iota_{z, w} f_{n, m}(z, w) = \sum_{j \in \mathbb{N}} \binom{n+j}{m} z^{-n-j-1} w^{n+j-1}, \quad (29)$$

$$f_{n, m}(z, w) = \frac{z^{-n}}{m!} \left(\frac{d}{dw} \right)^m \frac{w^n}{z-w}. \quad (30)$$

CORRELATION FUNCTIONS ON THE TORUS

One would like to ask now for generalizations at higher genus. In order to consider modular-invariance of n -point functions at genus one, Zhu introduced [47] a second "square-bracket" VOA $(V, Y[\cdot, \cdot], \mathbf{1}, \tilde{\omega})$ associated to a given VOA $(V, Y(\cdot, \cdot), \mathbf{1}, \omega)$. The new square bracket vertex operators are defined by a change of coordinates, namely

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1), \quad (31)$$

with $q_z = e^z$, while the new conformal vector is $\tilde{\omega} = \omega - \frac{c}{24} \mathbf{1}$. For v of $L(0)$ weight $wt(v) \in \mathbb{R}$ and $m \geq 0$,

$$v[m] = m! \sum_{i \geq m} c(wt(v), i, m) v(i), \quad (32)$$

$$\sum_{m=0}^i c(wt(v), i, m) x^m = \binom{wt(v) - 1 + x}{i}.$$

In particular we note that $v[0] = \sum_{i \geq 0} \binom{wt(v)-1}{i} v(i)$.

Recall [47] (see also [37, 40, 38, 39]), that at genus one, instead of matrix elements of the form (18), one considers traces over corresponding vertex operator algebra. Vertex operator algebra formal parameters are associated now with local coordinates around insertion points on the torus. For $v_1, \dots, v_n \in V$ the genus one n -point function has the form:

$$F_V^{(1)}(v_1, z_1; \dots; v_n, z_n; \tau) = \text{Tr}_V \left(Y(q_1^{L(0)} v_1, q_1) \dots Y(q_n^{L(0)} v_n, q_n) q^{L(0)-C/24} \right), \quad (33)$$

for $q = e^{2\pi i \tau}$ and $q_i = e^{z_i}$, where τ is the torus modular parameter.

Then the genus one Zhu recursion formula is given by

Theorem 0.4 [Zhu] *For any $v, v_1, \dots, v_n \in V$ we find for an $n + 1$ -point function*

$$\begin{aligned} F_V^{(1)}(v, z; v_1, z_1; \dots; v_n, z_n; \tau) &= \sum_{r=1}^n \sum_{m \geq 0} P_{m+1}(z - z_r, \tau) \cdot F_V^{(1)}(v_1, z_1; \dots; v[m]v_r, z_r; \dots; v_n, z_n; \tau) \\ &+ F_V^{(1)}(o(v); v_1, z_1; \dots; v_n, z_n; \tau), \end{aligned} \quad (34)$$

where

$$F_V^{(1)}(o(v); v_1, z_1; \dots; v_n, z_n; \tau) = \text{Tr}_V \left(o(v) Y(q_1^{L(0)} v_1, q_1) \dots Y(q_n^{L(0)} v_n, q_n) q^{L(0)-C/24} \right).$$

In this theorem $P_m(z, \tau)$ denote higher Weierstrass functions [45] defined by

$$P_m(z, \tau) = \frac{(-1)^m}{(m-1)!} \sum_{n \in \mathbb{Z}_{\neq 0}} \frac{n^{m-1} q^n}{1 - q^n}. \quad (35)$$

The higher genus versions of the genus zero and one Zhu reduction procedure and the formula (34) described in lemma 0.3 and theorem 0.4 are also available [24, 46].

