Upper bound theorem
for odd-dimensional flag manifolds

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UPPER BOUND THEOREM FOR ODD-DIMENSIONAL FLAG MANIFOLDS

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ABSTRACT. We prove that among all flag triangulations of manifolds of odd dimension $2r - 1$ with sufficiently many vertices the unique maximizer of the entries of the f-, h-, g- and γ-vector is the balanced join of r cycles. Our proof uses methods from extremal graph theory.

1. INTRODUCTION

The classification of face numbers (f-vectors) of various classes of simplicial complexes, especially triangulations of spheres, balls and manifolds, is a classical topic in enumerative combinatorics. The Charney–Davis conjecture [5] and its generalization by Gal [10] sparkled the interest in similar questions for the class of flag triangulations. In this paper we prove a general upper bound theorem for flag triangulations of odd-dimensional manifolds.

A simplicial complex K is *flag* if every set of vertices pairwise adjacent in the 1-skeleton of K spans a face of K or, equivalently, if K is the clique complex of its 1-skeleton. Flag complexes appear prominently in Gromov's approach to non-positive curvature (see [13] and [4] for an exposition). In this context Charney and Davis proposed their famous conjecture [5] that a certain linear combination of the face numbers of any odd-dimensional flag homology sphere is non-negative. Subsequently, Gal [10] introduced a modification of the f-vector called the γ-vector, which seems well-suited to the study of flag homology spheres, and is conjecturally non-negative. Since then a number of conjectures have been made about the structure of γ-vectors of flag spheres, with many of them verified in special cases [2, 3, 10, 14, 16, 18, 19]. Note that a flag complex is completely determined by its 1-skeleton, and thus its face vector is the clique vector of the underlying graph. Paradoxically, this only adds to the complexity of the problem. For example, face vectors of arbitrary simplicial complexes are characterized by the Kruskal–Katona theorem, while the clique vectors of general graphs are not so well understood [9].

Our contribution is an upper bound theorem for odd-dimensional flag homology manifolds, a class which includes flag simplicial manifolds and flag homology spheres. We exhibit a unique maximizer of any reasonable linear combination of face numbers. For any $r \geq 1$ and $n \geq 4r$ let $\mathbf{J}_r(n)$ be the n-vertex flag complex obtained as a join of r copies of the circle S^1, each one a cycle with $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$ vertices. This complex is a flag simplicial $(2r - 1)$-sphere. To phrase our main theorem we say that a real-valued function F defined on simplicial complexes is a *face*

function in dimension ℓ if it can be written as $F(K) = c_\ell f_\ell(K) + c_{\ell-1} f_{\ell-1}(K) + \ldots + c_0 f_0(K) + c_{-1}$ with $c_i \in \mathbb{R}$ and $c_\ell > 0$, where $f_i(K)$ is the number of i-dimensional faces of K.

Theorem 1 (Main theorem)**.** *For every even number $d \geq 4$ and every face function F in dimension ℓ, where $1 \leq \ell \leq \frac{d}{2} - 1$, there exists a constant n_0 for which the following holds. If M is a flag homology manifold of dimension $d - 1$ with $n \geq n_0$ vertices then*

$$F(M) \leq F(\mathbf{J}_{\frac{d}{2}}(n))$$

and equality holds if and only if M is isomorphic to $\mathbf{J}_{\frac{d}{2}}(n)$.

In this context the standard statistics based on face numbers are the f-vector $(f_{-1}, f_0, \ldots, f_{d-1})$, the h-vector $(h_0, h_1, \ldots, h_d)$, the g-vector $(g_0, g_1, \ldots, g_{\frac{d}{2}})$ and the γ-vector $(\gamma_0, \gamma_1, \ldots, \gamma_{\frac{d}{2}})$. Theorem 1 specializes to an upper bound statement for all of those simultaneously.

Corollary 2. *For every even number $d \geq 4$ there is a constant N_0 for which the following holds. If M is a flag homology manifold of dimension $d - 1$ with $n \geq N_0$ vertices then*

$$f_i(M) \leq f_i(\mathbf{J}_{\frac{d}{2}}(n)) \quad \text{for} \quad 1 \leq i \leq d - 1,$$
$$h_i(M) \leq h_i(\mathbf{J}_{\frac{d}{2}}(n)) \quad \text{for} \quad 2 \leq i \leq d - 2,$$
$$g_i(M) \leq g_i(\mathbf{J}_{\frac{d}{2}}(n)) \quad \text{for} \quad 2 \leq i \leq \frac{d}{2},$$
$$\gamma_i(M) \leq \gamma_i(\mathbf{J}_{\frac{d}{2}}(n)) \quad \text{for} \quad 2 \leq i \leq \frac{d}{2}.$$

Moreover, equality in any of these inequalities implies that M is isomorphic to $\mathbf{J}_{\frac{d}{2}}(n)$.

For all other values of the index i, as well as for face functions in dimension 0 or -1 in Theorem 1, the corresponding inequalities are trivially satisfied with equality for all M.

The only previously known case of Corollary 2 was $d = 4$ (for any n) due to Gal [10], with the uniqueness part (for large n) following from [2]. In this case all inequalities follow from $f_1(M) \leq f_1(\mathbf{J}_2(n))$. Our upper bound for the γ-vector confirms for large n a conjecture of Lutz and Nevo [17, Conjecture 6.3] and provides supporting evidence for a question of Nevo and Petersen [18, Problem 6.4] (see Section 5 for details). We also previously conjectured the upper bound on f_1 for arbitrary even d in [2].

For arbitrary (not necessarily flag) odd-dimensional homology manifolds tight upper bounds for f_i were obtained by Novik [20, Theorem 1.4]. In this case the maximum is attained by the boundary of the d-dimensional cyclic polytope with n vertices (the maximizer is not unique). For the subclass of simplicial spheres this had been known before by the celebrated upper bound theorem of Stanley [23]. In the flag case our result is new even for flag simplicial spheres.

2. PRELIMINARIES

We recommend the reader [24] and [20] as references for face numbers of triangulations of manifolds and spheres.

An *abstract simplicial complex K* with vertex set V is a collection $K \subseteq 2^V$ such that $\sigma \in K$ and $\tau \subseteq \sigma$ imply $\tau \in K$. The elements of K are called *faces.* The *dimension*

of σ is $|\sigma| - 1$ and the dimension of K is the maximal dimension of any of its faces. The *link* of a face σ is the subcomplex $\mathrm{lk}_K(\sigma) = \{\tau \in K : \tau \cap \sigma = \emptyset,\ \tau \cup \sigma \in K\}$.

A simplicial complex K is a *simplicial manifold* (resp. *simplicial sphere)* of dimension q if the geometric realization $|K|$ is homeomorphic to a connected, compact topological q-manifold without boundary (resp. to the sphere S^q). Most known results involving face numbers of simplicial manifolds hold for more general objects, which we now introduce. A simplicial complex K is a *homology manifold* if for any point $p \in |K|$ and any $i \neq \dim K$, $H_i(|K|, |K| - p; \mathbb{Z}) = 0$ and $H_{\dim K}(|K|, |K| - p; \mathbb{Z}) = \mathbb{Z}$. This is equivalent to saying that for every nonempty face $\sigma \in K$ the link $\mathrm{lk}_K(\sigma)$ has the homology of the sphere $S^{q-|\sigma|}$ (equivalence follows from the excision axiom, see [15, Lemma 3.3]). A *homology sphere* is a homology manifold K such that K itself has the homology of a sphere. It is easy to see that if K is a homology q-manifold then for every nonempty face $\sigma \in K$ the link $\mathrm{lk}_K(\sigma)$ is a homology $(q - |\sigma|)$-sphere. Clearly every simplicial manifold (resp. simplicial sphere) is a homology manifold (resp. homology sphere).

A complex K of dimension q is called *Eulerian* if for every face $\sigma \in K$ (including the empty one) the link $\mathrm{lk}_K(\sigma)$ has the same Euler characteristic as the sphere $S^{q-|\sigma|}$. Every homology manifold satisfies Poincaré duality; as a consequence the Euler characteristic of an odd-dimensional homology manifold M equals 0 and so M is Eulerian.

