DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **35**, Number **6**, June **2015**

doi:10.3934/dcds.2015.35.2405

pp. **2405–2421**

WEAK DIFFERENTIABILITY OF SCALAR HYSTERESIS OPERATORS

Martin Brokate

Fakultät für Mathematik, TU München
Boltzmannstr. 3
D 85747 Garching bei München, Germany

Pavel Krejčí

Institute of Mathematics, Academy of Sciences
Žitná 25
CZ 11567 Praha 1, Czech Republic

Abstract. Rate independent evolutions can be formulated as operators, called hysteresis operators, between suitable function spaces. In this paper, we present some results concerning the existence and the form of directional derivatives and of Hadamard derivatives of such operators in the scalar case, that is, when the driving (input) function is a scalar function.

1. **Introduction.** By standard terminology, a **hysteresis operator** $\mathcal{P}$ maps functions u defined on a time interval $[a,b]$ to functions $w = \mathcal{P}[u]$ defined on the same interval and has the property of **rate independence**, that is,

$$\mathcal{P}[u \circ \sigma] = (\mathcal{P}[u]) \circ \sigma \tag{1}$$

holds for a certain class of time transformations $\sigma : [a,b] \to [a,b]$, as well as the **Volterra** property, that is, $\mathcal{P}[u](t)$ depends only upon the values of u on $[a,t]$. Usually, $\mathcal{P}$ is parametrized by some initial value w_0 which represents the initial state of the system described by $\mathcal{P}$; we then write $w = \mathcal{P}[u; w_0]$.

Hysteresis operators may be specified explicitly, or they may arise implicitly as solution operators of rate independent evolutions. The simplest example of the former is the relay with two values $w(t) = \pm 1$ which switches from +1 to -1 or vice versa, according to whether the scalars $u(t)$ pass certain thresholds α resp. β. An example of the latter is the solution operator (the so-called **stop operator**) $(u; z_0) \mapsto z$ of the evolution variational inequality

$$\begin{aligned} \langle \dot z - \dot u \,,\, z - \zeta \rangle \le 0 \quad &\text{for all } \zeta \in Z, \text{ a.e. in } [a,b], \\ z(t) \in Z \quad \text{for all } t \in [a,b], &\quad z(a) = z_0 \in Z\,, \end{aligned} \tag{2}$$

where $u, z : [a,b] \to \mathbb{R}^m$ and $Z \subset \mathbb{R}^m$ is a closed convex constraint. It was introduced, in an equivalent formulation as a differential inclusion termed **sweeping process** (processus du rafle), by Moreau in [1, 2]. The **play operator** $w = \mathcal{P}[u; w_0]$ is related to (2) by

$$w(t) + z(t) = u(t)\,, \quad w_0 + z_0 = u(0)\,. \tag{3}$$

2010 *Mathematics Subject Classification.* Primary: 47J40, 47T20; Secondary: 34C55, 49J52.
Key words and phrases. Hysteresis, rate independence, differentiability, play, Preisach, Prandtl-Ishlinskii, evolution variational inequalities, accumulated maximum, gliding maximum.

The stop and the play operator serve as an important special case (the case of quadratic energy) in the class of rate independent evolutions treated in [3] for which the concept of an **energetic solution** has been developed [4].

The scalar play operator $\mathcal{P}_r$ with $m = 1$ and $Z = [-r, r]$, $r \geq 0$, was considered as an operator between function spaces for the first time in [5], where the Lipschitz continuity of $\mathcal{P}_r : C[a, b] \times \mathbb{R} \to C[a, b]$ has been proved. However, simple examples show that the play operator does not possess a classical (Fréchet) derivative. The question therefore arises whether the play operator is differentiable in a weaker sense. In the scalar case $m = 1$ we investigate the existence and some properties of the directional derivative, that is, of the limit

$$\lim_{\lambda \downarrow 0} \frac{\mathcal{P}_r[u + \lambda h; w_0 + \lambda q] - \mathcal{P}_r[u; w_0]}{\lambda}\,. \tag{4}$$

We prove that the scalar play operator is differentiable in the sense of Hadamard, if we weaken the norm in the range space, and that the limit (4) is a regulated function. Moreover, it is of bounded variation whenever the same holds for the function h.

The main idea we exploit to prove weak differentiability of the scalar play operator is the following. Near a given input $u \in C[a, b]$ we can represent the play operator $\mathcal{P}_r$ as a finite concatenation of accumulated maxima of the type

$$(Fu)(t) = \max_{s \in [a,t]} u(s)$$

which, in turn, is a convex real-valued functional on $C[a, b]$ for every fixed t. From the weak differentiability of the latter we obtain successively those of the accumulated maximum and of the play. This is done in Sections 3, 4 and 5 of the paper. Section 6 deals with the regularity of the limit (4) as a function of time.

It is well known [8, 9, 10] that, on the basis of the play operator, other rate independent operators can be constructed, for example the Prandtl-Ishlinskii and the Preisach operator. They allow for a flexible modelling of complex hysteresis behaviour, including e.g. nested hysteresis loops, and have found to be useful in various areas of mechanical and electrical engineering.

Below, we extend our results concerning the play operator to the Prandtl-Ishlinskii as well as the Preisach operator. This is done in Sections 7 and 8 of the paper.

The basic properties of scalar hysteresis operators have been studied quite some time ago, see the monographs [6, 7, 8, 9, 10]. For convenience, we mainly refer to [9] in this paper.

Let us close this introduction with two remarks.

Firstly, regulated functions constitute a rather large class of functions on which the play operator itself (and certain generalizations of it) are defined in a natural manner, see e.g. [11, 12], and this is useful in various contexts. In [13] regulated functions appear as a rate independent singular limit of a certain ODE under irregular oscillatory forcing, which can be interpreted as a limit under wbo-convergence [14]. For PDE's with hysteresis, however, regulated functions do not seem to be used so far.

Secondly, properties of the limit (4) are of immediate relevance when trying to compute a gradient of the "control to state" mapping in an optimal control problem whose dynamics involve the play operator. If, on the other hand, one is interested in the properties of an expression like

$$\frac{\mathrm{d}}{\mathrm{d}t} E(w(t))\,, \quad w = \mathcal{P}_r[u; w_0]\,,$$

for some energy functional E, the concept of a "chain rule inequality" has proved its usefulness in various contexts, which can be rather general, see e.g. [15].

2. **Notions of derivatives.**

We collect some classical notions of derivatives for mappings

$$F : U \to Y\,, \quad U \subset X\,,$$

where X and Y are normed spaces, and U is an open subset of X.

Definition 2.1. (i) The limit, if it exists,

$$F'(u; h) := \lim_{\lambda \downarrow 0} \frac{F(u + \lambda h) - F(u)}{\lambda}\,, \quad u \in U\,,\ h \in X\,, \tag{5}$$

is called the **directional derivative** of F at u in the direction h. It is an element of Y.
(ii) If the directional derivative satisfies

$$F'(u; h) = \lim_{\lambda \downarrow 0} \frac{F(u + \lambda h + r(\lambda)) - F(u)}{\lambda} \tag{6}$$

for all functions $r : [0, \lambda_0) \to X$ with $r(\lambda)/\lambda \to 0$ as $\lambda \to 0$, it is called the **Hadamard derivative** of F at u in the direction h.
(iii) If the directional derivative exists for all $h \in X$ and satisfies

$$\lim_{h \to 0} \frac{\|F(u + h) - F(u) - F'(u; h)\|}{\|h\|} = 0\,, \tag{7}$$

it is called the **Bouligand derivative** of F at u in the direction h.
(iv) If the Bouligand derivative has the form $F'(u; h) = Lh$ for some linear continuous mapping $L : X \to Y$, then L is called the **Fréchet derivative** of F at u.
(v) The mapping F is called directionally (resp. Hadamard, Bouligand, Fréchet) differentiable at u (resp. in U), if the corresponding derivative exists at u (resp. for all $u \in U$) for all directions $h \in X$. □

These notions are classical, but the terminology is not uniform in the literature. The following well known facts are elementary consequences of the definitions.