CLUSTER STRUCTURE FOR A VERTEX OPERATOR ALGEBRA

Now we are in a position to determine a cluster-like algebra structure for a vertex operator algebra. Let us introduce the cluster structure for a vertex operator algebra V provided by the following data. Fix a vertex operator algebra V . Choose n -marked points $p_i, i = 1, \dots, n$ on a compact Riemann surface [11]. In the vicinity of each marked point p_i define a local coordinate z_i with zero at p_i . Consider n -tuples $\mathbf{v} \equiv \{v_1, \dots, v_n\}$, of arbitrary states $v_i \in V$, and local corresponding vertex operators $\mathbf{Y}(\mathbf{v}, \mathbf{z}) \equiv \{Y(v_1, z_1), \dots, Y(v_n, z_n)\}$, with coordinates $\mathbf{z} \equiv \{z_1, \dots, z_n\}$ around $p_i, i = 1, \dots, n$. We define a *vertex operator cluster algebra seed*

$$(\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})), \quad (36)$$

where $F_n(\mathbf{v}, \mathbf{z}) \equiv F_n(v_1, z_1; \dots; v_n, z_n)$ is an n -point correlation function (matrix element (18) for the sphere case) for n states v_i . Now, define the mutation $\mu_k(v, m, z)$:

$$(\mathbf{v}', \mathbf{Y}(\mathbf{v}', \mathbf{z}), F'_n(\mathbf{v}', \mathbf{z})) = \mu_k(v, m, z) (\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})), \quad (37)$$

of the seed (36) in direction $k \in 1, \dots, n$ for $v \in V$, according to the Zhu reduction formula for corresponding Riemann surface genus, e.g., for the sphere as in (28), for the torus as in (34), etc. Namely, for \mathbf{v} , we define \mathbf{v}' as the mutation of \mathbf{v} in direction $k \in 1, \dots, n$ as

$$\mathbf{v}' = \mu_k(v, m, z) \mathbf{v} = (v_1, \dots, v(m)v_k, \dots, v_n), \quad (38)$$

for some $m \geq 0$. Note that due to the property (8) we get a finite number of terms as a result of the action of $v(m)$ on v_r . For the n -tuple of vertex operators we define

$$\mathbf{Y}(\mathbf{v}', \mathbf{z}) = \mu_k(v, m, z) \mathbf{Y}(\mathbf{v}, \mathbf{z}) = (Y(v_1, z_1), \dots, Y(v(m)v_k, z_k), \dots, Y(v_n, z_n)). \quad (39)$$

The mutation

$$F'_n(\mathbf{v}', \mathbf{z}) = \mu_k(v, m, z) F_n(\mathbf{v}, \mathbf{z}), \quad (40)$$

is defined by summing over mutations in all possible directions with auxiliary functions $f(\text{wt } v, m, k, z)$, $k \in 1, \dots, n$ and all $m \geq 0$:

$$\begin{aligned} F'_n(\mathbf{v}', \mathbf{z}) &= \mu_k(v, m, z) F_n(v_1, z_1; \dots; v_n, z_n) \\ &= \sum_{k=1}^n \sum_{m \geq 0} f(\text{wt } v, m, k, z) \cdot F_n(v_1, z_1; \dots; \mathbf{1}(m)v_k, z_k; \dots; v_n, z_n) + \widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}), \end{aligned} \quad (41)$$

where $\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z})$ denote higher terms in the Zhu reduction formula for a specific genus of a Riemann surfaces used in the consideration. In particular, for the genus zero case we have $f(\text{wt } v, m, k, z) = f_{v,m}(z, z_k)$ defined in (29)–(30) for some $m \geq 0$,

$$\widetilde{F}_n(v, z; \mathbf{v}, \mathbf{z}) = F_n^{(0)}(o(v); v, z) = \langle w', o(v) Y(v_1, z_1) \dots Y(v_n, z_n) w \rangle,$$

while for the genus one Riemann surface we take and $f(\text{wt } v, m, k, z) = P_{m+1}(z - z_k; \tau)$ given by (35), and

$$\widetilde{F}_n^{(1)}(v, z; \mathbf{v}, \mathbf{z}) = F_n^{(1)}(o(v); v, z) = \text{Tr}_V(o(v) Y(v_1, z_1) \dots Y(v_n, z_n)).$$