For a $(d-1)$-dimensional complex K with n vertices let $f_i(K)$ be the number of i-dimensional faces. The vector $(f_{-1}, f_0, \ldots, f_{d-1})$ is called the *f-vector* of K (note that $f_{-1}(K) = 1$ and $f_0(K) = n$). The *h*-vector $(h_0, h_1, \ldots, h_d)$ of K is a convenient modification of the f-vector defined by the identity

$$(2.1) \qquad \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i} = \sum_{i=0}^{d} h_i x^i.$$

Note $h_0(K) = 1$ and $h_1(K) = n - d$. An Eulerian simplicial complex satisfies the Dehn–Sommerville equations $h_i(K) = h_{d-i}(K)$ for $0 \leq i \leq d$. In that case one can define the *γ-vector* $(\gamma_0, \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor})$ of K by the identity

$$(2.2) \qquad \sum_{i=0}^{d} h_i x^i = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (x+1)^{d-2i}.$$

Here $\gamma_0(K) = 1$ and $\gamma_1(K) = n - 2d$. The γ-vector was first introduced by Gal [10] for flag homology spheres, for which it is conjectured to be non-negative. This conjecture generalizes the Charney–Davis conjecture, which in this language asserts that $\gamma_{\frac{d}{2}}(K)$ is non-negative for a $(d-1)$-dimensional flag homology sphere K with d even. Another classical invariant, studied mostly for simplicial spheres and balls, is the *g-vector* $(g_0, g_1, \ldots, g_{\lfloor \frac{d}{2} \rfloor})$ given by $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Suppose now that d is even and let M be a homology $(d-1)$-manifold. For any i the function $f_i(M)$ is clearly a face function in dimension i. For any $0 \leq i \leq d$ we have $h_i = \sum_{j=0}^{i} (-1)^{j-i} \binom{d-j}{i-j} f_{j-1}$, so $h_i(M)$ is a face function in dimension $i-1$. By the Dehn–Sommerville equations if $\frac{d}{2} \leq i \leq d$ then $h_i(M) = h_{d-i}(M)$ can be expressed as a face function in dimension $d-i-1$. For any $0 \leq i \leq d-1$ we have $f_i = \sum_{j=0}^{i+1} \binom{d-j}{i+1-j} h_j$. If $\frac{d}{2} \leq i \leq d-1$ the Dehn–Sommerville equations imply that

f_i is a linear combination of $(h_{\frac{d}{2}}, \ldots, h_0)$ with leading term $\binom{d/2}{i+1-d/2} h_{\frac{d}{2}}$, so by the previous observations $f_i(M)$ is equal to a face function in dimension $\frac{d}{2} - 1$. Finally both γ_i and g_i are linear combinations of $(h_i, \ldots, h_0)$ with leading term h_i, hence $\gamma_i(M)$ and $g_i(M)$ are face functions in dimension $i - 1$. Using these observations the proof of Corollary 2 from Theorem 1 is immediate.

Let us now move towards flag complexes. If $G = (V, E)$ is a finite, simple, undirected graph then the *clique number* $\omega = \omega(G)$ of G is the cardinality of the largest clique (complete subgraph) in G and the *clique vector* of G is the sequence $(e_0(G), e_1(G), \ldots, e_\omega(G))$, where $e_i(G)$ is the number of cliques of cardinality i (in particular $e_0(G) = 1$, $e_1(G) = |V|$ and $e_2(G) = |E|$). The *clique complex* of G, denoted $X(G)$, is the simplicial complex with vertex set V whose faces are all cliques in G. We have $\dim X(G) = \omega(G) - 1$ and $f_i(X(G)) = e_{i+1}(G)$. Note that the 1-skeleton of $X(G)$ is G. A simplicial complex is *flag* if it is the clique complex of a graph. A *flag homology manifold* (resp. *flag homology sphere)* is a flag complex which is a homology manifold (resp. a homology sphere).

By abuse of language we will say that G *triangulates a homology manifold* (resp. *sphere*) if $X(G)$ is a flag homology manifold (resp. sphere).

Fix $n, r \in \mathbb{N}$. We write $T_r(n)$ for the r-partite Turán graph of order n, that is a graph with n vertices partitioned into sets $V_1, V_2, \ldots, V_r$, each of size either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$, with no edge inside any V_i and with a complete bipartite graph between every two V_i and V_j, $i \neq j$. Further, for $n \geq 4r$ we define $J_r(n)$ to be the graph obtained from $T_r(n)$ by declaring that each of the parts V_i induces a cycle of length $|V_i|$. The condition $n \geq 4r$ guarantees that each part is a cycle of length at least 4, hence a flag triangulation of S^1. Of course we have $\mathbf{J}_r(n) = X(J_r(n))$ and this complex is a flag simplicial $(2r - 1)$-sphere.

We say that a real-valued function F defined on graphs is a *clique function of order k*, if F can be written as $F(G) = c_k e_k(G) + c_{k-1} e_{k-1}(G) + \cdots + c_1 e_1(G) + c_0$ where $c_i \in \mathbb{R}$ and $c_k > 0$. Theorem 1 can be equivalently rephrased as follows.

Theorem 3 (Main Theorem, Graph formulation)**.** *For every $r \geq 2$ and every clique function F of order k, where $2 \leq k \leq r$, there exists a constant n_0 for which the following holds. If G is a graph with $n \geq n_0$ vertices which triangulates a $(2r - 1)$-dimensional homology manifold then*

$$F(G) \leq F(J_r(n))$$

and equality holds if and only if G is isomorphic to $J_r(n)$.

Let us first fix some additional notation. The *neighborhood* of a vertex v in a graph G is the set $N_G(v) = \{w \colon vw \in E(G)\}$ and for a clique σ in G we define the *link of σ in G* as the induced subgraph $\mathrm{lk}_G(\sigma) = G[\bigcap_{v \in \sigma} N_G(v)]$. This notation is designed so that $X(\mathrm{lk}_G(\sigma)) = \mathrm{lk}_{X(G)}(\sigma)$. For a vertex $v \in V(G)$ and a subset $W \subset V(G)$ we write $\deg_G(v) = |N_G(v)|$ and $\deg_G(v, W) = |N_G(v) \cap W|$. The subscript G will be omitted if there is no risk of confusion.

2.1. **Properties of flag homology manifolds.** Below, we record two basic properties of flag homology manifolds that we need for our proof of the Main Theorem.

Lemma 4. *For every $r \geq 1$ there is a constant C_r such that every n-vertex graph G triangulating a $(2r - 1)$-dimensional homology manifold satisfies $e_{r+1}(G) \leq C_r n^r$.*

Proof. The Dehn–Sommerville relation $h_{r+1}(X(G)) = h_{r-1}(X(G))$ expressed in terms of face numbers implies that $f_r(X(G))$ is a linear combination of entries of the vector $(f_{r-1}(X(G)), \ldots, f_{-1}(X(G))) = (e_r(G), \ldots, e_0(G))$ with coefficients depending only on r. Since $e_i(G) \leq \binom{n}{i} \leq n^r$ for $0 \leq i \leq r$, we get $e_{r+1}(G) = f_r(X(G)) \leq C_r n^r$ for a suitable C_r. $\qquad\square$

Let $K_3^r := T_r(3r)$ denote the complete r-partite graph with all parts of size 3. A graph G is H-free if it does not contain H as a subgraph. The crucial geometric ingredient of our arguments is provided by the next lemma.

Lemma 5. *Fix $r \geq 1$. If G triangulates a homology sphere of dimension $2(r-1)$ then G is K_3^r-free.*

Proof. By a result of Galewski and Stern [11, Corollary 1.9], if $X(G)$ is a homology $2(r-1)$-sphere then the double suspension $\Sigma^2 X(G)$ is homeomorphic to S^{2r}. Now if G contained K_3^r then $\Sigma^2 X(G)$ would contain an embedded $X(K_3^{r+1})$, formed by the original K_3^r and any three of the four suspending vertices. That contradicts the theorem of van Kampen and Flores [25, 8] (see also [26, Section 2.4]) that $X(K_3^{r+1})$ is not embeddable in S^{2r}. $\qquad\square$

In our arguments we are going to apply Lemma 5 to links of faces in a homology manifold. For example, we get that if G triangulates a homology $(2r-1)$-manifold then for every vertex v the link $\mathrm{lk}_G(v)$ is K_3^r-free.