Lemma 2.2. *If F is directionally differentiable and locally Lipschitz continuous at $u \in U$, then it is Hadamard differentiable at u.* □

Lemma 2.3. *If F_1 and F_2 are Hadamard differentiable at u resp. $F_1(u)$, then $F_2 \circ F_1$ is Hadamard differentiable at u, and the chain rule*

$$(F_2 \circ F_1)'(u; h) = F_2'(F_1(u); F_1'(u; h)) \tag{8}$$

holds for all $h \in X$. □

3. **The maximum functional.** For $X = C[a,b]$, equipped with the maximum norm, we consider $\varphi : X \to \mathbb{R}$,

$$\varphi(u) = \max_{s\in[a,b]} u(s)\,. \tag{9}$$

It is well known (see e.g. [16]) that φ is directionally differentiable on X and that

$$\varphi'(u;h) = \max_{s\in M(u)} h(s)\,, \tag{10}$$

where

$$M(u) = \{\tau \in [a,b],\, u(\tau) = \varphi(u)\} \tag{11}$$

is the set where u attains its maximum. Moreover, φ is Hadamard differentiable due to Lemma 2.2, since φ is globally Lipschitz continuous with Lipschitz constant 1.

The following example shows that φ is not Bouligand differentiable on $C[a,b]$.

Example 3.1. Consider $u : [0,1] \to \mathbb{R}$ defined by $u(s) = 1 - s$. We have $\varphi(u) = 1$ and $M(u) = \{0\}$. Define $h_\lambda : [0,1] \to \mathbb{R}$ for $\lambda > 0$ by

$$h_\lambda(s) = \begin{cases} 2s\,, & s \le \lambda\,,\\ 2\lambda\,, & s > \lambda\,. \end{cases} \tag{12}$$

Then the function $u + h_\lambda$ attains its maximum at $s = \lambda$, and

$$\|h_\lambda\|_\infty = 2\lambda\,, \quad \varphi(u+h_\lambda) = 1+\lambda\,, \quad \varphi'(u;h_\lambda) = \max_{s\in M(u)} h_\lambda(s) = h_\lambda(0) = 0\,.$$

Consequently,

$$\frac{|\varphi(u+h_\lambda) - \varphi(u) - \varphi'(u;h_\lambda)|}{\|h_\lambda\|_\infty} = \frac{\lambda}{2\lambda} = \frac{1}{2}\,. \tag{13}$$

Thus, $\varphi : C[0,1] \to \mathbb{R}$ is not Bouligand differentiable at u. □

In order to treat ascending parts of the play operator, we will need the slightly more elaborate functional given by

$$\psi_+(u,p) = \max\{p\,,\, \max_{s\in[a,b]}(u(s) - r)\}\,, \tag{14}$$

where $r \ge 0$ is a fixed number.

Proposition 3.2. *For $X = C[a,b] \times \mathbb{R}$, the functional $\psi_+ : X \to \mathbb{R}$ given by (14) is Hadamard differentiable on X, and*

$$\psi'_+((u,p);(h,q)) = \max_{s\in M(u)} h(s) \tag{15}$$

if $\varphi(u) - r > p$, or if $\varphi(u) - r = p$ and $\max_{s\in M(u)} h(s) > q$,

$$\psi'_+((u,p);(h,q)) = 0\,, \quad \textit{otherwise,} \tag{16}$$

where as above $\varphi(u) = \max_{[a,b]} u$.

Proof. Since

$$\psi_+(u,p) = \max\{0\,,\, \max_{s\in[a,b]}(u(s) - r - p)\} + p\,,$$

we may write

$$\psi_+(u,p) - p = (g \circ \varphi \circ L)(u,p)\,,$$

where $g(x) = \max\{x, 0\}$, $g : \mathbb{R} \to \mathbb{R}$, denotes the positive part, and $L : C[a,b] \times \mathbb{R} \to C[a,b]$ is the continuous affine linear mapping given by $L(u,p) = u - r - p$. As g, φ, L are Hadamard differentiable on their respective domains, we conclude from

Lemma 2.3 that ψ_+ is Hadamard differentiable. Applying the chain rule (8) twice, we obtain the formula for the derivative as follows. As $L'((u,p);(h,q)) = h-q$ and $M(L(u,p)) = M(u)$, we get

$$(\varphi \circ L)'((u,p);(h,q)) = \varphi'(L(u,p);h-q) = \max_{\tau\in M(u)} (h-q)(\tau) = \max_{\tau\in M(u)} h(\tau) - q\,.$$

We have $g'(x;\alpha) = \alpha$ if $x>0$, or if $x=0$ and $\alpha>0$, and $g'(x;\alpha)=0$ otherwise. Setting $x = L(u,p) = u-r-p$, a second application of the chain rule yields

$$(g\circ\varphi\circ L)'((u,p);(h,q)) = \max_{\tau\in M(u)} h(\tau) - q$$

for pairs (u,p) as indicated in the assertion, whence the claim follows. □

For descending parts of the play operator, we work with

$$\psi_-(u,p) = \min\{p\,,\ \min_{s\in[a,b]}(u(s)+r)\}\,. \tag{17}$$

Since

$$\psi_-(u,p) = -\psi_+(-u,-p)\,, \tag{18}$$

we conclude from Proposition 3.2 that ψ_-, too, is Hadamard differentiable. Setting

$$m(u) = \{\tau\in[a,b],\ u(\tau) = \min_{s\in[a,b]} u(s)\}\,, \tag{19}$$

we obtain from (15), (16) and (18) that

$$\psi_-'((u,p);(h,q)) = \min_{s\in m(u)} h(s) \tag{20}$$

if $\min_{[a,b]} u+r<p$, or if $\min_{[a,b]} u+r = p$ and $\min_{s\in m(u)} h(s) < q$,

$$\psi_-'((u,p);(h,q)) = 0\,,\quad \text{otherwise.} \tag{21}$$

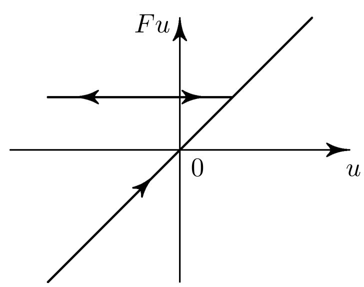

FIGURE 1. The accumulated maximum function

4. **The accumulated maximum.** We define the accumulated (or "gliding") maximum of a function $u\in C[a,b]$ as

$$\varphi_t(u) = \max_{s\in[a,t]} u(s)\,,\quad t\in[a,b]\,. \tag{22}$$

Setting

$$(Fu)(t) = \varphi_t(u) \tag{23}$$

we obtain an operator

$$F : C[a,b] \to C[a,b]\,. \tag{24}$$

Obviously, the function Fu is nondecreasing for every $u\in C[a,b]$. Since

$$|\varphi_t(u) - \varphi_t(v)| \le \max_{s\in[a,t]} |u(s)-v(s)|\,,\quad \text{for all } u,v\in C[a,b]\,,$$

we have

$$\|Fu - Fv\|_\infty \le \|u - v\|_\infty \,, \quad \text{for all } u, v \in C[a,b]\,. \tag{25}$$

