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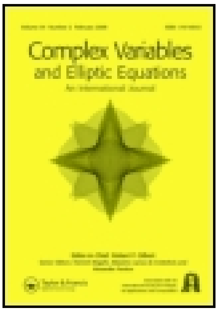

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Transmission problem for the Brinkman system

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Transmission problem for the Brinkman system

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L^2-solutions of the transmission problem, the Robin-transmission problem and the Dirichlet-transmission problem for the Brinkman system are studied by the integral equation method. Necessary and sufficient conditions for the solvability are given. The uniqueness of a solution is also studied.

Keywords: Brinkman system; transmission problem; single-layer potential; double layer potential

AMS Subject Classifications: 35Q35; 35Q30

1. Introduction

The integral equation method is one of traditional methods in hydrodynamics.[1–6] This method is especially fruitful for transmission problems.[4,7–12] In this paper, we study the following transmission problem: let $\Omega = \Omega_+ \subset R^m$, $m > 2$, be a bounded open set with Lipschitz boundary. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$, where $\overline{\Omega}_+$ is the closure of Ω_+. Let λ_+, λ_- and c_+ be non-negative constants and a_+, a_-, b_+ and b_- positive constants. We study the transmission problem for the Brinkman system

$$-\Delta \boldsymbol{u}_\pm + \lambda_\pm \boldsymbol{u}_\pm + \nabla p_\pm = 0, \qquad \nabla \cdot \boldsymbol{u}_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$a_+\boldsymbol{u}_+ - a_-\boldsymbol{u}_- = \boldsymbol{g}, \qquad b_+T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_-T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+\boldsymbol{u}_+ = \boldsymbol{f} \quad \text{on } \partial\Omega.$$

Here, $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$. We look for an L^2-solution of the problem, i.e. the non-tangential maximal functions of $\boldsymbol{u}_\pm$, $\nabla \boldsymbol{u}_\pm$ and $p_\pm$ are in $L^2(\partial\Omega)$ and the boundary conditions are fulfilled in the sense of the non-tangential limit. This problem was studied in [4] for $c_+ = 0$, $\lambda_\pm = 0$, and in [9] for $a_\pm = b_\pm = 1$, $c_+ = 0$. We study the transmission problem for arbitrary $\lambda_\pm$, $a_\pm$, $b_\pm$ and c_+.

In all preceding papers, the transmission problem is studied under additional condition concerning behaviour of $\boldsymbol{u}_-$ and p_- at infinity. To remove this additional condition, we study behaviour of a solution of the Brinkman system at infinity and we prove the theorem of Liouville's type. From this, we deduce that if the non-tangential maximal function corresponding to $\boldsymbol{u}_-$ and p_- is in $L^2(\partial\Omega)$, then there exist $\boldsymbol{u}_\infty \in R^m$, $p_\infty \in R^1$ such

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that $\boldsymbol{u}_-(\boldsymbol{x}) \to \boldsymbol{u}_\infty$, $p_-(\boldsymbol{x}) \to p_\infty$ as $|\boldsymbol{x}| \to \infty$, and $|\boldsymbol{u}_-(\boldsymbol{x}) - \boldsymbol{u}_\infty(\boldsymbol{x})| = O(|\boldsymbol{x}|^{2-m})$, $|\nabla \boldsymbol{u}_-| + |p_-(\boldsymbol{x}) - p_\infty| = O(|\boldsymbol{x}|^{1-m})$.

At the end, we study the Robin-transmission and the Dirichlet-transmission problems. Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $\Omega = \Omega_+$ be a bounded open set with Lipschitz boundary such that $\overline{\Omega} \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega}$, and by $\boldsymbol{n}_\pm$ the outward unit normal of $\Omega_\pm$. Let $\lambda_\pm$ and $c_\pm$ be non-negative constants, and $a_\pm$ and $b_\pm$ be positive constants. We study by the integral equation method the Robin-transmission problem for the Brinkman system

$$-\Delta \boldsymbol{u}_\pm + \lambda_\pm \boldsymbol{u}_\pm + \nabla p_\pm = 0, \quad \nabla \cdot \boldsymbol{u}_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$a_+\boldsymbol{u}_+ - a_-\boldsymbol{u}_- = \boldsymbol{g}, \quad b_+T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_-T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+\boldsymbol{u}_+ = \boldsymbol{f} \quad \text{on } \partial\Omega,$$

$$T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_- + c_-\boldsymbol{u}_- = \boldsymbol{h} \quad \text{on } \partial G.$$

Here, $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$, $\boldsymbol{h} \in L^2(\partial G)$. We look for an L^2-solution of the problem, i.e. the non-tangential maximal functions of $\boldsymbol{u}_\pm$, $\nabla \boldsymbol{u}_\pm$ and $p_\pm$ are in $L^2(\partial\Omega_-)$ and the boundary conditions are fulfilled in the sense of the non-tangential limit. (The integral representation of a solution gives that $\boldsymbol{u}_\pm \in H^{3/2}(\Omega, R^m)$, $p_\pm \in H^{1/2}(\Omega)$.) This problem was studied in [8] for $c_\pm = 0$, $a_\pm = b_\pm = 1$, $\lambda_+ = 0$.

Then, the regular Dirichlet-transmission problem is studied by the integral equation method:

$$-\Delta \boldsymbol{u}_\pm + \lambda_\pm \boldsymbol{u}_\pm + \nabla p_\pm = 0, \quad \nabla \cdot \boldsymbol{u}_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$a_+\boldsymbol{u}_+ - a_-\boldsymbol{u}_- = \boldsymbol{g}, \quad b_+T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_-T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+\boldsymbol{u}_+ = \boldsymbol{f} \quad \text{on } \partial\Omega,$$

$$\boldsymbol{u}_- = \boldsymbol{h} \quad \text{on } \partial G.$$

Here, $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$ and $\boldsymbol{h} \in W^{1,2}(\partial G)$. We look for an L^2-solution of the problem, i.e. the non-tangential maximal functions of $\boldsymbol{u}_\pm$, $\nabla \boldsymbol{u}_\pm$ and $p_\pm$ are in $L^2(\partial\Omega_-)$ and the boundary conditions are fulfilled in the sense of the non-tangential limit. (The integral representation of a solution gives that $\boldsymbol{u}_\pm \in H^{3/2}(\Omega, R^m)$, $p_\pm \in H^{1/2}(\Omega)$.) This problem was studied in [11] for $a_\pm = b_\pm = 1$, $c_+ = 0$.

2. Formulation of the transmission problem

Let $\Omega = \Omega_+ \subset R^m$, $m > 2$, be a bounded open set with Lipschitz boundary. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$, where $\overline{\Omega}_+$ is the closure of Ω_+. Denote by $\boldsymbol{n} = \boldsymbol{n}_+ = \boldsymbol{n}^\Omega$ the outward unit normal of Ω_+. Let λ_+, λ_-, c_+ be non-negative constants and a_+, a_-, b_+, b_- positive constants. We shall study the transmission problem for the Brinkman system

$$-\Delta \boldsymbol{u}_\pm + \lambda_\pm \boldsymbol{u}_\pm + \nabla p_\pm = 0, \quad \nabla \cdot \boldsymbol{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \tag{1}$$

$$a_+\boldsymbol{u}_+ - a_-\boldsymbol{u}_- = \boldsymbol{g}, \quad b_+T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_-T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+\boldsymbol{u}_+ = \boldsymbol{f} \quad \text{on } \partial\Omega. \tag{2}$$

If $\boldsymbol{u} = (u_1, \dots, u_m)$ is a velocity field, p is a pressure, denote

$$T(\boldsymbol{u}, p) = 2\hat{\nabla}\boldsymbol{u} - pI$$

the corresponding stress tensor. Here, I denotes the identity matrix and

$$\hat{\nabla}\boldsymbol{u} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right]$$

is the strain tensor, with $(\nabla \boldsymbol{u})^T$ as the matrix transposed to $\nabla \boldsymbol{u} = (\partial_j u_k), (k, j = 1, \dots, m)$. Denote $\nabla \cdot \boldsymbol{u} = \partial_1 u_1 + \dots + \partial_m u_m$ the divergence of $\boldsymbol{u}$.

Now, we define an L^2-solution of the transmission problem. Let G be an open set with Lipschitz boundary. If $\boldsymbol{x} \in \partial G$, $a > 0$ denote the non-tangential approach region of opening a at the point $\boldsymbol{x}$ by

$$\Gamma_a^G(\boldsymbol{x}) := \{\boldsymbol{y} \in G; |\boldsymbol{x} - \boldsymbol{y}| < (1 + a) \operatorname{dist}(\boldsymbol{y}, \partial G)\}.$$

If now $\boldsymbol{v}$ is a vector function defined in G, we denote the non-tangential maximal function of $\boldsymbol{v}$ on ∂G by

$$\boldsymbol{v}_G^*(\boldsymbol{x}) := \sup \left\{ |\boldsymbol{v}(\boldsymbol{y})|; \boldsymbol{y} \in \Gamma_a^G(\boldsymbol{x}) \right\}.$$

If $\boldsymbol{x} \in \partial G$, $\Gamma(\boldsymbol{x}) = \Gamma_a^G(\boldsymbol{x})$, then

$$\boldsymbol{v}(\boldsymbol{x}) = \lim_{\substack{\boldsymbol{y} \to \boldsymbol{x} \\ \boldsymbol{y} \in \Gamma(\boldsymbol{x})}} \boldsymbol{v}(\boldsymbol{y})$$

is the non-tangential limit of $\boldsymbol{v}$ with respect to G at $\boldsymbol{x}$.

Let $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$. *We say that* $\boldsymbol{u}_\pm$, $p_\pm$ *defined on* $\Omega_\pm$ *is an* L^2*-solution of the transmission problem* (1) *and* (2) *if* $\boldsymbol{u}_\pm$, $p_\pm$ *satisfy* (1); $\boldsymbol{u}_\pm^*$, $p_\pm^*$, $(\nabla \boldsymbol{u})_\pm^*$ *are from* $L^2(\partial\Omega, R^1)$; *for almost all* $\boldsymbol{x} \in \partial\Omega$ *there exist the non-tangential limits of* $\boldsymbol{u}_\pm$, $\nabla \boldsymbol{u}_\pm$, $p_\pm$ *at* $\boldsymbol{x}$ *and the condition* (2) *is fulfilled in the sense of the nontangential limit a.e. on* $\partial\Omega$.

3. The surface potentials

We shall look for a solution of the transmission problem by the integral equation method. The aim of this section is to assemble some basic facts on surface potentials for the Brinkman system.

