Curve of maximal values and its applications in stochastic bifucations and vortex visualization

Tomáš Vejchodský

vejchod@math.cas.cz

Joint work with Jakub Šístek

Institute of Mathematics
Academy of Sciences
Czech Republic

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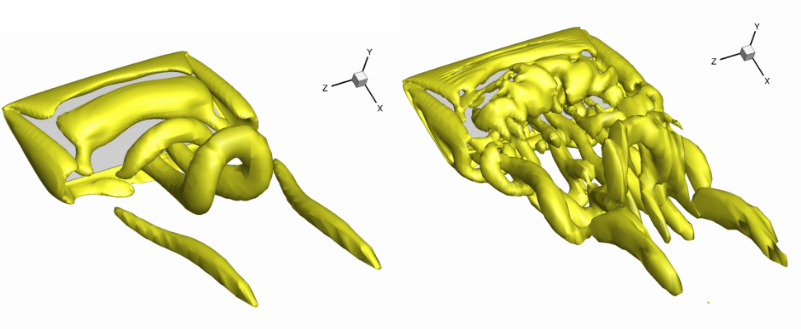
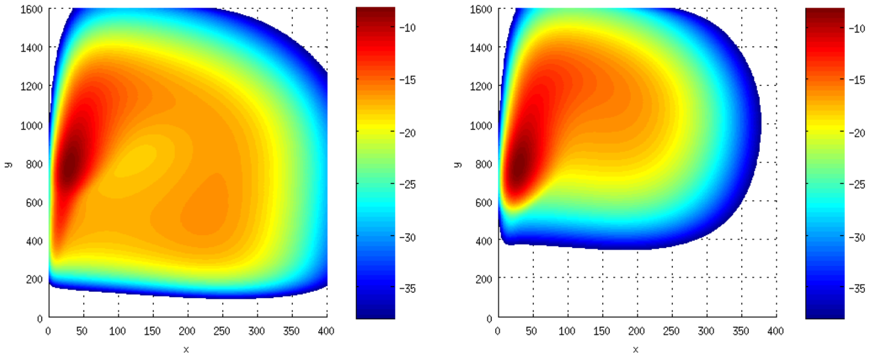
Vortex identification



[J. Šístek]

Stochastic bifurcations



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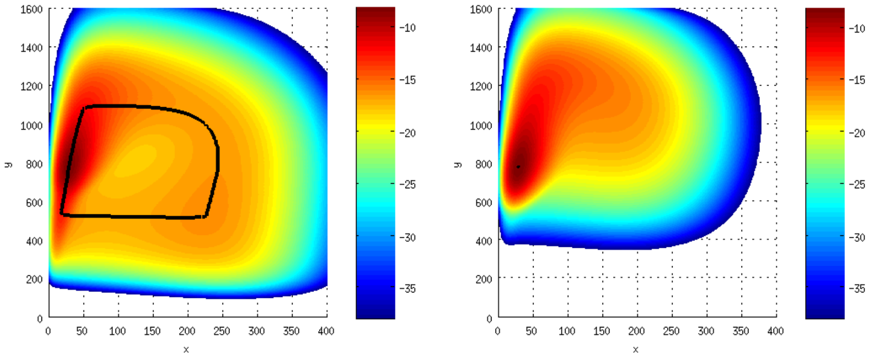
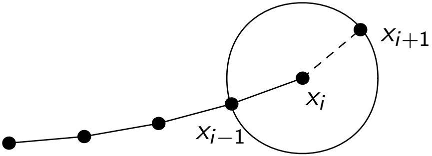


Algorithm 1. Go through local maxima

Input:

- Smooth function f
- Search radius r
- Initial point $\mathbf{x}_0$



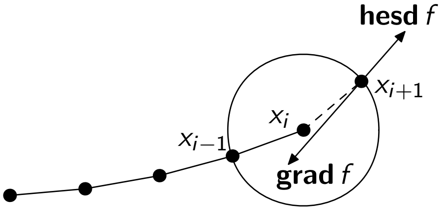
Algorithm:

- Find local maxima of f on $C_{\mathbf{x}_i,r}$
- Discard local maximum at $\mathbf{x}_{i-1}$
- If one local maximum left $\Rightarrow$ go there
- If more local maxima $\Rightarrow$ branching

Algorithm 2. Gradient aligns with Hessian direction.