For any fixed $t \in [a,b]$, the directional derivative of $\varphi_t : C[a,b] \to \mathbb{R}$ given in (10) yields that, for all $u, h \in C[a,b]$,

$$F^{PD}(u;h)(t) := \lim_{\lambda \downarrow 0} \frac{(F(u+\lambda h))(t) - (Fu)(t)}{\lambda} = \varphi_t'(u;h) = \max_{s \in M_t(u)} h(s)\,, \tag{26}$$

where

$$M_t(u) = \{\tau \in [a,t],\, u(\tau) = \varphi_t(u)\} \tag{27}$$

is the set where u attains its maximum on $[a,t]$. We call **pointwise directional derivative of F** the function $F^{PD}(u;h) : [a,b] \to \mathbb{R}$ obtained in this manner.

As the following example shows, $F^{PD}(u;h)$ in general does belong neither to $C[a,b]$ nor to $BV[a,b]$, the space of functions $u : [a,b] \to \mathbb{R}$ of bounded variation.

Example 4.1. On $[a,b] = [0,3]$, we consider the function u defined by

$$u(t) = \begin{cases} 1 - t\,, & t \in [0,1]\,, \\ t - 1\,, & t \in [1,3]\,. \end{cases}$$

We have

$$M_t(u) = \begin{cases} \{0\}\,, & t < 2\,, \\ \{0,2\}\,, & t = 2\,, \\ \{t\}\,, & t > 2\,. \end{cases}$$

According to (26), for every $h \in C[0,3]$ we get

$$F^{PD}(u;h)(t) = \max_{s \in M_t(u)} h(s) = \begin{cases} h(0)\,, & t < 2\,, \\ \max\{h(0), h(2)\}\,, & t = 2\,, \\ h(t)\,, & t > 2\,. \end{cases}$$

We observe that at $t = 2$, $F^{PD}(u;h)$ is right but not left continuous if $h(0) < h(2)$, left but not right continuous if $h(0) > h(2)$. Moreover, if the variation of h on $[2,3]$ is unbounded, the same is true for $F^{PD}(u;h)$. $\square$

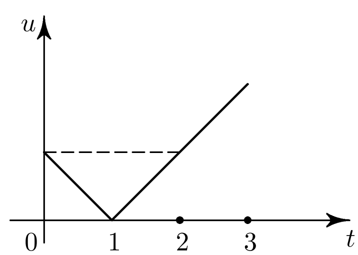

FIGURE 2. An illustration to Example 4.1

The regularity of $F^{PD}(u;h)$ in time exhibited in the foregoing example is typical. We refer to Proposition 6.8 below for a general result.

The example above also illustrates the fact that the convergence in (26) need not be uniform in t, because in that case the function $F^{PD}(u;h)$ has to be continuous, being the uniform limit of continuous functions. Thus, $F : C[a,b] \to Y$ is not directionally differentiable on $C[a,b]$ if we choose $Y = C[a,b]$, endowed with the maximum norm.

Proposition 4.2. *The accumulated maximum* $F : C[a,b] \to L^p(a,b)$ *is Hadamard differentiable for every* $1 \le p < \infty$.

Proof. For arbitrary $u, h \in C[a,b]$, due to (25) the difference quotients in (26) satisfy

$$\left\| \frac{(F(u+\lambda h)) - Fu}{\lambda} \right\|_\infty \le \frac{\|\lambda h\|_\infty}{|\lambda|} = \|h\|_\infty$$

and therefore converge in the norm of L^p for $\lambda \to 0$, by dominated convergence. Thus, F is directionally differentiable. By Lemma 2.2, F is Hadamard differentiable, as $F : C[a,b] \to L^p(a,b)$ is Lipschitz continuous due to (25). $\square$

Not surprisingly, the accumulated maximum

$$F : C[a,b] \to L^p(a,b)$$

is not Bouligand differentiable, no matter how p is chosen.

Example 4.3. As for the maximum, the functions $u, h_\lambda : [0,1] \to \mathbb{R}$ given by

$$u(s) = 1 - s\,, \quad h_\lambda(s) = \begin{cases} 2s\,, & s \le \lambda\,, \\ 2\lambda\,, & s > \lambda\,, \end{cases} \tag{28}$$

furnish a counterexample. We have

$$\varphi_t(u) = 1\,, \quad M_t(u) = \{0\}\,, \quad \varphi_t'(u; h_\lambda) = 0$$

for all t as before, as well as

$$\varphi_t(u + h_\lambda) = 1 + \lambda\,, \quad t \ge \lambda\,.$$

Therefore

$$\begin{aligned} \|F(u+h_\lambda) - F(u) - F^{PD}(u;h_\lambda)\|_p^p &\ge \int_\lambda^1 |\varphi_\tau(u+h_\lambda) - \varphi_\tau(u) - \varphi_\tau'(u;h_\lambda)|^p \,\mathrm{d}\tau \\ &= (1-\lambda)\lambda^p \end{aligned}$$

and thus

$$\frac{\|F(u+h_\lambda) - F(u) - F^{PD}(u;h_\lambda)\|_p}{\|h_\lambda\|_\infty} \ge (1-\lambda)^{1/p} \lambda \frac{1}{2\lambda} \to \frac{1}{2} \qquad \text{as } \lambda \downarrow 0\,. \tag{29}$$

Therefore, $F : C[a,b] \to L^p(a,b)$ is not Bouligand differentiable at u. $\square$

5. **The scalar play operator.** The original construction of the play operator resp. its “twin”, the stop operator, in [5], is based on piecewise monotone functions. A continuous function $u : [a,b] \to \mathbb{R}$ is called **piecewise monotone**, if the restriction of u to each interval $[t_i, t_{i+1}]$ of a suitably chosen partition $\Delta = \{t_i\}$, $a = t_0 < t_1 < \cdots < t_N = b$, then called a **monotonicity partition** of u, is either nondecreasing or nonincreasing. By $C_{pm}[a,b]$ we denote the space of all such functions.