For $\lambda \geq 0$ denote by $E^\lambda(\boldsymbol{x}) = \{E_{ij}^\lambda(\boldsymbol{x})\}_{i,j=1,\dots,m}$, $Q^\lambda(\boldsymbol{x}) = \{Q_j^\lambda(\boldsymbol{x})\}_{j=1,\dots,m}$ the fundamental matrix for the Brinkman system

$$-\Delta \boldsymbol{u} + \lambda \boldsymbol{u} + \nabla p = 0, \qquad \nabla \cdot \boldsymbol{u} = 0 \tag{3}$$

such that $E^\lambda(\boldsymbol{x}) \to 0$, $Q^\lambda(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$. If j is fixed, $\boldsymbol{u} = (E_{1j}, \dots, E_{mj})$, $p = Q_j$ then $\boldsymbol{u}$, p is a solution of the Brinkman system (3) in $R^m \setminus \{0\}$. If $\lambda = 0$ then the fundamental matrix for the Stokes system is given by

$$E_{ij}^0(\boldsymbol{x}) = \frac{1}{2\omega_m} \left[\delta_{ij} \frac{|\boldsymbol{x}|^{2-m}}{m-2} + \frac{x_i x_j}{|\boldsymbol{x}|^m} \right], \qquad Q_j^0(\boldsymbol{x}) = \frac{x_j}{\omega_m |\boldsymbol{x}|^m},$$

where ω_m denotes the surface of the unit sphere in R^m. (See [6] or [4].) The fundamental matrix for $\lambda > 0$ is studies in Chapter 2 of [6]:

$$\begin{aligned}
Q^\lambda(\boldsymbol{x}) &= Q^0(\boldsymbol{x}), \\
E_{ij}^\lambda(\boldsymbol{x}) &= \frac{1}{\omega_m} \left[\frac{\delta_{ij}}{|\boldsymbol{x}|^{m-2}} A_1\left(\sqrt{\lambda}|\boldsymbol{x}|\right) + \frac{x_i x_j}{|\boldsymbol{x}|^m} A_2\left(\sqrt{\lambda}|\boldsymbol{x}|\right) \right], \\
A_1(t) &= \frac{t^{m/2-1} K_{m/2-1}(t)}{2^{m/2-1}\Gamma(m/2)} + \frac{t^{m/2-2} K_{m/2}(t)}{2^{m/2-1}\Gamma(m/2)} - \frac{1}{t^2},
\end{aligned}$$

$$A_2(t) = \frac{m}{t^2} - \frac{t^{m/2-1} K_{m/2+1}(t)}{2^{m/2-1}\Gamma(m/2)},$$

where K_ν is the modified Bessel function of order ν. If $\lambda > 0$ then

$$|E^\lambda(\boldsymbol{x})| = O\left(|\boldsymbol{x}|^{-m}\right), \quad |\nabla E^\lambda(\boldsymbol{x})| = O\left(|\boldsymbol{x}|^{1-m}\right) \qquad \text{as } |\boldsymbol{x}| \to \infty.$$

Since $E^\lambda \in \mathcal{C}^\infty\left(R^m \setminus \{0\}; R^{m\times m}\right)$, $Q^\lambda \in \mathcal{C}^\infty(R^m \setminus \{0\}; R^m)$, we can define for $\boldsymbol{\Psi} \in L^2(\partial\Omega, R^m)$ the single-layer potential with density $\boldsymbol{\Psi}$ by

$$\left(E_\Omega^\lambda \boldsymbol{\Psi}\right)(\boldsymbol{x}) = \int\limits_{\partial\Omega} E^\lambda(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{\Psi}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}_{m-1}(\boldsymbol{y}) \tag{4}$$

and the corresponding pressure by

$$\left(Q_\Omega^\lambda \boldsymbol{\Psi}\right)(\boldsymbol{x}) = \int\limits_{\partial\Omega} Q^\lambda(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{\Psi}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}_{m-1}(\boldsymbol{y}). \tag{5}$$

Then, $E_\Omega^\lambda \boldsymbol{\Psi} \in \mathcal{C}^\infty(R^m \setminus \partial\Omega, R^m)$, $Q_\Omega^\lambda \boldsymbol{\Psi} \in \mathcal{C}^\infty\left(R^m \setminus \partial\Omega, R^1\right)$, $\nabla Q_\Omega^\lambda \boldsymbol{\Psi} - \Delta E_\Omega^\lambda \boldsymbol{\Psi} + \lambda E_\Omega^\lambda \boldsymbol{\Psi} = 0$, $\nabla \cdot E_\Omega^\lambda \boldsymbol{\Psi} = 0$ in $R^m \setminus \partial\Omega$.

$E_\Omega^\lambda \boldsymbol{\Psi}$ can be defined for almost all $\boldsymbol{x} \in \partial\Omega$ and $E_\Omega^\lambda \boldsymbol{\Psi}(\boldsymbol{x})$ is the non-tangential limit of $E_\Omega^\lambda \boldsymbol{\Psi}$. The non-tangential maximal function of $E_\Omega^\lambda \boldsymbol{\Psi}$, $\nabla E_\Omega^\lambda \boldsymbol{\Psi}$ and $Q_\Omega^\lambda \boldsymbol{\Psi}$ with respect to Ω_+ and Ω_- is in $L^2(\partial\Omega)$ (see [13], Lemma 2.1.4). Moreover, E_Ω^λ is a bounded linear operator from $L^2(\partial\Omega, R^m)$ to $W^{1,2}(\partial\Omega, R^m)$. (For $\lambda = 0$ see [4], for $\lambda > 0$ see for example [8].)

Denote

$$K_\Omega^\lambda(\boldsymbol{y}, \boldsymbol{x}) = -T_{\boldsymbol{x}}\left(E^\lambda(\boldsymbol{x}-\boldsymbol{y}), \quad Q^\lambda(\boldsymbol{x}-\boldsymbol{y})\right)\boldsymbol{n}^\Omega(\boldsymbol{x}).$$

For $\boldsymbol{\Psi} \in L^2(\partial\Omega, R^m)$ define

$$K'_{\Omega,\lambda}\boldsymbol{\Psi}(\boldsymbol{x}) = \lim_{\epsilon \searrow 0} \int\limits_{\partial\Omega \setminus B(\boldsymbol{x};\epsilon)} K_\Omega^\lambda(\boldsymbol{y}, \boldsymbol{x})\,\boldsymbol{\Psi}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}_{m-1}(\boldsymbol{y}),$$

where $B(\boldsymbol{x};\epsilon) = \{\boldsymbol{y};\, |\boldsymbol{x}-\boldsymbol{y}| < \epsilon\}$. Then, $K'_{\Omega,\lambda}$ is a bounded linear operator on $L^2(\partial\Omega, R^m)$. If $\boldsymbol{\Psi} \in L^2(\partial\Omega, R^m)$, then there exist the non-tangential limits $\left[\nabla E_\Omega^\lambda \boldsymbol{\Psi}(\boldsymbol{x})\right]_\pm$, $\left[Q_\Omega^\lambda \boldsymbol{\Psi})(\boldsymbol{x})\right]_\pm$ of $\nabla E_\Omega^\lambda \boldsymbol{\Psi}$, $Q_\Omega^\lambda \boldsymbol{\Psi}$ with respect to $\Omega_\pm$ at almost all $\boldsymbol{x} \in \partial\Omega$, and

$$\left[T\left(E_\Omega^\lambda \boldsymbol{\Psi}, Q_\Omega^\lambda \boldsymbol{\Psi}\right)\right]_+ \boldsymbol{n}^\Omega = \frac{1}{2}\boldsymbol{\Psi} - K'_{\Omega,\lambda}\boldsymbol{\Psi}, \tag{6}$$

$$\left[T\left(E_\Omega^\lambda \boldsymbol{\Psi}, Q_\Omega^\lambda \boldsymbol{\Psi}\right)\right]_- \boldsymbol{n}^\Omega = -\frac{1}{2}\boldsymbol{\Psi} - K'_{\Omega,\lambda}\boldsymbol{\Psi}. \tag{7}$$

(For $\lambda = 0$ see [4], for $\lambda > 0$ see for example [8]. See also [14].)

Now, we define a double layer potential. For $\boldsymbol{\Psi} \in L^2(\partial\Omega, R^m)$ define in $R^m \setminus \partial\Omega$

$$\left(D_\Omega^\lambda \boldsymbol{\Psi}\right)(\boldsymbol{x}) = \int\limits_{\partial\Omega} K_\Omega^\lambda(\boldsymbol{x}, \boldsymbol{y})\,\boldsymbol{\Psi}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}_{m-1}(\boldsymbol{y}), \tag{8}$$

and the corresponding pressure by

$$\left(\Pi_\Omega^\lambda \boldsymbol{\Psi}\right)(\boldsymbol{x}) = \int\limits_{\partial\Omega} \Pi_\Omega^\lambda(\boldsymbol{x}, \boldsymbol{y})\,\boldsymbol{\Psi}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}_{m-1}(\boldsymbol{y}), \tag{9}$$

where

$$\Pi_\Omega^\lambda(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\omega_m}\left\{-(\boldsymbol{y}-\boldsymbol{x})\frac{2m(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}^\Omega(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|^{m+2}} + \frac{2\boldsymbol{n}^\Omega(\boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{x}|^m} - \lambda\frac{|\boldsymbol{x}-\boldsymbol{y}|^{2-m}}{m-2}\boldsymbol{n}^\Omega(\boldsymbol{y})\right\}.$$

Then, $D_\Omega^\lambda\boldsymbol{\Psi}\in\mathcal{C}^\infty(R^m\setminus\partial\Omega,R^m)$, $\Pi_\Omega^\lambda\boldsymbol{\Psi}\in\mathcal{C}^\infty(R^m\setminus\partial\Omega,R^1)$ and $\nabla\Pi_\Omega^\lambda\boldsymbol{\Psi}-\Delta D_\Omega^\lambda\boldsymbol{\Psi}+\lambda D_\Omega^\lambda\boldsymbol{\Psi}=0$, $\nabla\cdot D_\Omega^\lambda\boldsymbol{\Psi}=0$ in $R^m\setminus\partial\Omega$.