The mutation $\mu_k(v, m, z)$ defined by (38), (39), (41) is an involution, i.e.,

$$\mu_k(v, m, z) \mu_k(v, m, z) (\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})) = (\mathbf{v}, \mathbf{Y}(\mathbf{v}, \mathbf{z}), F_n(\mathbf{v}, \mathbf{z})),$$

subject a few conditions. As the first condition, one can take $v(m)v(m)v_k = v_k$, $k = 1, \dots, n$ for the actions (38–39). The simplest case, in particular, when $v \in V_k$ when $v(m) = o(v) \equiv v(\text{wt } v - 1)$ (see (16)), then, due to the property (20), $v(m)v(m) : V_m \longrightarrow V_m$, when $k - m - 1 = 0$. Secondly, since the vacuum state $\mathbf{1} \in V_0$, and $Y(\mathbf{1}, z) = \text{Id}_V$ (15), we can take $v = \mathbf{1}$ in the mutation $\mu_k(v, m, z)$ in order to keep the rank of the resulting cluster matrix $F_n(\mathbf{v}', \mathbf{z}')$. Thus we have

$$\begin{aligned} F'_n(\mathbf{v}, \mathbf{z}) &= \mu_k(\mathbf{1}, m, z) F_n(v_1, z_1; \dots; v_n, z_n) \\ &= \sum_{k=1}^n \sum_{m \geq 0} f(0, m, k, z) \cdot F_n(v_1, z_1; \dots; \mathbf{1}(m)v_k, z_k; \dots; v_n, z_n) + \widetilde{F}_n(v, z) \\ &= F_n(\mathbf{1}, z; \mathbf{v}, \mathbf{z}) = F_n(\mathbf{v}, \mathbf{z}). \end{aligned}$$

Thus, in this case, the mutation is obviously an involution. In more sophisticated cases when $v \neq \mathbf{1}$, one can impose further conditions on v_k , $k = 1, \dots, n$ to make (41) an involution.

Note that when we sum in (41) over mutations in all possible directions $k \in 1, \dots, n$ and all $m \geq 0$, we obtain a correlation function (matrix element for the sphere) of rank $n + 1$ (see (28) and (34)) with extra $v \in V$ inserted at a point p with corresponding local coordinate z :

$$F_{n+1}^{(g)}(v, z; v_1, z_1; \dots; v_n, z_n; \tau) = \sum_{k=1}^n \sum_{m \geq 0} f(\text{wt } v, m, k, z) \cdot F_n^{(g)}(v_1, z_1; \dots; v(m)v_k \dots; v_n, z_n; \tau) + \widetilde{F}_n^{(g)}(v, z; \mathbf{v}, \mathbf{z}). \quad (42)$$

When we reduce $F_n^{(g)}(v_1, z_1; \dots; v(m)v_k \dots; v_n, z_n)$ in (41) to the partition function $F_0^{(g)}$ (i.e., the zero point function) according to the Zhu reduction formulas ((28) or (34)), we obtain multiple action of modes $\prod_{m \geq 0} v_r(m)$ on various v_k as well as products of $f(\text{wt } v_r, m_r, r, z_r)$ functions as a result of action on z_k . In that case we obtain a cluster structure of infinite type. The cluster algebra structure given by (37)–(40) thus serves as a counterpart for ordinary cluster algebra structure over a non-commutative variables $v_1, \dots, v_n \in V$. As we mentioned before, the Zhu reduction formula is available for genus two correlation functions [24] and supposed to be expandable to higher genus [38, 46]. Thus we add the superscript (g) in (41). One can use this construction in order to describe a conformal field theory [2, 4, 13] corresponding to a vertex operator algebra. Starting from the partition function (or the zero-point function) $F_0^{(g)}$, we generate recursively, according to the cluster algebra rules (37)–(41), all higher n -point function seeds ($n = 1, \dots, n$), extending the zero-rank seed to (36).

We can expect that the cluster algebra structure for a vertex operator algebra is related to ordinary cluster algebra structure over rational functions $\mathbb{F}(x_1, \dots, x_n)$ via correlation functions on the Riemann sphere. To finish this paper we conjecture that for a vertex operator algebra V , the cluster algebra structure defined by mutations (37)–(41) of a seed (36) in provides an ordinary cluster algebra structure over rational functions. Since (under certain conditions) [47, 33, 36] genus one correlations functions are modular forms [30], a cluster algebra structure (37)–(41) with z_i defined on the torus give us an "elliptic" version of a cluster algebra over modular forms. One can show for instance that the cluster algebra for a vertex operator algebra on the sphere generates the cluster algebras of rank one and two over rational functions.