For arbitrary $r \ge 0$, the play operator $\mathcal{P}_r$ is constructed as follows. (For more details, we refer to section 2.3 of [9].) Given a function $u \in C_{pm}[a,b]$ and an initial value $w_0 \in \mathbb{R}$, we define a function $w : [a,b] \to \mathbb{R}$ successively on the intervals $[t_i, t_{i+1}]$ of a monotonicity partition Δ of u by

$$\begin{aligned} w(a) &= \max\{u(a) - r\,,\, \min\{u(a) + r, w_0\}\}\,, \\ w(t) &= \max\{u(t) - r\,,\, \min\{u(t) + r, w(t_i)\}\}\,, \quad t_i < t \le t_{i+1}\,, \quad 0 \le i < N\,. \end{aligned} \tag{30}$$

Note that $w(a) = w_0$ if $|u(a) - w_0| \le r$. In this manner, we obtain an operator

$$w = \mathcal{P}_r[u; w_0]\,, \quad \mathcal{P}_r : C_{pm}[a,b] \times \mathbb{R} \to C_{pm}[a,b]\,.$$

It satisfies

$$\max_{s\in[a,t]} |\mathcal{P}_r[u;w_0](s) - \mathcal{P}_r[v;y_0](s)| \le \max\{\max_{s\in[a,t]} |u(s)-v(s)|\,,\, |w_0-y_0|\} \tag{31}$$

for all $t \in [a,b]$, $u,v \in C_{pm}[a,b]$ and $w_0, y_0 \in \mathbb{R}$. Therefore, $\mathcal{P}_r$ can be uniquely extended to a Lipschitz continuous operator

$$\mathcal{P}_r : C[a,b] \times \mathbb{R} \to C[a,b] \tag{32}$$

which satisfies

$$\|\mathcal{P}_r[u;w_0] - \mathcal{P}_r[v;y_0]\| \le \max\{\|u-v\|\,,\, |w_0-y_0|\}\,. \tag{33}$$

For $r=0$, $\mathcal{P}_0$ reduces to the identity, $\mathcal{P}_0[u;w_0] = u$.

The trajectories $\{(u(t),w(t)) : t\in[a,b]\}$ lie within the subset $A = \{|u-w| \le r\}$ of the plane $\mathbb{R}^2$ whose boundary consists of the straight lines $u - w = \pm r$.

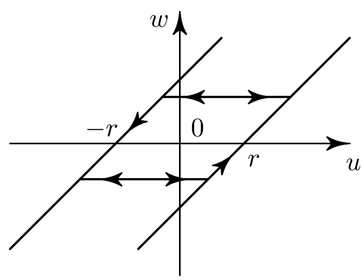

FIGURE 3. The scalar play operator

Let $(u,w_0) \in C[a,b]\times\mathbb{R}$ be given, let $w = \mathcal{P}_r[u;w_0]$, $r>0$, and consider the sets of times

$$I_0 = \{t\in[a,b] : |u(t)-w(t)| < r\}\,,$$
$$I_+ = \{t\in[a,b] : u(t)-w(t) = r\}\,, \quad I_- = \{t\in[a,b] : u(t)-w(t) = -r\}$$

where the trajectory lies in the interior of A resp. on the right resp. on the left part of ∂A. It is intuitively clear that w should be nondecreasing on $I_+\cup I_0$ and nonincreasing on $I_-\cup I_0$.

Lemma 5.1. *Let $I = [\alpha,\beta] \subset I_+\cup I_0$. Then $w = \mathcal{P}_r[u;w_0]$ satisfies*

$$w(t) = \psi_+(u(t), w(\alpha); t, \alpha) \tag{34}$$

for all $t\in I$, where

$$\psi_+(u,p;t,\alpha) = \max\{p\,,\, \max_{s\in[\alpha,t]} (u(s)-r)\}\,. \tag{35}$$

In particular, w is nondecreasing on I.

Proof. Since $w(\alpha) \ge u(\alpha) - r$, (34) obviously holds for $t=\alpha$. Now let $t\in(\alpha,\beta]$ be arbitrary. On $[\alpha,t]$, let $\{u_n\}$ be a sequence of piecewise linear interpolants of u with $u_n(\alpha) = u(\alpha)$ satisfying $u_n \to u$ uniformly, and set $u_n = u$ on $[a,\alpha]$. Then $w_n = \mathcal{P}_r[u_n;w_0] \to w$ uniformly, and $u_n - w_n > -r$ on I for n sufficiently large. Therefore, if $\Delta = \{\tau_i\}$ is a monotonicity partition for u_n on I, we have

$$w_n(s) = \max\{u_n(s) - r\,,\, w_n(\tau_i)\}\,, \quad s\in[\tau_i,\tau_{i+1}]\,,$$

and it follows by induction over i that

$$w_n(t) = \psi_+(u_n(t), w_n(\alpha); t, \alpha)\,.$$

Letting $n \to \infty$ we obtain (35). □

In an analogous manner, if $I = [\alpha, \beta] \subset I_- \cup I_0$ we obtain that w is nonincreasing on I and satisfies

$$w(t) = \psi_-(u(t), w(\alpha); t, \alpha) \tag{36}$$

for all $t \in I$, where

$$\psi_-(u, p; t, \alpha) = \min\{p\,, \min_{s\in[\alpha,t]}(u(s) + r)\}\,. \tag{37}$$

We now construct a specific monotonicity partition Δ_w for w which consists of intervals of the type encountered in the foregoing lemma. Set

$$I_0 = \{t \in [a, b] : |u(t) - w(t)| < r\}\,, \quad I_= = \{t \in [a, b] : u(t) = w(t)\}\,. \tag{38}$$

For any $t \in [a, b]$, define

$$\tau_\pm(t) = \min(\{s \in [t, b] : s \in I_\pm\} \cup \{b\})\,. \tag{39}$$

Set $\tau_0 = t_0 = a$. If I_+ or I_- are empty, set $t_1 = b$ and $\Delta_w = \{t_0, t_1\} = \{a, b\}$. Otherwise, either $\tau_+(\tau_0) < \tau_-(\tau_0)$ holds, or vice versa. In the former case we set $\tau_1 = \tau_-(\tau_0)$ and

$$t_1 = \max\{s : \tau_0 < s < \tau_1\,,\, s \in I_=\}\,. \tag{40}$$

It follows that

$$[t_0, t_1] \subset I_0 \cup I_+\,, \quad [t_1, \tau_1) \subset I_0\,.$$

If $\tau_1 \notin I_-$, we set $t_2 = b$ and are done. Otherwise, we continue setting $\tau_2 = \tau_+(\tau_1)$ and

$$t_1 = \max\{s : \tau_1 < s < \tau_2\,,\, s \in I_=\}\,.$$

It follows that

$$[t_1, t_2] \subset I_0 \cup I_-\,, \quad [t_2, \tau_2) \subset I_0\,.$$

If $\tau_2 \notin I_+$, we set $t_3 = b$ and are done; otherwise, we continue in this manner. Since

$$|\tau_{k+1} - \tau_k| \geq \delta_I := \min\{|\tau - \sigma| : \tau \in I_+\,,\, \sigma \in I_-\} > 0 \tag{41}$$

whenever $\tau_k, \tau_{k+1} \in (a, b)$, the process terminates after a finite number of steps with a partition Δ_w, $a = t_0 < \cdots < t_N = b$, satisfying $t_k \in I_=$ for $1 < k < N$ as well as

$$[t_k, t_{k+1}] \cap I_- = \emptyset \qquad \text{or} \qquad [t_k, t_{k+1}] \cap I_+ = \emptyset \tag{42}$$

for all k, $0 \leq k < N$.

Lemma 5.2. *Let $u \in C[a, b]$, $w_0 \in \mathbb{R}$, $w = \mathcal{P}_r[u; w_0]$ and $r > 0$. Then there exists a $\delta > 0$ such that for all $v \in C[a, b]$ and $y_0 \in \mathbb{R}$ with $\|v - u\|_\infty < \delta$ and $|y_0 - w_0| < \delta$, the function $y = \mathcal{P}_r[v; y_0]$ is piecewise monotone, and the partition Δ_w constructed above is a monotonicity partition for y. Moreover, if v_{Δ_w} denotes the piecewise linear interpolant of v on Δ_w, we have*

$$\mathcal{P}_r[v; y_0](t_k) = \mathcal{P}_r[v_{\Delta_w}; y_0](t_k) \tag{43}$$

for all points t_k of Δ_w.