2.2. **Extremal graph theory.** The remaining tools for our proof come entirely from extremal graph theory. An approach to face enumeration via extremal graph theory was pioneered in [2] where we classified all flag homology 3-manifolds M with a sufficiently large number n of vertices which are almost extremal for f_1 or γ_2. Thus, the main technical contribution of our current work is in connecting further tools from extremal graph theory (namely Zykov's inequalities (Theorem 8) and the Removal lemma (Theorem 9)) to the area of face enumeration.

The following definition introduces a distance — sometimes called the *edit distance* — on the set of n-vertex graphs.

Definition 6. We say that two graphs with the same number of n vertices are ϵ-*close* if there exists an identification of their vertex sets, so that then one graph can be obtained from the other by editing (i.e., adding or deleting) less than ϵn^2 edges.

The celebrated Stability Theorem of Erdős and Simonovits [6, 22] below says that a K_{r+1}-free graph whose number of edges is close to the Turán bound must actually be close to the Turán graph in the edit distance.

Theorem 7. *Suppose that $r \geq 2$ and $\epsilon > 0$ are given. Then there exists $\delta > 0$ such that whenever H is an n-vertex, K_{r+1}-free graph with $e_2(H) > (1-\delta)e_2(T_r(n))$ then H is ϵ-close to $T_r(n)$.*

We will also make use of the following result.

Theorem 8. *Let $r \geq 1$ and suppose than H is an n-vertex, K_{r+1}-free graph. Then we have*

$$1 = \frac{e_1(H)}{e_1(T_r(n))} \geq \frac{e_2(H)}{e_2(T_r(n))} \geq \ldots \geq \frac{e_r(H)}{e_r(T_r(n))} \ .$$

Theorem 8 generalizes the result of Zykov [27] that $e_k(H) \leq e_k(T_r(n))$, which in turn generalizes Turán's Theorem stating $e_2(H) \leq e_2(T_r(n))$. A nice proof of Theorem 8 using symmetrization can be found in [12, Theorem 3.1].

Let us now motivate the Removal lemma. A graph of order n can contain at most $\binom{n}{r+1} = \Theta(n^{r+1})$ copies of K_{r+1}. If the graph is not complete then of course it contains less copies. However, we think of the graph H as "essentially K_{r+1}-free" if $e_{r+1}(H) = o(n^{r+1})$. It is then tempting to say that by removing a few edges we can delete all the copies of K_{r+1}. This is true, yet far from trivial, and a subject of the famous Removal lemma, a form of which first appeared in [21], and which was later formulated in its full strength in [7].

Theorem 9. *Suppose that $r \geq 1$ and $\alpha > 0$ are given. Then there exists $\beta > 0$ such that whenever H is an n-vertex graph with $e_{r+1}(H) \leq \beta n^{r+1}$ then by deleting a suitable set of less than αn^2 edges H can be made K_{r+1}-free.*

2.3. **Outline of the proof of Theorem 3.** Suppose G triangulates a homology $(2r - 1)$-manifold and the number of vertices n is large. First, note that if $k \leq r$ then $e_k(T_r(n)) \approx \binom{r}{k} \left(\frac{n}{r}\right)^k$ and $e_k(J_r(n)) = e_k(T_r(n)) + O(n^{k-1}) = \binom{r}{k} \left(\frac{n}{r}\right)^k + O(n^{k-1})$. Now if G is such that $e_k(G) \leq (1 - \alpha)e_k(T_r(n))$, for a fixed (but arbitrary) $\alpha > 0$ then the inequality $F(G) \leq F(J_r(n))$ follows just by comparing the terms of order n^k in F. That leaves us only with the case where $e_k(G) \approx e_k(T_r(n))$, in which case we can also deduce $e_2(G) \approx e_2(T_r(n))$ by Theorem 8. By Lemma 4, G is "sparse in $(r + 1)$-cliques", i.e., it has only $O(n^r) = o(n^{r+1})$ many K_{r+1}'s, yet at the same time very dense (close to the maximal number of edges allowed for a K_{r+1}-free graph by Turán's Theorem). At this point Theorem 7 shows that G must be similar to $T_r(n)$. Additional geometric properties of $X(G)$ allow us to conclude from there that $F(G)$ is maximized by $F(J_r(n))$.

2.4. **Organisation of the paper.** As said earlier, the difficult cases Theorem 3 are those when G is close to $T_r(n)$. We will analyze their structure more closely in the next section. In Section 4 we give a proof of Theorem 3. We then conclude with open problems stemming from this work in Section 5.

3. ANALYSIS OF ALMOST EXTREMAL GRAPHS

For any $r \geq 1$ denote $[r] = \{1, \ldots, r\}$. We denote by $H[X]$ the subgraph of H induced by a set of vertices X and by $H[X, Y]$ the bipartite sugbraph of H with parts $X, Y \subset V(H)$, $X \cap Y = \emptyset$.

In this section we deal with almost extremal cases, that is, with triangulations of homology $(2r - 1)$-manifolds that are close to $T_r(n)$. These graphs fall into the class of (η, r)-extremal graphs introduced below.

Definition 10. Let $0 \leq \eta < 1$ and $r \geq 1$ be given. We say that an n-vertex graph H is (η, r)-*extremal* if the vertices of H can be partitioned into sets $V_0, V_1, \ldots, V_r$ such that

(a) $|V_0| \leq \frac{1}{30r^r}\eta n$ and $\lfloor (1 - \frac{1}{30r}\eta)\frac{n}{r} \rfloor \leq |V_i| \leq \lceil (1 + \frac{1}{30r}\eta)\frac{n}{r} \rceil$ for $i \in [r]$,

(b) $H[V_i]$ is triangle-free, for $i \in [r]$,

(c) $H[V_i]$ has maximum degree at most 2, for $i \in [r]$,

(d) for each $i, j \in [r]$, $i \neq j$, and any $v \in V_i$ we have $\deg_H(v, V_j) \geq (1 - \eta)|V_j|$,

(e) each vertex of V_0 is either of Type 1 or Type 2, where

we say a vertex v is of *Type 1* if there exist two distinct indices $g, h \in [r]$ such that $\deg_H(v, V_g) \leq 2$ and $\deg_H(v, V_h) \leq (1 - \frac{1}{2}\eta)|V_h|$, and it is of *Type 2* if there exist two distinct indices $g, h \in [r]$ such that $\deg_H(v, V_g) \leq 3r\eta|V_g|$ and $\deg_H(v, V_h) \leq 3r\eta|V_h|$.

For small η, graphs with the above structure resemble $J_r(n)$ up to some error. That is, we allow that a small fraction of edges missing in each $H[V_i, V_j]$, that the parts are slightly unbalanced and we admit a small set of exceptional vertices V_0. In the next definition we introduce a class of graphs that resemble $J_r(n)$ even better.

Definition 11. We say that a graph is *r-radical* if it is $(0, r)$-extremal, and for each $i \in [r]$ every vertex of $H[V_i]$ has degree 2.

If H is r-radical then $V_0 = \varnothing$, each V_i is of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$ for $i \in [r]$ and each graph $H[V_i, V_j]$ is complete bipartite for $i, j \in [r], i \neq j$. An r-radical graph is (η, r)-extremal for any $0 \leq \eta < 1$. Note that if H is any n-vertex r-radical graph then $F(H) = F(J_r(n))$ for every clique function F.

Lemma 12. *If H is an r-radical graph with n vertices which triangulates a homology $(2r - 1)$-manifold then H is isomorphic to $J_r(n)$.*

Proof. For all $i = 1, \ldots, r - 1$ pick any edge in $H[V_i]$. The endpoints of these $r - 1$ edges form a clique of order $2r - 2$ whose link is $H[V_r]$. However, in a homology $(2r - 1)$-manifold the link of a face of size $2r - 2$ is a homology 1-sphere, that is a cycle. It means that $H[V_r]$ is a cycle. The same argument shows that all $H[V_i]$ are cycles and therefore H is isomorphic to $J_r(n)$. □

The next lemma is used to find copies of K_3^r in (η, r)-extremal graphs.