Define

$$K_{\Omega,\lambda}\boldsymbol{\Psi}(\boldsymbol{x}) = \lim_{\epsilon\searrow 0}\int_{\partial\Omega\setminus B(\boldsymbol{x};\epsilon)} K_\Omega^\lambda(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\Psi}(\boldsymbol{y})\,\mathrm{d}\mathcal{H}_{m-1}(\boldsymbol{y}),\quad \boldsymbol{x}\in\partial\Omega.$$

Then, $K_{\Omega,\lambda}$ is a bounded linear operator on $L^2(\partial\Omega;R^m)$ (adjoint to $K'_{\Omega,\lambda}$). There exists the non-tangential limit $[D_\Omega^\lambda\boldsymbol{\Psi}]_+(\boldsymbol{x})$ of $D_\Omega^\lambda\boldsymbol{\Psi}$ with respect to Ω_+ and the non-tangential limit $[D_\Omega^\lambda\boldsymbol{\Psi}]_-(\boldsymbol{x})$ of $D_\Omega^\lambda\boldsymbol{\Psi}$ with respect to Ω_- for almost all $\boldsymbol{x}\in\partial\Omega$ and

$$\left[D_\Omega^\lambda\boldsymbol{\Psi}\right]_+(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{\Psi}(z)+K_{\Omega,\lambda}\boldsymbol{\Psi}(z),\quad \left[D_\Omega^\lambda\boldsymbol{\Psi}\right]_-(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{\Psi}(z)+K_{\Omega,\lambda}\boldsymbol{\Psi}(z). \tag{10}$$

If $\boldsymbol{\Psi}\in W^{1,2}(\partial\Omega,R^m)$ then $\left[|D_\Omega^\lambda\boldsymbol{\Psi}|\right]^*_{\Omega_\pm}+\left[|\nabla D_\Omega^\lambda\boldsymbol{\Psi}|\right]^*_{\Omega_\pm}\in L^2(\partial\Omega)$ and at almost all points of $\partial\Omega$ there exist the non-tangential limits of $\nabla D_\Omega^\lambda\boldsymbol{\Psi}$ with respect to Ω_+ and with respect to Ω_-. Moreover, $\left[T\left(D_\Omega^\lambda\boldsymbol{\Psi},\Pi_\Omega^\lambda\boldsymbol{\Psi}\right)\right]_+\boldsymbol{n}^\Omega = \left[T\left(D_\Omega^\lambda\boldsymbol{\Psi},\Pi_\Omega^\lambda\boldsymbol{\Psi}\right)\right]_-\boldsymbol{n}^\Omega$. (For $\lambda=0$ see [4], for $\lambda>0$ see for example [8].)

4. Behaviour at infinity

PROPOSITION 4.1 *Let $\lambda\geq 0$, $u_1,\dots,u_k$ and p be tempered distributions in R^k, $k\geq 2$, $\boldsymbol{u}=(u_1,\dots,u_k)$. If $-\Delta\boldsymbol{u}+\lambda\boldsymbol{u}+\nabla p=0$, $\nabla\cdot\boldsymbol{u}=0$ in the sense of distributions in R^k, then $u_1,\dots,u_k$ and p are polynomials.*

Proof Denote by $\mathcal{F}f$ the Fourier transformation of f. Since $-\Delta\boldsymbol{u}+\lambda\boldsymbol{u}+\nabla p=0$, $\nabla\cdot\boldsymbol{u}=0$, the Fourier transformation gives

$$|\boldsymbol{x}|^2\mathcal{F}\boldsymbol{u}(\boldsymbol{x})+\lambda\mathcal{F}\boldsymbol{u}(\boldsymbol{x})+\boldsymbol{x}\mathcal{F}p(\boldsymbol{x})=0, \tag{11}$$

$$\boldsymbol{x}\cdot\mathcal{F}\boldsymbol{u}(\boldsymbol{x})=0. \tag{12}$$

Using (11) and (12)

$$0=\boldsymbol{x}\cdot\left[\left(|\boldsymbol{x}|^2+\lambda\right)\mathcal{F}\boldsymbol{u}+\boldsymbol{x}\mathcal{F}p(\boldsymbol{x})\right]=|\boldsymbol{x}|^2\mathcal{F}p(\boldsymbol{x}).$$

Thus, $\mathcal{F}p=0$ on $R^k\setminus\{0\}$. If $\boldsymbol{x}\in R^k\setminus\{0\}$ then

$$0=|\boldsymbol{x}|^2\mathcal{F}\boldsymbol{u}(\boldsymbol{x})+\lambda\mathcal{F}\boldsymbol{u}(\boldsymbol{x})+\boldsymbol{x}\mathcal{F}p(\boldsymbol{x})=\left(|\boldsymbol{x}|^2+\lambda\right)\mathcal{F}\boldsymbol{u}.$$

Therefore, $\mathcal{F}u_j=0$ in $R^k\setminus\{0\}$. According to [15], Chapter II, Section 10, there exist $n\in N_0$ and constants a_α such that

$$\mathcal{F}u_j=\sum_{|\alpha|\leq n}a_\alpha\partial^\alpha\delta_0.$$

Set

$$P_j(x) = \sum_{|\alpha| \le n} a_\alpha (-ix)^\alpha.$$

Then

$$\mathcal{F} P_j = \sum_{|\alpha| \le n} a_\alpha \mathcal{F}\left[(-ix)^\alpha 1\right] = \sum_{|\alpha| \le n} a_\alpha \partial^\alpha \delta_0 = \mathcal{F} u_j.$$

Since the Fourier transform is an isomorphism on the space of tempered distributions, we infer that $u_j = P_j$. Similarly for p. ☐

Proposition 4.2 *Let $\boldsymbol{u}$, p be a bounded solution of the Brinkman system $-\Delta \boldsymbol{u} + \lambda \boldsymbol{u} + \nabla p = 0$, $\nabla \cdot \boldsymbol{u} = 0$ in $R^m \setminus F$, where F is a compact subset of R^m, $m > 2$, $\lambda \ge 0$. Then, there exist $p_\infty \in R^1$, $\boldsymbol{u}_\infty \in R^m$ such that $p(\boldsymbol{x}) \to p_\infty$, $\boldsymbol{u}(\boldsymbol{x}) \to \boldsymbol{u}_\infty$ as $|\boldsymbol{x}| \to \infty$. Moreover, $|p(\boldsymbol{x}) - p_\infty| = O(|\boldsymbol{x}|^{1-m})$, $|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_\infty| = O\left(|\boldsymbol{x}|^{2-m}\right)$, $|\nabla \boldsymbol{u}(\boldsymbol{x})| = O\left(|\boldsymbol{x}|^{1-m}\right)$ as $|\boldsymbol{x}| \to \infty$. If $\lambda > 0$ then $\boldsymbol{u}_\infty = 0$.*

Proof Fix $\varphi \in \mathcal{C}^\infty(R^m)$ such that $\varphi = 0$ on a neighbourhood of F and $\varphi = 1$ on $R^m \setminus B(0; r)$ for some $r > 0$. Define $\tilde{\boldsymbol{u}} = \varphi \boldsymbol{u}$, $\tilde{p} = \varphi p$ on $R^m \setminus F$; $\tilde{\boldsymbol{u}} = 0$, $\tilde{p} =$ on F. Denote $(f_1, \ldots, f_m)^T = -\Delta \tilde{\boldsymbol{u}} + \lambda \tilde{\boldsymbol{u}} + \nabla \tilde{p}$, $f_{m+1} = \nabla \cdot \tilde{\boldsymbol{u}}$, $\boldsymbol{f} = (f_1, \ldots, f_{m+1})^T$. Define the $(m+1) \times (m+1)$ matrix function $\tilde{E}^\lambda$ by $\tilde{E}^\lambda_{ij} = E^\lambda_{ij}$, $\tilde{E}^\lambda_{m+1,j} = \tilde{E}^\lambda_{j,m+1} = Q^\lambda_j$ for $i, j \le m$, $\tilde{E}_{m+1,m+1}(\boldsymbol{x}) = \delta(x) + \lambda |\boldsymbol{x}|^{2-m} / [(m-2)\omega_m]$. Denote $(v_1, \ldots, v_m, q)^T = \tilde{E}^\lambda * \boldsymbol{f}$, $\boldsymbol{v} = (v_1, \ldots, v_m)^T$, where $*$ means the convolution. Then, $-\Delta \boldsymbol{v} + \lambda \boldsymbol{v} + \nabla q = (f_1, \ldots, f_m)^T$, $\nabla \cdot \boldsymbol{v} = f_{m+1}$ by [6], Section 2.1. According to a behaviour of $\tilde{E}^\lambda$ at infinity we see that $|\boldsymbol{v}(\boldsymbol{x})| = O\left(|\boldsymbol{x}|^{2-m}\right)$, $|\nabla \boldsymbol{v}(\boldsymbol{x})| + |q(\boldsymbol{x})| = O\left(|\boldsymbol{x}|^{1-m}\right)$ as $|\boldsymbol{x}| \to \infty$. Since the functions $u_j - v_j$, $p - q$ are bounded, they are tempered distributions (see [16], Example 14.22). Since $-\Delta\left(\tilde{\boldsymbol{u}} - \boldsymbol{v}\right) + \lambda\left(\tilde{\boldsymbol{u}} - \boldsymbol{v}\right) + \nabla(\tilde{p} - q) = 0$, $\nabla \cdot (\boldsymbol{u} - \boldsymbol{v}) = 0$ in R^m, Proposition 4.1 gives that $\tilde{u}_j - v_j$, $\tilde{p} - q$ are polynomials. Since $\tilde{u}_j - v_j$, $\tilde{p} - q$ are bounded there exist $p_\infty \in R^1$, $\boldsymbol{u}_\infty \in R^m$ such that $\tilde{p} - q = p_\infty$, $\tilde{\boldsymbol{u}} - \boldsymbol{v} = \boldsymbol{u}_\infty$. If $\lambda > 0$ then $0 = -\Delta(\tilde{\boldsymbol{u}} - \boldsymbol{v}) + \lambda(\tilde{\boldsymbol{u}} - \boldsymbol{v}) + \nabla(\tilde{p} - q) = \lambda \boldsymbol{u}_\infty$ and thus $\boldsymbol{u}_\infty = 0$. ☐

5. Solution of the transmission problem

Put $\tilde{b}_\pm = b_\pm / a_\pm$, $\tilde{c}_+ = c_+ / a_+$. If $\tilde{\boldsymbol{u}}_\pm = a_\pm \boldsymbol{u}_\pm$, $\tilde{p}_\pm = a_\pm p_\pm$ then $\boldsymbol{u}_\pm$, $p_\pm$ is an L^2-solution of the transmission problem (1) and (2) if and only if $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the transmission problem

$$\Delta \tilde{\boldsymbol{u}}_\pm + \lambda_\pm \tilde{\boldsymbol{u}}_\pm + \nabla \tilde{p}_\pm = 0, \quad \nabla \cdot \tilde{\boldsymbol{u}}_\pm = 0 \quad \text{in } \Omega_\pm, \tag{13}$$

$$\tilde{\boldsymbol{u}}_+ - \tilde{\boldsymbol{u}}_- = \boldsymbol{g}, \quad \tilde{b}_+ T\left(\tilde{\boldsymbol{u}}_+, \tilde{p}_+\right) \boldsymbol{n} - \tilde{b}_- T\left(\tilde{\boldsymbol{u}}_-, \tilde{p}_-\right) \boldsymbol{n} + \tilde{c}_+ \tilde{\boldsymbol{u}}_+ = \boldsymbol{f} \quad \text{on } \partial\Omega. \tag{14}$$