Input:

- Smooth function f
- Search radius r
- Initial point $\mathbf{x}_0$



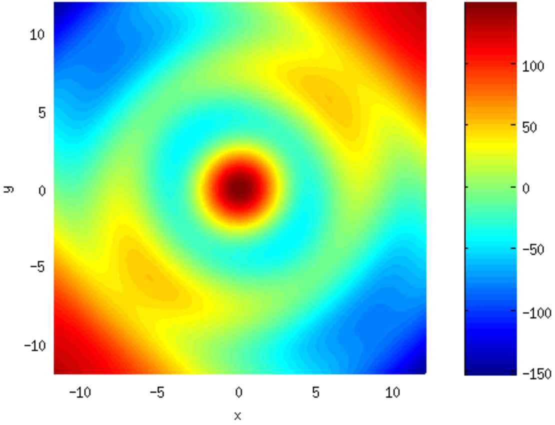
Algorithm:

- On $C_{\mathbf{x}_i,r}$ find points, where $\textbf{grad}\, f = \alpha\, \textbf{hesd}\, f$
- Discard the point at $\mathbf{x}_{i-1}$
- If one point left $\Rightarrow$ go there
- If more points $\Rightarrow$ branching

Definition:

hesd f ... eigenvector corresponding to the largest eigenvalue of the Hessian of f

Example



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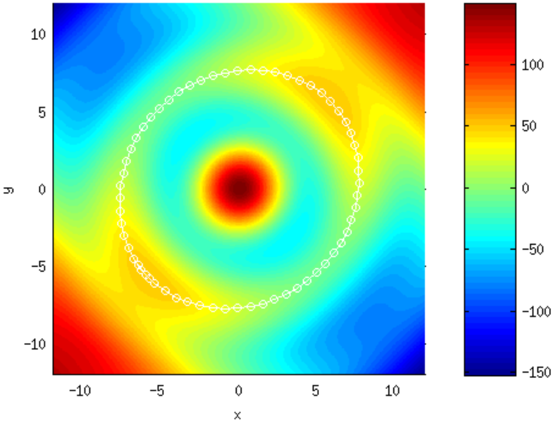

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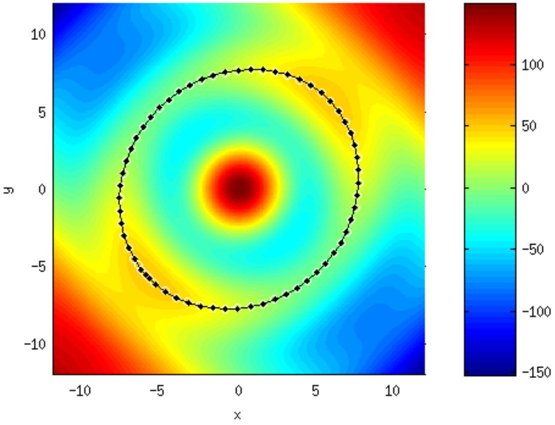


Why the two algorithms agree?

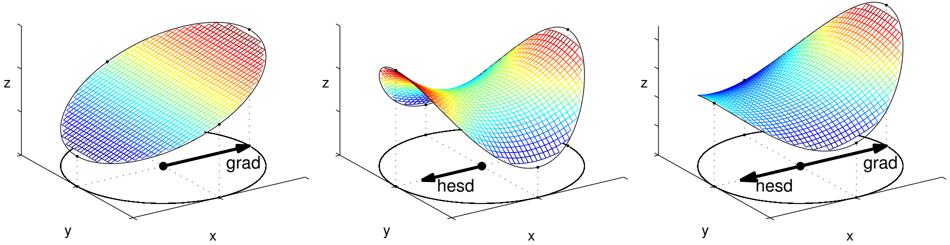
Lemma:

- f is quadratic (in the neighbourhood of $\mathbf{x}_0$)
- Eigenvalues of Hessian $H(\mathbf{x}_0)$ are not equal
- $\mathbf{grad}\, f(\mathbf{x}_0) = \alpha\, \mathbf{hesd}\, f(\mathbf{x}_0)$

$\Rightarrow$ f has two local maxima $\mathbf{x}_1^{\max}$ and $\mathbf{x}_2^{\max}$ on a circle around $\mathbf{x}_0$ and $\mathbf{grad}\, f(\mathbf{x}_0) = \alpha\, \mathbf{hesd}\, f(\mathbf{x}_0) = \beta(\mathbf{x}_2^{\max} - \mathbf{x}_1^{\max})$

Proof:

$$f(\mathbf{x}_0) + \mathbf{grad}\, f(\mathbf{x}_0)\cdot\mathbf{x} \quad + \tfrac{1}{2}\mathbf{x}^T H(\mathbf{x}_0)\mathbf{x} \quad = f(\mathbf{x}_0 + \mathbf{x})$$