APPENDIX:

The definition of maps ι_{i_1, \dots, i_n}

Here we introduce some auxiliary notions following [10]. Let

$$S = \left\{ \sum_{i=1}^n a_i z_i, a_i \in \mathbb{F}, a_i \text{ not all zero} \right\} \subset \mathbb{F}[z_1, \dots, z_n], \quad (43)$$

where $\mathbb{F}[z_1, \dots, z_n]$ is the field of rational functions. Let $\mathbb{F}[z_1, \dots, z_n]_S$ be a subring of $\mathbb{F}[z_1, \dots, z_n]$ obtained by inverting the products of elements of S . Let (i_1, \dots, i_n) be a permutation of the set $(1, \dots, n)$. Recursively define maps

$$\iota_{(i_1, \dots, i_n)} : \mathbb{F}[z_1, \dots, z_n]_S \longrightarrow \mathbb{F} \left[[z_1, z_1^{-1}], \dots, [z_n, z_n^{-1}] \right], \quad (44)$$

by the following rule. For $n-1$, define ι_1 be the inclusion map. Thus $S = \{a_1 z_1, a_1 \in \mathbb{F}^\times\}$, and $\mathbb{F}[z_1]_S = \mathbb{F}[z_1, z_1^{-1}]$. Assume we defined the maps $\iota_{(i_1, \dots, i_{n-1})}$. Now let us define $\iota_{(i_1, \dots, i_n)}$. Let $f(z_1, \dots, z_n) \in \mathbb{F}[z_1, \dots, z_n]_S$. Then it is of the form

$$f(z_1, \dots, z_n) = \frac{g(z_1, \dots, z_n)}{\prod_{k=1}^r \left(\sum_{j=2}^n a_{k_j} z_{i_j} \right) \prod_{l=1}^s \left(\sum_{j=1}^n b_{l_j} z_{i_j} \right)}, \quad (45)$$

where $g(z_1, \dots, z_n) \in \mathbb{F}[z_1, \dots, z_n]$, the denominator is non-vanishing, and $b_{l_j} \neq 0$, $l = 1, \dots, s$. Then we expand $\left(\prod_{l=1}^s \left(\sum_{j=1}^n b_{l_j} z_{i_j} \right) \right)^{-1}$ as a power series in z_{i_2}, \dots, z_{i_n} since $b_{l_j} \neq 0$. Let us call this series $h(z_1, \dots, z_n)$. Then for each $t \in \mathbb{Z}$ the of $z_{i_1}^t$ in $g(z_1, \dots, z_n)h(z_1, \dots, z_n)$ is a polynomial in z_{i_2}, \dots, z_{i_n} , which we denote by $g_t(z_1, \dots, z_n)$. Since we assume

that the maps $\iota_{(i_2, \dots, i_n)} \left(\frac{g(z_{i_2}, \dots, z_{i_n})}{\prod_{k=1}^r \left(\sum_{j=2}^n a_{k_j} z_{i_j} \right)} \right)$ are defined let us put

$$\iota_{(i_1, \dots, i_n)} f(z_1, \dots, z_n) = \sum_{t \in \mathbb{Z}} \frac{g_t(z_{i_2}, \dots, z_{i_n})}{\prod_{k=1}^r \left(\sum_{j=2}^n a_{k_j} z_{i_j} \right)} z_{i_1}^t. \quad (46)$$

For instance, suppose $n = 2$ and \mathbb{F} is algebraically closed. Then all non-vanishing homogeneous polynomials in two variables are inverted

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{z_{i_2}^s \prod_{l=1}^s (b_{l_1} z_{i_1} + b_{l_2} z_{i_2})},$$

and ι_{i_1, i_2} is the expansion in negative powers of z_{i_1} or in positive powers of z_{i_2} . The maps ι_{i_1, \dots, i_n} are injective.

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