Let $\boldsymbol{\Phi} \in W^{1,2}\left(\partial\Omega, R^m\right)$, $\boldsymbol{\Psi} \in L^2\left(\partial\Omega, R^m\right)$. Put

$$\tilde{\boldsymbol{u}}_\pm = D^{\lambda_\pm}_\Omega \boldsymbol{\Phi} + E^{\lambda_\pm}_\Omega \boldsymbol{\Psi}, \quad \tilde{p}_\pm = \Pi^{\lambda_\pm}_\Omega \boldsymbol{\Phi} + Q^{\lambda_\pm}_\Omega \boldsymbol{\Psi} \quad \text{in } \Omega_\pm, \tag{15}$$

$$\tau_1^{\lambda_+, \lambda_-}(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = \boldsymbol{\Phi} + K_{\Omega,\lambda_+} \boldsymbol{\Phi} - K_{\Omega,\lambda_-} \boldsymbol{\Phi} + E^{\lambda_+}_\Omega \boldsymbol{\Psi} - E^{\lambda_-}_\Omega \boldsymbol{\Psi},$$

$$\begin{aligned} \tau_2^{\lambda_+, \lambda_-, \tilde{b}_+, \tilde{b}_-, \tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = {} & \tilde{b}_+ \left[\boldsymbol{\Psi} - K'_{\Omega,\lambda_+}\right] - \tilde{b}_- \left[-\boldsymbol{\Psi} - K_{\Omega,\lambda_-}\right] + \tilde{c}_+ E^{\lambda_+}_\Omega \boldsymbol{\Psi} \\ & + \tilde{b}_+ \left[T\left(D^{\lambda_+}_\Omega \boldsymbol{\Phi}, \Pi^{\lambda_+}_\Omega \boldsymbol{\Phi}\right)\right]_+ \boldsymbol{n}^\Omega - \tilde{b}_- \left[T\left(D^{\lambda_-}_\Omega \boldsymbol{\Phi}, \Pi^{\lambda_-}_\Omega \boldsymbol{\Phi}\right)\right]_- \boldsymbol{n}^\Omega. \end{aligned}$$

The operator $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} = \left[\tau_1^{\lambda_+,\lambda_-}, \tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}\right]$ is a bounded linear operator on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$. The functions $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ given by (15) are an L^2-solution of the transmission problem (13) and (14) such $\tilde{\boldsymbol{u}}_-(\boldsymbol{x}) \to 0$, $\tilde{p}_-(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$ if and only if $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,c_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = [\boldsymbol{g}, \boldsymbol{f}]$.

Lemma 5.1 *Denote* $\mathcal{R}_m = \{\boldsymbol{v}(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}; \boldsymbol{b} \in R^m, A = (a_{ij})$ *an antisymmetric matrix, i.e.* $a_{ij} = -a_{ji}\}$ *the space of rigid motions. Let* $\boldsymbol{u} \in \mathcal{R}_m$, $M = \{\boldsymbol{x}; \boldsymbol{u}(\boldsymbol{x}) = 0\}$. *If* $\mathcal{H}_{m-1}(M) > 0$ *then* $\boldsymbol{u} \equiv 0$.

Proof There exists a matrix $A = (a_{ij})$ with $a_{ij} = -a_{ji}$ and $\boldsymbol{b} \in R^m$ such that $\boldsymbol{u}(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$. Suppose first $a_{ij} \neq 0$ for some indices i, j. Denote $L_i = \{\boldsymbol{x}; a_{i1}x_1 + \cdots + a_{im}x_m + b_i = 0\}$, $L_j = \{\boldsymbol{x}; a_{j1}x_1 + \cdots + a_{jm}x_m + b_i = 0\}$. Since $a_{ii} = a_{jj} = 0$, $a_{ji} = -a_{ij} \neq 0$ we have $\mathcal{H}_{m-1}(L_i \cap L_j) = 0$. This contradicts to $M \subset L_i \cap L_j$. Hence, $A = 0$ and $\boldsymbol{u}$ is constant. $M \neq \emptyset$ forces $\boldsymbol{u} \equiv 0$. □

Proposition 5.2 *Let* $\boldsymbol{u}_\pm$, $p_\pm$ *be an* L^2*-solution for the transmission problem* (1) *and* (2). *If* $\boldsymbol{f} = 0$, $\boldsymbol{g} = 0$ *and* $\boldsymbol{u}_-(\boldsymbol{x}) \to 0$, $p_-(\boldsymbol{x}) \to 0$ *as* $|\boldsymbol{x}| \to \infty$ *then* $\boldsymbol{u}_\pm \equiv 0$, $p_\pm \equiv 0$.

Proof $|p(\boldsymbol{x})| = O(|\boldsymbol{x}|^{1-m})$, $|\boldsymbol{u}(\boldsymbol{x})| = O(|\boldsymbol{x}|^{2-m})$, $|\nabla\boldsymbol{u}(\boldsymbol{x})| = O(|\boldsymbol{x}|^{1-m})$ as $|\boldsymbol{x}| \to \infty$ (see Proposition 4.2). Using Green's formula

$$
\begin{aligned}
0 &= \int_{\partial\Omega} \boldsymbol{u}_+ \cdot [b_+ T(\boldsymbol{u}_+, p_+)\boldsymbol{n} - b_- T(\boldsymbol{u}_-, p_-)\boldsymbol{n} + c_+\boldsymbol{u}_+] \, \mathrm{d}\mathcal{H}_{m-1} \\
&= b_+ \int_{\partial\Omega_-} \boldsymbol{u}_+ \cdot T(\boldsymbol{u}_+, p_+)\, \boldsymbol{n}^{\Omega_+} \, \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial\Omega_-} c_+|\boldsymbol{u}_+|^2 \, \mathrm{d}\mathcal{H}_{m-1} \\
&\quad + \lim_{r\to\infty} b_- \frac{a_-}{a_+} \int_{\partial(\Omega_- \cap B(0;r))} \boldsymbol{u}_- \cdot T(\boldsymbol{u}_-, p_-)\, \boldsymbol{n}^{\Omega_-} \\
&= b_+ \int_{\Omega_+} \left[2|\hat{\nabla}\boldsymbol{u}_+|^2 + \lambda_+|\boldsymbol{u}_+|^2\right] \\
&\quad + \int_{\partial\Omega_-} c_+|\boldsymbol{u}_+|^2 \, \mathrm{d}\mathcal{H}_{m-1} + \frac{b_- a_-}{a_+} \int_{\Omega_+} \left[2|\hat{\nabla}\boldsymbol{u}_+|^2 + \lambda_-|\boldsymbol{u}_+|^2\right] \mathrm{d}\mathcal{H}_m.
\end{aligned}
$$

Denote $u = u_\pm$ on $\Omega_\pm$. Then, $\hat{\nabla}\boldsymbol{u} = 0$ in $R^m \setminus \partial\Omega$. Denote by $\omega_0, \omega_1, \ldots, \omega_k$ all components of $R^m \setminus \partial\Omega$, where ω_0 is the unbounded component. According to [17], Lemma 3.1, there exist antisymmetric matrices A^j and vectors $\boldsymbol{B}^j$ such that $\boldsymbol{u}(\boldsymbol{x}) = A^j\boldsymbol{x} + \boldsymbol{B}^j$ in ω_j. Since $\boldsymbol{u}(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$, we deduce that $\boldsymbol{u} = 0$ in ω_0. If $\partial\omega_0 \cap \partial\omega_j \neq \emptyset$, then the condition $a_+\boldsymbol{u}_+ = a_-\boldsymbol{u}_-$ gives that $A^j\boldsymbol{x} + \boldsymbol{B}^j = 0$ on $\partial\omega_0 \cap \partial\omega_j$. Lemma 5.1 gives that $A^j\boldsymbol{x} + \boldsymbol{B}^j \equiv 0$. We can continue by this way and prove that $\boldsymbol{u} = 0$. □

Proposition 5.3 *The operator* $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}$ *is an isomorphism on the space* $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$.

Proof The operator $\tau^{0,0,\tilde{b}_+,\tilde{b}_-,0}$ is a Fredholm operator with index 0 on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$ by [4]. If $\lambda \geq 0$ then $K_{\Omega,\lambda} - K_{\Omega,0}$ is compact on $W^{1,2}(\partial\Omega, R^m)$, $K'_{\Omega,\lambda} - K'_{\Omega,0}$ is compact on $L^2(\partial\Omega, R^m)$, $E^\lambda_\Omega - E^0_\Omega$ is a compact operator from $L^2(\partial\Omega, R^m)$ to $W^{1,2}(\partial\Omega, R^m)$ (see [8], Theorem 3.4). Since E^0_Ω is a bounded operator from $L^2(\partial\Omega, R^m)$ to $W^{1,2}(\partial\Omega, R^m)$, it is a compact linear operator on $L^2(\partial\Omega, R^m)$. Thus $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} - \tau^{0,0,\tilde{b}_+,\tilde{b}_-,0}$ is a compact operator on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$. Hence, $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,c_+}$ is a Fredholm operator with index 0. Therefore, it is enough to prove that $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}$ is injective.

