DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **35**, Number **6**, June **2015**

doi:10.3934/dcds.2015.35.2465

pp. **2465–2495**

A NEW PHASE FIELD MODEL FOR MATERIAL FATIGUE IN AN OSCILLATING ELASTOPLASTIC BEAM

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Dedicated to Prof. Jürgen Sprekels on the occasion of his 65th birthday: we wish him many happy years of fruitful research.

Abstract. We pursue the study of fatigue accumulation in an oscillating elastoplastic beam under the additional hypothesis that the material can partially recover by the effect of melting. The full system consists of the momentum and energy balance equations, an evolution equation for the fatigue rate, and a differential inclusion for the phase dynamics. The main result consists in proving the existence and uniqueness of a strong solution.

Introduction. It was shown in [16] that the Kirchhoff-Love method of reducing the 3D problem of transversal oscillations of a solid elastoplastic beam with the single yield von Mises plasticity law leads to the beam equation with a multiyield hysteresis Prandtl-Ishlinskii constitutive operator. The present authors have used in [7] (see also [5, 11]) the Prandtl-Ishlinskii formalism to propose a model for the cyclic fatigue accumulation in a transversally oscillating beam and to study its properties; results have been obtained correspondingly also for the plate, see [6, 8, 1]. Here, we extend the model by taking into account the possibility of partial fatigue recovery by the effect of melting when a solid-liquid phase transition takes place.

The fatigue accumulation law is still based on the observation that there exists a proportionality between accumulated fatigue and dissipated energy, see [2, 9]. Unlike [7] and similarly to [9], we assume that out of all dissipative components in the energy balance, only the purely plastic dissipation produces damage. This makes

2010 *Mathematics Subject Classification.* 47J40, 74R20, 74K10, 74N30, 82B26.

Key words and phrases. Elastoplastic beam, Prandtl-Ishlinskii operator, material fatigue, thermo-elasto-plasticity, phase transition.

Supported by GAČR Grant P201/10/2315, and by the institutional support for the development of research organizations IČ 47813059 and 67985840. The work of Michela Eleuteri was supported by the FP7-IDEAS-ERC-StG Grant #256872 (EntroPhase) and by GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica).

the mathematical problem easier: the system of equations then does not develop singularities in finite time and a unique regular solution is proved to exist on every bounded time interval. On the other hand we consider here an additional difficulty - we assume that the weight function φ in the definition of the Prandtl-Ishlinskii operator depends also on the fatigue parameter m; this has been considered also in [8] and [1].

The unknowns of the problem are the transversal displacement $w \in \mathbb{R}$ of the beam, the absolute temperature $\theta > 0$, the fatigue variable $m \geq 0$, and the phase variable $\chi \in [0,1]$. The full system of equations consists of the momentum balance equation (the simply supported beam equation with a fatigue dependent hysteresis operator), the energy balance equation with the no-flux boundary conditions, the fatigue accumulation equation and the phase transition equation. The model is derived in detail in Section 1.

The problem is rigorously stated in Section 2, where we also check the thermodynamic consistency of the system and collect some preliminary material in Section 3. In Section 4 we carry out formally the a priori estimates that allow us to construct the solution of the full system. In Section 5, we apply these ideas to a spatially discrete scheme and derive estimates that are sufficient for proving that the space discrete approximations converge to a solution of the original problem in appropriate function spaces. The main existence and uniqueness Theorem 2.2 is proved in Section 6.

1. **The model.**

1.1. **Governing equations.**

We consider a transversally inhomogeneous beam of length 1, and denote by $x \in [0,1]$ the longitudinal variable, by $t \in [0,T]$ the time variable, by $w(x,t)$ the transversal displacement of the point x at time t, by $\varepsilon(x,t) = w_{xx}(x,t)$ the linearized curvature, and by $\sigma(x,t)$ the bending moment. We assume a thermo-visco-elasto-plastic scalar constitutive law in the form

$$\sigma = B\varepsilon + P[m,\varepsilon] + \nu\varepsilon_t - \beta(\theta - \theta_{\mathrm{ref}})\,, \tag{1.1}$$

where $B > 0$ is a constant hardening modulus, $m \geq 0$ is a scalar time and space dependent parameter describing the accumulation of fatigue, where $m = 0$ corresponds to zero fatigue, $P[m,\varepsilon]$ is a fatigue dependent Prandtl-Ishlinskii constitutive operator of elastoplasticity defined below in Subsection 1.2, ν is the viscosity coefficient, β is the thermal bending coefficient related to a layered structure of the beam, $\theta > 0$ is the absolute temperature, and θ_{ref} is a fixed referential temperature (more specifically, the melting temperature). Following [16], Newton's law of motion is formally written as

$$\rho w_{tt} - \alpha w_{xxtt} + \sigma_{xx} = F(x,t), \tag{1.2}$$

where $\alpha = \rho l^2/12$ and $l > 0$ is the thickness of the beam, ρ the mass density and F is the external load.

With the constitutive law (1.1), we associate the *free energy operator*

$$\mathcal{F}(\varepsilon,\theta,\chi) = c\theta(1 - \log(\theta/\theta_{\mathrm{ref}})) + \frac{B}{2}\varepsilon^2 + V[m,\varepsilon] - \beta(\theta - \theta_{\mathrm{ref}})\varepsilon - \frac{L}{\theta_{\mathrm{ref}}}(\theta - \theta_{\mathrm{ref}})\chi + I_{[0,1]}(\chi)\,, \tag{1.3}$$

where $V[m,\varepsilon]$ is the fatigue dependent Prandtl-Ishlinskii potential (1.19), c (the specific heat capacity) and L (the latent heat) are given constants, and $I_{[0,1]}$ is the indicator function of the interval $[0,1]$. The *entropy operator* $\mathcal{S}$ and *internal energy* *operator* $\mathcal{U}$ then read

$$\mathcal{S}(\varepsilon,\theta,\chi) = -\frac{\partial \mathcal{F}}{\partial \theta} = c\log(\theta/\theta_{\mathrm{ref}}) + \beta\varepsilon + \frac{L}{\theta_{\mathrm{ref}}}\chi\,, \tag{1.4}$$

$$\mathcal{U}(\varepsilon,\theta,\chi) = \mathcal{F}(\varepsilon,\theta) + \theta\mathcal{S}(\varepsilon,\theta) = c\theta + \frac{B}{2}\varepsilon^2 + V[m,\varepsilon] + \beta\theta_{\mathrm{ref}}\varepsilon + L\chi + I_{[0,1]}(\chi)\,. \tag{1.5}$$

We consider the first and the second principles of thermodynamics in the form

$$\mathcal{U}(\varepsilon,\theta,\chi)_t + q_x = \varepsilon_t\sigma + g\,, \tag{1.6}$$

$$\mathcal{S}(\varepsilon,\theta,\chi)_t + \left(\frac{q}{\theta}\right)_x \ge \frac{g}{\theta}\,, \tag{1.7}$$

where $q = -\kappa\theta_x$ is the heat flux with a constant heat conductivity $\kappa > 0$, and g is the heat source density. Note that (1.6) is the energy conservation law, (1.7) is the Clausius-Duhem inequality.

The evolution of the phase variable χ is governed by the inclusion $-\gamma\chi_t \in \partial_\chi\mathcal{F}$, that is,

$$-\gamma\chi_t \in \partial I_{[0,1]}(\chi) - \frac{L}{\theta_{\mathrm{ref}}}(\theta - \theta_{\mathrm{ref}}), \tag{1.8}$$

where $\gamma > 0$ is a characteristic time of phase transition, and $\partial I_{[0,1]}$ is the subdifferential of the indicator function $I_{[0,1]}$. Indeed, we necessarily have $\chi \in [0,1]$, and we interpret $\chi = 0$ as the solid phase, $\chi = 1$ as liquid, and the intermediate values correspond to the relative liquid content in a mixture of the two.

Let $D[m,\varepsilon]$ be the dissipation operator defined in (1.20) associated with the Prandtl-Ishlinskii operator $P[m,\varepsilon]$. The analysis of the so-called *rainflow method of cyclic fatigue accumulation* in elastoplastic materials carried out in [2] has shown a close relation between accumulated fatigue and dissipated energy, similarly as in [9]. Here, we assume in addition that partial recovery of the damaged material is possible under strong local melting. Mathematically, this is expressed in terms of the evolution equation for the fatigue variable m

$$m_t(x,t) \in -\partial I_{[0,\infty)}(m) - h(\chi_t(t)) + \int_0^1 \lambda(x-y)D[m,\varepsilon](y,t)\,\mathrm{d}y, \tag{1.9}$$

where h is a nonnegative nondecreasing function vanishing for negative arguments, see Hypothesis 2.1 (vi), λ is a nonnegative smooth function with (small) compact support and $D[m,\varepsilon]$ is the fatigue dependent dissipation operator, see (1.20). The subdifferential $\partial I_{[0,\infty)}$ of the indicator function $I_{[0,\infty)}$ ensures that the fatigue parameter remains nonnegative.

The meaning of (1.9) is simple. If no phase transition takes place or if the material solidifies, that is, $\chi_t \le 0$, then fatigue at a point x increases proportionally to the energy dissipated in a neighborhood of the point x. On the other hand, under strong melting if χ grows faster than the plastic dissipation rate, the fatigue may decrease until it possibly reaches the unperturbed state $m = 0$.

1.2. **Hysteresis operators.** Let us first recall the definition of the stop.

Definition 1.1. Let $u \in W^{1,1}(0,T)$ and a closed connected set $Z \subset \mathbb{R}$ be given. The variational inequality

$$\left.\begin{array}{lll} u(t) = z(t) + \xi(t) & & \forall t \in [0,T], \\ z(t) \in Z & & \forall t \in [0,T], \\ \dot\xi(t)(z(t) - y) \ge 0 & \text{a.e.} & \forall y \in Z\,, \\ z(0) = z^0 \in Z, & & \end{array}\right\} \tag{1.10}$$

defines the stop and play operators $\mathfrak{s}_Z$ and $\mathfrak{p}_Z$ by the formula

$$z(t) = \mathfrak{s}_Z[z^0, u](t)\,, \quad \xi(t) = \mathfrak{p}_Z[z^0, u](t)\,. \tag{1.11}$$

For a canonical choice of $Z = [-r, r]$ with some $r > 0$ and for the initial condition $z(0) = Q_r(u(0))$, where Q_r is the projection of $\mathbb{R}$ onto the interval $[-r, r]$, we simply write

$$z(t) = \mathfrak{s}_r[u](t)\,, \quad \xi(t) = \mathfrak{p}_r[u](t)\,. \tag{1.12}$$

A simple proof of the following easy properties of the play and stop can be found e.g. in [13, Proposition II.1.1].

Proposition 1.2. *Let $u_1, u_2 \in W^{1,1}(0,T)$, a closed connected set $Z \subset \mathbb{R}$, and data $z_1^0, z_2^0 \in Z$ be given, $z_i = \mathfrak{s}_Z[z_i^0, u_i]$, $\xi_i = u_i - z_i$, $i = 1, 2$. Then $z_i, \xi_i \in W^{1,1}(0,T)$, and*

(i) $(z_1(t) - z_2(t))(\dot u_1(t) - \dot u_2(t)) \ \ge\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}(z_1(t) - z_2(t))^2$ *a.e.;*

(ii) $|\dot\xi_1(t) - \dot\xi_2(t)| + \dfrac{\mathrm{d}}{\mathrm{d}t}|z_1(t) - z_2(t)| \ \le\ |\dot u_1(t) - \dot u_2(t)|$ *a.e.;*

(iii) $|z_1(t) - z_2(t)| \ \le\ |z_1^0 - z_2^0| + 2 \max_{0 \le \tau \le t} |u_1(\tau) - u_2(\tau)| \quad \forall t \in [0,T]$*;*

(iv) $\dot\xi_i(t)\dot u_i(t) = \dot\xi_i(t)^2$ *a.e.*

The variational inequality (1.10) can be equivalently written as the inclusion $\dot z(t) + \partial I_Z(z(t)) \ni \dot u(t)$. This enables us to rewrite the differential inclusions (1.8) and (1.9) for the phase variable χ and fatigue variable m with a choice $\chi^0(x) \in [0,1]$, $m^0(x) \ge 0$ of initial conditions in the form

$$\chi(x,t) \;=\; \mathfrak{s}_{[0,1]}[\chi^0(x), A(x,\cdot)](t), \tag{1.13}$$

$$m(x,t) \;=\; \mathfrak{s}_{[0,\infty)}[m^0(x), S(x,\cdot)](t), \tag{1.14}$$

where

$$A(x,t) \;:=\; \int_0^t \frac{1}{\gamma}\left(\frac{L}{\theta_{\mathrm{ref}}}(\theta - \theta_{\mathrm{ref}})\right)(x,\tau)\ \mathrm{d}\tau, \tag{1.15}$$

$$S(x,t) \;:=\; \int_0^t \left(-h(\chi_t(\tau)) + \int_0^1 \lambda(x-y) D[m,\varepsilon](y,\tau)\,\mathrm{d}y\right)(x,\tau)\ \mathrm{d}\tau. \tag{1.16}$$

The advantage of this representation is that now, χ and m are defined by equations involving, by virtue of Proposition 1.2, only operators that are Lipschitz continuous in $C[0,T]$ and in $W^{1,1}(0,T)$.