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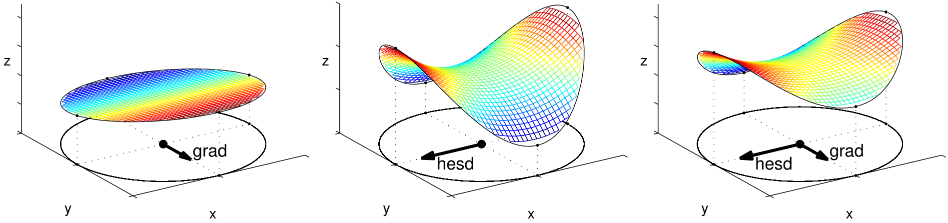
But:

If $\alpha\, \mathbf{hesd}\, f(\mathbf{x}_0) = \beta(\mathbf{x}_2^{\max} - \mathbf{x}_1^{\max}) \not\Rightarrow \mathbf{grad}\, f(\mathbf{x}_0) = \alpha\, \mathbf{hesd}\, f(\mathbf{x}_0)$



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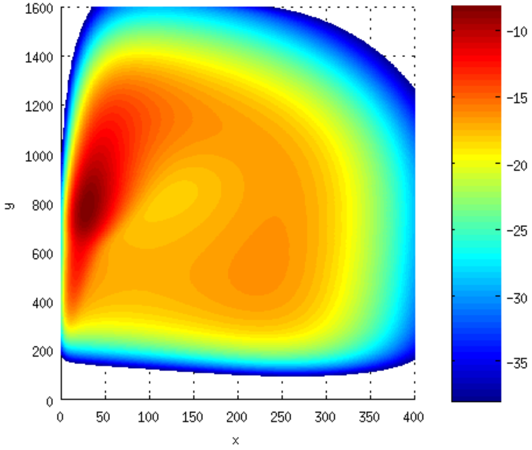

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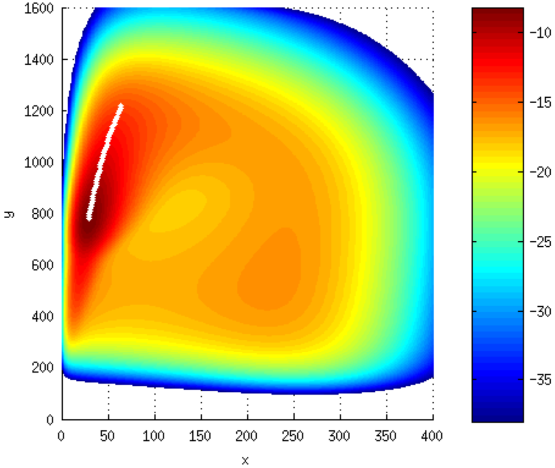

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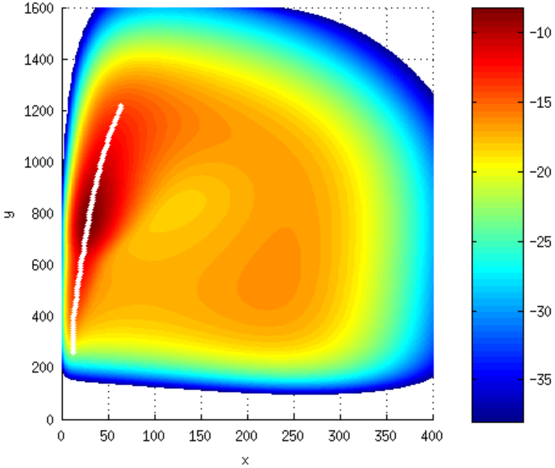

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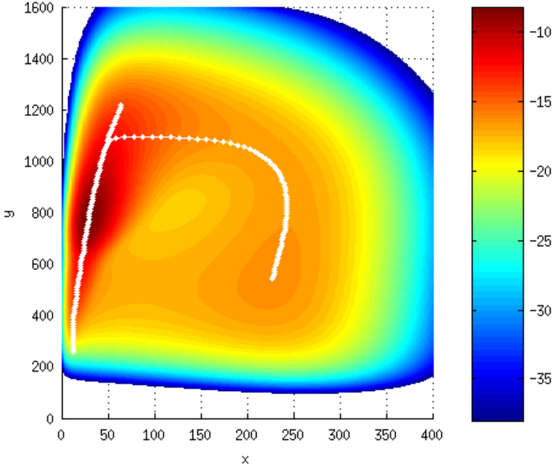