Let $(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \in W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$, $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = 0$. Let $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ be given by (15). Then, $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the problem (13) and (14) with $\boldsymbol{g} = 0$, $\boldsymbol{f} = 0$ such that $\tilde{\boldsymbol{u}}_-(\boldsymbol{x}) \to 0$, $\tilde{p}_-(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$. Proposition 5.2 gives that $\tilde{\boldsymbol{u}}_\pm = 0$, $\tilde{p}_\pm = 0$. Thus $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the problem (13),

$$\tilde{\boldsymbol{u}}_+ - \tilde{\boldsymbol{u}}_- = 0, \quad T(\tilde{\boldsymbol{u}}_+, \tilde{p}_+)\boldsymbol{n} - T(\tilde{\boldsymbol{u}}_-, \tilde{p}_-)\boldsymbol{n} = 0 \quad \text{on } \partial\Omega.$$

Denote $\tilde{\lambda}_+ = \lambda_-$, $\tilde{\lambda}_- = \lambda_+$,

$$\boldsymbol{v}_+ = D^{\lambda_-}_\Omega \boldsymbol{\Phi} + E^{\lambda_-}_\Omega \boldsymbol{\Psi}, \quad q_+ = \Pi^{\lambda_-}_\Omega \boldsymbol{\Phi} + Q^{\lambda_-}_\Omega \boldsymbol{\Psi}, \quad \text{in } \Omega_+,$$

$$\boldsymbol{v}_- = -D^{\lambda_+}_\Omega \boldsymbol{\Phi} - E^{\lambda_+}_\Omega \boldsymbol{\Psi}, \quad q_+ = -\Pi^{\lambda_+}_\Omega \boldsymbol{\Phi} - Q^{\lambda_+}_\Omega \boldsymbol{\Psi}, \quad \text{in } \Omega_-.$$

Using boundary behaviour of potentials, we obtain on $\partial\Omega$

$$\boldsymbol{v}_+ = \boldsymbol{\Phi} + \tilde{\boldsymbol{u}}_- = \boldsymbol{\Phi},$$

$$\boldsymbol{v}_- = -[-\boldsymbol{\Phi} + \tilde{\boldsymbol{u}}_+] = \boldsymbol{\Phi},$$

$$\left[T(\boldsymbol{v}_+, q_+)\boldsymbol{n}^\Omega\right]_+ = \boldsymbol{\Psi} + \left[T(\tilde{\boldsymbol{u}}_-, \tilde{p}_-)\boldsymbol{n}^\Omega\right]_- = \boldsymbol{\Psi},$$

$$\left[T(\boldsymbol{v}_-, q_-)\boldsymbol{n}^\Omega\right]_- = -[-\boldsymbol{\Psi} + \left[T(\tilde{\boldsymbol{u}}_+, \tilde{p}_+)\boldsymbol{n}^\Omega\right]_+ = \boldsymbol{\Psi}.$$

Therefore, $\boldsymbol{v}_\pm$, $q_\pm$ is a solution of the transmission problem

$$-\Delta\boldsymbol{v}_\pm + \tilde{\lambda}_\pm \boldsymbol{v}_\pm + \nabla q_\pm = 0, \quad \nabla \cdot \boldsymbol{v}_\pm = 0 \quad \text{in } \Omega_\pm,$$

$$\boldsymbol{v}_+ - \boldsymbol{v}_- = 0, \quad T(\boldsymbol{v}_+, q_+)\boldsymbol{n} - T(\boldsymbol{v}_-, q_-)\boldsymbol{n} = 0 \quad \text{on } \partial\Omega,$$

$$\boldsymbol{v}_-(\boldsymbol{x}) \to 0, \quad q_-(\boldsymbol{x}) \to 0 \quad \text{as } |\boldsymbol{x}| \to \infty.$$

Proposition 5.2 gives that $\boldsymbol{v}_\pm \equiv 0$, $q_\pm \equiv 0$. We have on $\partial\Omega$

$$\boldsymbol{\Phi} = \boldsymbol{v}_+ = 0,$$

$$\boldsymbol{\Psi} = \left[T(\boldsymbol{v}_+, q_+)\boldsymbol{n}^\Omega\right]_+ = 0.$$

□

THEOREM 5.4 *Let* $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$. *Then, there exists an* L^2*-solution of the transmission problem* (1) *and* (2). *If* $\boldsymbol{u}_\pm$, $p_\pm$ *is an* L^2*-solution of the problem then there exist* $p_\infty \in R^1$, $\boldsymbol{u}_\infty \in R^m$ *such that* $\boldsymbol{u}_-(\boldsymbol{x}) \to \boldsymbol{u}_\infty$, $p_-(\boldsymbol{x}) \to p_\infty$ *as* $|\boldsymbol{x}| \to \infty$. *If* $\lambda_- > 0$ *then* $\boldsymbol{u}_\infty = 0$. *Fix* $p_\infty \in R^1$, $\boldsymbol{u}_\infty \in R^m$. *If* $\lambda_- > 0$ *suppose that* $\boldsymbol{u}_\infty = 0$. *Then, there exists a unique* L^2*-solution* $\boldsymbol{u}_\pm$, $p_\pm$ *of the transmission problem* (1) *and* (2) *such that* $\boldsymbol{u}_-(\boldsymbol{x}) \to \boldsymbol{u}_\infty$, $p_-(\boldsymbol{x}) \to p_\infty$ *as* $|\boldsymbol{x}| \to \infty$.

Proof If $\boldsymbol{u}_\pm$, $p_\pm$ is an L^2-solution of the problem then there exist $p_\infty \in R^1$, $\boldsymbol{u}_\infty \in R^m$ such that $\boldsymbol{u}_-(\boldsymbol{x}) \to \boldsymbol{u}_\infty$, $p_-(\boldsymbol{x}) \to p_\infty$ as $|\boldsymbol{x}| \to \infty$. If $\lambda_- > 0$ then $\boldsymbol{u}_\infty = 0$. (See Proposition 4.2).

Fix $p_\infty \in R^1$, $\boldsymbol{u}_\infty \in R^m$. If $\lambda_- > 0$ suppose that $\boldsymbol{u}_\infty = 0$. Put $\boldsymbol{u}_- = \boldsymbol{v}_- + \boldsymbol{u}_\infty$, $\boldsymbol{u}_+ = \boldsymbol{v}+$, $p_- = q_- + p_\infty$, $p_+ = q_+$. Then, $\boldsymbol{u}_\pm$, $p_\pm$ is a solution of the problem (1) and (2), $\boldsymbol{u}_-(\boldsymbol{x}) \to \boldsymbol{u}_\infty$, $p_-(\boldsymbol{x}) \to p_\infty$ if and only if $\boldsymbol{v}_\pm$, $q_\pm$ is a solution of the transmission problem (1),

$$a_+\boldsymbol{v}_+ - a_-\boldsymbol{v}_- = \boldsymbol{g} + a_-\boldsymbol{u}_\infty, \; b_+T(\boldsymbol{v}_+, q_+)\boldsymbol{n} - b_-T(\boldsymbol{v}_-, q_-)\boldsymbol{n} + c_+\boldsymbol{v}_+ = \boldsymbol{f} - b_-p_\infty\boldsymbol{n},$$

$\boldsymbol{v}_-(\boldsymbol{x}) \to 0$, $q_-(\boldsymbol{x}) \to 0$. According to Proposition 5.3 there exist $\boldsymbol{\Phi} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{\Psi} \in L^2(\partial\Omega, R^m)$ such that

$$\boldsymbol{v}_\pm = a_\pm^{-1}\left[D_\Omega^{\lambda_\pm}\boldsymbol{\Phi} + E_\Omega^{\lambda_\pm}\boldsymbol{\Psi}\right], \quad q_\pm = a_\pm^{-1}\left[\Pi_\Omega^{\lambda_\pm}\boldsymbol{\Phi} + Q_\Omega^{\lambda_\pm}\boldsymbol{\Psi}\right] \quad \text{in } \Omega_\pm$$

is a solution of the problem. The uniqueness of a solution follows from Proposition 5.2. ☐

6. Robin-transmission problem

Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $\Omega = \Omega_+$ be a bounded open set with Lipschitz boundary such that $\overline{\Omega} \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega}$, and by $\boldsymbol{n}_\pm$ the outward unit normal of $\Omega_\pm$. Let $\lambda_\pm$ and $c_\pm$ be non-negative constants and $a_\pm$ and $b_\pm$ be positive constants. We shall study the Robin-transmission problem for the Brinkman system (1) and (2) accompanied with the condition

$$T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_- + c_-\boldsymbol{u}_- = \boldsymbol{h} \quad \text{on } \partial G. \tag{16}$$

Let $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$, $\boldsymbol{h} \in L^2(\partial G, R^m)$. *We say that* $\boldsymbol{u}_\pm$, $p_\pm$ *defined on* $\Omega_\pm$ *is an* L^2*-solution of the Robin-transmission problem* (1), (2) *and* (16) *if* $\boldsymbol{u}_\pm$, $p_\pm$ *satisfy* (1)*;* $\boldsymbol{u}_\pm^*$, $p_\pm^*$, $(\nabla\boldsymbol{u})_\pm^*$ *are from* $L^2(\partial\Omega_\pm, R^1)$*; for almost all* $\boldsymbol{x} \in \partial\Omega_\pm$ *there exist the non-tangential limits of* $\boldsymbol{u}_\pm$, $\nabla\boldsymbol{u}_\pm$ *and* $p_\pm$ *at* $\boldsymbol{x}$ *and the conditions* (2) *and* (16) *are fulfilled in the sense of the non-tangential limit a.e. on* $\partial\Omega_-$.

Put $\tilde{b}_\pm = b_\pm/a_\pm$ and $\tilde{c}_+ = c_+/a_\pm$. If $\tilde{\boldsymbol{u}}_\pm = a_\pm\boldsymbol{u}_\pm$, $\tilde{p}_\pm = a_\pm p_\pm$ then $\boldsymbol{u}_\pm$, $p_\pm$ is an L^2-solution of the Robin-transmission problem (1), (2) and (16) if and only if $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the Robin-transmission problem (13) and (14),

$$T(\tilde{\boldsymbol{u}}_-, \tilde{p}_-)\boldsymbol{n}_- + c_-\tilde{\boldsymbol{u}}_- = a_-\boldsymbol{h} \quad \text{on } \partial G. \tag{17}$$

Let $\boldsymbol{\Phi} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{\Psi} \in L^2(\partial\Omega, R^m)$, $\boldsymbol{\Theta} \in L^2(\partial G, R^m)$. Let $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ be given by (15),

$$\tilde{\boldsymbol{u}}_- = D_\Omega^{\lambda_-}\boldsymbol{\Phi} + E_\Omega^{\lambda_-}\boldsymbol{\Psi} + E_G^{\lambda_-}\boldsymbol{\Theta}, \quad \tilde{p}_- = \Pi_\Omega^{\lambda_-}\boldsymbol{\Phi} + Q_\Omega^{\lambda_-}\boldsymbol{\Psi} + Q_G^{\lambda_-}\boldsymbol{\Theta} \quad \text{in } \Omega_-. \tag{18}$$

Then $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the Robin-transmission problem (13), (14) and (17) if and only if

$$R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}(\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = [\boldsymbol{g}, \boldsymbol{f}, a_-\boldsymbol{h}],$$

where

$$\begin{aligned} &R^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+,c_-}(\boldsymbol{\Phi},\boldsymbol{\Psi},\boldsymbol{\Theta}) \\ &\quad= \Big[\tau_1^{\lambda_+,\lambda_-}(\boldsymbol{\Phi},\boldsymbol{\Psi}) - E_G^{\lambda_-}\boldsymbol{\Theta},\ \tau_2^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(\boldsymbol{\Phi},\boldsymbol{\Psi}) - \tilde b_- T\left(E_G^{\lambda_-}\boldsymbol{\Theta},\ Q_G^{\lambda_-}\boldsymbol{\Theta}\right)\boldsymbol{n}_+, \\ &\qquad \frac{1}{2}\boldsymbol{\Theta} - K'_{G,\lambda_-}\boldsymbol{\Theta} + T\left(E_\Omega^{\lambda_-}\boldsymbol{\Psi} + D_\Omega^{\lambda_-}\boldsymbol{\Phi},\ Q_G^{\lambda_-}\boldsymbol{\Psi}\right)\boldsymbol{n}_- \\ &\qquad + c_-\left(E_G^{\lambda_-}\boldsymbol{\Theta} + E_\Omega^{\lambda_-}\boldsymbol{\Psi} + D_\Omega^{\lambda_-}\boldsymbol{\Phi}\right)\Big]. \end{aligned}$$

Lemma 6.1 *The operator* $R^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+,c_-}$ *is a Fredholm operator with index* 0 *on* $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$.