The variational inequality (1.10) is also used to model single-yield elastoplasticity. In this case, the constraint $Z = [-r, r]$ is the admissible stress domain, the input $u = \varepsilon$ is the strain, and the output $z = \sigma_r := \mathfrak{s}_r[\varepsilon]$ is the stress. We can rewrite (1.10) equivalently in "energetic" form

$$\dot\varepsilon(t)\sigma_r(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\sigma_r^2(t)\right) + r|\dot\xi(t)|. \tag{1.17}$$

Indeed, $\dot\varepsilon(t)\sigma_r(t)$ is the power supplied to the system, part of it is used for the increase of the potential $\frac{1}{2}\sigma_r^2(t)$, and the rest $r|\dot\xi(t)|$ is dissipated.

The Prandtl-Ishlinskii model is constructed as a linear combination of stops with all possible yield points $r > 0$. Here, given a measurable function $\varphi : [0,\infty) \times (0,\infty) \to [0,\infty)$ satisfying Hypothesis 2.1 (i) below, we define the fatigue dependent Prandtl-Ishlinskii operator $P : (W^{1,1}(0,T))^2 \to W^{1,1}(0,T)$ by the integral

$$P[m,\varepsilon](t) = \int_0^\infty \varphi(m(t),r)\ \mathfrak{s}_r[\varepsilon](t)\,\mathrm{d}r\,. \tag{1.18}$$

Eq. (1.17) enables us to establish the energy balance for the Prandtl-Ishlinskii operator (1.18). Indeed, if we define the Prandtl-Ishlinskii potential

$$V[m,\varepsilon](t) = \frac{1}{2}\int_0^\infty \varphi(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r\,, \tag{1.19}$$

and the dissipation operator

$$D[m,\varepsilon](t) = \int_0^\infty r\varphi(m,r)|\mathfrak{p}_r[\varepsilon]_t(t)|\,\mathrm{d}r\,, \tag{1.20}$$

we can write the Prandtl-Ishlinskii energy balance in the form

$$\dot\varepsilon(t)P[m,\varepsilon](t) = \frac{\mathrm{d}}{\mathrm{d}t}V[m,\varepsilon](t) + D[m,\varepsilon](t) - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r \quad \text{a.e.} \tag{1.21}$$

As a consequence of Proposition 1.2 (iv), we have

$$D[m,\varepsilon](t) \le |\dot\varepsilon(t)|\int_0^\infty r\varphi(m,r)\,\mathrm{d}r\,. \tag{1.22}$$

2. **Statement of the problem.** For any $T > 0$, we denote $\Omega_T := (0,1)\times(0,T)$, $u(x,t) = \int_0^t \sigma(x,\tau)\,\mathrm{d}\tau$, $f(x,t) = \int_0^t F(x,\tau)\,\mathrm{d}\tau + \rho w_t(x,0) - \alpha w_{xxt}(x,0)$. We rewrite the equations (1.1), (1.2), (1.6), (1.8), (1.9) as the system

$$\begin{aligned}
u_t &= Bw_{xx} + P[m,w_{xx}] + \nu w_{xxt} - \beta(\theta-\theta_{\mathrm{ref}}), && (2.1)\\
\rho w_t - \alpha w_{xxt} &= -u_{xx} + f(x,t), && (2.2)\\
c\theta_t - \kappa\theta_{xx} &= D[m,w_{xx}] + \nu w_{xxt}^2 - \beta\theta w_{xxt} - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[w_{xx}]\,\mathrm{d}r\\
&\quad -L\chi_t + g(\theta,x,t), && (2.3)\\
-\gamma\chi_t &\in \partial I_{[0,1]}(\chi) - \frac{L}{\theta_{\mathrm{ref}}}(\theta-\theta_{\mathrm{ref}}), && (2.4)\\
m_t &\in -\partial I_{[0,\infty)}(m) - h(\chi_t) + \int_0^1 \lambda(x-y)D[m,w_{xx}](y,t)\,\mathrm{d}y, && (2.5)
\end{aligned}$$

for unknown functions u, w, θ, m, χ with initial and boundary conditions

$$\left.\begin{aligned}
w(x,0) &= u(x,0) = 0\,,\\
m(x,0) &= m^0(x) = 0\,,\\
\theta(x,0) &= \theta^0(x),\\
\chi(x,0) &= \chi^0(x),
\end{aligned}\right\} \tag{2.6}$$

$$\left.\begin{aligned}
w(0,t) &= u(0,t) = w(1,t) = u(1,t) = 0\,,\\
\theta_x(0,t) &= \theta_x(1,t) = 0\,.
\end{aligned}\right\} \tag{2.7}$$

The zero initial conditions for w and m are motivated by the fact that it is difficult to determine the initial degree of fatigue for a material with unknown loading history, and the most transparent hypothesis consists in assuming that no deformation (and therefore no fatigue) has taken place prior to the time $t = 0$.

The data are required to fulfill the following hypotheses:

Hypothesis 2.1. (i) *P is a Prandtl-Ishlinskii operator (1.18) with a measurable distribution function $\varphi : [0,\infty) \times (0,\infty) \to [0,\infty)$, locally Lipschitz continuous in the first variable, and there exist $\tilde\varphi, \varphi^* \in L^1(0,\infty)$ such that $\varphi(m,r) \le \tilde\varphi(r)$, $0 \le -\varphi_m(m,r) \le \varphi^*(r)$, $|\varphi_{mm}(m,r)| \le \varphi^*(r)$ a.e., with $\tilde M := \int_0^\infty r\tilde\varphi(r)\,\mathrm{d}r < \infty$, $M := \int_0^\infty r^2\varphi^*(r)\,\mathrm{d}r < \infty$.*

(ii) *$B, \nu, \beta, \theta_{\rm ref}, \rho, \alpha, c, \kappa, L, \gamma$ are given positive constants.*

(iii) *$\lambda : \mathbb{R} \to [0,\infty)$ is a C^1 function with compact support, $\Lambda := \max\{\lambda(x) + |\lambda'(x)|\,,\ x \in \mathbb{R}\}$.*

(iv) *$f \in L^2(\Omega_T)$ is a given function for some fixed $T > 0$, such that $f_x, f_{tt}, f_{xt} \in L^2(\Omega_T)$.*

(v) *$\theta^0 \in L^\infty(0,1)$ and $\chi^0 \in W^{1,2}(\Omega)$ are such that $\theta^0 \ge \theta_* > 0$, $\theta^0_x \in L^2(0,1)$, $\chi^0(x) \in [0,1]$ for all $x \in [0,1]$.*

(vi) *$h : \mathbb{R} \to [0,\infty)$ is a nondecreasing Lipschitz continuous function such that $h(z) \le bz^2$, $0 \le h'(z) \le a$ a.e. for $z \in \mathbb{R}$, and a, b are positive constants such that $bM \le \gamma$, where M is as in (i) and γ is the relaxation coefficient from (2.4).*

(vii) *$g : [0,\infty)\times\Omega_T \to \mathbb{R}$ is a Carathéodory function such that $g_0(x,t) := g(0,x,t) \ge 0$, $g_0 \in L^2(\Omega_T)$, and $|g_\theta(\theta,x,t)| \le g_1$ a.e. with g_1 a constant.*

The assumption that $\varphi(m,r)$ decreases with increasing fatigue m corresponds to the observation that the stiffness of the material decreases with increasing fatigue. Also the assumption that $g_0(x,t) \ge 0$ makes sense. Note that g is the heat source density, so that at zero temperature, we cannot remove heat from the system.

We now check that regular solutions of (2.1)–(2.7) satisfy (1.6)–(1.7) with $\varepsilon = w_{xx}$ and σ given by (1.1), which implies the thermodynamic consistency of the system. Indeed, we have by (2.3) and (1.21)

$$\mathcal{U}(\varepsilon,\theta,\chi)_t + q_x - \varepsilon_t\sigma = g$$

and by (1.4), (2.3), and (2.4)

$$\begin{aligned} &\mathcal{S}(\varepsilon,\theta,\chi)_t + \left(\frac{q}{\theta}\right)_x - \frac{g}{\theta} \\ = \ &\frac{\kappa\theta_x^2}{\theta^2} + \frac{\nu\varepsilon_t^2}{\theta} + \frac{1}{\theta}\left(D[m,\varepsilon](t) - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r + \gamma\chi_t^2\right). \end{aligned}$$

By Hypothesis 2.1 (i) and (2.5),

$$D[m,\varepsilon](t) - \frac{1}{2}m_t\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[\varepsilon](t)\,\mathrm{d}r + \gamma\chi_t^2 \ge \gamma\chi_t^2 - \frac{M}{2}h(\chi_t),$$

hence, by Hypothesis 2.1 (vi)

$$\mathcal{S}(\varepsilon,\theta,\chi)_t + \left(\frac{q}{\theta}\right)_x \ge \frac{\kappa\theta_x^2}{\theta^2} + \frac{\nu\varepsilon_t^2}{\theta} + \frac{\gamma\chi_t^2}{2\theta} \ge 0,$$

provided we check that the absolute temperature θ stays positive. In Subsection 5.1, we will find a positive lower bound for the discrete approximations of the temperature, which is independent of the discretization parameter, and therefore is preserved in the limit and implies the positivity of the temperature.

The main result of this paper reads as follows.

Theorem 2.2. *Let Hypothesis 2.1 hold. Then there exists a unique solution to the system (2.1)–(2.7) in Ω_T such that $\theta(x,t) > 0$ for all $(x,t) \in \Omega_T$, and with the regularity*

- $w_{xxxt}, w_{xxtt}, \theta_t, \theta_{xx}, u_{tt}, u_{xxt} \in L^2(\Omega_T)$,
- $\theta, m_t, \chi_t \in L^\infty(\Omega_T)$.

3. **Function spaces, interpolation.** Let $p, q, s \in [1, \infty]$ be such that $q > s$, and let $|\cdot|_p$ denote the norm in $L^p(0,1)$, $||\cdot||_p$ the norm in $L^p(\Omega_T)$.

The *Gagliardo-Nirenberg inequality* states that there exists a constant $C > 0$ such that for every $v \in W^{1,p}(0,1)$ we have

$$|v|_q \le C\left(|v|_s + |v|_s^{1-\varrho}|v'|_p^{\varrho}\right), \quad \varrho = \frac{\frac{1}{s}-\frac{1}{q}}{1+\frac{1}{s}-\frac{1}{p}}. \tag{3.1}$$

In fact, (3.1) is straightforward: If we introduce an auxiliary parameter $r = 1 + s(1-\frac{1}{p})$ and use the chain rule $\frac{\mathrm{d}}{\mathrm{d}x}|v(x)|^r \le r|v(x)|^{r-1}|v'(x)|$ a.e., we obtain from Hölder's inequality the estimate

$$|v|_\infty \le |v|_r + C|v|_s^{1-(1/r)}|v'|_p^{1/r}.$$

Combined with the obvious interpolation inequality $|v|_h \le |v|_\infty^{1-(s/h)}|v|_s^{s/h}$ for $h = q$ (and for $h = r$, if $r > s$), this yields (3.1).

Let now $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ be a vector, and let $\mathbf{D}\mathbf{v} := (n(v_1 - v_0), \ldots, n(v_n - v_{n-1}))$ be the associated vector of difference quotients of $\mathbf{v}$. We denote

$$|\mathbf{v}|_p := \left(\frac{1}{n}\sum_{k=0}^{n}|v_k|^p\right)^{1/p}, \quad |\mathbf{D}\mathbf{v}|_p := \left(n^{p-1}\sum_{k=1}^{n}|v_k - v_{k-1}|^p\right)^{1/p}. \tag{3.2}$$

The discrete counterpart of (3.1) reads

$$|\mathbf{v}|_q \le C\left(|\mathbf{v}|_s + |\mathbf{v}|_s^{1-\varrho}|\mathbf{D}\mathbf{v}|_p^{\varrho}\right), \quad \varrho = \frac{\frac{1}{s}-\frac{1}{q}}{1+\frac{1}{s}-\frac{1}{p}}, \tag{3.3}$$

and can be easily derived from (3.1) by defining v as e.g. equidistant piecewise linear interpolations of v_k.