Lemma 13. *Fix $r \geq 1$ and $\eta > 0$. Suppose H is a graph with $n \geq 2r\eta^{-1}$ vertices and a partition $V(H) = V_0 \sqcup V_1 \sqcup \ldots \sqcup V_r$ which satisfies conditions (a) and (d) of Definition 10. Let $w_1, w_2, w_3 \in V_1$ be any three fixed vertices. For $i \in \{2, \ldots, r\}$ let $A_i \subseteq V_i$ be sets with $|A_i| \geq 3r\eta|V_i|$. Then the subgraph of H induced by $\{w_1, w_2, w_3\} \cup \bigcup_{i=2}^{r} A_i$ contains a K_3^r with 3 vertices in each part $V_i, i \in [r]$.*

Proof. We will construct by induction 3-element subsets $\{w_1^i, w_2^i, w_3^i\} \subseteq A_i$ such that for each $l \in [r]$ the subgraph of H induced by w_j^i with $j = 1, 2, 3$ and $i = 1, \ldots, l$ contains K_3^l. For $l = r$ this proves the lemma. When $i = 1$ the vertices $w_j^1 = w_j$ are already given.

Suppose we have constructed the vertices $\{w_1^i, w_2^i, w_3^i\}_{i=1}^{l}$ for some $l \leq r - 1$. By condition (d) the common neighborhood N_{l+1} of these $3l$ vertices satisfies $|N_{l+1} \cap V_{l+1}| \geq (1 - 3l\eta)|V_{l+1}|$. It follows that

$$
\begin{aligned}
|A_{l+1} \cap N_{l+1}| &\geq |A_{l+1}| - |V_{l+1} \setminus N_{l+1}| \geq 3r\eta|V_{l+1}| - 3l\eta|V_{l+1}| \\
&\geq 3\eta|V_{l+1}| \geq 3\eta\frac{n}{2r} \geq 3,
\end{aligned}
$$

where the last line uses condition (a) of Definition 10 and the bound $n \geq 2r\eta^{-1}$. It means that we can pick three distinct vertices $w_1^{l+1}, w_2^{l+1}, w_3^{l+1} \in A_{l+1} \cap N_{l+1}$ and the induction step is complete. □

We can now prove that graphs triangulating homology $(2r - 1)$-manifolds are (η, r)-extremal as soon as they are sufficiently close to $T_r(n)$.

Lemma 14. *For every $r \geq 2$ and $0 < \eta < \frac{1}{7r}$ set $\epsilon = \frac{\eta^2}{120r^{r+3}}$. If a graph H with $n \geq 2r\eta^{-1}$ vertices triangulates a homology $(2r-1)$-manifold and H is ϵ-close to $T_r(n)$ then H is (η, r)-extremal.*

Proof. If H is ϵ-close to $T_r(n)$ then the vertices of H can be partitioned into r sets $X_1, \ldots, X_r$, each of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$, such that

$$\sum_{i<j} \bar{e}_2(H[X_i, X_j]) + \sum_i e_2(H[X_i]) \leq \epsilon n^2.$$

Here $\bar{e}_2(H[X_i, X_j])$ is the number of edges missing between X_i and X_j, that is $\bar{e}_2(H[X_i, X_j]) = |X_i| \cdot |X_j| - e_2(H[X_i, X_j])$. For every $i \neq j$ let

$$X_{i,j} = \{v \in X_i \colon \deg(v, X_j) \leq (1 - \tfrac{2}{3}\eta)|X_j|\}.$$

Every vertex in $X_{i,j}$ contributes to the number of missing edges $\bar{e}(H[X_i, X_j])$ as follows

$$\epsilon n^2 \geq \bar{e}_2(H[X_i, X_j]) \geq \sum_{v \in X_{i,j}} (|X_j| - \deg(v, X_j)) \geq \tfrac{2}{3}\eta|X_j| \cdot |X_{i,j}| \geq \tfrac{1}{2}\eta\tfrac{n}{r} \cdot |X_{i,j}|,$$

hence $|X_{i,j}| \leq 2\epsilon r\eta^{-1}n$. Consider a new partition $V(H) = Y_0 \sqcup Y_1 \sqcup \ldots \sqcup Y_r$,

$$Y_0 = \bigcup_{i \neq j} X_{i,j}, \quad Y_i = X_i \setminus Y_0 \text{ for } i \in [r].$$

We have $|Y_0| \leq r^2 \cdot 2\epsilon r\eta^{-1}n = \frac{1}{60r^r}\eta n$ and, for $i \in [r]$,

$$\lceil \tfrac{n}{r} \rceil \geq |X_i| \geq |Y_i| \geq |X_i| - |Y_0| \geq (1 - \tfrac{1}{30r}\eta)\tfrac{n}{r}.$$

By definition, for every vertex $v \in Y_0$ there exists an index $j \in [r]$ such that $\deg(v, X_j) \leq (1 - \tfrac{2}{3}\eta)|X_j|$. Let $Z_j \subset Y_0$ consists of those vertices for which j is the only such index, formally:

$$Z_j = \left\{ v \in Y_0 \colon \deg(v, X_k) \leq (1 - \tfrac{2}{3}\eta)|X_k| \text{ iff } k = j \text{ for } k \in [r] \right\}.$$

We now define the final partition of $V(H)$ as

$$V_0 = Y_0 \setminus \bigcup_j Z_j, \quad V_i = Y_i \cup Z_i \text{ for } i \in [r].$$

We claim that the partition $V(H) = V_0 \sqcup V_1 \sqcup \ldots \sqcup V_r$ witnesses (η, r)-extremality of H. We have $|V_0| \leq |Y_0| \leq \frac{1}{30r^r}\eta n$ and $|V_i| \geq |Y_i| \geq (1 - \frac{1}{30r}\eta)\frac{n}{r}$ for $i \in [r]$. Moreover,

$$|V_i| = |Y_i| + |Z_i| \leq |Y_i| + |Y_0| \leq \lceil \tfrac{n}{r} \rceil + \tfrac{1}{30r^r}\eta n \leq (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r}.$$

That proves condition (*a*) of Definition 10. Next we verify condition (*d*). Pick any vertex $v \in V_i$, $i \in [r]$. Regardless of whether $v \in Y_i$ or $v \in Z_i$ we have that $\deg(v, X_j) \geq (1 - \tfrac{2}{3}\eta)|X_j|$ for all $j \neq i$. That yields

$$\begin{aligned}
\deg(v, V_j) &\geq \deg(v, Y_j) \geq \deg(v, X_j) - |Y_0| \geq (1 - \tfrac{2}{3}\eta)|X_j| - |Y_0| \\
&\geq (1 - \tfrac{2}{3}\eta - \tfrac{\eta}{30r^{r-1}})\tfrac{n}{r} \geq (1 - \tfrac{2}{3}\eta - \tfrac{\eta}{30r^{r-1}})(1 + \tfrac{1}{30r}\eta)^{-1}|V_j| \\
&\geq (1 - \eta)|V_j|.
\end{aligned}$$

To prove property (*b*), suppose, without loss of generality, that $H[V_1]$ contains a triangle $t = \{w_1, w_2, w_3\}$. For $i = 2, \ldots, r$ let $A_i = N_H(w_1) \cap N_H(w_2) \cap N_H(w_3) \cap V_i$. By the already proven property (*d*) we have $|A_i| \geq (1 - 3\eta)|V_i| \geq 3\eta r|V_i|$ (the last inequality uses $\eta < \frac{1}{7r}$). Since $n \geq 2r\eta^{-1}$ Lemma 13 now yields that the link $\mathrm{lk}_H(t)$ contains K_3^{r-1} as a subgraph. This is a contradiction to Lemma 5, since $\mathrm{lk}_H(t)$ triangulates a homology sphere of dimension $2r - 1 - 3 = 2(r - 2)$.

Similarly, to prove (*c*), suppose $v \in V_1$ has three distinct neighbors $w_1, w_2, w_3 \in V_1$. Applying Lemma 13 with w_1, w_2, w_3 and $A_i = N_H(v) \cap V_i$ for $i = 2, \ldots, r$, where $|A_i| \geq (1 - \eta)|V_i| \geq 3\eta r|V_i|$, we get that $\mathrm{lk}_H(v)$ contains a K_3^r. This contradicts the fact that $\mathrm{lk}_H(v)$ triangulates a homology $2(r-1)$-sphere.