Proof $R : (\boldsymbol{\Phi},\boldsymbol{\Psi},\boldsymbol{\Theta}) \mapsto [\tau_1^{\lambda_+,\lambda_-}(\boldsymbol{\Phi},\boldsymbol{\Psi}), \tau_2^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(\boldsymbol{\Phi},\boldsymbol{\Psi}), \frac{1}{2}\boldsymbol{\Theta} - K'_{G,0}\boldsymbol{\Theta}]$ is a Fredholm operator with index 0 on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$ by [4] and Proposition 5.3. If $\lambda \geq 0$ then $K'_{G,\lambda} - K'_{G,0}$ is compact on $L^2(\partial G, R^m)$, $E_G^\lambda - E_G^0$ is a compact operator from $L^2(\partial G, R^m)$ to $W^{1,2}(\partial G, R^m)$ (see [8], Theorem 3.4). Since E_G^0 is a bounded operator from $L^2(\partial G, R^m)$ to $W^{1,2}(\partial G, R^m)$, it is a compact linear operator on $L^2(\partial G, R^m)$. Thus, $R^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+,c_-} - R$ is a compact operator. Hence, $R^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+,c_-}$ is a Fredholm operator with index 0. ☐

Lemma 6.2 *Let* $\tilde{\boldsymbol{u}}_+$, $\tilde p_+$ *be given by* (15), *and* $\tilde{\boldsymbol{u}}_-$, $\tilde p_-$ *by* (18). *If* $\tilde{\boldsymbol{u}}_\pm = 0$, $\tilde p_\pm = 0$ *in* $\Omega_\pm$ *then* $\boldsymbol{\Phi} = 0$, $\boldsymbol{\Psi} = 0$, $\boldsymbol{\Theta} = 0$.

Proof Define

$$\boldsymbol{v} = D_\Omega^{\lambda_-}\boldsymbol{\Phi} + E_\Omega^{\lambda_-}\boldsymbol{\Psi} + E_G^{\lambda_-}\boldsymbol{\Theta}, \quad q = \Pi_\Omega^{\lambda_-}\boldsymbol{\Phi} + Q_\Omega^{\lambda_-}\boldsymbol{\Psi} + Q_G^{\lambda}\boldsymbol{\Theta} \quad \text{in } \omega = R^m \setminus \overline{G}.$$

Continuity of a single-layer potential gives that $\boldsymbol{v} = \boldsymbol{u}_- = 0$ on ∂G. Since $\boldsymbol{v}(\boldsymbol{x}) = O(|\boldsymbol{x}|^{2-m})$, $|\nabla\boldsymbol{v}(\boldsymbol{x})| + |q(\boldsymbol{x})| = O(|\boldsymbol{x}|^{1-m})$ as $|\boldsymbol{x}| \to \infty$ then Green's formula gives

$$0 = \int_{\partial\omega} \boldsymbol{v}\cdot T(\boldsymbol{v}, q)\boldsymbol{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1} = \int_\omega \left[|2\hat\nabla\boldsymbol{v}|^2 + \lambda_-|\boldsymbol{v}|^2\right] \mathrm{d}\mathcal{H}_m.$$

Since $\hat\nabla\boldsymbol{v} = 0$ we have $\boldsymbol{v} \in \mathrm{R}_m$ by [17], Lemma 3.1. Behaviour of potentials at infinity gives that $\boldsymbol{v}(\boldsymbol{x}) \to 0$ as $|\boldsymbol{x}| \to \infty$. This forces that $\boldsymbol{v} \equiv 0$. Since $\nabla q = \Delta\boldsymbol{v} - \lambda_-\boldsymbol{v} = 0$ we deduce that q is constant. Behaviour of potentials at infinity gives that $q \equiv 0$.

By virtue of (6) and (7)

$$\boldsymbol{\Theta} = T(\tilde{\boldsymbol{u}}_-, \tilde p_-)\boldsymbol{n}_- - T(\boldsymbol{v}, q)\boldsymbol{n}_- = 0.$$

Denote $\omega_+ = \Omega_+$, $\omega_- = R^m \setminus \overline{\omega}_+$. If $\tilde{\boldsymbol{u}}_\pm$, $\tilde p_\pm$ is given by (15) in $\omega_\pm$ then $\tilde{\boldsymbol{u}}_\pm$, $\tilde p_\pm$ is an L^2-solution of the transmission problem

$$-\Delta\tilde{\boldsymbol{u}}_\pm + \lambda_\pm\tilde{\boldsymbol{u}}_\pm + \nabla\tilde p_\pm = 0, \quad \nabla\cdot\tilde{\boldsymbol{u}}_\pm = 0 \quad \text{in } \omega_\pm,$$

$$\tilde{\boldsymbol{u}}_+ - \tilde{\boldsymbol{u}}_- = 0, \quad \tilde b_+ T(\tilde{\boldsymbol{u}}_+, \tilde p_+)\boldsymbol{n}_+ - \tilde b_- T(\tilde{\boldsymbol{u}}_-, \tilde p_-)\boldsymbol{n}_+ + \tilde c_+\tilde{\boldsymbol{u}}_+ = 0 \quad \text{on } \partial\omega_+.$$

In particular, $\tau_2^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(\boldsymbol{\Phi},\boldsymbol{\Psi}) = 0$. Proposition 5.3 gives that $\boldsymbol{\Phi} = 0$, $\boldsymbol{\Psi} = 0$. ☐

PROPOSITION 6.3 *Let* $\boldsymbol{u}_\pm$, $p_\pm$ *be an* L^2*-solution of the Robin-transmission problem* (1), (2) *and* (16) *with* $\boldsymbol{g} = 0$, $\boldsymbol{f} = 0$, $\boldsymbol{h} = 0$.

- *If* $\lambda_+ + \lambda_- + c_+ + c_- > 0$ *then* $\boldsymbol{u}_\pm \equiv 0$, $p_\pm \equiv 0$.
- *If* $\lambda_+ + \lambda_- + c_+ + c_- = 0$ *then* $p_\pm \equiv 0$ *and there exists a rigid motion* $\boldsymbol{v} \in \mathcal{R}_m$ *such that* $\boldsymbol{u}_\pm = \boldsymbol{v}/a_\pm$.

Proof Using Green's formula

$$
\begin{aligned}
0 = b_-^{-1} &\int_{\partial\Omega} \boldsymbol{u}_- \cdot [b_+ T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_- T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+\boldsymbol{u}_+]\, \mathrm{d}\mathcal{H}_{m-1} \\
&+ \int_{\partial G} \boldsymbol{u}_- \cdot [T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_- + c_-\boldsymbol{u}_-]\, \mathrm{d}\mathcal{H}_{m-1} = \int_{\Omega_-} \left[2|\hat{\nabla}\boldsymbol{u}_-|^2 + \lambda_-|\boldsymbol{u}_-|^2\right] \mathrm{d}\mathcal{H}_m \\
&+ \frac{a_+b_+}{a_-b_-} \int_{\Omega_+} \left[2|\hat{\nabla}\boldsymbol{u}_+|^2 + \lambda_+|\boldsymbol{u}_+|^2\right] \mathrm{d}\mathcal{H}_m + \int_{\partial G} c_-|\boldsymbol{u}_-|^2\, \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial\Omega} \frac{c_+a_+|\boldsymbol{u}_+|^2}{a_-}\, \mathrm{d}\mathcal{H}_{m-1}.
\end{aligned}
$$

Thus $\hat{\nabla}\boldsymbol{u}_\pm = 0$, $\lambda_\pm\boldsymbol{u}_\pm = 0$ in $\Omega_\pm$, $c_+\boldsymbol{u}_+ = 0$ on $\partial\Omega$, $c_-\boldsymbol{u}_- = 0$ on ∂G. Define $\boldsymbol{v} = a_\pm\boldsymbol{u}_\pm$ on $\Omega_\pm$. Denote by $\omega_1, \dots, \omega_k$ all components of $G \setminus \partial\Omega$. According to [17], Lemma 3.1 there exist antisymmetric matrices A^j and vectors $\boldsymbol{B}^j$ such that $\boldsymbol{v}(\boldsymbol{x}) = A^j\boldsymbol{x} + \boldsymbol{B}^j$ in ω_j. If $\partial\omega_j \cap \partial\omega_i \neq \emptyset$, $\omega_j \subset \Omega_+$, $\omega_i \subset \Omega_-$ then $a_+\boldsymbol{u}_+ - a_-\boldsymbol{u}_- = 0$ gives $(A^j\boldsymbol{x} + \boldsymbol{B}^j) - (A^i\boldsymbol{x} + \boldsymbol{B}^i) = 0$ on $\partial\omega_j \cap \partial\omega_i$. Lemma 5.1 gives that $(A^j\boldsymbol{x} + \boldsymbol{B}^j) - (A^i\boldsymbol{x} + \boldsymbol{B}^i) = 0$ in R^m. Thus, $\boldsymbol{v} \in \mathcal{R}_m$. If $\lambda_+ + \lambda_- + c_+ + c_- > 0$ then Lemma 5.1 gives that $\boldsymbol{v} \equiv 0$.