4. **Formal estimates.** In order to explain the estimation technique, we first proceed formally, assuming that the positivity of temperature is already established. Rigorous proofs based on a space semi-discrete scheme will be carried out in Section 5. For the sake of simplicity we set from now on

$$\mathcal{K}[m, w_{xx}](x,t) := -\frac{1}{2}\int_0^\infty \varphi_m(m,r)\mathfrak{s}_r^2[w_{xx}](x,t)\,\mathrm{d}r \tag{4.1}$$

$$\mathcal{D}[m, w_{xx}](x,t) := \int_0^1 \lambda(x-y)D[m, w_{xx}](y,t)\,\mathrm{d}y. \tag{4.2}$$

Due to the fact that by Definition 1.1 we have $|\mathfrak{s}_r[w_{xx}](t)| \le r$ and from Hypothesis 2.1 (i) we deduce

$$0 \le \mathcal{K}[m, w_{xx}] \le \frac{1}{2}\int_0^\infty r^2\varphi^*(r)\,\mathrm{d}r = \frac{M}{2}. \tag{4.3}$$

Finally, due to Hypothesis 2.1 (i) and (iii) and (1.22), we have

$$0 \le \mathcal{D}[m, w_{xx}](x,t) \le \Lambda\tilde{M}\,|w_{xxt}(t)|_1. \tag{4.4}$$

We will denote in the sequel by C any constant possibly depending only on the constants in Hypothesis 2.1 and on the initial data of the problem.

4.1. **The energy estimate.** We multiply (2.1) by w_{xxt}, obtaining

$$- u_t\, w_{xxt} + Bw_{xx}\, w_{xxt} + P[m, w_{xx}]w_{xxt} + \nu w_{xxt}^2 - \beta(\theta - \theta_{\mathrm{ref}})w_{xxt} = 0; \qquad (4.5)$$

we differentiate (2.2) in time and multiply by w_t, getting

$$\rho w_{tt} w_t - \alpha w_{xxtt} w_t + u_{xxt} w_t - f_t w_t = 0, \qquad (4.6)$$

and finally we sum up (4.5), (4.6) and (2.3), all integrated in space. The first term in (4.5) simplifies with the third term in (4.6) due to integration by parts and our choice of boundary conditions; moreover the viscosity terms and the term $\beta\theta$ cancel out. Concerning the term with hysteresis, using (1.21) we deduce

$$P[m, w_{xx}]w_{xxt} = \frac{\mathrm{d}}{\mathrm{d}t} V[m, w_{xx}] + D[m, w_{xx}] + m_t\, \mathcal{K}[m, w_{xx}] \qquad (4.7)$$

and thus in the sum of (4.5), (4.6) and (2.3) what remains is just the term containing V. More precisely we have the *energy balance*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(\frac{1}{2} B w_{xx}^2 + V[m, w_{xx}] + \beta\theta_{\mathrm{ref}} w_{xx} + \frac{1}{2}\rho w_t^2 + \frac{1}{2}\alpha w_{xt}^2 + c\theta + L\chi \right) \mathrm{d}x \\ = \;& \int_0^1 (f_t\, w_t\, + g)\, \mathrm{d}x, \end{aligned} \qquad (4.8)$$

and Gronwall's argument together with Hypothesis 2.1 (iv) and (vii) gives the estimate

$$\forall t \in [0, T] : \; |w_{xx}(t)|_2 + |w_t(t)|_2 + |w_{xt}(t)|_2 + |\theta(t)|_1 \le C. \qquad (4.9)$$

4.2. **The Dafermos estimate.** We test the equation for the temperature (2.3) by $\theta^{-1/3}$ and obtain, using notations (4.1) and (4.2), that

$$\begin{aligned} 0 \;=\; & \int_0^1 -c\theta_t \theta^{-1/3}\, \mathrm{d}x + \int_0^1 \kappa\theta_{xx}\theta^{-1/3}\, \mathrm{d}x + \int_0^1 m_t \mathcal{K}[m, w_{xx}]\theta^{-1/3}\, \mathrm{d}x \\ & + \int_0^1 D[m, w_{xx}]\theta^{-1/3}\, \mathrm{d}x + \int_0^1 \nu w_{xxt}^2 \theta^{-1/3}\, \mathrm{d}x - \int_0^1 \beta w_{xxt}\theta^{2/3}\, \mathrm{d}x \\ & - \int_0^1 L\chi_t \theta^{-1/3}\, \mathrm{d}x + \int_0^1 g\, \theta^{-1/3}\, \mathrm{d}x \\ =: \;& T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8. \end{aligned} \qquad (4.10)$$

We keep the terms $T_5 = \nu \int_0^1 w_{xxt}^2 \theta^{-1/3}\, \mathrm{d}x$, $T_6 = -\beta \int_0^1 w_{xxt}\theta^{2/3}\, \mathrm{d}x$, and

$$T_2 = \int_0^1 \kappa\theta_{xx}\theta^{-1/3}\, \mathrm{d}x = \frac{\kappa}{3} \int_0^1 \left[\theta^{-2/3}\theta_x\right]^2 \mathrm{d}x = 3\kappa \int_0^1 \left[(\theta^{1/3})_x\right]^2 \mathrm{d}x, \qquad (4.11)$$

where we have integrated by parts and used the boundary conditions (2.7). All the other terms will be estimated from below. First,

$$T_1 = -c \int_0^1 \theta_t \theta^{-1/3}\, \mathrm{d}x = -\frac{3c}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (\theta^{2/3})\, \mathrm{d}x.$$

The quantity

$$T_3 := \int_0^1 m_t\, \mathcal{K}[m, w_{xx}]\theta^{-1/3}\, \mathrm{d}x \qquad (4.12)$$

is non-negative whenever $m_t \geq 0$. On the other hand, if $m_t < 0$, then by (2.5), (4.2), (4.4), and Hypothesis 2.1 (vi) we have

$$m_t = -h(\chi_t) + \mathcal{D}[m, w_{xx}] \geq -a\chi_t. \tag{4.13}$$

Now the assumption $m_t < 0$ implies that $\chi_t > 0$. However, by (2.4), we then have that

$$\gamma\chi_t = \frac{L}{\theta_{\mathrm{ref}}}(\theta - \theta_{\mathrm{ref}}) \leq \frac{L}{\theta_{\mathrm{ref}}}\theta \qquad a.e. \tag{4.14}$$

Combining the above inequalities with (4.3), we obtain for $m_t < 0$ that

$$T_3 \geq -\frac{MLa}{2\gamma\theta_{\mathrm{ref}}}\int_0^1 \theta^{2/3}\,\mathrm{d}x. \tag{4.15}$$

We obviously have

$$T_4 = \int_0^1 D[m, w_{xx}]\theta^{-1/3}\,\mathrm{d}x \geq 0.$$

The term

$$T_7 := -L\int_0^1 \chi_t\theta^{-1/3}\,\mathrm{d}x$$

can be treated in a similar way as the term T_3 and using (4.14) for $\chi_t \neq 0$ we get

$$T_7 \geq -\frac{L^2}{\gamma\theta_{\mathrm{ref}}}\int_0^1 \theta^{2/3}\,\mathrm{d}x.$$

Finally, we find a lower bound for T_8 by Hypothesis 2.1 (vii) as follows:

$$T_8 = \int_0^1 g(\theta,x,t)\,\theta^{-1/3}\,\mathrm{d}x \geq \int_0^1 (g(\theta,x,t) - g(0,x,t))\,\theta^{-1/3}\,\mathrm{d}x \geq -g_1\int_0^1 \theta^{2/3}\,\mathrm{d}x.$$

Coming back to (4.10), integrating it in time, we deduce

$$\begin{aligned} 3\kappa\int_0^t\int_0^1 \left[(\theta^{1/3})_x\right]^2\,\mathrm{d}x\,\mathrm{d}\tau + \nu\int_0^t\int_0^1 w_{xxt}^2\theta^{-1/3}\,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{3c}{2}\int_0^1 \theta^{2/3}\,\mathrm{d}x \\ + C_1\int_0^t\int_0^1 \theta^{2/3}\,\mathrm{d}x\ \mathrm{d}\tau + \beta\int_0^t\int_0^1 |w_{xxt}|\theta^{2/3}\,\mathrm{d}x\ \mathrm{d}\tau, \end{aligned} \tag{4.16}$$

where we put

$$C_1 := \frac{L}{\gamma\theta_{\mathrm{ref}}}\left(\frac{aM}{2} + L\right) + g_1.$$

The first two terms on the right hand side of (4.16) are bounded due to (4.9). The last term we estimate by Hölder inequality as follows

$$\begin{aligned} \beta\int_0^t\int_0^1 |w_{xxt}|\theta^{2/3}\,\mathrm{d}x\ \mathrm{d}\tau &= \beta\int_0^t\int_0^1 \theta^{5/6}\theta^{-1/6}|w_{xxt}|\,\mathrm{d}x\ \mathrm{d}\tau \\ &\leq \beta\int_0^t \left[\left(\int_0^1 \theta^{5/3}\,\mathrm{d}x\right)^{1/2}\left(\int_0^1 w_{xxt}^2\theta^{-1/3}\,\mathrm{d}x\right)^{1/2}\right]\mathrm{d}\tau \\ &\leq \frac{\nu}{2}\int_0^t\int_0^1 w_{xxt}^2\theta^{-1/3}\,\mathrm{d}x\ \mathrm{d}\tau + \frac{\beta^2}{2\nu}\int_0^t\int_0^1 \theta^{5/3}\,\mathrm{d}x\ \mathrm{d}\tau, \end{aligned}$$

and (4.16) yields that

$$\int_0^t\int_0^1 \left[(\theta^{1/3})_x\right]^2\,\mathrm{d}x\ \mathrm{d}\tau + \int_0^t\int_0^1 w_{xxt}^2\theta^{-1/3}\,\mathrm{d}x\ \mathrm{d}\tau \leq C\left(1 + \int_0^t\int_0^1 \theta^{5/3}\,\mathrm{d}x\ \mathrm{d}\tau\right). \tag{4.17}$$

Now we apply (3.1) with $v = \theta^{1/3}$, $s = 3$, $q = 5$, $p = 2$, $\rho = 4/25$ and notice that

$$|\theta^{1/3}|_3 = \left(\int_0^1 (\theta^{1/3})^3 \,\mathrm{d}x\right)^{1/3} \le C$$

due to (4.9). We therefore have

$$\left(\int_0^1 (\theta^{1/3})^5 \,\mathrm{d}x\right)^{1/5} = |\theta^{1/3}|_5 \le C\left(|\theta^{1/3}|_3 + |\theta^{1/3}|_3^{21/25} \left|(\theta^{1/3})_x\right|_2^{4/25}\right)$$

so that

$$\int_0^t \int_0^1 \theta^{5/3} \,\mathrm{d}x \,\mathrm{d}\tau \le C\left(1 + \int_0^t \left[\int_0^1 \left[(\theta^{1/3})_x\right]^2 \,\mathrm{d}x\right]^{2/5} \mathrm{d}\tau\right).$$

Combining this last estimate with (4.17), we deduce

$$\int_0^t \int_0^1 \left[(\theta^{1/3})_x\right]^2 \,\mathrm{d}x \,\mathrm{d}\tau + \int_0^t \int_0^1 w_{xxt}^2 \theta^{-1/3} \,\mathrm{d}x \,\mathrm{d}\tau \le C. \tag{4.18}$$

Applying again the Gagliardo-Nirenberg inequality with the choices $v = \theta^{1/3}$, $s = 3$, $q = 8$, $p = 2$, $\rho = 1/4$, we obtain that

$$\int_0^1 \theta^{8/3} \,\mathrm{d}x = |\theta^{1/3}|_8^8 \le C\left(|\theta^{1/3}|_3 + |\theta^{1/3}|_3^{3/4} \left|(\theta^{1/3})_x\right|_2^{1/4}\right)^8$$

and this, after time integration, together with (4.9) and (4.18) brings the estimate

$$\|\theta\|_{8/3} = \int_0^T \int_0^1 \theta^{8/3} \,\mathrm{d}x \,\mathrm{d}t \le C. \tag{4.19}$$