We now turn to verifying (*e*). Let us start with an auxiliary claim.

Claim. Let $v \in V_0$ be any vertex and suppose $j \in [r]$ is any index such that $\deg(v, X_j) \leq (1 - \frac{2}{3}\eta)|X_j|$. Then $\deg(v, V_j) \leq (1 - \frac{1}{2}\eta)|V_j|$.

Proof. We have

$$
\begin{aligned}
\deg(v, V_j) \quad &\leq \quad \deg(v, Y_j) + |Z_j| \leq \deg(v, X_j) + |Y_0| \\
&\leq \quad (1 - \tfrac{2}{3}\eta)\lceil \tfrac{n}{r} \rceil + \tfrac{1}{30r^r}\eta n \leq (1 - \tfrac{2}{3}\eta + \tfrac{1}{15r^{r-1}}\eta)\tfrac{n}{r} \\
&\leq \quad (1 - \tfrac{2}{3}\eta + \tfrac{1}{15r^{r-1}}\eta)(1 - \tfrac{1}{30r}\eta)^{-1}|V_j| \leq (1 - \tfrac{1}{2}\eta)|V_j|.
\end{aligned}
$$

$\square$

Now suppose that some vertex $v \in V_0$ is not of Type 2. Then, without loss of generality, $\deg(v, V_i) > 3\eta r|V_i|$ for $i = 2, \ldots, r$. Suppose that $\deg(v, V_1) \geq 3$ and let $w_1, w_2, w_3 \in N_H(v) \cap V_1$ be three distinct vertices. We already proved properties (*a*) and (*d*), so we can apply Lemma 13 with w_1, w_2, w_3 and $A_i = N_H(v) \cap V_i$ for $i = 2, \ldots, r$ to conclude that $\mathrm{lk}_H(v)$ contains a K_3^r, a contradiction to Lemma 5. Therefore, $\deg(v, V_1) \leq 2$. By the definition of V_0, there exist an index $j \neq 1$ such $\deg(v, X_j) \leq (1 - \frac{2}{3}\eta)|X_j|$. The above Claim then gives that $\deg(v, V_j) \leq (1 - \frac{1}{2}\eta)|V_j|$. This proves that v is of Type 1. Condition (*e*) follows.

This completes the proof of the lemma. $\square$

Our last lemma says that for among (η, r)-extremal graphs, the graph $J_r(n)$ maximizes any clique function of order up to r (for sufficiently large n). Note that in this part of the proof we do not assume that H triangulates a homology manifold.

Lemma 15. *Let $r \geq 2$ and let F be a clique function of order k, $2 \leq k \leq r$. Set $\eta = \frac{1}{14r^r}$. Then there exists a number m_0 such that the following holds. If H is a graph with $n \geq m_0$ vertices then $F(H) \leq F(J_r(n))$ and equality is attained only when H is r-radical.*

Proof. Let the clique function be $F(G) = c_k e_k(G) + c_{k-1} e_{k-1}(G) + \ldots + c_1 e_1(G) + c_0$. The value of m_0 will be chosen during the proof in such a way that (3.1), (3.2), (3.3), (3.4) and (3.5) are satisfied for all $n \geq m_0$. Among all (η, r)-extremal graphs with n vertices, let us consider a graph H that maximizes $F(H)$. We will show that H is r-radical.

Claim. For each $i, j \in [r]$, $i \neq j$, the bipartite graph $H[V_i, V_j]$ is complete.

Proof. Suppose for a contradiction and without loss of generality that there exist vertices $v_1 \in V_1$ and $v_2 \in V_2$ that do not form an edge. Let us now add that edge

to H. Observe that the modified graph H' is still (η, r)-extremal. We will now find a lower bound for the number of cliques in H' which contain the edge $v_1 v_2$. By condition (d), v_1 and v_2 are both adjacent to at least $(1 - 2\eta)|V_3| \geq (1 - 3\eta)\frac{n}{r}$ vertices v_3 in V_3. In general, given vertices $v_1 \in V_1, v_2 \in V_2, \ldots, v_\ell \in V_\ell$ there are at least $(1 - \ell\eta)|V_{\ell+1}| \geq (1 - (\ell+1)\eta)\frac{n}{r}$ vertices $v_{\ell+1}$ in $V_{\ell+1}$ adjacent to each of $v_1, v_2, \ldots, v_\ell$. This sequential extension gives at least $((1 - k\eta)\frac{n}{r})^{k-2} > (\frac{1}{2} \cdot \frac{n}{r})^{k-2}$ many k-cliques containing both v_1 and v_2 in H'. For each $t = 2, \ldots, k-1$, the number of t-cliques increased by at most n^{t-2}, and the number of vertices did not change. So, in total,

$$
(3.1) \qquad F(H') - F(H) \geq c_k \left(\frac{n}{2r}\right)^{k-2} - \sum_{t=2}^{k-1} |c_t| n^{t-2} > 0 ,
$$

since the coefficients c_t are fixed and $n \geq m_0$ is large enough. This is a contradiction to the assumption that H maximizes F. □

Claim. The set V_0 does not contain any Type 1 vertex.

Proof. Suppose that $v \in V_0$ is a Type 1 vertex. Let g and h be the two indices as in the definition of Type 1 in Definition 10. Since the average size of the sets V_i, $i \in [r]$, is $\frac{n - |V_0|}{r}$, there is an index $j \in [r]$ so that $|V_j| < \frac{n}{r}$. We construct a new graph H' by deleting v (and its incident edges) from V_0 and introducing a new vertex w into the set V_j. We make w adjacent to all the vertices in $\bigcup_{i \in [r] \setminus j} V_i$, and to no other. The modified graph H' is (η, r)-extremal.

The vertex w is contained in at least $\binom{r-1}{k-1} \lfloor (1 - \frac{1}{30r}\eta)\frac{n}{r} \rfloor^{k-1}$ many k-cliques in H'. Indeed, we can choose an arbitrary $(k-1)$-element set $\{p_1, p_2, \ldots, p_{k-1}\} \subset [r] \setminus j$, and this choice gives us at least $\lfloor (1 - \frac{1}{30r}\eta)\frac{n}{r} \rfloor^{k-1}$ choices of vertices $w_1 \in V_{p_1}, \ldots, w_{k-1} \in V_{p_{k-1}}$. By the previous Claim, for any such choice $\{w, w_1, \ldots, w_{k-1}\}$ is a clique.

Let us now upper-bound the number of k-cliques in H containing v. The number of cliques containing v and some other vertex of V_0 is at most $|V_0| \cdot n^{k-2} \leq \frac{1}{30r^r}\eta n^{k-1}$. The number of k-cliques through v and through a vertex from the set V_g is at most $2n^{k-2}$ by the definition of Type 1. By Definition 10(c) and (d), if $k \geq 3$ the number of cliques containing v and at least two vertices from a fixed V_i, $i \in [r]$, is at most $e_2(H[V_i]) \cdot n^{k-3} \leq |V_i| \cdot n^{k-3}$. Therefore the number of k-cliques that touch some of the sets V_i in at least two vertices is upper bounded by n^{k-2}. It remains to upper-bound the number of k-cliques in H through v that contain no vertex from $(V_0 \setminus \{v\}) \cup V_g$, and that intersect each of the sets V_i in at most one vertex. Trivially, this number is at most $\binom{r-1}{k-1} \lceil (1 + \frac{1}{30r}\eta)\frac{n}{r} \rceil^{k-1}$. However, the fact that $\deg(v, V_h) \leq (1 - \frac{1}{2}\eta)|V_h|$ allows us to refine this upper-bound to

$$
\binom{r-2}{k-1} \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil^{k-1} + \binom{r-2}{k-2} \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil^{k-2} (1 - \tfrac{1}{2}\eta) \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil
$$

$$
= \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil^{k-1} \left(\binom{r-1}{k-1} - \tfrac{1}{2}\eta \cdot \binom{r-2}{k-2} \right)
$$

$$
= \binom{r-1}{k-1} \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil^{k-1} (1 - \tfrac{\eta}{2} \cdot \tfrac{k-1}{r-1}) .
$$