Proof. For perturbations (v, y_0) of (u, w_0) with $\delta > 0$ small enough, property (42) of the partition Δ_w remains valid when we replace I_- and I_+ by the corresponding contact sets of the trajectory (v, y), since v and y are continuous and $\mathcal{P}_r$ is Lipschitz continuous. Due to Lemma 5.1, Δ_w is a monotonicity partition not only for w, but for any such function y. Since both sides of (43) are equal to $\psi_\pm(v(t_{k+1}), y(t_k); t_{k+1}, t_k)$, all assertions follow. □

Proposition 5.3. *Let $r > 0$. Then for every $t \in [a, b]$ the mapping*

$$(u, w_0) \mapsto w(t) = \mathcal{P}_r[u; w_0](t) \tag{44}$$

is Hadamard differentiable from $C[a, b] \times \mathbb{R}$ to $\mathbb{R}$.

Proof. For $(u, w_0) \in C[a, b] \times \mathbb{R}$, let $\Delta_w = \{t_k\}$ be the partition constructed above. According to Lemma 5.1 and Lemma 5.2, in a sufficiently small δ-neighbourhood of (u, w_0), the mapping $(v, y_0) \mapsto y(t)$ defined by (44) can be represented as a finite concatenation of mappings

$$\begin{aligned} (v, y_0) &\mapsto y^{(0)} = \psi_0(v, y_0; t_0, t_0)\,, \\ (v, y^{(k)}) &\mapsto y^{(k+1)} = \psi_{k+1}(v, y^{(k)}; t_{k+1}, t_k)\,, \\ (v, y^{(M)}) &\mapsto y(t) = \psi_{M+1}(v, y^{(M)}; t, t_M)\,, \end{aligned} \tag{45}$$

where ψ_k stands for either ψ_+ or ψ_- and $a = t_0 < \cdots < t_M < t$. As ψ_+ and ψ_- are Hadamard differentiable by Proposition 3.2, the assertion follows from the chain rule, Lemma 2.3. □

Corollary 5.4. *The play operator $\mathcal{P}_r : C[a, b] \times \mathbb{R} \to C[a, b]$ possesses a pointwise directional derivative*

$$\mathcal{P}_r^{PD}([u; w_0]; [h; q])(t) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda}\Big(\mathcal{P}_r[u + \lambda h; w_0 + \lambda q](t) - \mathcal{P}_r[u; w_0](t)\Big) \tag{46}$$

for every $(u, w_0), (h, q) \in C[a, b] \times \mathbb{R}$ and every $t \in [a, b]$. □

As for the accumulated maximum, in order to obtain weak differentiability in function spaces we have to use a larger range space with a weaker norm, since in general the difference quotients do not converge uniformly in t, and the pointwise derivative may be discontinuous.

Proposition 5.5. *The play operator $\mathcal{P}_r : C[a, b] \times \mathbb{R} \to L^p(a, b)$ is Hadamard differentiable for every $1 \le p < \infty$.*

Proof. This is completely analogous to the proof of Proposition 4.2. The difference quotients are bounded,

$$\Big\| \frac{1}{\lambda}\Big(\mathcal{P}_r[u + \lambda h; w_0 + \lambda q] - \mathcal{P}_r[u; w_0]\Big)\Big\|_\infty \le \max\{\|h\|_\infty\,,\, |q|\}\,,$$

thus converge in L^p by dominated convergence, and the directional derivative is Hadamard due to Lemma 2.2. □

Like the accumulated maximum, the play operator is not Bouligand differentiable w.r.t. the norms used in Proposition 5.5. To prove this, one may easily adapt Example 4.1 given above.

A formula for the derivative can be obtained from the chain rule 2.3. One has to differentiate the formulas in the concatenation procedure (45), using Proposition 3.2 for ψ_+ and its analogue for ψ_-. We refrain from writing down the resulting expressions.

6. **Time regularity of the pointwise derivative.** We now consider the time regularity of the pointwise derivative of the play. We first want to prove that if $h \in C[a,b]$ has bounded variation, then the variation of the difference quotients for $\mathcal{P}_r$ can be bounded uniformly, and hence the pointwise limit $\mathcal{P}_r^{PD}$ has bounded variation, as a consequence of Helly's theorem.

For $u : [a,b] \to \mathbb{R}$ and a partition $\Delta = \{t_i\}$, $0 \le i \le N$, of $[a,b]$ we define

$$\mathrm{var}_\Delta(u) = \sum_{i=0}^{N-1} |v(t_{i+1}) - v(t_i)|\,, \quad \mathrm{var}(u) = \sup_\Delta \mathrm{var}_\Delta(u)\,,$$

where the sup ranges over all partitions Δ of $[a,b]$.

Lemma 6.1. *Let $u, v : [a,b] \to \mathbb{R}$ be piecewise linear and continuous. Then*

$$\mathrm{var}(\mathcal{P}_r[v;0] - \mathcal{P}_r[u;0]) \le \mathrm{var}(v-u) + |v(0) - u(0)|\,. \tag{47}$$

Proof. See Proposition 2.3.9 and formula (3.39) in [9]. □

We next obtain a Lipschitz estimate for the variation of the play operator, as a slight generalization of Proposition 2.3.11 in [9].

Proposition 6.2. *Let $u, v \in C[a,b]$, $w_0, y_0 \in \mathbb{R}$ and $r > 0$. If $v - u \in BV[a,b]$, then $w = \mathcal{P}_r[u;w_0]$ and $y = \mathcal{P}_r[v;y_0]$ satisfy*

$$\mathrm{var}(y-w) \le \mathrm{var}(v-u) + |v(0) - u(0)| + |y_0 - w_0|\,. \tag{48}$$

Proof. We first consider the case $w_0 = y_0 = 0$. Let Δ_w and Δ_y be the monotonicity partitions constructed above for w and y, respectively. Let $\Delta = \{t_k\}$ be an arbitrary refinement of $\Delta_w \cup \Delta_y$, let v_Δ and u_Δ be the piecewise linear interpolates for v and w on Δ, set

$$w_\Delta = \mathcal{P}_r[u_\Delta;0]\,, \quad y_\Delta = \mathcal{P}_r[v_\Delta;0]\,.$$

According to Lemma 5.1 and Lemma 5.2,

$$w(t) = w_\Delta(t)\,, \quad y(t) = y_\Delta(t)\,, \quad \text{for all } t \in \Delta\,. \tag{49}$$

Therefore, $\mathrm{var}_\Delta(y-w) = \mathrm{var}(y_\Delta - w_\Delta)$, and from Lemma 6.1 we obtain

$$\begin{aligned}\mathrm{var}_\Delta(y-w) &= \mathrm{var}(y_\Delta - w_\Delta) \le \mathrm{var}(v_\Delta - u_\Delta) + |v_\Delta(0) - u_\Delta(0)| \\ &= \mathrm{var}((v-u)_\Delta) + |v(0) - u(0)| \le \mathrm{var}(v-u) + |v(0) - u(0)|\,.\end{aligned}$$