To derive a further estimate, we sum again (4.5) and (4.6), and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{B}{2} w_{xx}^2 + \beta\theta_{\mathrm{ref}} w_{xx} + \frac{\rho}{2} w_t^2 + \frac{\alpha}{2} w_{xt}^2\right) \mathrm{d}x + \int_0^1 \nu w_{xxt}^2 \,\mathrm{d}x \\ = \int_0^1 (\beta\theta w_{xxt} - P[m, w_{xx}]\, w_{xxt} + f_t w_t) \,\mathrm{d}x.\end{aligned}$$

We estimate the first term on the right hand side using the inequality $\beta\theta w_{xxt} \le \frac{\beta^2}{2\nu}\theta^2 + \frac{\nu}{2} w_{xxt}^2$ and the previous estimate (4.19). In the second term, $P[m, w_{xx}]$ is bounded by Hypothesis 2.1 (i), and the third term is handled using Hypothesis 2.1 (iv). This finally gives the additional estimate

$$\|w_{xxt}\|_2 \le C. \tag{4.20}$$

4.3. **Higher order estimates.** We differentiate (2.1) in space, obtaining

$$u_{xt} = B w_{xxx} + P[m, w_{xx}]_x + \nu w_{xxxt} - \beta\theta_x. \tag{4.21}$$

We integrate by parts in space, recalling that since $u(0,t) = u(1,t) = 0$ by (2.7), then $u_t(0,t) = u_t(1,t) = 0$. We deduce by (2.1) and (2.2)

$$\begin{aligned}&\int_0^1 (B w_{xxx} + P[m, w_{xx}]_x + \nu w_{xxxt} - \beta\theta_x)^2 \,\mathrm{d}x \\ \overset{(4.21)}{=} &\int_0^1 u_{xt}^2 \,\mathrm{d}x = \int_0^1 u_t(-u_{xxt}) \,\mathrm{d}x \\ = &\int_0^1 (\rho w_{tt} - \alpha w_{xxtt} - f_t)\,(B w_{xx} + P[m, w_{xx}] + \nu w_{xxt} - \beta(\theta - \theta_{\mathrm{ref}})) \,\mathrm{d}x.\end{aligned} \tag{4.22}$$

The implication $(a+b)^2 = X \Rightarrow a^2/2 \le X + 3b^2$ for all $a, b, X \in \mathbb{R}$ brings

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (\nu w_{xxxt} + B w_{xxx})^2 \,\mathrm{d}x + \nu\rho \int_0^1 w_{xt} w_{xtt} \,\mathrm{d}x + \nu\alpha \int_0^1 w_{xxt} w_{xxtt} \,\mathrm{d}x \\
\le\ & 3\int_0^1 (P[m, w_{xx}]_x^2 + \theta_x^2)\,\mathrm{d}x \\
& - \int_0^1 f_t (B w_{xx} + P[m, w_{xx}] + \nu w_{xxt} - \beta(\theta - \theta_{\mathrm{ref}}))\,\mathrm{d}x \\
& + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (\rho w_t - \alpha w_{xxt})\,(B w_{xx} + P[m, w_{xx}] - \beta(\theta - \theta_{\mathrm{ref}}))\,\mathrm{d}x \\
& - \int_0^1 (\rho w_t - \alpha w_{xxt})\,(B w_{xxt} + P[m, w_{xx}]_t - \beta\theta_t)\,\mathrm{d}x.
\end{aligned} \tag{4.23}$$

First of all, using Hypothesis 2.1 (i) and (iv), (4.9), (4.19) and (4.20) we estimate

$$\int_0^T \int_0^1 |f_t||B w_{xx} + P[m, w_{xx}] + \nu w_{xxt} - \beta(\theta - \theta_{\mathrm{ref}})|\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{4.24}$$

Furthermore, by (4.9) and Hypothesis 2.1 (i) we have

$$\int_0^1 (\rho w_t - \alpha w_{xxt})\,(B w_{xx} + P[m, w_{xx}] - \beta(\theta - \theta_{\mathrm{ref}}))\,\mathrm{d}x \le C(1 + |w_{xxt}|_2)(1 + |\theta|_2). \tag{4.25}$$

Note that by (4.19) we have

$$|\theta|_2^2 - |\theta^0|_2^2 = 2\int_0^t \int_0^1 \theta\theta_t \,\mathrm{d}x\,\mathrm{d}\tau \le C\left(\int_0^t |\theta_t|_2^2 \,\mathrm{d}\tau\right)^{1/2}, \tag{4.26}$$

and (4.25) yields that

$$\begin{aligned}
&\int_0^1 (\rho w_t - \alpha w_{xxt})\,(B w_{xx} + P[m, w_{xx}] - \beta(\theta - \theta_{\mathrm{ref}}))\,\mathrm{d}x \\
&\quad\le\ C(1 + |w_{xxt}|_2)\left(1 + \int_0^t |\theta_t|_2^2 \,\mathrm{d}\tau\right)^{1/4} \\
&\quad\le\ \frac{\nu\alpha}{4}|w_{xxt}|_2^2 + C\left(1 + \int_0^t |\theta_t|_2^2 \,\mathrm{d}\tau\right)^{1/2}.
\end{aligned} \tag{4.27}$$

Finally, still by (4.9) and (4.20),

$$\begin{aligned}
&\left|\int_0^t \int_0^1 (\rho w_t - \alpha w_{xxt})\,(B w_{xxt} + P[m, w_{xx}]_t - \beta\theta_t)\,\mathrm{d}x\,\mathrm{d}\tau\right| \\
&\quad\le\ C\left(1 + \int_0^t |P[m, w_{xx}]_t|_2^2 + |\theta_t|_2^2 \,\mathrm{d}\tau\right)^{1/2}.
\end{aligned} \tag{4.28}$$

Note that we have for a.e. $(x,t) \in \Omega_T$ that

$$|P[m, w_{xx}]_t(x,t)| \le \left| m_t \int_0^\infty \varphi_m(m, r)\mathfrak{s}_r[w_{xx}]\ \mathrm{d}r\right| + \left|\int_0^\infty \varphi(m, r)(\mathfrak{s}_r[w_{xx}])_t \,\mathrm{d}r\right|.$$

By (1.10) we have $|\mathfrak{s}_r[w_{xx}]| \le r$, and Proposition 1.2 (iv), together with (2.4), (2.5), (1.22), and Hypothesis 2.1 (i), (iii), (vi) yield

$$\begin{aligned}
&|\mathfrak{s}_r[w_{xx}]_t| \le |w_{xxt}|,\ \ |m_t| \le |-h(\chi_t) + \mathcal{D}[m, w_{xx}]| \\
\le\ & C(|\chi_t| + |w_{xxt}|_1),\ \ |\chi_t| \le C(1 + \theta)\ \ \text{a.e.}
\end{aligned} \tag{4.29}$$

From Hypothesis 2.1 (i) we thus obtain the pointwise bound

$$|P[m, w_{xx}]_t| \le C(1 + \theta + |w_{xxt}|_1) \quad \text{a.e.,} \tag{4.30}$$

and from (4.28), using (4.19) and (4.20) we conclude that

$$\left| \int_0^t \int_0^1 (\rho w_t - \alpha w_{xxt})\,(B w_{xxt} + P[m, w_{xx}]_t - \beta\theta_t)\,\mathrm{d}x\,\mathrm{d}\tau \right| \le C \left(1 + \int_0^t |\theta_t|_2^2\,\mathrm{d}\tau \right)^{1/2}. \tag{4.31}$$

We now integrate (4.23) in time from 0 to t for $t \in (0, T)$ and combine the result with (4.24), (4.27), and (4.31) to obtain

$$\begin{aligned} &\int_0^1 \left(w_{xt}^2 + w_{xxt}^2 + w_{xxx}^2\right)\,\mathrm{d}x + \int_0^t \int_0^1 (w_{xxx}^2 + w_{xxxt}^2)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad\le\ C \left(1 + \int_0^t \int_0^1 (P[m, w_{xx}]_x^2 + \theta_x^2)\,\mathrm{d}x\,\mathrm{d}\tau + \left(\int_0^t |\theta_t|_2^2\,\mathrm{d}\tau \right)^{1/2} \right). \end{aligned} \tag{4.32}$$

Here we had to estimate the initial values

$$\int_0^1 (w_{xt}^2 + w_{xxt}^2 + w_{xxx}^2)(x, 0)\,\mathrm{d}x,$$

which can be done as follows: We have by (2.6) $w_{xxx}(x, 0) = 0$. Eq. (2.2) for $t = 0$ reads

$$\rho w_t(x, 0) - \alpha w_{xxt}(x, 0) = f(x, 0),$$

and testing this identity by $w_{xxt}(x, 0)$ we see, using Hypothesis 2.1 (iv) that

$$\int_0^1 (w_{xt}^2 + w_{xxt}^2 + w_{xxx}^2)(x, 0)\,\mathrm{d}x \le C. \tag{4.33}$$

Finally, we deal with the term

$$\begin{aligned} &P[m, w_{xx}]_x(x, t) \\ &= \int_0^\infty \left(\varphi_m(m, r)\; m_x\; \mathfrak{s}_r[w_{xx}]\right)(x, t)\,\mathrm{d}r + \int_0^\infty \left(\varphi(m, r)\mathfrak{s}_r[w_{xx}]_x\right)(x, t)\,\mathrm{d}r. \end{aligned} \tag{4.34}$$

For all x, h, and t, we have by Proposition 1.2 (iii)

$$|\mathfrak{s}_r[w_{xx}](x + h, t) - \mathfrak{s}_r[w_{xx}](x, t)| \le 2 \max_{\tau \in [0,t]} |w_{xx}(x + h, \tau) - w_{xx}(x, \tau)|,$$

which implies

$$|\mathfrak{s}_r[w_{xx}]_x(x, t)| \le\ 2 \max_{\tau \in [0,t]} |w_{xxx}(x, \tau)| \quad \text{a.e.}$$

By (1.13), (1.15) and Proposition 1.2 (ii), we have

$$|\chi_t(x + h, t) - \chi_t(x, t)| + \frac{\mathrm{d}}{\mathrm{d}t}|\chi(x + h, t) - \chi(x, t)| \le 2|A_t(x + h, t) - A_t(x, t)| \quad \text{a.e.,} \tag{4.35}$$

hence

$$\begin{aligned} &\int_0^t |\chi_t(x + h, \tau) - \chi_t(x, \tau)|\,\mathrm{d}\tau \\ &\quad\le\ C \left(|\chi(x + h, 0) - \chi(x, 0)| + \int_0^t |\theta(x + h, \tau) - \theta(x, \tau)|\,\mathrm{d}\tau \right), \end{aligned}$$

which entails for a.e. $x \in (0, 1)$ that

$$\int_0^t |\chi_{xt}(x, \tau)|\,\mathrm{d}\tau \le\ C \left(|\chi_x^0(x)| + \int_0^t |\theta_x(x, \tau)|\,\mathrm{d}\tau \right),$$

and in a similar way we obtain from (1.14), (1.16), Hypothesis 2.1 (i), (vi), (1.22), Proposition 1.2 (ii), and (4.20) that

$$
\begin{aligned}
|m_x(x,t)| &\le \int_0^t |m_{xt}(x,\tau)|\,\mathrm{d}\tau \le C\left(\int_0^t |\chi_{xt}(x,\tau)|\,\mathrm{d}\tau + \int_0^t |w_{xxt}|_1(\tau)\,\mathrm{d}\tau\right)\\
&\le C\left(1+\int_0^t |\chi_{xt}(x,\tau)|\,\mathrm{d}\tau\right),
\end{aligned}
$$

where we also used that $m(x,0)=0$. Therefore, from (4.35), using Hypothesis 2.1 (i),(iii) and (2.6), we get for a.e. $(x,t)\in\Omega_T$ that

$$
\begin{aligned}
|P[m,w_{xx}]_x(x,t)| &\le C\left(1+|\chi_x^0(x)|+\int_0^t |\theta_x(x,\tau)|\,\mathrm{d}\tau + \max_{\tau\in[0,t]} |w_{xxx}(x,\tau)|\right)\\
&\le C\left(1+|\chi_x^0(x)|+\int_0^t |\theta_x(x,\tau)|\,\mathrm{d}\tau + \int_0^t |w_{xxxt}(x,\tau)|\,\mathrm{d}\tau\right).
\end{aligned}
\tag{4.36}
$$

Combining (4.32) and (4.36) and applying Gronwall's lemma we deduce

$$
|w_{xxt}(t)|_2^2 + |w_{xxx}(t)|_2^2 + ||w_{xxxt}^2||_2^2 \le C\left(1+||\theta_x||_2^2+||\theta_t||_2\right), \tag{4.37}
$$

where we also used Hypothesis 2.1 (v) to estimate $|\chi_x^0(x)|$.