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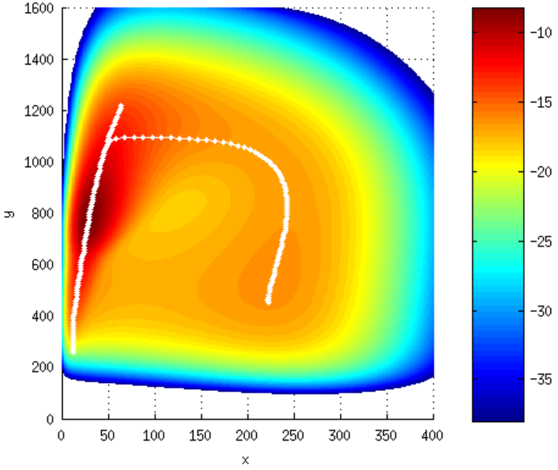

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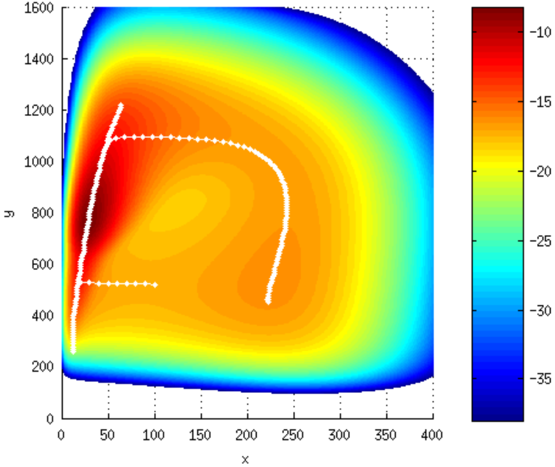

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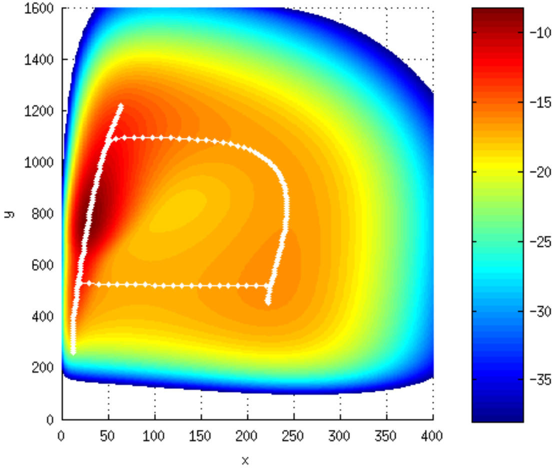

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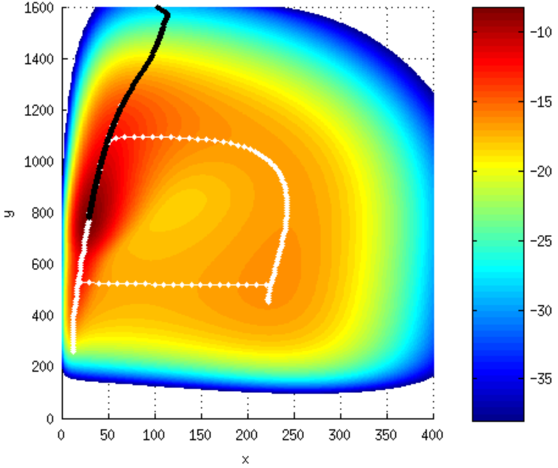

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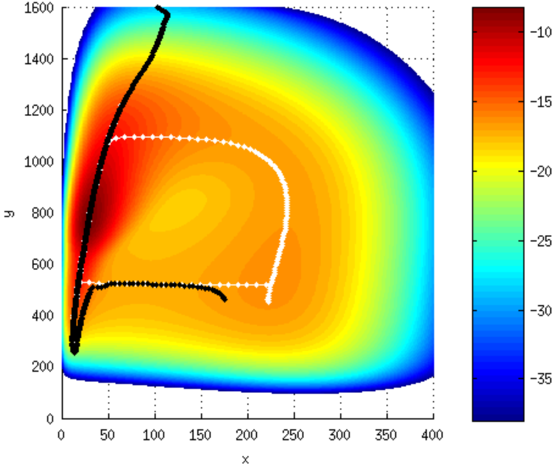

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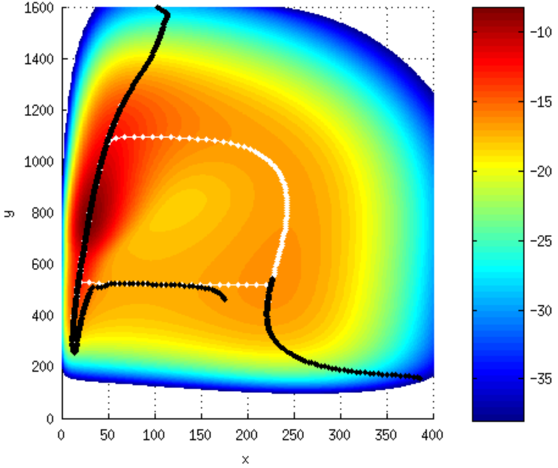