Passing to the supremum with respect to all such partitions Δ on the left side, we obtain (48) for the special case $w_0 = y_0 = 0$. Using the formula (see Theorem 2.3.2 in [9])

$$\mathcal{P}_r[u;w_0] = \mathcal{P}_r[u - w_0;0] + w_0\,, \tag{50}$$

we see that, for arbitrary initial values $w_0, y_0 \in \mathbb{R}$,

$$\begin{aligned}\mathrm{var}(\mathcal{P}_r[v;y_0] - \mathcal{P}_r[u;w_0]) &= \mathrm{var}(\mathcal{P}_r[v - y_0;0] - \mathcal{P}_r[u - w_0;0]) \\ &\le \mathrm{var}((v - y_0) - (u - w_0)) + |(v(0) - y_0) - (u(0) - w_0)| \\ &\le \mathrm{var}(v-u) + |v(0) - u(0)| + |y_0 - w_0|\,.\end{aligned}$$

□

Proposition 6.3. *Let $u \in C[a,b]$, $w_0 \in \mathbb{R}$, $r > 0$. If $h \in C[a,b] \cap BV[a,b]$ and $q \in \mathbb{R}$, then the pointwise directional derivative of the play operator satisfies*

$$\mathcal{P}_r^{PD}([u;w_0];[h;q]) \in BV[a,b]\,. \tag{51}$$

Proof. We estimate the variation of the difference quotients, using Proposition 6.2. For arbitrary $\lambda > 0$ we get

$$\begin{aligned}
&\operatorname{var}\Big(\frac{1}{\lambda}\big(\mathcal{P}_r[u+\lambda h; w_0+\lambda q] - \mathcal{P}_r[u;w_0]\big)\Big)\\
&\quad \le \frac{1}{\lambda}\Big(\operatorname{var}(u+\lambda h - u) + |u(0)+\lambda h(0) - u(0)| + |w_0+\lambda q - w_0|\Big)\\
&\quad = \frac{1}{\lambda}\Big(\operatorname{var}(\lambda h) + |\lambda h(0)| + |\lambda q|\Big) = \operatorname{var}(h) + |h(0)| + |q|\,.
\end{aligned}$$

As the rightmost expression does not depend on λ, Helly's theorem implies that the pointwise limit of the difference quotients (which we already know to be bounded uniformly in λ by $\max\{\|h\|_\infty, |q|\}$) has bounded variation. □

We have already seen in Example 4.1 that the pointwise directional derivative can have unbounded variation if the variation of h is unbounded. However, we will prove that, for general $h \in C[a,b]$, the pointwise directional derivative of the play is a regulated function.

A bounded function $u : [a,b] \to \mathbb{R}$ is called **regulated**, if the one-sided limits

$$u(t+) := \lim_{\lambda\downarrow 0} u(t+\lambda)\,, \quad u(t-) := \lim_{\lambda\downarrow 0} u(t-\lambda)$$

exist for all $t \in [a,b]$, with the convention $u(a-) := u(a)$, $u(b+) := u(b)$. By $G[a,b]$ we denote the space of all regulated functions on $[a,b]$. It is well known that $G[a,b]$ endowed with the supremum norm is a Banach space, and that the piecewise constant functions (when they are allowed to have arbitrary values at their discontinuity points) are dense in $G[a,b]$. As a consequence, $BV[a,b]$ is a dense subset of $G[a,b]$.

Our main tool is a generalization to $G[a,b]$ of Helly's theorem for BV functions. Such a generalization has been introduced in [17] and further developed in [14]. For our purposes it is convenient to use the following concept from [14].

Definition 6.4. Let U be a bounded subset of $G[a,b]$. We say that U has **uniformly bounded oscillation**, if there exists a nonincreasing function $N : (0,\infty) \to (0,\infty)$ such that the following assertion holds for every $\delta > 0$: If $u \in U$ and if $(a_1,b_1),\dots,(a_M,b_M)$ is a system of M pairwise disjoint subintervals of $[a,b]$ such that

$$|u(b_k) - u(a_k)| \ge \delta\,, \quad \text{for all } 1 \le k \le M\,, \tag{52}$$

then we must have

$$M \le N(\delta)\,. \tag{53}$$

Proposition 6.5. *Let $\{u_n\}$ be a bounded sequence in $G[a,b]$ which has uniformly bounded oscillation. Then there exists a subsequence $\{u_{n_k}\}$ and a function $u \in G[a,b]$ such that $u_{n_k} \to u$ pointwise.*

Proof. See [14], Theorem 2.2 and Proposition 2.3. □

Proposition 6.6. *Let $u,h \in C[a,b]$, $w_0, q \in \mathbb{R}$ and $r > 0$. Then the pointwise directional derivative of the play operator satisfies*

$$\mathcal{P}_r^{PD}([u;w_0];[h;q]) \in G[a,b]\,. \tag{54}$$

Proof. We may assume that $\operatorname{var}(h) = +\infty$, otherwise Proposition 6.3 applies. In view of Proposition 6.5 it suffices to prove that the set $U = \{d_\lambda : \lambda > 0\}$ of the difference quotients

$$d_\lambda = \frac{1}{\lambda}\big(\mathcal{P}_r[u + \lambda h; w_0 + \lambda q] - \mathcal{P}_r[u; w_0]\big)$$

has uniformly bounded oscillation. First, U is bounded by $\max\{\|h\|_\infty, |q|\}$ as has been shown above. The idea of the proof is to approximate h by a sequence $\{h_n\}$ of BV functions and to employ the BV estimate of Proposition 6.2. Let $h_n \in C[a,b] \cap BV[a,b]$ such that $h_n \to h$ uniformly. Since $\operatorname{var}(h) = +\infty$, $\{\operatorname{var}(h_n)\}$ is unbounded by Helly's theorem. Passing to a subsequence if necessary, we may assume that $\|h_n\|_\infty \le \|h\|_\infty + 1$ for all n, that $\|h_n - h\|_\infty$ is decreasing and that $\operatorname{var}(h_n)$ is increasing as $n \to \infty$. Set

$$d_{\lambda,n} = \frac{1}{\lambda}\big(\mathcal{P}_r[u + \lambda h_n; w_0 + \lambda q] - \mathcal{P}_r[u; w_0]\big)\,.$$

We have

$$\|d_{\lambda,n} - d_\lambda\|_\infty \le \frac{1}{\lambda}\big(\mathcal{P}_r[u + \lambda h_n; w_0 + \lambda q] - \mathcal{P}_r[u + \lambda h; w_0 + \lambda q]\big) \le \|h_n - h\|_\infty \tag{55}$$

for all λ and all n. We now construct the function N as required in Definition 6.4. Fix $\delta > 0$ and choose $n_0(\delta)$ as the smallest number such that

$$\|h_n - h\|_\infty \le \frac{\delta}{4}\,, \quad \text{for all } n \ge n_0(\delta)\,.$$

Then n_0 is nonincreasing, and in view of (55)

$$\|d_{\lambda,n} - d_\lambda\|_\infty \le \frac{\delta}{4}\,, \quad \text{for all } n \ge n_0(\delta) \text{ and all } \lambda > 0\,. \tag{56}$$

We set

$$N(\delta) = \frac{2}{\delta}\Big(\operatorname{var}(h_{n_0(\delta)}) + \|h\|_\infty + 1 + |q|\Big)\,. \tag{57}$$

Now we prove that (52) implies (53) in Definition 6.4. Choose an arbitrary $\lambda > 0$ and an arbitrary system $a \le a_1 < b_1 \le \cdots \le a_M < b_M \le b$ such that

$$|d_\lambda(b_j) - d_\lambda(a_j)| \ge \delta\,, \quad \text{for all } 1 \le j \le M\,.$$