Since $\nabla p_\pm = \Delta\boldsymbol{u}_\pm - \lambda_\pm\boldsymbol{u}_\pm = 0$, there exist constant $d_1, \dots, d_k$ such that $p = d_j$ on ω_j, where $p = p_\pm$ on $\Omega_\pm$. If $\partial\omega_j \cap \partial\omega_i \neq \emptyset$, $\omega_j \subset \Omega_+$, $\omega_i \subset \Omega_-$ then $0 = b_+T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_-T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+\boldsymbol{u}_+ = (b_id_i - b_+d_j)\boldsymbol{n}_+$. Therefore, there is a constant d such that $p_\pm = d/b_\pm$. On ∂G we have $0 = T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_- = -d\boldsymbol{n}_-/b_-$. This gives $d = 0$. □

THEOREM 6.4 *Let* $\lambda_+ + \lambda_- + c_+ + c_- > 0$. *Then,* $R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}$ *is an isomorphism on* $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$. *Let* $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$, $\boldsymbol{h} \in L^2(\partial G, R^m)$. *Then, there exists a unique* L^2*-solution* $\boldsymbol{u}_\pm$, $p_\pm$ *of the Robin-transmission problem* (1), (2) *and* (16). *Moreover,* $\boldsymbol{u}_\pm \in H^{3/2}(\Omega_\pm, R^m)$, $p_\pm \in H^{1/2}(\Omega_\pm)$ *and*

$$
\begin{aligned}
&\|\boldsymbol{u}_+\|_{H^{3/2}(\Omega_+)} + \|\boldsymbol{u}_-\|_{H^{3/2}(\Omega_-)} + \|p_+\|_{H^{1/2}(\Omega_+)} + \|p_-\|_{H^{1/2}(\Omega_1)} \\
&\quad \le C\left[\|\boldsymbol{g}\|_{W^{1,2}(\partial\Omega,R^m)} + \|\boldsymbol{f}\|_{L^2(\partial\Omega,R^m)} + \|\boldsymbol{h}\|_{L^2(\partial G,R^m)},\right]
\end{aligned}
$$

where C *does not depend on* $\boldsymbol{g}$, $\boldsymbol{f}$ *and* $\boldsymbol{h}$.

Proof $R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}$ is a Fredholm operator with index 0 by Lemma 6.1. Let $R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}(\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = 0$. Let $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\boldsymbol{u}}_-$, $\tilde{p}_-$ by (18). Then $\tilde{\boldsymbol{u}}_\pm = 0$, $\tilde{\boldsymbol{p}}_\pm = 0$ by Proposition 6.3. Lemma 6.2 gives $\boldsymbol{\Phi} = 0$, $\boldsymbol{\Psi} = 0$, $\boldsymbol{\Theta} = 0$. Since $R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}$ is a Fredholm operator with index 0, we infer that $R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}$ is an isomorphism.

Let $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$, $\boldsymbol{h} \in L^2(\partial G, R^m)$ be fixed. Put

$$
(\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = (R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-})^{-1}[\boldsymbol{g}, \boldsymbol{f}, a_-\boldsymbol{h}].
$$

Define $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ by (15), and $\tilde{\boldsymbol{u}}_-$, $\tilde{p}_-$ by (18). Then, $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the Robin-transmission problem (13), (14) and (17). Denoting $\boldsymbol{u}_\pm = \tilde{\boldsymbol{u}}_\pm / a_\pm$, $p_\pm = \tilde{p}_\pm / a_\pm$ we obtain an L^2 solution of the problem (1), (2) and (16). The uniqueness follows from Proposition 6.3. The rest is a consequence of the fact that $E^{\lambda_\pm}_{\Omega_\pm} : L^2(\partial\Omega_\pm, R^m) \to H^{3/2}(\Omega_\pm, R^m)$, $D^{\lambda_\pm}_{\Omega_\pm} : W^{1,2}(\partial\Omega_\pm, R^m) \to H^{3/2}(\Omega_\pm, R^m)$, $Q^{\lambda_\pm}_{\Omega_\pm} : L^2(\partial\Omega_\pm, R^m) \to H^{1/2}(\Omega_\pm, R^m)$, $\Pi^{\lambda_\pm}_{\Omega_\pm} : W^{1,2}(\partial\Omega_\pm, R^m) \to H^{1/2}(\Omega_\pm, R^m)$ are bounded linear operators (see [8] and [4]). ☐

THEOREM 6.5 *Let* $\lambda_+ = \lambda_- = c_+ = c_- = 0$, $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$, $\boldsymbol{h} \in L^2(\partial G, R^m)$. *Then, there exists an* L^2*-solution* $\boldsymbol{u}_\pm$, $p_\pm$ *of the Robin-transmission problem* (1), (2) *and* (16) *if and only if*

$$\int\limits_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{f} \, \mathrm{d}\mathcal{H}_{m-1} + \int\limits_{\partial G} b_- \boldsymbol{v} \cdot \boldsymbol{h} \, \mathrm{d}\mathcal{H}_{m-1} = 0 \quad \forall \boldsymbol{v} \in \mathcal{R}_m. \tag{19}$$

The general from of an L^2*-solution of the problem* (1), (2) *and* (16) *is*

$$\boldsymbol{u}_\pm + \boldsymbol{v}/a_\pm, \; p\pm, \qquad \boldsymbol{v} \in \mathcal{R}_m. \tag{20}$$

Proof Let $\boldsymbol{u}_\pm$, $p_\pm$ be an L^2-solution of the Robin-transmission problem (1), (2) and (16), $\boldsymbol{v} \in \mathcal{R}_m$. Then,

$$\int\limits_{\partial\Omega_\pm} \boldsymbol{v} \cdot T(\boldsymbol{u}_\pm, p_\pm)\boldsymbol{n}^{\Omega_\pm} \, \mathrm{d}\mathcal{H}_{m-1} = 0$$

(see [4]). Thus

$$0 = b_+ \int\limits_{\partial\Omega_+} \boldsymbol{v} \cdot T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ + b_- \int\limits_{\partial\Omega_-} \boldsymbol{v} \cdot T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_- = \int\limits_{\partial\Omega} \boldsymbol{v} \cdot f + \int\limits_{\partial G} b_- \boldsymbol{v} \cdot h.$$

Denote by X^{b_-} the space of $[\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h}] \in X = W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$ satisfying (19). We have proved that $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) \subset X^{\tilde{b}_-}$. Therefore, $\operatorname{codim} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) \geq \operatorname{codim} X^{\tilde{b}_-} = \dim \mathcal{R}_m$.

Let $[\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}] \in \operatorname{Ker} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}$. Let $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\boldsymbol{u}}_-$, $\tilde{p}_-$ by (18). According to Proposition 6.3, there exists $\boldsymbol{v} \in \mathcal{R}_m$ such that $\tilde{\boldsymbol{u}}_\pm = \boldsymbol{v}$, $\tilde{p}_\pm = 0$. If $\boldsymbol{v} = 0$ then $\boldsymbol{\Phi} = 0$, $\boldsymbol{\Psi} = 0$, $\boldsymbol{\Theta} = 0$ by Lemma 6.2. Thus, $\dim \operatorname{Ker} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0} \leq \dim \mathcal{R}_m$. Since $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}$ is a Fredholm operator with index 0 by Lemma 6.1, we deduce that $\dim \operatorname{Ker} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0} = \operatorname{codim} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) = \dim \mathcal{R}_m$. Therefore, $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) = X^{\tilde{b}_-}$.

Let now $[\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h}] \in X$. We have proved that there exist $[\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}]$ such that $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}[\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}] = [\boldsymbol{g}, \boldsymbol{f}, a_-\boldsymbol{h}]$. Let $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\boldsymbol{u}}_-$, $\tilde{p}_-$ by (18), $\boldsymbol{u}_\pm = \tilde{\boldsymbol{u}}_\pm / a_\pm$, $p_\pm = \tilde{p}_\pm / a_\pm$. Then, $\boldsymbol{u}_\pm$, $p_\pm$ is an L^2-solution of the Robin-transmission problem (1), (2) and (16). Easy calculation yields that (20) gives another solution of the problem. Proposition 6.3 gives that each solution of the problem is of the form (20). ☐

7. Regular Dirichlet-transmission problem

Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $\Omega = \Omega_+$ be a nonempty bounded open set with Lipschitz boundary such that $\overline{\Omega} \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega}$, and by $\boldsymbol{n}_\pm$ the outward unit normal of $\Omega_\pm$. Let $\lambda_\pm$ and c_+ be non-negative constants and $a_\pm$ and $b_\pm$ be positive constants. We shall study the regular Dirichlet-transmission problem for the Brinkman system (1) and (2) accompanied with the condition

$$\boldsymbol{u}_- = \boldsymbol{h} \quad \text{on } \partial G. \tag{21}$$

Let $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$ and $\boldsymbol{h} \in W^{1,2}(\partial G, R^m)$. *We say that* $\boldsymbol{u}_\pm$, $p_\pm$ *defined on* $\Omega_\pm$ *is an* L^2*-solution of the regular Dirichlet-transmission problem* (1), (2) *and* (21) *if* $\boldsymbol{u}_\pm$, $p_\pm$ *satisfy* (1); $\boldsymbol{u}^*_\pm$, $p^*_\pm$, $(\nabla \boldsymbol{u})^*_\pm$ *are from* $L^2\left(\partial\Omega_\pm, R^1\right)$; *for almost all* $\boldsymbol{x} \in \partial\Omega_\pm$ *there exist the non-tangential limits of* $\boldsymbol{u}_\pm$, $\nabla\boldsymbol{u}_\pm$, $p_\pm$ *at* $\boldsymbol{x}$ *and the conditions* (2) *and* (21) *are fulfilled in the sense of the non-tangential limit a.e. on* $\partial\Omega_-$.

Put $\tilde{b}_\pm = b_\pm/a_\pm$, $\tilde{c}_+ = c_+/a_\pm$. If $\tilde{\boldsymbol{u}}_\pm = a_\pm \boldsymbol{u}_\pm$, $\tilde{p}_\pm = a_\pm p_\pm$ then $\boldsymbol{u}_\pm$, $p_\pm$ is an L^2-solution of the regular Dirichlet-transmission problem (1), (2) and (21) if and only if $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the regular Dirichlet-transmission problem (13) and (14):

$$\tilde{\boldsymbol{u}}_- = a_- \boldsymbol{h} \quad \text{on } \partial G. \tag{22}$$

Let $\boldsymbol{\Phi} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{\Psi} \in L^2(\partial\Omega, R^m)$ and $\boldsymbol{\Theta} \in L^2(\partial G, R^m)$. Let $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\boldsymbol{u}}_-$, $\tilde{p}_-$ be given by (18). Then, $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the regular Dirichlet-transmission problem (13), (14) and (22) if and only if

$$R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = [\boldsymbol{g}, \boldsymbol{f}, a_- \boldsymbol{h}],$$

where

$$R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = \Big[\tau_1^{\lambda_+,\lambda_-}(\boldsymbol{\Phi}, \boldsymbol{\Psi}) - E_G^{\lambda_-}\boldsymbol{\Theta}\ , \tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \\ - \tilde{b}_- T(E_G^{\lambda_-}\boldsymbol{\Theta}, Q_G^{\lambda_-}\boldsymbol{\Theta})\boldsymbol{n}_+,\ D_\Omega^{\lambda_-}\boldsymbol{\Phi} + E_\Omega^{\lambda_-}\boldsymbol{\Psi} + E_G^{\lambda_-}\boldsymbol{\Theta}\Big].$$

Proposition 7.1 *Let* $\boldsymbol{u}_\pm$, $p_\pm$ *be an* L^2*-solution of the regular Dirichlet-transmission problem* (1), (2) *and* (21) *with* $\boldsymbol{g} = 0$, $\boldsymbol{f} = 0$, $\boldsymbol{h} = 0$. *Then, there exists a constant* c *such that* $\boldsymbol{u}_\pm = 0$, $p_\pm = c/b_\pm$.