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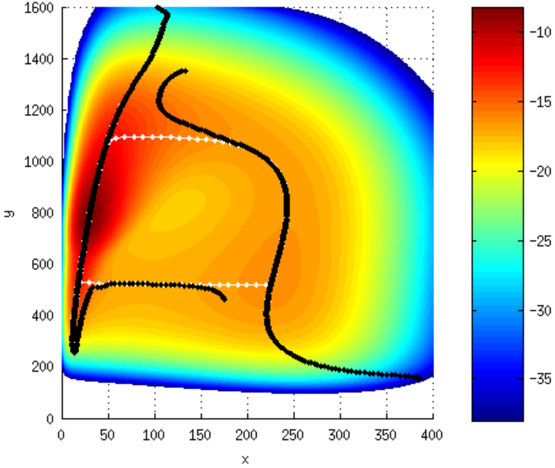

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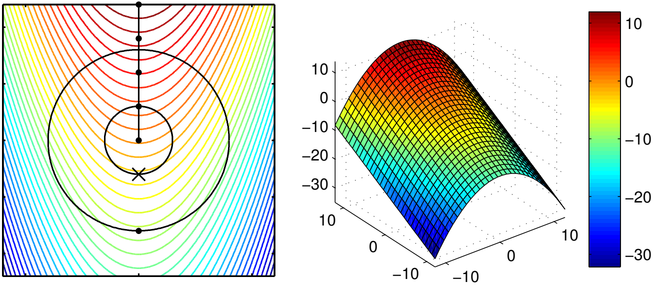
Algorithm 1. Go through local maxima

Problems:

- Dependence on the search circle size.

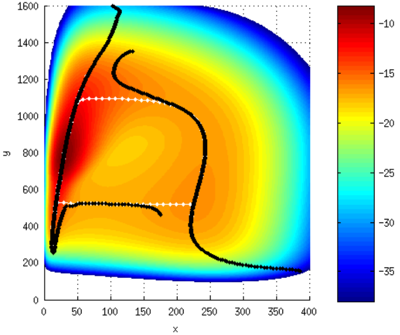


- One local maximum on the circle $\Rightarrow$???
- No local maximum $\Rightarrow$ f is constant.
- Noise in f $\Rightarrow$ spurious local maxima.
- Auxiliary parameters ε.

Algorithm 2. Gradient aligns with Hessian direction.

Problems:

- It finds curves of minimal values as well.



- If **grad** $f = 0 \Rightarrow$ go in **hesd** f direction?
- If **hesd** $f = 0 \Rightarrow$ go in **grad** f direction?
- If both **grad** $f = 0$ and **hesd** $f = 0 \Rightarrow f$ is constant.
- Calculation of gradient and Hessian from discrete data, smoothing.
- Auxiliary parameters ε.

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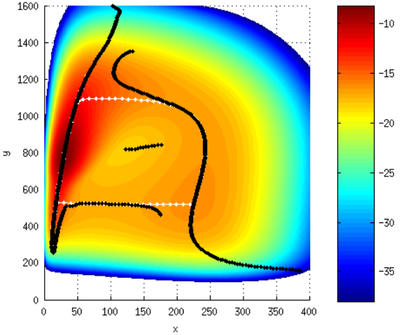

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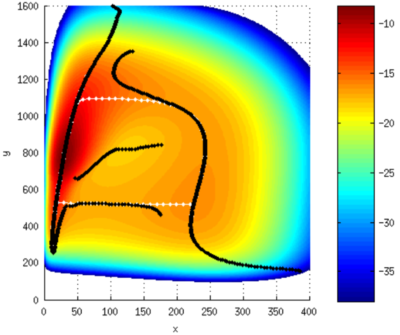

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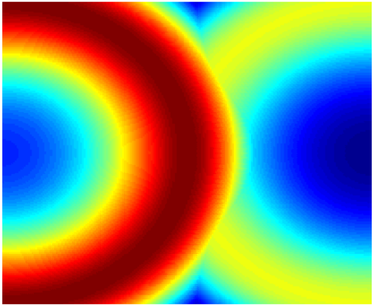
Future work and further problems

- Dependence on the direction



- Branching
- 3D
- Real problems of vortex identification

Thank you for your attention

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