For $n = n_0(\delta)$ we get from (56) that

$$|d_{\lambda,n}(b_j) - d_{\lambda,n}(a_j)| \ge |d_\lambda(b_j) - d_\lambda(a_j)| - 2\|d_{\lambda,n} - d_\lambda\|_\infty \ge \delta - \frac{\delta}{2} = \frac{\delta}{2}\,, \tag{58}$$

thus

$$\sum_{j=1}^{M} |d_{\lambda,n}(b_j) - d_{\lambda,n}(a_j)| \ge M\frac{\delta}{2}$$

for $n = n_0(\delta)$. From Proposition 6.2 with $v = u + \lambda h_{n_0(\delta)}$ and $y_0 = w_0 + \lambda q$ it follows that

$$M\frac{\delta}{2} \le \operatorname{var}(d_{\lambda,n_0(\delta)}) \le \operatorname{var}(h_{n_0(\delta)}) + |h_{n_0(\delta)}(0)| + |q|$$

so $M \le N(\delta)$ from (57) as required. Thus, U has uniformly bounded oscillation. □

The foregoing results also apply to the accumulated maximum, as the latter can be represented by the play operator on bounded subsets of $C[a,b]$. More precisely, the following result holds.

Lemma 6.7. *We have*

$$F(u) = \mathcal{P}_r[u; u(a) - r] + r = \mathcal{P}_r[u - u(a) + r; 0] + u(a) \tag{59}$$

for all $u \in C[a,b]$ *with* $\|u\|_\infty < r$ *.*

Proof. Let $u \in C[a,b]$, set $w_0 = u(a) - r$ and $w = \mathcal{P}_r[u; w_0]$. Then $u(a) - w_0 = r$, thus $w(a) = w_0$ and $a \in I_+$. If $I_- \neq \emptyset$ then there exist $a \le t_+ < t_- \le b$ with $t_+ \in I_+$, $t_- \in I_-$ and $(t_+, t_-) \subset I_0$. It follows that $w(t_+) = w(t_-)$ and therefore $u(t_+) - u(t_-) = 2r$, so $\|u\|_\infty \ge r$. It follows that $I_- = \emptyset$ and $I_+ \cup I_0 = [a,b]$ whenever $\|u\|_\infty < r$. By Lemma 5.1,

$$w(t) = \psi_+(u(t), w(a); t, a) = \max\{u(a) - r\,,\ \max_{s\in[a,t]} (u(s) - r)\} + r = \max_{s\in[a,t]} u(s)$$

holds for all $t \in [a,b]$. ☐

Proposition 6.8. *The pointwise directional derivative* $F^{PD}(u;h)$ *of the accumulated maximum belongs to* $G[a,b]$ *for every* $u, h \in C[a,b]$*. If moreover* $h \in BV[a,b]$*, then* $F^{PD}(u;h) \in BV[a,b]$*.*

Proof. In view of Lemma 6.7, this is a consequence of Propositions 6.3, 6.6 and of the chain rule. ☐

7. The Prandtl-Ishlinskii operator.

The scalar Prandtl-Ishlinskii operator goes back to the scalar approximations proposed for the constitutive relations in elastoplasticity in [18, 19]. In terms of the family $\{\mathcal{P}_r\}_{r\ge 0}$ of play operators, it can be written in the form

$$\mathcal{P}[u; w_0](t) = \int_0^\infty \mathcal{P}_r[u; w_0(r)](t)\,\mathrm{d}\mu(r)\,. \tag{60}$$

Here, $u \in C[a,b]$ is the driving function as before. The function $w_0 : \mathbb{R}_+ \to \mathbb{R}$, $\mathbb{R}_+ := [0,\infty)$, generates the initial values and is assumed to be measurable and bounded; let us denote by W_0 the space of all such functions. The measure μ is a regular signed Borel measure on $\mathbb{R}_+$, assumed here to be finite for simplicity. Under these assumptions (see e.g. [9]), the function

$$(t, r) \mapsto \mathcal{P}_r[u; w_0(r)](t)$$

is continuous w.r.t. t on $[a,b]$ for fixed r, and measurable w.r.t. r on $\mathbb{R}_+$ for fixed t, and the function $\mathcal{P}[u; w_0]$ is continuous on $[a,b]$. Moreover,

$$\|\mathcal{P}[v; y_0] - \mathcal{P}[u; w_0]\|_\infty \le |\mu|(\mathbb{R}_+) \max\{\|v - u\|_\infty\,,\ \|y_0 - w_0\|_\infty\}\,. \tag{61}$$

For $h \in C[a,b]$ and $q \in W_0$ we obtain that

$$\frac{1}{\lambda}\Big(\mathcal{P}[u + \lambda h; w_0 + \lambda q] - \mathcal{P}[u; w_0]\Big)(t) = \int_0^\infty d_\lambda(t,r)\,\mathrm{d}\mu(r)\,, \tag{62}$$

where

$$d_\lambda(t,r) = \frac{1}{\lambda}\Big(\mathcal{P}_r[u + \lambda h; w_0 + \lambda q] - \mathcal{P}_r[u; w_0]\Big)(t)\,. \tag{63}$$

Since μ is finite and $|d_\lambda(t,r)| \le \max\{\|h\|_\infty\,,\ |q|\}$ for all λ, t, r as before, by dominated convergence we may pass to the limit $\lambda \downarrow 0$ under the integral in (62) and arrive at the following result.

Proposition 7.1. *Under the assumptions outlined below (60), the Prandtl-Ishlinskii operator defined in (60) possesses a pointwise directional derivative at every point* (u, w_0) *in every direction* (h, q) *in* $C[a, b] \times W_0$ *, given by*

$$\mathcal{P}^{PD}([u; w_0]; [h; q])(t) = \int_0^\infty \mathcal{P}_r^{PD}([u; w_0]; [h; q])(t)\, \mathrm{d}\mu(r)\,. \tag{64}$$

Moreover, $\mathcal{P} : C[a, b] \times W_0 \to L^p(a, b)$ *is Hadamard differentiable for* $1 \le p < \infty$ *.*

Proof. We have $d_\lambda(t, r) \to \mathcal{P}_r^{PD}([u; w_0]; [h; q])(t)$ as $\lambda \downarrow 0$ by Corollary 5.4. As $\mathcal{P}$ is Lipschitz continuous by virtue of (61), Hadamard differentiability again follows from Lemma 2.2. □

Proposition 7.2. *Under the assumptions outlined below (60), the pointwise directional derivative (64) of the Prandtl-Ishlinskii operator belongs to* $G[a, b]$ *, and we have*

$$\mathrm{var}(\mathcal{P}^{PD}([u; w_0]; [h; q])) \le \int_0^\infty \mathrm{var}(\mathcal{P}_r^{PD}([u; w_0]; [h; q]))\, \mathrm{d}\mu(r)\,, \tag{65}$$

which may be finite or infinite.