It remains to estimate the $W^{1,2}$-norm of θ both in space and time. In the first step, we test (2.3) by θ and obtain, using (4.29), (1.22), and Hypothesis 2.1 (i), (vii), that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta^2\,\mathrm{d}x + \int_0^1 \theta_x^2\,\mathrm{d}x\\
&\le C\int_0^1 \left(|w_{xxt}|\theta + \theta|w_{xxt}|^2 + \theta^2|w_{xxt}| + |m_t|\theta + |\chi_t|\theta + \theta|g|\right)\,\mathrm{d}x\\
&\le C\left(1+\int_0^1 \left(\theta^2 + w_{xxt}^2 + \theta w_{xxt}^2 + \theta^2|w_{xxt}|\right)\,\mathrm{d}x\right).
\end{aligned}
\tag{4.38}
$$

By Hölder's inequality and (4.19), we have

$$
\int_0^t\int_0^1 \theta^2|w_{xxt}|\,\mathrm{d}x\,\mathrm{d}t \le \|\theta\|_{8/3}^2\left(\int_0^t\int_0^1 w_{xxt}^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/4} \le C\left(\int_0^t\int_0^1 w_{xxt}^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/4},
$$

and

$$
\int_0^t\int_0^1 \theta w_{xxt}^2\,\mathrm{d}x\,\mathrm{d}t \le \|\theta\|_2^2\left(\int_0^t\int_0^1 w_{xxt}^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \le C\left(\int_0^t\int_0^1 w_{xxt}^4\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.
$$

Exploiting once more (4.19) and (4.20), we finally obtain integrating (4.38) with respect to t that

$$
|\theta(t)|_2^2 + ||\theta_x||_2^2 \le C\left(1+||w_{xxt}||_4^2\right). \tag{4.39}
$$

On the other hand, testing (2.3) by θ_t we deduce

$$
\int_0^1 \theta_t^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_x^2\,\mathrm{d}x \le C\left(1+\int_0^1 (m_t^2+w_{xxt}^2+\chi_t^2)\,\mathrm{d}x + \int_0^1 w_{xxt}^4\,\mathrm{d}x + \int_0^1 \theta^2 w_{xxt}^2\,\mathrm{d}x\right)
$$

and by a similar argument as above we obtain

$$
\|\theta_t\|_2^2 + \sup_{t\in[0,T]} |\theta_x(t)|_2^2 \le C\left(1+\|w_{xxt}\|_4^4 + \|\theta\|_4^4\right). \tag{4.40}
$$

We have by (4.19) that

$$
\|\theta\|_4^4 \le \|\theta\|_\infty^{4/3}\|\theta\|_{8/3}^{8/3} \le C\|\theta\|_\infty^{4/3}. \tag{4.41}
$$

We now apply the Gagliardo-Nirenberg inequality (3.1) with $q = \infty$, $s = 1$, $p = 2$, and $\gamma = 2/3$, to deduce

$$|\theta|_\infty \le C\left(|\theta|_1 + |\theta|_1^{1/3}|\theta_x|_2^{2/3}\right).$$

Using (4.9) we obtain

$$\|\theta\|_\infty \le C\left(1 + \sup_{t\in[0,T]} |\theta_x|_2^{2/3}\right). \tag{4.42}$$

It follows from (4.40), (4.41), (4.42) that

$$\|\theta_t\|_2^2 + \sup_{t\in[0,T]} |\theta_x(t)|_2^2 \le C\left(1 + \|w_{xxt}\|_4^4\right), \tag{4.43}$$

which is what we were looking for. Coming back to (4.37), using (4.39) and (4.43) we deduce in particular

$$\sup_{t\in[0,T]} |w_{xxt}(t)|_2^2 + \|w_{xxxt}\|_2^2 \le C(1 + \|w_{xxt}\|_4^2). \tag{4.44}$$

We estimate the right hand side of (4.44) using the Gagliardo-Nirenberg inequality (3.1) with $q = 4$, $s = p = 2$, $\gamma = 1/4$, and obtain

$$|w_{xxt}(t)|_4 \le C\left(|w_{xxt}(t)|_2 + |w_{xxt}(t)|_2^{3/4}\,|w_{xxxt}(t)|_2^{1/4}\right),$$

and this implies, by Hölder's inequality and (4.20) that

$$\begin{aligned} \|w_{xxt}\|_4^4 &\le C \sup_{t\in[0,T]} |w_{xxt}(t)|_2^2 \left(||w_{xxt}||_2^2 + \|w_{xxt}\|_2\|w_{xxxt}\|_2\right) \\ &\le C \sup_{t\in[0,T]} |w_{xxt}(t)|_2^2 \left(1 + \|w_{xxxt}\|_2\right). \end{aligned} \tag{4.45}$$

Using this last estimate and coming back to (4.44) we get

$$||w_{xxt}||_4^2 \le C \sup_{t\in[0,T]} |w_{xxt}(t)|_2 \left(1 + ||w_{xxxt}||_2^{1/2}\right) \le \left(1 + ||w_{xxt}||_4^{3/2}\right),$$

which enables us to conclude that

$$\|w_{xxt}\|_4 \le C, \tag{4.46}$$

and consequently by (4.42)–(4.43)

$$\|\theta\|_\infty^2 + \|\theta_t\|_2^2 + \sup_{t\in[0,T]} |\theta_x(t)|_2^2 \le C. \tag{4.47}$$

Coming back to (4.29) and (4.37), we deduce the following additional estimate

$$\|m_t\|_\infty + \|\chi_t\|_\infty + \sup_{t\in[0,T]} \left(|w_{xxt}(t)|_2 + |w_{xxx}(t)|_2\right) + \|w_{xxxt}\|_2 \le C. \tag{4.48}$$

Finally, we differentiate (2.1) in t and test by w_{xxtt}, differentiate (2.2) twice in t and test by w_{tt}, sum up the results, eliminating the terms in u_{tt} by integrating by parts. We have to estimate the initial values

$$\int_0^1 \left(|w_{tt}^2(x,0)| + |w_{xtt}^2(x,0)|\right) \mathrm{d}x.$$

To do that, we proceed similarly as in (4.33). We have by (2.6) and (2.1)

$$u_t(x,0) = \nu w_{xxt}(x,0) - \beta(\theta^0(x) - \theta_{\mathrm{ref}}),$$

hence

$$u_{xt}(x,0) = \nu w_{xxxt}(x,0) - \beta\theta_x^0(x). \tag{4.49}$$

On the other hand, by (2.2),

$$\rho w_{xt}(x,0) - \alpha w_{xxxt}(x,0) = f_x(x,0), \tag{4.50}$$

and

$$\rho w_{tt}(x,0) - \alpha w_{xxtt}(x,0) = -u_{xxt}(x,0) + f_t(x,0). \tag{4.51}$$

We estimate the initial value $f_x(x,0)$ in $L^2(0,1)$ using Hypothesis 2.1 (iv) and the formula $f_x(x,0) = f_x(x,t) - \int_0^t f_{xt}(x,\tau)\,\mathrm{d}\tau$ square integrated in space and time, and similarly for f_t. We further test (4.50) by $w_{xxxt}(x,0)$, use (4.33), and obtain

$$\int_0^1 w_{xxxt}^2(x,0)\,\mathrm{d}x \le C. \tag{4.52}$$

Hence, by (4.49) and Hypothesis 2.1 (v)

$$\int_0^1 u_{xt}^2(x,0)\,\mathrm{d}x \le C. \tag{4.53}$$

Testing (4.51) by $w_{tt}(x,0)$ and integrating by parts we finally obtain

$$\begin{aligned} &\int_0^1 (\rho|w_{tt}^2(x,0)| + \alpha|w_{xtt}^2(x,0)|)\,\mathrm{d}x \\ \le\ &\int_0^1 (|u_{xt}(x,0)|\,|w_{xtt}(x,0)| + |f_t(x,0)|\,|w_{tt}(x,0)|)\,\mathrm{d}x, \end{aligned} \tag{4.54}$$

which implies the desired estimate

$$\int_0^1 (|w_{tt}^2(x,0)| + |w_{xtt}^2(x,0)|)\,\mathrm{d}x \le C. \tag{4.55}$$

This enables us to conclude that

$$\sup_{t\in[0,T]} (|w_{tt}(t)|_2 + |w_{xtt}(t)|_2) + \|w_{xxtt}\|_2 \le C. \tag{4.56}$$

5. **Approximation.** Here, we make rigorous the estimates derived formally in the previous section. From now on, the values of all physical constants are set to 1 for simplicity.

We choose an integer $n \in \mathbb{N}$, and consider the space discrete approximations of (2.1)–(2.4) for $k = 1, \dots n-1$:

$$\dot{u}_k = \varepsilon_k + P[m_k, \varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta_{\mathrm{ref}}\,, \tag{5.1}$$

$$\dot{w}_k - \dot{\varepsilon}_k = -n^2(u_{k+1} - 2u_k + u_{k-1}) + f_k\,, \tag{5.2}$$

$$\varepsilon_k = n^2(w_{k+1} - 2w_k + w_{k-1})\,, \tag{5.3}$$

$$\dot{\theta}_k = n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1}) + \dot{m}_k\,\mathcal{K}_k + D_k + \dot{\varepsilon}_k^2 - \theta_k\dot{\varepsilon}_k - \dot{\chi}_k + g_k(\theta_k, t)\,, \tag{5.4}$$

$$m_k = \mathfrak{s}_{[0,\infty)}[0, S_k], \qquad S_k(t) = \int_0^t (-h(\dot{\chi}_k) + \mathcal{D}_k^*)(\tau)\,\mathrm{d}\tau, \tag{5.5}$$

$$\chi_k = \mathfrak{s}_{[0,1]}[\chi_k^0, A_k], \qquad A_k(t) = \int_0^t (\theta_k - \theta_{\mathrm{ref}})(\tau)\,\mathrm{d}\tau, \tag{5.6}$$

where

$$
\begin{aligned}
&\mathcal{K}_k(t) = -\frac{1}{2}\int_0^\infty \varphi_m(m_k(t),r)\mathfrak{s}_r^2[\varepsilon_k](t)\,\mathrm{d}r \in \left[0,\frac{M}{2}\right],\\
&D_k(t) = \int_0^\infty \varphi(m_k(t),r)\,\mathfrak{s}_r[\varepsilon_k](t)(\varepsilon_k - \mathfrak{s}_r[\varepsilon_k])_t(t)\,\mathrm{d}r \ge 0\,,\\
&\mathcal{D}_k^*(t) = \frac{1}{n}\sum_{j=1}^{n-1}\lambda_{k-j}D_j(t) \ge 0\,,\\
&\lambda_i = \lambda(i/n)\,,\\
&f_k(t) = n\int_{(k-1)/n}^{k/n} f(x,t)\,\mathrm{d}x\,,\\
&g_k(\theta,t) = \begin{cases} n\displaystyle\int_{(k-1)/n}^{k/n} g(\theta,x,t)\,\mathrm{d}x & \text{for } \theta\ge 0\\ g_k(0,t) & \text{for } \theta<0. \end{cases}
\end{aligned}
$$

We prescribe initial conditions for $k = 1,\dots,n-1$

$$
\left.\begin{aligned}
&w_k(0) = u_k(0) = 0\,,\\
&\theta_k(0) = \theta_k^0 := \theta^0(k/n)\,,\\
&m_k(0) = 0\,,\\
&\chi_k(0) = \chi_k^0 := n\int_{(k-1)/n}^{k/n}\chi^0(x)\,\mathrm{d}x\,,
\end{aligned}\right\} \tag{5.7}
$$

and “boundary conditions”

$$
\left.\begin{aligned}
&w_0 = w_n = u_0 = u_n = 0\,,\\
&\theta_0 = \theta_1,\ \theta_n = \theta_{n-1}\,.
\end{aligned}\right\} \tag{5.8}
$$

This is a system of ODEs for u_k, w_k, θ_k. We proceed as follows:

We claim that (5.1)–(5.6) admits a $W^{1,\infty}$ solution in an interval $[0,T_n]$. First, denoting by $\mathbf{w}$ the vector $(w_1,\dots,w_{n-1})$, and $\boldsymbol{\varepsilon} = (\varepsilon_1,\dots,\varepsilon_{n-1})$, we have, by (5.3), $-\boldsymbol{\varepsilon} = S\mathbf{w}$ with a positive definite matrix S, which has the form

$$
S = n^2\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
-1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\
0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2
\end{bmatrix}
$$

Hence, the left hand side of (5.2) reads $(I+S)\dot{\mathbf{w}}$. By (5.2), $\dot{\boldsymbol{\varepsilon}}$ is itself a Lipschitz continuous mapping of $\mathbf{u} = (u_1,\dots,u_{n-1})$. Using Proposition 1.2 (ii) we see that (5.1)–(5.4) can be considered as an ODE system in u_k, w_k, θ_k, with a locally Lipschitz continuous and locally bounded right hand side and the existence and uniqueness of a local solution in an interval $[0,T_n]$ follows from the standard theory of ODEs, and the solution belongs to $W^{1,\infty}(0,T_n)$.