Putting these bounds together, we get

$$
\begin{aligned}
& e_k(H') - e_k(H) \\
& \geq \binom{r-1}{k-1} \lfloor (1 - \tfrac{1}{30r}\eta)\tfrac{n}{r} \rfloor^{k-1} - \binom{r-1}{k-1} \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil^{k-1} (1 - \tfrac{\eta}{2} \cdot \tfrac{k-1}{r-1}) \\
& \qquad\qquad - \tfrac{1}{30r^r}\eta n^{k-1} - 3n^{k-2} .
\end{aligned}
$$

Using the inequality $\lfloor (1 - \tfrac{1}{30r}\eta)\tfrac{n}{r} \rfloor > \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil (1 - \tfrac{1}{10r}\eta)$ we can write

$$
\begin{aligned}
& e_k(H') - e_k(H) \\
& > \binom{r-1}{k-1} \lceil (1 + \tfrac{1}{30r}\eta)\tfrac{n}{r} \rceil^{k-1} [(1 - \tfrac{1}{10r}\eta)^{k-1} - 1 + \tfrac{\eta}{2} \cdot \tfrac{k-1}{r-1}] \\
& \qquad\qquad - \tfrac{1}{30r^r}\eta n^{k-1} - 3n^{k-2} .
\end{aligned}
$$

By Bernoulli's inequality the coefficient in the square brackets is at least

$$
1 - \frac{k-1}{10r}\eta - 1 + \frac{\eta}{2} \cdot \frac{k-1}{r-1} > -\frac{k-1}{10r}\eta + \frac{k-1}{2r}\eta = \frac{2}{5} \cdot \frac{k-1}{r}\eta .
$$

That gives

$$
\begin{aligned}
e_k(H') - e_k(H) & > \binom{r-1}{k-1} \frac{n^{k-1}}{r^{k-1}} \cdot \frac{2}{5} \cdot \frac{k-1}{r}\eta - \frac{1}{30r^r}\eta n^{k-1} - 3n^{k-2} \\
& = \frac{1}{r^r}\eta n^{k-1} \left(\tfrac{2}{5}\binom{r-1}{k-1}(k-1)r^{r-k} - \tfrac{1}{30} \right) - 3n^{k-2} \\
& \geq \frac{1}{r^r}\eta n^{k-1}(\tfrac{2}{5} - \tfrac{1}{30}) - 3n^{k-2} = \frac{11}{30r^r}\eta n^{k-1} - 3n^{k-2} .
\end{aligned}
$$

The number of cliques of size t changed by at most n^{t-1} for $t = 2, \ldots, k-1$. That implies

$$
(3.2) \qquad F(H') - F(H) > \tfrac{11}{30r^r}\eta c_k n^{k-1} - 3c_k n^{k-2} - \sum_{t=2}^{k-1} |c_t| n^{t-1} > 0
$$

since $n \geq m_0$ is sufficiently large. That contradicts the maximality of H and proves the claim. $\qquad\square$

Claim. The set V_0 does not contain any Type 2 vertex.

Proof. We proceed similarly as in the previous case. Suppose that $v \in V_0$ is a Type 2 vertex. Let g and h be the two indices as in Definition 10. We delete v from V_0 and introduce a new vertex w in some set V_j, $j \in [r]$ with $|V_j| < \tfrac{n}{r}$ which we make adjacent to all the vertices in $\bigcup_{i \in [r]\setminus\{j\}} V_i$, and to no other. Let H' be the resulting (η, r)-extremal graph. As before, the new vertex w belongs to at least $\binom{r-1}{k-1} \lfloor (1 - \tfrac{1}{30r}\eta)\tfrac{n}{r} \rfloor^{k-1}$ cliques of size k in H'.

Next we upper-bound the number of cliques containing v in H. The number of k-cliques through v and through a vertex from the set $V_g \cup V_h \cup V_0 \setminus \{v\}$ is at most $(|V_g \cap N_H(v)| + |V_h \cap N_H(v)| + |V_0|)n^{k-2} \leq 7\eta n^{k-1}$ by the definition of Type 2. The number of k-cliques through v that touch at least two vertices in some V_i is at most n^{k-2}, as in the previous claim. Last, the number of k-cliques through v that

do not intersect $V_g \cup V_h \cup (V_0 \setminus \{v\})$ and contain at most one vertex from each V_i is upper-bounded by

$$\binom{r-2}{k-1}\lceil(1+\tfrac{1}{30r}\eta)\tfrac{n}{r}\rceil^{k-1} \quad = \quad \frac{r-k}{r-1}\binom{r-1}{k-1}\lceil(1+\tfrac{1}{30r}\eta)\tfrac{n}{r}\rceil^{k-1}$$

(in particular it must be 0 when $k = r$). Proceeding as in the proof of the previous claim we get

$$e_k(H') - e_k(H)$$
$$\geq \quad \binom{r-1}{k-1}\lfloor(1-\tfrac{1}{30r}\eta)\tfrac{n}{r}\rfloor^{k-1} - \tfrac{r-k}{r-1}\binom{r-1}{k-1}\lceil(1+\tfrac{1}{30r}\eta)\tfrac{n}{r}\rceil^{k-1} - 7\eta n^{k-1} - n^{k-2}$$
$$\geq \quad \binom{r-1}{k-1}\lceil(1+\tfrac{1}{30r}\eta)\tfrac{n}{r}\rceil^{k-1}[(1-\tfrac{1}{10r}\eta)^{k-1} - \tfrac{r-k}{r-1}] - 7\eta n^{k-1} - n^{k-2}.$$

The expression in the square brackets is at least

$$1 - \frac{k-1}{10r}\eta - 1 + \frac{k-1}{r-1} > (k-1)\left(\frac{1}{r} - \frac{1}{10r}\eta\right) > \frac{9(k-1)}{10r} \geq \frac{9}{10r}\,.$$

Hence we get

$$e_k(H') - e_k(H) \quad \geq \quad n^{k-1}\left(\frac{9}{10r^k}\binom{r-1}{k-1} - 7\eta\right) - n^{k-2} > \frac{1}{3r^r}n^{k-1} - n^{k-2}\,,$$

where we used $7\eta \leq \frac{1}{2r^r}$, and finally

$$(3.3) \qquad F(H') - F(H) > \frac{1}{3r^r}c_k n^{k-1} - c_k n^{k-2} - \sum_{t=2}^{k-1}|c_t|n^{t-1} > 0\,,$$

because $n \geq m_0$. $\qquad\qquad\square$

Thus, by the three claims above, the vertex set of H is partitioned into sets $V_1, \ldots, V_r$, all pairs of which form complete bipartite graphs. Recall that the graphs $H[V_i]$ are triangle-free and of maximum degree at most 2.

Claim. For each $i \in [r]$, we have $e_2(H[V_i]) = |V_i|$.

Proof. The condition that the maximum degree of $H[V_i]$ is at most 2 implies that $e_2(H[V_i]) \leq |V_i|$. Suppose now that $e_2(H[V_i]) < |V_i|$. We replace the subgraph $H[V_i]$ with the graph consisting of a path with $e_2(H[V_i])$ edges followed by $|V_i| - e_2(H[V_i]) - 1$ isolated vertices. Let H' be the resulting graph. Note that H' is (η, r)-extremal, and since $H[V_i]$ was triangle-free we have $e_\ell(H') = e_\ell(H)$ for all ℓ. Next, we create H'' by adding one edge to $H'[V_i]$, so that we get a longer path or a cycle. We still have that H'' is (η, r)-extremal.

The number of k-cliques increased from H' to H'' by at least $\binom{r-1}{k-2}\lfloor(1-\tfrac{1}{30r}\eta)\tfrac{n}{r}\rfloor^{k-2} \geq (\frac{n}{2r})^{k-2}$. At the same time, the total number of cliques of order $t = 2, \ldots, k-1$ increased by at most n^{t-2}. Hence

$$(3.4) \qquad F(H'') - F(H) = F(H'') - F(H') \geq \left(\frac{n}{2r}\right)^{k-2} - \sum_{t=2}^{k-1}|c_t|n^{t-2} > 0$$

for $n \geq m_0$, a contradiction to the supposed maximality of H. $\qquad\square$

Claim. For each $1 \leq i < j \leq r$, we have $|V_i| - 1 \leq |V_j| \leq |V_i| + 1$.