Proof. Since by dominated convergence we may pass to the limits $t_n \to t\pm$ in (64) with t_n inserted for t, it follows from Proposition 6.6 that the pointwise directional derivative is a regulated function. Moreover, (64) implies that

$$\mathrm{var}_\Delta(\mathcal{P}^{PD}([u; w_0]; [h; q])) \le \int_0^\infty \mathrm{var}_\Delta(\mathcal{P}_r^{PD}([u; w_0]; [h; q]))\, \mathrm{d}\mu(r)\,,$$

whence (65) follows by passing to the supremum with respect to Δ. □

8. The Preisach operator.

In [20], Preisach proposed a scalar model for the constitutive law of ferromagnetism which allows for nested hysteresis loops. It can be written in terms of the family $\{\mathcal{P}_r\}_{r \ge 0}$ of play operators in the form [21, 22]

$$\mathcal{P}[u; w_0](t) = \int_0^\infty k(r, \mathcal{P}_r[u; w_0(r)](t))\, \mathrm{d}\mu(r)\,, \tag{66}$$

for some function k which arises from the Preisach density function in the usual formulation of the Preisach model as a linear superposition of relays, see [10, 9]. Setting $k(r, s) = s$, the Prandtl-Ishlinskii operator is seen to be a special case of the Preisach operator.

Proposition 8.1. *Let the assumptions outlined below (60) hold, and let* $k = k(r, s)$ *be measurable as a function of* r *and continuously differentiable as a function of* s *. Then the pointwise directional derivative of the Preisach operator exists and belongs to* $G[a, b]$ *at every point* (u, w_0) *in every direction* (h, q) *in* $C[a, b] \times W_0$ *. We have*

$$\mathcal{P}^{PD}([u; w_0]; [h; q])(t) = \int_0^\infty \partial_s k(r, t) \cdot \mathcal{P}_r^{PD}([u; w_0]; [h; q])(t)\, \mathrm{d}\mu(r)\,, \tag{67}$$

where we have used the abbreviation

$$\partial_s k(r, t) = \partial_s k(r, \mathcal{P}_r[u; w_0(r)](t))\,.$$

Moreover, $\mathcal{P} : C[a, b] \times W_0 \to L^p(a, b)$ *is Hadamard differentiable for* $1 \le p < \infty$ *.*

Proof. For fixed r and t, the mapping from $C[a,b] \times W_0$ to $\mathbb{R}$ given by

$$(u; w_0) \mapsto \mathcal{P}_r[u; w_0(r)](t) \mapsto k(r, \mathcal{P}_r[u; w_0(r)](t))$$

has a directional derivative in any direction $(h,q) \in C[a,b] \times W_0$ given by the integrand on the right side of (67), since $\mathcal{P}_r$ is pointwise directionally differentiable and k is C^1 in s. As in the proof of Proposition 7.1 we obtain (67), passing to the limit in the corresponding difference quotients by dominated convergence. As $\mathcal{P}$ is Lipschitz continuous, see [9], Hadamard differentiability again follows from Lemma 2.2. $\square$

With arguments completely analogous to those in the proof of Proposition 7.2, we obtain the corresponding result for the Preisach operator.

Proposition 8.2. *Under the assumptions of Proposition 8.1, the pointwise directional derivative (67) of the Preisach operator belongs to $G[a,b]$, and we have*

$$\mathrm{var}(\mathcal{P}^{PD}([u; w_0]; [h; q])) \le \int_0^\infty \mathrm{var}(g_r)\, \mathrm{d}\mu(r)\,, \tag{68}$$

where

$$g_r(t) = \partial_s k(r, \mathcal{P}_r[u; w_0(r)](t)) \cdot \mathcal{P}_r^{PD}([u; w_0]; [h; q])(t)\,.$$

$\square$

Acknowledgments. This work was supported by the GA ČR Grant P201/10/2315 during the visit of the first author at the Institute of Mathematics of the Czech Academy of Sciences.

REFERENCES

[1] J.-J. Moreau, Problème d'evolution associé à un convexe mobile d'un espace hilbertien, *C. R. Acad. Sci. Paris Sér. A-B,* **276** (1973), A791–A794. (In French.)

[2] J.-J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, *J. Diff. Eq.,* **26** (1977), 347–374.

[3] A. Mielke, Evolution of rate independent systems, in *Evolutionary Equations, vol. II* (eds. C.M. Dafermos and and E. Feireisl), Handbook of Differential Equations, Elsevier/North Holland, Amsterdam, **II** (2005), 461–559.

[4] A. Mielke, F. Theil and V. I. Levitas, A variational formulation of rate-independent phase transformations using an extremum principle, *Arch. Ration. Mech. Anal.,* **162** (2002), 137–177.

[5] M. A. Krasnosel'skiĭ, B. M. Darinskiĭ, I. V. Emelin, P. P. Zabrejko, E. A. Lifshits and A. V. Pokrovskiĭ, An operator-hysterant, *Dokl. Akad. Nauk SSSR,* **190** (1970), 34–37; *Soviet Math. Dokl.,* **11** (1970), 29–33.

[6] M. A. Krasnosel'skiĭ and A. V. Pokrovskiĭ, *Systems with Hysteresis*, Nauka, Moscow, 1983. (In Russian.)

[7] M. A. Krasnosel'skiĭ and A. V. Pokrovskiĭ, *Systems with Hysteresis*, Springer, Heidelberg, 1989.

[8] A. Visintin, *Differential models of hysteresis*, Springer, Berlin, 1994.

[9] M. Brokate and J. Sprekels, *Hysteresis and Phase Transitions*, Springer, New York, 1996.

[10] P. Krejčí, *Hysteresis, Convexity and Dissipation in Hyperbolic Equations*, Gakkōtosho, Tokyo, 1996.

[11] P. Dupuis and H. Ishii, On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications, *Stochastics Stochastics Rep.,* **35** (1991), 31–62.

[12] P. Krejčí and Ph. Laurençot, Generalized variational inequalities, *J. Convex Anal.,* **9** (2002), 159–183.

[13] P. Krejčí, Hysteresis in singularly perturbed problems, in *Singular Perturbations and Hysteresis* (eds. M.P. Mortell, R.E. O'Malley, A. Pokrovskiĭ, and V. Sobolev), SIAM, (2005), 73–100.

[14] M. Brokate and P. Krejčí, Duality in the space of regulated functions and the play operator, *Math. Z.,* **245** (2003), 667–688.

[15] R. Rossi, A. Mielke and G. Savaré, A metric approach to a class of doubly nonlinear evolution equations and applications, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5),* **7** (2008), 97–169.

[16] I. V. Girsanov, *Lectures on Mathematical Theory of Extremum Problems*, Springer, Berlin, 1972.

[17] D. Fraňková, Regulated functions, *Math. Bohem.,* **116** (1991), 20–59.

[18] L. Prandtl, Ein Gedankenmodell zur kinetischen Theorie der festen Körper, *Z. f. angew. Math.,* **8** (1928), 85–106. (In German.)

[19] A. Yu. Ishlinskiĭ, Some applications of statistical methods to describing deformations of bodies, *Izv. AN SSSR Techn. Ser.,* **9** (1928), 583–590. (In Russian.)

[20] F. Preisach, Über die magnetische Nachwirkung, *Z. Phys.*, **94** (1935), 277–302. (In German.)

[21] P. Krejčí, On Maxwell equations with the Preisach operator: The one-dimensional time-periodic case, *Apl. Mat.,* **34** (1989), 364–374.

[22] M. Brokate, Some BV properties of the Preisach hysteresis operator, *Appl. Anal.,* **32** (1989), 229–252.

Received January 2014; revised May 2014.

E-mail address: brokate@ma.tum.de
E-mail address: krejci@math.cas.cz