In the sequel, we will systematically use the "summation by parts formula"

$$\sum_{k=1}^{n-1} \xi_k(\eta_{k+1} - 2\eta_k + \eta_{k-1}) + \sum_{k=1}^{n} (\xi_k - \xi_{k-1})(\eta_k - \eta_{k-1}) = \xi_n(\eta_n - \eta_{n-1}) - \xi_0(\eta_1 - \eta_0) \tag{5.9}$$

for all vectors $(\xi_0, \dots, \xi_n)$, $(\eta_0, \dots, \eta_n)$.

5.1. **Positivity of the temperature.** In this subsection, we prove that θ_k remain positive in the whole range of existence. As a first step, we test (5.4) by $-\theta_k^-$, where θ_k^- is the negative part of θ_k.

We have

$$-\frac{1}{n}\sum_{k=1}^{n-1} \dot\theta_k \theta_k^- = \frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{n-1} (\theta_k^-)^2.$$

On the other hand, by (5.9) we also have

$$\begin{aligned} -n\sum_{k=1}^{n-1} (\theta_{k+1} - 2\theta_k + \theta_{k-1})\theta_k^- &= n\sum_{k=1}^{n} (\theta_k - \theta_{k-1})(\theta_k^- - \theta_{k-1}^-) \\ &\le -n\sum_{n=1}^{n} (\theta_k^- - \theta_{k-1}^-)^2 \le 0, \end{aligned}$$

Moreover, $D_k(t) \ge 0$ and $g_k(\theta, t) \ge 0$ for $\theta \le 0$ by Hypothesis 2.1 (vii), hence

$$-\left(D_k(t) + \dot\varepsilon_k^2(t) + g_k(\theta_k, t)\right)\theta_k^- \le 0.$$

Now we deal with the phase term. We have that

$$\left.\begin{array}{ll} \dot\chi_k(t)\theta_k^-(t) = 0 & \text{if } \dot\chi_k(t) = 0, \\ \dot\chi_k(t)\theta_k^-(t) = (\theta_k(t) - \theta_{\mathrm{ref}})\theta_k^-(t) \le 0 & \text{otherwise.} \end{array}\right\} \tag{5.10}$$

Finally, if $\dot m_k(t) \ne 0$, then

$$-\dot m_k(t)\mathcal{K}_k(t)\theta_k^-(t) = \left(h(\dot\chi_k(t)) - \mathcal{D}_k^*(t)\right)\theta_k^-(t)\mathcal{K}_k(t) \le h(\dot\chi_k(t))\theta_k^-(t)\mathcal{K}_k(t) \le 0$$

by virtue of (5.10) and Hypothesis 2.1 (vi). Summarizing the above computations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2n}\sum_{k=1}^{n-1} (\theta_k^-)^2 \le \frac{1}{n}\sum_{k=1}^{n-1} (\theta_k^-)^2 \dot\varepsilon_k \le \frac{K_{\varepsilon,n}}{n}\sum_{k=1}^{n-1} (\theta_k^-)^2,$$

where we put

$$K_{\varepsilon,n} := \max\{|\dot\varepsilon_k(t)| : k = 1, \dots, n-1, \ \ t \in [0, T_n]\}$$

and Gronwall's argument yields $\theta_k^-(t) = 0$ for all k and $t \in [0, T_n]$.

We now prove that in fact, $\theta_k(t)$ are bounded away from 0 for all k and all $t \in [0, T_n]$. First of all we notice that if $\dot\chi_k \ne 0$ then

$$-\dot\chi_k = -\theta_k + \theta_{\mathrm{ref}} \ge -\theta_k.$$

We moreover have

$$\dot m_k \ge -h(\dot\chi_k) \ge -a\dot\chi_k \ge -a\theta_k,$$

from which we deduce

$$\dot m_k \mathcal{K}_k \ge -\frac{Ma}{2}\theta_k.$$

Using the estimates above together with

$$\dot\varepsilon_k^2 - \theta_k\dot\varepsilon_k \ge -\frac{1}{4}\theta_k^2$$

we obtain from (5.4) that

$$\dot{\theta}_k - n^2(\theta_{k+1} - 2\theta_k + \theta_{k-1}) \geq -\psi(\theta_k),$$

where we set

$$\psi(z) := \frac{1}{4}z^2 + a\left(1 + \frac{M}{2}\right) z.$$

Let p be the solution of the differential equation

$$\dot{p} + \psi(p) = 0\,, \quad p(0) = \theta_*,$$

with $\theta_* > 0$ from Hypothesis 2.1 (v). It is easy to check that

$$p(t) = \frac{\mu\theta_* e^{-\mu t}}{\delta\theta_*(1 - e^{-\mu t}) + \mu}\,, \qquad \text{with} \qquad \delta = \frac{1}{4}, \qquad \mu = a\left(1 + \frac{M}{2}\right).$$

Then

$$(\dot{p} - \dot{\theta}_k) - n^2((p - \theta_{k+1}) - 2(p - \theta_k) + (p - \theta_{k-1})) + \psi(p) - \psi(\theta_k) \leq 0\,. \tag{5.11}$$

Testing (5.11) by $(p - \theta_k)^+$ and using (5.9), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{n-1}((p-\theta_k)^+)^2 + n^2\sum_{k=1}^{n}((p-\theta_k)^+ - (p-\theta_{k-1})^+)^2 + (\psi(p) - \psi(\theta_k))(p-\theta_k)^+ \leq 0\,, \tag{5.12}$$

hence, as ψ is nondecreasing for positive arguments,

$$\sum_{k=1}^{n-1}((p - \theta_k)^+)^2(t) \leq \sum_{k=1}^{n-1}((p - \theta_k)^+)^2(0) = 0,$$

so that $\theta_k(t) \geq p(t) > 0$ for all k and all $t \in [0, T_n]$, which is the desired result.

5.2. **Discrete energy estimate.** We test (5.1) by $\dot{\varepsilon}_k$, differentiate (5.2) in t and test it by $\dot{w}_k$, and sum up over $k = 1, \dots n-1$. From (5.3), with a repeated use of (5.8) and (5.9), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2n}\sum_{k=1}^{n-1}\dot{w}_k^2 + \frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})^2\right) \\ &+\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k(\varepsilon_k + P[m_k, \varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta_{\mathrm{ref}}) = \frac{1}{n}\sum_{k=1}^{n-1}\dot{f}_k\dot{w}_k\,. \end{aligned} \tag{5.13}$$

We add (5.14) to (5.4), which yields, by virtue of (1.21),

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2}\varepsilon_k^2 + V[m_k, \varepsilon_k] + \theta_{\mathrm{ref}}\varepsilon_k + \frac{1}{2}\dot{w}_k^2 + \theta_k + \chi_k\right)\right. \\ &\left.+\frac{n}{2}\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})^2\right) = \frac{1}{n}\sum_{k=1}^{n-1}(\dot{f}_k\dot{w}_k + g_k). \end{aligned} \tag{5.14}$$

We estimate the right hand side of (5.14) using the *discrete Hölder inequality*

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k\eta_k \leq \left(\frac{1}{n}\sum_{k=1}^{n}|\xi_k|^p\right)^{1/p}\left(\frac{1}{n}\sum_{k=1}^{n}|\eta_k|^{p'}\right)^{1/p'} \tag{5.15}$$

for all vectors $(\xi_1, \dots, \xi_n)$, $(\eta_1, \dots, \eta_n)$, and for $1/p + 1/p' = 1$. We have

$$\frac{1}{n}\sum_{k=1}^{n-1} \dot{f}_k^2(t) \le \int_0^1 f_t^2(x,t)\,\mathrm{d}x\,,$$

hence, by (5.15), Hypothesis 2.1 (vii), and Gronwall's lemma,

$$\frac{1}{n}\sum_{k=1}^{n-1}\Big(\dot{w}_k^2 + \varepsilon_k^2 + \theta_k\Big)(t) + n\sum_{k=1}^{n}(\dot{w}_k - \dot{w}_{k-1})^2(t) \le C\,, \tag{5.16}$$

We conclude in particular that the approximate solutions exist globally and $T_n = T$.

5.3. **The discrete Dafermos estimate.** We test (5.4) by $\theta_k^{-1/3}$ and we proceed similarly as in Subsection 4.2. The integration by parts is replaced by the elementary inequality

$$-(x-y)(x^{-1/3} - y^{-1/3}) \ge 3(x^{1/3} - y^{1/3})^2,$$

with the choice $x = \theta_k, y = \theta_{k-1}$. We obtain for all $t \in [0,T]$ after summing up from $k = 1, ..., n-1$ and integrating in time the following counterpart of (4.16):

$$\begin{aligned}&\int_0^t \left(3n\sum_{k=1}^{n}\left(\theta_k^{1/3} - \theta_{k-1}^{1/3}\right)^2 + \frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\,\theta_k^{-1/3}\right)\mathrm{d}\tau \qquad (5.17)\\ \le\ &\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k|\theta_k^{2/3}\,\mathrm{d}\tau + \frac{3}{2n}\sum_{k=1}^{n-1}\theta_k^{2/3}(t) + C_1\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{2/3}\,\mathrm{d}\tau.\end{aligned}$$

The last two terms on the right hand side are bounded by virtue of (5.16). By (5.15),

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}|\dot{\varepsilon}_k|\theta_k^{2/3}\,\mathrm{d}\tau \le \left(\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{5/3}\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{-1/3}\dot{\varepsilon}_k^2\,\mathrm{d}\tau\right)^{1/2},$$

hence,

$$\int_0^t \left(\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\theta_k^{-1/3} + 3n\sum_{k=1}^{n}\left(\theta_k^{1/3} - \theta_{k-1}^{1/3}\right)^2\right)\mathrm{d}\tau \le C\left(1 + \int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{5/3}\,\mathrm{d}\tau\right). \tag{5.18}$$

We now apply the inequality (3.3) as in the formal case, which implies that

$$\int_0^t \left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{-1/3}\dot{\varepsilon}_k^2 + n\sum_{k=1}^{n}\left(\theta_k^{1/3} - \theta_{k-1}^{1/3}\right)^2\right)\mathrm{d}\tau \le C\,. \tag{5.19}$$

Using (3.3) again we obtain

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}(\tau)\,\mathrm{d}\tau \le C\,, \tag{5.20}$$

and, as a consequence of (5.14),

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(\tau)\,\mathrm{d}\tau \le C\,. \tag{5.21}$$

5.4. **Higher order discrete estimates.** We define $\varepsilon_0, \varepsilon_n$ for $k = 0, k = n$, as solutions to the differential equations

$$\left.\begin{aligned} \varepsilon_0 + P[m_0, \varepsilon_0] + \dot\varepsilon_0 - \theta_0 + \theta_{\mathrm{ref}} &= 0\,,\\ \varepsilon_n + P[m_n, \varepsilon_n] + \dot\varepsilon_n - \theta_n + \theta_{\mathrm{ref}} &= 0\,, \end{aligned}\right\} \tag{5.22}$$

with initial conditions $\varepsilon_0(0) = \varepsilon_n(0) = 0$. The values of m_0, m_n are chosen as in (5.5), where we choose for χ the natural "boundary" conditions compatible with (5.8), that is,

$$\chi_0(t) = \chi_1(t), \quad \chi_n(t) = \chi_{n-1}(t).$$

Then (5.1) holds for all $k = 0, \dots, n$, and we have

$$\dot u_k - \dot u_{k-1} = \varepsilon_k - \varepsilon_{k-1} + P[m_k, \varepsilon_k] - P[m_{k-1}, \varepsilon_{k-1}] + \dot\varepsilon_k - \dot\varepsilon_{k-1} - \theta_k + \theta_{k-1} \tag{5.23}$$

for all $k = 1, \dots n$. By (5.9) we have

$$n\sum_{k=1}^{n} (\dot u_k - \dot u_{k-1})^2 = -n\sum_{k=1}^{n-1} \dot u_k(\dot u_{k+1} - 2\dot u_k + \dot u_{k-1}),$$

hence, by (5.1)–(5.2) and (5.23),

$$\begin{aligned} &n\sum_{k=1}^{n} (\varepsilon_k - \varepsilon_{k-1} + P[m_k, \varepsilon_k] - P[m_{k-1}, \varepsilon_{k-1}] + \dot\varepsilon_k - \dot\varepsilon_{k-1} - \theta_k + \theta_{k-1})^2\\ &\quad = \frac{1}{n}\sum_{k=1}^{n-1} (\ddot w_k - \ddot\varepsilon_k - \dot f_k)(\varepsilon_k + P[m_k, \varepsilon_k] + \dot\varepsilon_k - \theta_k + \theta_{\mathrm{ref}}). \end{aligned} \tag{5.24}$$

