What is the value of the trace constant?

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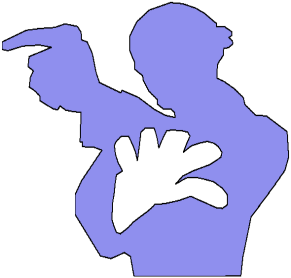
European
Finite
Element
Fair
Prague
5–6 June
2015

What is the value of the trace constant of the clown?

$$\|v\|_{L^2(\partial\Omega)} \leq C_{\mathrm{T}} \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$



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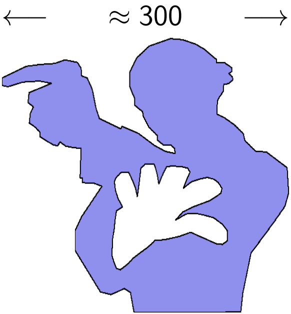

Let us vote

(a) $C_{\mathrm{T}} \approx 2.52$

(b) $C_{\mathrm{T}} \approx 25.2$

(c) $C_{\mathrm{T}} \approx 252$

How to compute C_{T}?

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Smallest eigenvalue:

$$\lambda_1 = \inf_{0 \ne v \in H^1(\Omega)} \frac{\|v\|^2_{H^1(\Omega)}}{\|v\|^2_{L^2(\partial\Omega)}} \le \frac{\|v\|^2_{H^1(\Omega)}}{\|v\|^2_{L^2(\partial\Omega)}} \quad \forall v \in H^1(\Omega)$$

$$\|v\|^2_{L^2(\partial\Omega)} \le \frac{1}{\lambda_1} \|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

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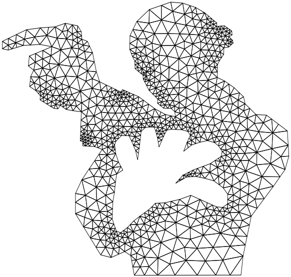
$$\|v\|^2_{L^2(\partial\Omega)} \le \frac{1}{\lambda_1}\|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega)$$

Optimal constant: $C_{\mathrm{T}} = \dfrac{1}{\sqrt{\lambda_1}}$

Solving the eigenproblem by FEM



Define

$$V_h = \{v_h \in H^1(\Omega) : v_h|_K \in P^1(K)\ \forall K \in \mathcal{T}_h\}$$

Find $0 \neq u_{h,i} \in V_h$ and $\lambda_{h,i} \in \mathbb{R}$:

$$(\nabla u_{h,i}, \nabla v_h)_\Omega + (u_{h,i}, v_h)_\Omega = \lambda_{h,i}(u_{h,i}, v_h)_{\partial\Omega} \quad \forall v_h \in V_h$$

But: $\lambda_i \leq \lambda_{h,i}$

$$\Rightarrow\ C_{\mathrm{T},h} = \frac{1}{\sqrt{\lambda_{h,1}}} \leq \frac{1}{\sqrt{\lambda_1}} = C_{\mathrm{T}}$$

How to get a lower bound $X \le \lambda_1$?

Algorithm

- Find suitable $\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega)$
- Compute:

$$A = \left(\|\nabla u_{h,1} - \mathbf{q}\|^2_{L^2(\Omega)} + \|u_{h,1} - \mathrm{div}\,\mathbf{q}\|^2_{L^2(\Omega)} \right)^{1/2} / \|u_{h,1}\|_{L^2(\partial\Omega)}$$

$$B = \|\mathbf{q} \cdot \mathbf{n}_\Omega - \lambda_{h,1} u_{h,1}\|_{L^2(\partial\Omega)} / \|u_{h,1}\|_{L^2(\partial\Omega)}$$

$$X_2 = \frac{1}{2} \left(-A + \sqrt{A^2 + 4(\lambda_{h,1} - B)} \right)$$

- Then $X_2^2 \le \lambda_1$

Assumption: $\lambda_{h,1}$ must be (relatively) closer to λ_1 than to λ_2.

[I. Šebestová, T.V., SIAM J. Numer. Anal., 2014]

How to find suitable $\mathbf{q}$?

Optimal: $\mathbf{q} = \operatorname*{arg\,min}_{\mathbf{w} \in \mathbf{H}(\mathrm{div},\Omega)} J(\mathbf{w})$, where

$$J(\mathbf{w}) = \left(\|\nabla u_{h,1} - \mathbf{w}\|^2_{L^2(\Omega)} + \|u_{h,1} - \operatorname{div} \mathbf{w}\|^2_{L^2(\Omega)} \right)^{1/2} + C_{\mathrm{T}} \|\mathbf{w} \cdot \mathbf{n}_\Omega - \lambda_{h,1} u_{h,1}\|_{L^2(\partial\Omega)}$$

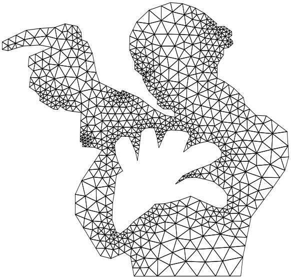
Practical:

$$\tilde{J}(\mathbf{w}) = \left(1 + \frac{1}{\rho}\right) \left(\|\nabla u_{h,1} - \mathbf{w}\|^2_{L^2(\Omega)} + \|u_{h,1} - \operatorname{div} \mathbf{w}\|^2_{L^2(\Omega)} \right) + (1 + \rho) \frac{1}{\lambda_{h,1}} \|\mathbf{w} \cdot \mathbf{n}_\Omega - \lambda_{h,1} u_{h,1}\|^2_{L^2(\partial\Omega)}$$

Numerical results

Initial mesh:

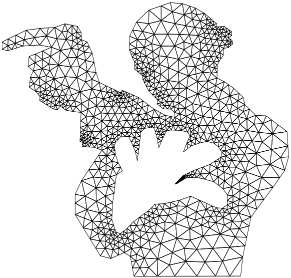
$N_{\mathrm{DOF}} = 871$



$2.0359 \leq C_{\mathrm{T}} \leq 3.0742$

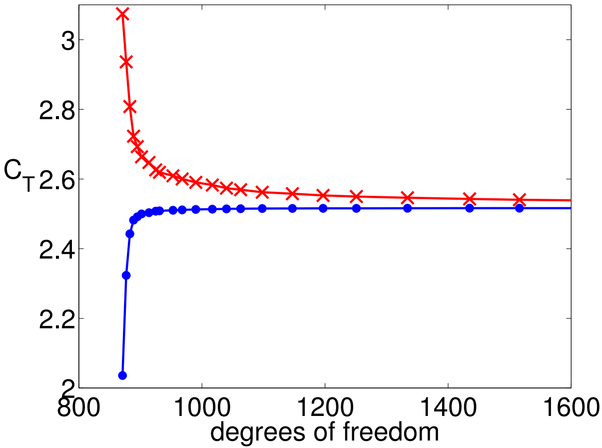
After 44 adaptive steps:

$N_{\mathrm{DOF}} = 33863$



$2.5164 \leq C_{\mathrm{T}} \leq 2.5194$

Adaptive convergence



Conclusions

- Two-sided bounds for the trace constant
- Good for all types of
  - trace inequalities
  - Friedrichs inequalities
  - Poincaré inequalities

  As soon as
  - the operator is symmetric and elliptic
  - there is compactness

Thank you for your attention

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