Proof. Consider the class of all (η, r)-extremal graphs G with sufficiently many vertices which are partitioned into classes $V(G) = V_0(G) \sqcup V_1(G) \sqcup \ldots \sqcup V_r(G)$ which satisfy all the previous claims, i.e. $V_0(G) = \emptyset$, $G[V_i(G), V_j(G)]$ is complete bipartite for $i, j \in [r]$, $i \neq j$, each $G[V_i(G)]$ is triangle-free and $e_2(G[V_i(G)]) = |V_i(G)|$ for $i \in [r]$. Let $\sigma_j(x_1, \ldots, x_r) = \sum_{1 \leq i_1 < \cdots < i_j \leq r} x_{i_1} \cdots x_{i_j}$ denote the j-th elementary symmetric polynomial in r variables. For a graph G in the above class we have

$$e_\ell(G) = \sum_{i=0}^{\ell} \binom{i}{\ell-i} \sigma_i\left(|V_1(G)|, \ldots, |V_r(G)|\right), \quad \ell = 0, \ldots, 2r.$$

It follows that there are constants $c'_k, \ldots, c'_0$ depending only on F and r such that $c'_k = c_k > 0$ and $F(G) = \sum_{i=0}^{k} c'_i \sigma_i(|V_1(G)|, \ldots, |V_r(G)|)$ for all graphs G in the class. Now suppose, without loss of generality, that in the maximizer H we have $|V_1(H)| - |V_2(H)| \geq 2$. Take any graph H' in the same class with parts of size $(|V_1| - 1, |V_2| + 1, |V_3|, \ldots, |V_r|)$. For any numbers $x_1, \ldots, x_r$ have

$$\sigma_j(x_1 - 1, x_2 + 1, x_3, \ldots, x_r) - \sigma_j(x_1, \ldots, x_r) = (x_1 - x_2 - 1)\sigma_{j-2}(x_3, \ldots, x_r)$$

and hence

$$F(H') - F(H) = (|V_1| - |V_2| - 1) \sum_{i=2}^{k} c'_i \sigma_{i-2}(|V_3|, \ldots, |V_r|)$$

$$(3.5) \qquad \geq c'_k \binom{r-2}{k-2} \left(\frac{n}{2r}\right)^{k-2} - \sum_{t=2}^{k-1} |c'_t| \binom{r-2}{t-2} \lceil (1 + \tfrac{1}{30r}\eta)\frac{n}{r}\rceil^{t-2} > 0$$

for sufficiently large $n \geq m_0$, again a contradiction to the maximality of H. $\qquad \square$

The claims above clearly prove the lemma. $\qquad \square$

4. PROOF OF THE MAIN THEOREM

We can now prove Theorem 3. Fix $r \geq k \geq 2$ and a clique function $F(G) = \sum_{i=0}^{k} c_i e_i(G)$ with $c_k > 0$.

Let $\eta = \frac{1}{14r^r}$ and m_0 be the constants provided by Lemma 15 given r and F. Let $\epsilon = \frac{\eta^2}{120r^{r+3}}$ be the constant provided by Lemma 14 given r and η.

Let δ be the constant from Theorem 7 for input parameters r and $\frac{1}{2}\epsilon$. Define $\alpha = \min\{\frac{1}{4}\binom{r}{k}\frac{1}{r^k}\delta, \frac{1}{2}\epsilon\}$ and let β be the constant from Theorem 9 for input r and α. Let m_1 be such that for $n \geq m_1$ we have $(1 - \frac{1}{4}\delta)F(T_r(n)) < F(J_r(n))$. Let m_2 be such that for each $n \geq m_2$ and each n-vertex graph G the condition $F(G) > (1 - \frac{1}{4}\delta)F(T_r(n))$ implies $e_k(G) > (1 - \frac{1}{2}\delta)e_k(T_r(n))$. The existence of m_1 and m_2 follows by observing that for n-vertex graphs G we have $F(G) = c_k e_k(G) + O(n^{k-1})$ and moreover $F(T_r(n)) = c_k \binom{r}{k}\left(\frac{n}{r}\right)^k + O(n^{k-1})$, $F(J_r(n)) = c_k \binom{r}{k}\left(\frac{n}{r}\right)^k + O(n^{k-1})$. Finally let C_r be the constant from Lemma 4. We claim that Theorem 3 holds for $n_0 = \max\{m_0, m_1, m_2, C_r\beta^{-1}, 2r\eta^{-1}\}$.

Suppose H is any graph with $n \geq n_0$ vertices which triangulates a homology $(2r - 1)$-manifold. First, suppose that $F(H) \leq (1 - \frac{1}{4}\delta)F(T_r(n))$. Since $n \geq m_1$ this implies $F(H) < F(J_r(n))$, and the result is proved (in that case, equality is impossible).

That leaves us with the case $F(H) > (1 - \frac{1}{4}\delta)F(T_r(n))$. Since $n \geq m_2$ we get $e_k(H) > (1 - \frac{1}{2}\delta)e_k(T_r(n))$. By Lemma 4 we have $e_{r+1}(H) \leq C_r n^r \leq \beta n^{r+1}$. Theorem 9 now shows that we can remove at most αn^2 edges from H to obtain a K_{r+1}-free subgraph G with the same vertex set. The removal of one edge destroys at most n^{k-2} cliques of size k, therefore

$$e_k(G) \geq e_k(H) - \alpha n^k \geq (1 - \frac{1}{2}\delta)e_k(T_r(n)) - \frac{1}{4}\delta\binom{r}{k}\frac{1}{r^k}n^k \geq (1 - \delta)e_k(T_r(n)),$$

where in the last step we used $e_k(T_r(n)) \geq \frac{1}{2}\binom{r}{k}\frac{n^k}{r^k}$. Theorem 8 now gives $e_2(G) \geq (1 - \delta)e_2(T_r(n))$. By Theorem 7 the graph G is $\frac{1}{2}\epsilon$-close to $T_r(n)$. Since H arises from G by adding at most $\alpha n^2 \leq \frac{1}{2}\epsilon n^2$ edges, we conclude that H is ϵ-close to $T_r(n)$. From Lemma 14, we have that H is (η, r)-extremal. As $n \geq m_0$, Lemma 15 now shows that $F(H) \leq F(J_r(n))$. That ends the proof of the inequality.

If $F(H) = F(J_r(n))$ then by Lemma 15 the graph H is r-radical. Since H triangulates a homology $(2r - 1)$-manifold, Lemma 12 yields that H is isomorphic to $J_r(n)$. That proves the uniqueness part.

5. CONJECTURES

First of all, it is natural to expect that the conclusion of Corollary 2 holds for flag triangulations of any size, not just sufficiently large. For the γ-vector this was conjectured in [17]. Moreover, we conjecture that the extremum is stable, in the sense that if $F(M)$ is sufficiently close to $F(\mathbf{J}_{\frac{d}{2}}(n))$ then M is still a join of cycles of total length n and of individual lengths close to $\frac{n}{d/2}$, but not necessarily all equal (see also [2, Conjecture 5.1]).

As mentioned in the introduction, Gal [10] conjectures that for flag homology spheres M the γ-vector $\gamma(M)$ is non-negative. This is known to be true in a number of special cases (see [14, 18, 19] and the references therein). One method of showing non-negativity is to exhibit a simplicial complex of which $\gamma(M)$ is the f-vector. In particular, Nevo and Petersen [18, Problem 6.4] asked if for every n-vertex flag homology sphere M there exists a graph G such that the γ-vector of M is the clique vector of G. Our result $\gamma_i(M) \leq \gamma_i(\mathbf{J}_r(n))$ supports this claim in odd dimension $2r - 1$. Indeed, one checks that $\gamma_1(M) = n - 4r$ and $\gamma_i(\mathbf{J}_r(n)) = e_i(T_r(n - 4r))$. If the conjectural graph G exists, then it is K_{r+1}-free, has $n - 4r$ vertices, and thus by Zykov's theorem $\gamma_i(M) = e_i(G) \leq e_i(T_r(n - 4r)) = \gamma_i(\mathbf{J}_r(n))$, which is what we showed.

Having proved that $F(M) \leq F(J_r(n))$ it is tempting to conjecture, for the classical enumeration vectors (i.e. the f-, h-, g- or γ-vector), a generalization in the spirit of Theorem 8. We pose this as an open problem.