This yields, as a counterpart to (4.23),

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n}{2}\sum_{k=1}^{n} (\dot w_k - \dot w_{k-1})^2 + \frac{1}{2n}\sum_{k=1}^{n-1} \dot\varepsilon_k^2 + n\sum_{k=1}^{n} (\varepsilon_k - \varepsilon_{k-1})^2\right)\\ &+ n\sum_{k=1}^{n} \big((\varepsilon_k - \varepsilon_{k-1})^2 + (\dot\varepsilon_k - \dot\varepsilon_{k-1})^2\big)\\ &\le Cn\sum_{k=1}^{n} \big((P[m_k, \varepsilon_k] - P[m_{k-1}, \varepsilon_{k-1}])^2 + (\theta_k - \theta_{k-1})^2\big)\\ &+ \frac{1}{n}\sum_{k=1}^{n-1} |\dot f_k|\, |\varepsilon_k + P[m_k, \varepsilon_k] + \dot\varepsilon_k - \theta_k + \theta_{\mathrm{ref}}|\\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1} (\dot w_k - \dot\varepsilon_k)(\varepsilon_k + P[m_k, \varepsilon_k] - \theta_k + \theta_{\mathrm{ref}})\right)\\ &+ \frac{1}{n}\sum_{k=1}^{n-1} |\dot w_k - \dot\varepsilon_k||\dot\varepsilon_k + P[m_k, \varepsilon_k]_t - \dot\theta_k|. \end{aligned} \tag{5.25}$$

As in (4.30), we have $|P[m_k, \varepsilon_k]_t| \le C(1 + \theta_k + \frac{1}{n}\sum_{k=1}^{n-1} |\dot\varepsilon_j|)$, and this enables us to estimate the terms on the right hand side of (5.25) as follows:

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1} |\dot w_k - \dot\varepsilon_k||\dot\varepsilon_k + P[m_k, \varepsilon_k]_t - \dot\theta_k|\,\mathrm{d}\tau \le C\left(1 + \left(\int_0^t \frac{1}{n}\sum_{k=1}^{n-1} \dot\theta_k^2\,\mathrm{d}\tau\right)^{1/2}\right),$$

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1} |\dot{f}_k|\,|\varepsilon_k + P[m_k,\varepsilon_k] + \dot{\varepsilon}_k - \theta_k + \theta_{\mathrm{ref}}|\,\mathrm{d}\tau \le C, \tag{5.26}$$

$$\begin{aligned} & \frac{1}{n}\sum_{k=1}^{n-1}(\dot{w}_k - \dot{\varepsilon}_k)(\varepsilon_k + P[m_k,\varepsilon_k] - \theta_k + \theta_{\mathrm{ref}}) \\ \le\ & C\left(1 + \left(\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2\right)^{1/2}\right)\left(1 + \left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right)^{1/2}\right) \\ \le\ & \frac{1}{4n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2 + C\left(1 + \frac{1}{n}\int_0^t\sum_{k=1}^{n-1}\theta_k^2\,\mathrm{d}\tau\right)^{1/2}, \end{aligned} \tag{5.27}$$

where we have used (5.16), (5.20), (5.21) and Hypothesis 2.1 (i), similarly to (4.31) and (4.24). Moreover, we have by Proposition 1.2 and Hypothesis 2.1 (i)

$$\begin{aligned} & |P[m_k,\varepsilon_k](t) - P[m_{k-1},\varepsilon_{k-1}](t)| \\ =\ & \left|\int_0^\infty (\varphi(m_k,r)\mathfrak{s}_r[\varepsilon_k] - \varphi(m_{k-1},r)\mathfrak{s}_r[\varepsilon_{k-1}])\,\mathrm{d}r\right| \\ =\ & \left|\int_0^\infty (\varphi(m_k,r) - \varphi(m_{k-1},r))\,\mathfrak{s}_r[\varepsilon_k]\,\mathrm{d}r \right. \\ & \left. + \int_0^\infty (\varphi(m_{k-1},r)(\mathfrak{s}_r[\varepsilon_k] - \mathfrak{s}_r[\varepsilon_{k-1}]))\,\mathrm{d}r\right| \\ \le\ & C\left(|m_k - m_{k-1}| + \max_{\tau\in[0,t]}|\varepsilon_k(\tau) - \varepsilon_{k-1}(\tau)|\right), \end{aligned} \tag{5.28}$$

where we have by (5.5), Proposition 1.2 (ii), Hypothesis 2.1 (i), (iii), (vi), (1.22), (5.6), (5.15), and (5.21) that

$$\begin{aligned} |m_k - m_{k-1}| \ \le\ & C\int_0^t\left(|\dot{\chi}_k - \dot{\chi}_{k-1}| + \frac{1}{n}\sum_{j=1}^{n}|\lambda_{k-j} - \lambda_{k-j-1}|D_j(\tau)\right)\mathrm{d}\tau \\ \le\ & C\int_0^t\left(|\dot{\chi}_k - \dot{\chi}_{k-1}| + \frac{1}{n}\sum_{j=1}^{n}|\dot{\varepsilon}_j||\lambda_{k-j} - \lambda_{k-j-1}|\right)\mathrm{d}\tau \\ \le\ & C\left(\int_0^t|\dot{\chi}_k - \dot{\chi}_{k-1}|\,\mathrm{d}\tau + \frac{1}{n^2}\int_0^t\sum_{j=1}^{n}|\dot{\varepsilon}_j|\,\mathrm{d}\tau\right) \\ \le\ & C\left(|\chi_k^0 - \chi_{k-1}^0| + \int_0^t|\theta_k - \theta_{k-1}|\,\mathrm{d}\tau + \frac{1}{n}\right). \end{aligned} \tag{5.29}$$

We estimate the initial conditions as in (4.33), and integrating (5.25) in time we conclude from the above considerations that

$$\frac{1}{2n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(t) + n\sum_{k=1}^{n}(\varepsilon_k - \varepsilon_{k-1})^2(t) + \int_0^t n\sum_{k=1}^{n}(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau)\,\mathrm{d}\tau$$

$$
\leq C\left(1+\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2(t)+\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2(\tau)\,\mathrm{d}\tau\right)^{1/2}+\int_0^t n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(\tau)\,\mathrm{d}\tau\right.
$$
$$
\left.+\int_0^t\int_0^\tau n\sum_{k=1}^{n}(\dot\varepsilon_k-\dot\varepsilon_{k-1})^2(\tau')\,\mathrm{d}\tau'\,\mathrm{d}\tau\right).
$$

Gronwall's argument and (5.20) then yields the following counterpart to (4.37)

$$
\begin{aligned}
&\frac{1}{2n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(t)+n\sum_{k=1}^{n}(\varepsilon_k-\varepsilon_{k-1})^2(t)+\int_0^t n\sum_{k=1}^{n}(\dot\varepsilon_k-\dot\varepsilon_{k-1})^2(\tau)\,\mathrm{d}\tau\\
&\quad\leq C\left(1+\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2\,\mathrm{d}\tau\right)^{1/2}+\int_0^t n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(\tau)\,\mathrm{d}\tau\right).
\end{aligned}
\tag{5.30}
$$

We now test (5.4) by θ_k and obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\right)+n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2\leq C\left(\frac{1}{n}\sum_{k=1}^{n-1}\left(\dot\varepsilon_k^2+\theta_k+|\dot\varepsilon_k|(1+\theta_k)\right)\theta_k\right),
$$

where, by (5.20)

$$
\begin{aligned}
\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2\theta_k\,\mathrm{d}\tau &\leq \left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2\,\mathrm{d}\tau\right)^{1/2}\\
&\leq C\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau\right)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}|\dot\varepsilon_k|\theta_k^2\,\mathrm{d}\tau &\leq \left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau\right)^{1/4}\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}\,\mathrm{d}\tau\right)^{3/4}\\
&\leq C\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau\right)^{1/4},
\end{aligned}
$$

hence, by analogy to (4.39),

$$
\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^2(t)+\int_0^t n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(\tau)\,\mathrm{d}\tau\leq C\left(1+\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau\right)^{1/2}\right). \tag{5.31}
$$

Finally, we test (5.4) by $\dot\theta_k$ and obtain from Hölder's inequality that

$$
\begin{aligned}
&\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2(\tau)\,\mathrm{d}\tau+n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(t)\\
&\leq C\left(1+\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau+\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}\tau\right).
\end{aligned}
\tag{5.32}
$$

Using (5.20) we have

$$\begin{aligned}\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^4\,\mathrm{d}\tau &\le \left(\max_{\tau\in[0,t]}\max_{i=1,\dots,n}\theta_i^{4/3}(\tau)\right)\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\theta_k^{8/3}\,\mathrm{d}\tau\\ &\le C\max_{\tau\in[0,t]}\max_{i=1,\dots,n}\theta_i^{4/3}(\tau)\end{aligned}$$

and from (3.3) with $q=\infty$, $s=1$, $p=2$, $\varrho=2/3$ it follows

$$\max_{i=1,\dots,n}\theta_i(\tau)\le C\left(1+\left(n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(\tau)\right)^{1/3}\right). \tag{5.33}$$

We thus infer from (5.32) that

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\dot\theta_k^2(\tau)\,\mathrm{d}\tau + n\sum_{k=1}^{n}(\theta_k-\theta_{k-1})^2(t)\le C\left(1+\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau\right). \tag{5.34}$$

Combining (5.30) with (5.31) and (5.34) yields

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(t)+\int_0^t n\sum_{k=1}^{n}(\dot\varepsilon_k-\dot\varepsilon_{k-1})^2(\tau)\,\mathrm{d}\tau\le C\left(1+\left(\int_0^t\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^4\,\mathrm{d}\tau\right)^{1/2}\right). \tag{5.35}$$

Using the vector notation (3.2), we have by (5.22) and (5.20) that

$$\begin{aligned}|\dot\varepsilon(t)|_2^2=\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(t)+\frac{1}{n}(\dot\varepsilon_0^2(t)+\dot\varepsilon_n^2(t)) &\le \frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(t)+\frac{C}{n}\left(1+\sum_{k=1}^{n-1}\theta_k^2(t)\right)\\ &\le C+\frac{1}{n}\sum_{k=1}^{n-1}\dot\varepsilon_k^2(t),\end{aligned}$$

and we rewrite (5.35) as (compare with (4.44))

$$\max_{\tau\in[0,t]}|\dot\varepsilon(\tau)|_2^2+\int_0^t|\mathbf{D}\dot\varepsilon(\tau)|_2^2\,\mathrm{d}\tau\le C\left(1+\left(\int_0^t|\dot\varepsilon(\tau)|_4^4\,\mathrm{d}\tau\right)^{1/2}\right). \tag{5.36}$$

We estimate the right hand side of (5.36) using (3.3) as follows:

$$|\dot\varepsilon(\tau)|_4\le C\big(|\dot\varepsilon(\tau)|_2+|\dot\varepsilon(\tau)|_2^{3/4}|\mathbf{D}\dot\varepsilon(\tau)|_2^{1/4}\big).$$

We have $\int_0^t|\dot\varepsilon(\tau)|_2^2\,\mathrm{d}\tau\le C$ by virtue of (5.21), hence

$$\begin{aligned}&\int_0^t|\dot\varepsilon(\tau)|_4^4\,\mathrm{d}\tau\\ &\le C\max_{\tau\in[0,t]}|\dot\varepsilon(\tau)|_2^2\left(\int_0^t|\dot\varepsilon(\tau)|_2^2\,\mathrm{d}\tau+\left(\int_0^t|\dot\varepsilon(\tau)|_2^2\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t|\mathbf{D}\dot\varepsilon(\tau)|_2^2\,\mathrm{d}\tau\right)^{1/2}\right)\\ &\le C\max_{\tau\in[0,t]}|\dot\varepsilon(\tau)|_2^2\left(1+\int_0^t|\mathbf{D}\dot\varepsilon(\tau)|_2^2\,\mathrm{d}\tau\right)^{1/2}.\end{aligned} \tag{5.37}$$