Proof Using Green's formula

$$\begin{aligned} 0 &= b_-^{-1} \int_{\partial\Omega} \boldsymbol{u}_- \cdot [b_+ T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_- T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+ \boldsymbol{u}_+]\, \mathrm{d}\mathcal{H}_{m-1} \\ &\quad + \int_{\partial G} \boldsymbol{u}_- \cdot T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_-\, \mathrm{d}\mathcal{H}_{m-1} \\ &= \int_{\Omega_-} \left[2|\hat{\nabla}\boldsymbol{u}_-|^2 + \lambda_-|\boldsymbol{u}_-|^2\right] \mathrm{d}\mathcal{H}_m \\ &\quad + \frac{a_+ b_+}{a_- b_-} \int_{\Omega_+} \left[2|\hat{\nabla}\boldsymbol{u}_+|^2 + \lambda_+|\boldsymbol{u}_+|^2\right] \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \frac{c_+ a_+ |\boldsymbol{u}_+|^2}{a_-}\, \mathrm{d}\mathcal{H}_{m-1}. \end{aligned}$$

Thus $\hat{\nabla}\boldsymbol{u}_\pm = 0$. According to [17], Lemma 3.1, there exist an antisymmetric matrix A and a vector $\boldsymbol{B}$ such that $\boldsymbol{u}_-(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{B}$. Since $\boldsymbol{u}_- = 0$ on ∂G, Lemma 5.1 gives that $\boldsymbol{u}_- = 0$. Since $\nabla p_- = \Delta\boldsymbol{u}_- - \lambda_-\boldsymbol{u}_- = 0$, there exists a constant c such that $p_- = c/b_-$. Let ω be a component of Ω_+. According to [17], Lemma 3.1 there exist an antisymmetric matrix A and a vector $\boldsymbol{B}$ such that $\boldsymbol{u}_+(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{B}$ in ω. Since $\boldsymbol{u}_+ = a_-\boldsymbol{u}_-/a_+ = 0$ on $\partial\omega$, Lemma 5.1 gives that $\boldsymbol{u}_+ = 0$ in ω. Since $\nabla p_+ = \Delta\boldsymbol{u}_+ - \lambda_+\boldsymbol{u}_+ = 0$, there exists a constant C such that $p_+ = C$ in ω. We have $0 = b_+T(\boldsymbol{u}_+, p_+)\boldsymbol{n}_+ - b_-T(\boldsymbol{u}_-, p_-)\boldsymbol{n}_+ + c_+\boldsymbol{u}_+ = -b_+C\boldsymbol{n}_+ + b_-(c/b_-)\boldsymbol{n}_+$ on $\partial\omega$. Hence, $p_+ = C = c/b_+$. □

THEOREM 7.2 *Let* $\boldsymbol{g} \in W^{1,2}(\partial\Omega, R^m)$, $\boldsymbol{f} \in L^2(\partial\Omega, R^m)$, $\boldsymbol{h} \in W^{1,2}(\partial G, R^m)$. *There exists an* L^2*-solution* $\boldsymbol{u}_\pm$, $p_\pm$ *of the regular Dirichlet-transmission problem* (1), (2) *and* (21) *if and only if*

$$\int_{\partial\Omega} \boldsymbol{n}_+ \cdot \boldsymbol{g}\, \mathrm{d}\mathcal{H}_{m-1} + a_- \int_{\partial G} \boldsymbol{n}_- \cdot \boldsymbol{h}\, \mathrm{d}\mathcal{H}_{m-1} = 0. \tag{23}$$

The general form of a solution of the problem is $\boldsymbol{u}_\pm$, $p_\pm + c/b_\pm$, *where c is a constant.*

Proof Suppose that $\boldsymbol{u}_\pm$, $p_\pm$ be an L^2-solution $\boldsymbol{u}_\pm$, $p_\pm$ of the regular Dirichlet-transmission problem (1), (2) and (21). Then,

$$0 = a_+ \int_{\partial\Omega} \boldsymbol{n}_+ \cdot \boldsymbol{u}_+ + a_- \int_{\partial G} \boldsymbol{n}_- \cdot \boldsymbol{u}_- = \int_{\partial\Omega} \boldsymbol{n}_+ \cdot \boldsymbol{g}\, \mathrm{d}\mathcal{H}_{m-1} + a_- \int_{\partial G} \boldsymbol{n}_- \cdot \boldsymbol{h}\, \mathrm{d}\mathcal{H}_{m-1}.$$

$R : (\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) \mapsto [\tau_1^{\lambda_+,\lambda_-}(\boldsymbol{\Phi}, \boldsymbol{\Psi}), \tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}), E_G^0\boldsymbol{\Theta}]$ is a Fredholm operator with index 0 from $X = W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$ to the space $Y = W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times W^{1,2}(\partial G, R^m)$ by [4] and Proposition 5.3. If $\lambda \geq 0$, then $E_G^\lambda - E_G^0$ is a compact operator from $L^2(\partial G, R^m)$ to $W^{1,2}(\partial G, R^m)$ (see [8], Theorem 3.4). Thus, $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} - R$ is a compact operator. Hence, $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}$ is a Fredholm operator from X to Y with index 0. Denote by $Z(a_-)$ the set of all $[\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h}] \in Y$ satisfying (23). We have proved that $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(X) \subset Z(1)$. Thus, codim $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(X) \geq 1$.

Let now $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = 0$. Let $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\boldsymbol{u}}_-$, $\tilde{p}_-$ be given by (18). Then, $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the regular Dirichlet-transmission problem (13), (14) and (22) with $\boldsymbol{g} = 0$, $\boldsymbol{f} = 0$, $\boldsymbol{h} = 0$. Proposition 7.1 gives that there exists a constant c such that $\boldsymbol{u}_\pm = 0$, $p_\pm = c/\tilde{b}_\pm$. If $c = 0$ then $\boldsymbol{\Phi} = 0$, $\boldsymbol{\Psi} = 0$, $\boldsymbol{\Theta} = 0$ by Lemma 6.2. Therefore, dim Ker $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} \leq 1$. Hence $1 \leq$ codim $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(X) =$ dim Ker $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} \leq 1$. This forces $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(X) = Z(1)$.

Suppose now that (23) is fulfilled. We have proved that there exists $[\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}] \in X$ such that $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = [\boldsymbol{g}, \boldsymbol{f}, a_-\boldsymbol{h}]$. Let $\tilde{\boldsymbol{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\boldsymbol{u}}_-$, $\tilde{p}_-$ be given by (18). Then, $\tilde{\boldsymbol{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the regular Dirichlet-transmission problem (13), (14) and (22). So $\boldsymbol{u}_\pm = \tilde{\boldsymbol{u}}_\pm/a_\pm$, $p_\pm = \tilde{p}_\pm/a_\pm$ is an L^2-solution of (1), (2) and (21). If c is a constant, then easy calculation gives that $\boldsymbol{u}_\pm$, $p_\pm + c/b_\pm$ is a solution of the problem, too. Proposition 7.1 gives that each solution of the problem has this form. □

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References

[1] Fabes EB, Kenig CE, Verchota GC. The Dirichlet problem for the Stokes system on Lipschitz domains. Duke Math. J. 1988;57:769–793.

[2] Maremonti P, Russo R, Starita G. On the Stokes equations: the boundary value problem. In: Maremonti P, editor. Advances in fluid dynamics. Caserta: Dipartimento di Matematica Seconda Università di Napoli; 1999. p. 69–140.

[3] Maz'ya V, Mitrea M, Shaposhnikova T. The inhomogeneous Dirichlet problem for the Stokes system in Lipschitz domains with unit normal close to VMO^*. Funct. Anal. Appl. 2009;43: 217–235.

[4] Mitrea M, Wright M. Boundary value problems for the Stokes system in arbitrary Lipschitz domains. Paris: Astérisque; 2012.

[5] Odquist FKG. Über die Randwertaufgaben in der Hydrodynamik zäher Flüssigkeiten [On boundary value problems in hydrodynamics of viscid fluids]. Math. Z. 1930;32:329–375.

[6] Varnhorn W. The Stokes equations. Berlin: Akademie Verlag; 1994.

[7] Chang TK, Pahk DH. Spectral properties for layer potentials associated to the Stokes equation in Lipschitz domains. Manuscripta Math. 2009;130:359–373.

[8] Kohr M, Lanza de Cristoforis M, Wendland WL, Nonlinear Neumann-transmission problems for Stokes and Brinkman equations on Euclidean Lipschitz domains. Potential Anal. 2013;38: 1123–1171.

[9] Kohr M, Pintea C, Wendland WL, Brinkman-type operators on Riemannian manifolds: transmission problems in Lipschitz and C^1 domains. Potential Anal. 2010;32:229–273.

[10] Kohr M, Pintea C, Wendland WL, Stokes-Brinkman transmission problems on Lipschitz and C^1 domains in Riemannian manifolds. Commun. Pure Appl. Anal. 2010;9:493–537.

[11] Kohr M, Pintea C, Wendland WL, Dirichlet-transmission problems for general Brinkman operators in Lipschitz and C^1 domains in Riemannian manifolds. Discret. Contin. Dyn. S. 2011;15:999–1018.

[12] Kohr M, Raja Sekhar GP, Ului EM, Wendland WL. Two-dimensional Stokes-Brinkmann cell model – a boundary integral formulation. Appl. Anal. 2012;91:251–275.

[13] Kenig CE. Recent progress on boundary value problems on Lipschitz domains. Pseudodifferential operators and applications. Proc. Symp., Notre Dame/ Indiana 1984. Proc. Symp. Pure Math. 1985;43:175–205.

[14] Mitrea M, Taylor M. Navier-Stokes equations on Lipschitz domains in Riemannian manifolds. Math. Ann. 2011;321:955–987.

[15] Shilov GE. Mathematical analysis. Second special course. Moskva: Nauka; 1965 (Russian).

[16] Duistermaat JJ, Kolk JAC. Distributions. Theory and applications. New York: Birkhäuser; 2010.

[17] Medková D. Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system. Acta Appl. Math. 2011;116:281–304.