Problem 16. Let $(v_1, \ldots, v_r)$ be any of $(f_0, \ldots, f_{r-1})$, $(h_1, \ldots, h_r)$, $(g_1, \ldots, g_r)$ or $(\gamma_1, \ldots, \gamma_r)$. Is it true that for sufficiently large n the inequalities

$$\text{(5.1)} \qquad 1 = \frac{v_1(M)}{v_1(\mathbf{J}_r(n))} \geq \frac{v_2(M)}{v_2(\mathbf{J}_r(n))} \geq \ldots \geq \frac{v_r(M)}{v_r(\mathbf{J}_r(n))}$$

hold for any flag homology $(2r - 1)$-manifold (or sphere) M with n vertices?

If $v_i = \gamma_i$ and M is a homology $(2r - 1)$-sphere the positive answer to Problem 16 would follow directly from the conjecture of Nevo and Petersen mentioned

earlier. In this case the inequalities (5.1) are equivalent to those of Theorem 8 for some graph G such that $\gamma(M) = e(G)$.

It is very likely that $\mathbf{J}_n(r)$ is the maximizer of face numbers for a wider class of $(2r - 1)$-dimensional *flag weak pseudomanifolds*. A weak pseudomanifold of dimension $d - 1$ is a pure $(d - 1)$-dimensional simplicial complex in which every face of dimension $d - 2$ belongs to exactly two maximal faces. For $X(G)$ the condition translates to saying that every maximal clique in G has size d and for every clique σ of size $d - 1$ the link $\mathrm{lk}_G(\sigma)$ consists of two isolated vertices.

Problem 17. Let $r \geq 2$. Is it true that:

  (*i*) for every n-vertex flag weak $(2r - 1)$-pseudomanifold M with n sufficiently large we have $f_i(M) \leq f_i(\mathbf{J}_r(n))$ for $i = 1, \ldots 2r - 1$? This is open even for $i = 1$.
  (*ii*) for every $\beta > 0$ there is a constant n_0 such that for every flag weak $(2r - 1)$-pseudomanifold M with $n \geq n_0$ vertices we have $f_r(M) \leq \beta n^{r+1}$?

Recall that for homology manifolds condition (*ii*), guaranteed by Lemma 4 and ultimately by the middle Dehn–Sommerville equation, is the weakest possible assumption which allows us to initiate the stability method for dense graphs. The first author proved in [1] that for families of flag weak $(2r - 1)$-pseudomanifolds which satisfy a stronger condition $f_r(M) \leq Cn^r$ for some fixed C we have $f_1(M) \leq f_1(\mathbf{J}_r(n))$ for sufficiently large n.

In even dimensions the situation seems to be more complicated. For $r \geq 1$ let $J_r^*(n)$ be the graph obtained from $J_r(n - 2)$ by adding two new vertices adjacent to all of $J_r(n - 2)$. Then the clique complex $\mathbf{J}_r^*(n) := X(J_r^*(n))$ is a flag simplicial $2r$-sphere.

Conjecture 18. *Fix $r \geq 2$. For every flag homology $2r$-sphere M with n vertices we have $f_i(M) \leq f_i(\mathbf{J}_r^*(n))$ for $i = 0, \ldots, 2r$ and $\gamma_i(M) \leq \gamma_i(\mathbf{J}_r^*(n))$ for $i = 0, \ldots, r$.*

This statement is obviously true when $r = 1$, since $(f_0(M), f_1(M), f_2(M)) = (n, 2n - 4, 3n - 6)$ and all the inequalities are equalities for every M. Also for $r \geq 2$ the conjecture (if true) cannot be augmented by a uniqueness statement. To see this, consider the subgraph of $J_r^*(n)$ induced by $V_i \cup \{a, b\}$, where V_i is any part of $V(J_r(n - 2)) = V_1 \sqcup \cdots \sqcup V_r$ and a, b are the two additional vertices. It is a flag triangulation of S^2 as the suspension of a cycle. Upon replacing this subgraph by any other flag triangulation of S^2 with the same number of vertices one gets a flag simplicial $2r$-sphere with the same face numbers as $\mathbf{J}_r^*(n)$.

REFERENCES

[1] M. Adamaszek. An upper bound theorem for a class of flag weak pseudomanifolds, 2013. arXiv:1303.5603.

[2] M. Adamaszek and J. Hladký. Dense flag triangulations of 3-manifolds via extremal graph theory. *Trans. Amer. Math. Soc.*, 367(4):2743–2764, 2015.

[3] N. Aisbett. Frankl–Füredi–Kalai inequalities on the γ-vectors of flag nestohedra. *Discrete & Computational Geometry*, 51(2):323–336, 2014.

[4] R. Charney. Metric geometry: connections with combinatorics. *Proceedings of FPSCA Conference, DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, 24:55–69, 1996.

[5] R. Charney and M. Davis. The Euler characteristic of a non-positively curved, piecewise Euclidean manifold. *Pacific J. Math.*, 171:117–137, 1995.

[6] P. Erdős. On some new inequalities concerning extremal properties of graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 77–81. Academic Press, New York, 1968.

[7] P. Erdős, P. Frankl, and V. Rödl. The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent. *Graphs Combin.*, 2(2):113–121, 1986.

[8] A. Flores. Über die Existenz n-dimensionaler Komplexe die nicht im den $\mathbb{R}^{2n}$ topologisch einbettar sind. *Ergeb. Math. Kolloq.*, 5:17–24, 1933.

[9] A. Frohmader. Face vectors of flag complexes. *Israel J. Math.*, 164:153–164, 2008.

[10] Ś. R. Gal. Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.*, 34(2):269–284, 2005.

[11] D. E. Galewski and R. J Stern. Classification of simplicial triangulations of topological manifolds. *Annals of Mathematics*, 111:1–34, 1980.

[12] A. Goodarzi. Convex hull of face vectors of colored complexes. *European Journal of Combinatorics*, 36:247–250, 2014.

[13] M. Gromov. Hyperbolic groups. In *Essays in Group Theory (S. M. Gersten, M.S.R.I. Publ. 8 eds.)*, pages 75–264. Springer–Verlag, 1987.

[14] K. Karu. The cd-index of fans and posets. *Compositio Mathematica*, 142(3):701–718, 2006.

[15] J. R. Munkres. Topological results in combinatorics. *Michigan Math. J.*, 31(1):113–128, 1984.

[16] S. Murai and E. Nevo. On the cd-index and γ-vector of S*-shellable CW-spheres. *Mathematische Zeitschrift*, 271(3-4):1309–1319, 2012.

[17] E. Nevo and F. Lutz. Stellar theory for flag complexes, 2013. arxiv:1302.5197, to appear in Math. Scand.

[18] E. Nevo and T K. Petersen. On γ-vectors satisfying the Kruskal–Katona inequalities. *Discrete & Computational Geometry*, 45(3):503–521, 2011.

[19] E. Nevo, T. K. Petersen, and B. E. Tenner. The γ-vector of a barycentric subdivision. *Journal of Combinatorial Theory, Series A*, 118(4):1364–1380, 2011.

[20] I. Novik. Upper bound theorems for homology manifolds. *Israel Journal of Mathematics*, 108(1):45–82, 1998.

[21] I. Z. Ruzsa and E. Szemerédi. Triple systems with no six points carrying three triangles. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 939–945. North-Holland, Amsterdam-New York, 1978.

[22] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.

[23] R. P Stanley. The upper bound conjecture and cohen-macaulay rings. *Stud. Appl. Math*, 54:135–142, 1975.

[24] R.P. Stanley. *Combinatorics and Commutative Algebra*. Progress in Mathematics. Birkhäuser Boston, 2004.

[25] E.R. van Kampen. Komplexe in euklidischen räumen. *Abh. Math. Sem. Univ. Hamburg*, 9:72–78, 1932.

[26] U. Wagner. Minors in random and expanding hypergraphs. *Proc. 27th Annual ACM Symposium on Computational Geometry (SoCG)*, pages 351–360, 2011.

[27] A. A. Zykov. On some properties of linear complexes. *Mat. Sbornik N.S.*, 24(66):163–188, 1949.

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