Combining (5.36) with (5.37) yields

$$|\dot\varepsilon(t)|_2^2+\int_0^t|\mathbf{D}\dot\varepsilon(\tau)|_2^2\,\mathrm{d}\tau\le C. \tag{5.38}$$

Therefore there exist a constant $C > 0$ such that

$$\frac{1}{n}\sum_{k=1}^{n-1}\dot{\varepsilon}_k^2(t) + \int_0^t n\sum_{k=1}^{n}(\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau)\,\mathrm{d}\tau + \int_0^t \frac{1}{n}\sum_{k=1}^{n-1}(\dot{\varepsilon}_k^4 + \varepsilon_k^4)(\tau)\,\mathrm{d}\tau \le C\,, \tag{5.39}$$

$$\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}(\theta_k^4 + \dot{\theta}_k^2)(\tau)\,\mathrm{d}\tau + n\sum_{k=1}^{n}(\theta_k - \theta_{k-1})^2(t) \le C \tag{5.40}$$

for $t \in [0,T]$. From (5.40) and (5.33) it follows that $\max_{t\in[0,T]}\max_{i=1,\dots,n}\theta_i(t) \le C$, hence $\max_{t\in[0,T]}\max_{i=1,\dots,n}|\dot{\chi}_i(t)| \le C$ and, also by virtue of (5.38),

$$\max_{t\in[0,T]}\max_{i=1,\dots,n}(\theta_i(t) + |\dot{\chi}_i(t)| + |\dot{m}_i(t)|) \le C. \tag{5.41}$$

By comparison of the terms in (5.4), we also have

$$\int_0^t n^3\sum_{k=1}^{n-1}(\theta_{k+1} - 2\theta_k + \theta_{k-1})^2(\tau)\,\mathrm{d}\tau \le C\,, \tag{5.42}$$

and similarly for u_k by (5.2). Finally, we differentiate (5.1) once in t and test by $\ddot{\varepsilon}_k$, (5.2) twice in t and test by $\ddot{w}_k$, and sum the two equations up. Using (5.39)–(5.40) and treating the initial conditions as in (4.49)–(4.55), we get the estimate

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2(t) + n\sum_{k=1}^{n}(\ddot{w}_k - \ddot{w}_{k-1})^2(t) + \int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\ddot{\varepsilon}_k^2(\tau)\,\mathrm{d}\tau \le C\,. \tag{5.43}$$

6. **Proof of Theorem 2.2.**

6.1. **Existence.** For a generic sequence $\{\varphi_k : k = 0,1,\dots,n\}$ we put $\Delta_k\varphi = n(\varphi_k - \varphi_{k-1})$, and $\Delta_k^2\varphi = n^2(\varphi_{k+1} - 2\varphi_k + \varphi_{k-1})$, and define piecewise constant, piecewise linear, and piecewise quadratic interpolations

$$\bar{\varphi}^{(n)}(x) = \begin{cases} \varphi_k & \text{for } \ x \in \left[\frac{k-1}{n}, \frac{k}{n}\right),\ k = 1,\dots,n-1\,, \\ \varphi_{n-1} & \text{for } \ x \in \left[\frac{n-1}{n}, 1\right], \end{cases} \tag{6.1}$$

$$\hat{\varphi}^{(n)}(x) = \ \varphi_{k-1} + \big(x - \tfrac{k-1}{n}\big)\Delta_k\varphi \quad \text{for } \ x \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right), \qquad k = 1,\dots,n\,, \tag{6.2}$$

$$\tilde{\varphi}^{(n)}(x) = \begin{cases} \frac{1}{2}(\varphi_{k-1} + \varphi_k) + \big(x - \frac{k-1}{n}\big)\Delta_k\varphi + \frac{1}{2}\big(x - \frac{k-1}{n}\big)^2\Delta_k^2\varphi \\ \ \text{for } \ x \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \quad k = 1,\dots,n-1\,, \\ \frac{1}{2}(\varphi_{n-1} + \varphi_n) + \big(x - \frac{n-1}{n}\big)\Delta_n\varphi + \frac{1}{2}\big(x - \frac{n-1}{n}\big)^2\Delta_{n-1}^2\varphi \\ \ \text{for } \ x \in \left[\frac{n-1}{n}, 1\right]. \end{cases} \tag{6.3}$$

We also define

$$\lambda^{(n)}(x,y) = \lambda_{k-j} \qquad \text{for } \ (x,y) \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \times \left[\frac{j-1}{n}, \frac{j}{n}\right). \tag{6.4}$$

For functions $\bar{\varepsilon}^{(n)}$, $\bar{\theta}^{(n)}$, $\bar{u}^{(n)}$, $\bar{w}^{(n)}$, $\hat{\varepsilon}^{(n)}$, $\tilde{\theta}^{(n)}$, $\tilde{u}^{(n)}$, $\tilde{w}^{(n)}$, we have derived estimates (5.39)–(5.43) that we rewrite in the form

$$\big|\bar{\varepsilon}_t^{(n)}(t)\big|_2^2 + \int_0^t \big|\hat{\varepsilon}_{xt}^{(n)}(\tau)\big|_2^2\,\mathrm{d}\tau + \int_0^t \Big(\big|\bar{\varepsilon}_t^{(n)}(\tau)\big|_4^4 + \big|\bar{\varepsilon}^{(n)}(\tau)\big|_4^4\Big)\,\mathrm{d}\tau \le C, \tag{6.5}$$

$$\int_0^t \left(\left|\bar\theta_t^{(n)}(\tau)\right|_2^2 + \left|\bar\theta^{(n)}(\tau)\right|_4^4\right) \mathrm{d}\tau + \left|\hat\theta_x^{(n)}(t)\right|_2^2 \le C, \tag{6.6}$$

$$\int_0^t \left|\tilde\theta_{xx}^{(n)}(\tau)\right|_2^2 \mathrm{d}\tau \le C, \tag{6.7}$$

$$\left|\bar w_{tt}^{(n)}(t)\right|_2^2 + \left|\hat w_{xtt}^{(n)}(t)\right|_2^2 + \int_0^t \left|\bar\varepsilon_{tt}^{(n)}(\tau)\right|_2^2 \mathrm{d}\tau \le C, \tag{6.8}$$

and by (5.2)–(5.3),

$$\int_0^t \left(\left|\tilde w_{xxt}^{(n)}(\tau)\right|_2^2 + \left|\tilde u_{xxt}^{(n)}(\tau)\right|_2^2\right) \mathrm{d}\tau \le C. \tag{6.9}$$

System (5.1)–(5.6) has the form

$$\bar u_t^{(n)} = \bar\varepsilon^{(n)} + P[\bar m^{(n)}, \bar\varepsilon^{(n)}] + \bar\varepsilon_t^{(n)} - (\bar\theta^{(n)} - \theta_{\mathrm{ref}}), \tag{6.10}$$

$$\bar w_t^{(n)} - \bar\varepsilon_t^{(n)} = -\tilde u_{xx}^{(n)} + \bar f^{(n)}, \tag{6.11}$$

$$\bar\varepsilon^{(n)} = \tilde w_{xx}^{(n)}, \tag{6.12}$$

$$\bar\theta_t^{(n)} = \tilde\theta_{xx}^{(n)} + \bar m_t^{(n)} \bar K^{(n)} + \bar D^{(n)} + (\bar\varepsilon_t^{(n)})^2 - \bar\theta^{(n)}\bar\varepsilon_t^{(n)} + \bar g^{(n)}(\bar\theta^{(n)}) - \bar\chi_t^{(n)}, \tag{6.13}$$

$$\bar\chi^{(n)}(x,t) = \mathfrak{s}_{[0,1]}[\bar\chi^{(n)}(0), \bar A^{(n)}(x,\cdot)](t), \tag{6.14}$$

$$\bar m^{(n)}(x,t) = \mathfrak{s}_{[0,\infty)}[0, \bar S^{(n)}(x,\cdot)](t), \tag{6.15}$$

$$\bar A^{(n)}(x,t) = \int_0^t \frac{1}{\gamma}\left(\frac{L}{\theta_{\mathrm{ref}}}(\bar\theta^{(n)} - \theta_{\mathrm{ref}})\right)(x,\tau)\, \mathrm{d}\tau, \tag{6.16}$$

$$\bar S^{(n)}(x,t) = \int_0^t \left(-h(\bar\chi_t^{(n)}(x,\tau)) + \int_0^1 \lambda^{(n)}(x,y)\bar D^{(n)}(y,\tau)\,\mathrm{d}y\right)(x,\tau)\, \mathrm{d}\tau, \tag{6.17}$$

$$\bar D^{(n)}(x,t) = \int_0^\infty \varphi(\bar m^{(n)}, r)\ \mathfrak{s}_r[\bar\varepsilon^{(n)}](\bar\varepsilon^{(n)} - \mathfrak{s}_r[\bar\varepsilon^{(n)}])_t(x,t)\,\mathrm{d}r, \tag{6.18}$$

$$\bar K^{(n)}(x,t) = -\frac{1}{2}\int_0^\infty \varphi_m(\bar m^{(n)}, r)\mathfrak{s}_r^2[\bar\varepsilon^{(n)}]\,\mathrm{d}r, \tag{6.19}$$

with $\bar\chi^{(n)}(0)$ chosen in agreement with (5.7). We further have

$$\int_0^t \left|\hat\varepsilon_{tt}(\tau)\right|_2^2 \mathrm{d}\tau \le \int_0^t \frac{2}{n}\sum_{k=0}^{n} \ddot\varepsilon_k^2(\tau)\,\mathrm{d}\tau \le \int_0^t \left(\frac{2}{n}\sum_{k=1}^{n-1}\ddot\varepsilon_k^2(\tau) + \frac{2}{n}(\ddot\varepsilon_0^2(\tau) + \ddot\varepsilon_n^2(\tau))\right) \mathrm{d}\tau.$$

By (5.22), (5.41), we have for $k = 0$ and $k = n$

$$\ddot\varepsilon_k^2(\tau) \le C(1 + \dot\theta_k^2(\tau)),$$

hence, using the "boundary condition" (5.8),

$$\int_0^t \left|\hat\varepsilon_{tt}(\tau)\right|_2^2 \mathrm{d}\tau \le C\Big(1 + \int_0^t \frac{2}{n}\sum_{k=1}^{n-1}\left(\ddot\varepsilon_k^2 + \dot\theta_k^2\right)(\tau)\,\mathrm{d}\tau\Big) \le C. \tag{6.20}$$

From (6.5), (6.20), and from Sobolev embedding theorems it follows that there exists $\varepsilon \in W^{1,2}(\Omega_T)$ such that $\varepsilon_{xt}, \varepsilon_{tt} \in L^2(\Omega_T)$, and a subsequence of $\{\hat\varepsilon^{(n)}\}$, still indexed by n, such that

$$\hat\varepsilon^{(n)} \to \varepsilon \ \text{ strongly in } C(\overline{\Omega}_T), \quad \hat\varepsilon_t^{(n)} \to \varepsilon_t \ \text{ strongly in } L^p(\Omega_T)$$

for all $p > 1$. Furthermore,

$$|\bar{\varepsilon}_t^{(n)} - \hat{\varepsilon}_t^{(n)}|^2(x,t) \le |\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1}|^2(t)$$

for $x \in [(k-1)/n, k/n]$, hence

$$\int_0^t \int_0^1 |\bar{\varepsilon}_t^{(n)} - \hat{\varepsilon}_t^{(n)}|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \le \int_0^t \frac{1}{n}\sum_{k=1}^n (\dot{\varepsilon}_k - \dot{\varepsilon}_{k-1})^2(\tau)\,\mathrm{d}\tau \le \frac{C}{n^2}\,,$$

so that

$$\bar{\varepsilon}_t^{(n)} \to \varepsilon_t \ \text{ strongly in } L^2(\Omega_T). \tag{6.21}$$

Similarly,

$$|\bar{\varepsilon}^{(n)} - \hat{\varepsilon}^{(n)}|^2(x,t) \le \max_{k=1,\dots,n} |\varepsilon_k - \varepsilon_{k-1}|^2(t) \le \sum_{k=1}^n (\varepsilon_k - \varepsilon_{k-1})^2(t) \le \frac{C}{n},$$

hence

$$\bar{\varepsilon}^{(n)} \to \varepsilon \ \text{ strongly in } L^\infty(\Omega_T). \tag{6.22}$$

Using also (6.11)–(6.13), we check in the same way that there exist $u, w, \theta \in C(\overline{\Omega}_T)$ such that, selecting again a subsequence if necessary,

$$\left.\begin{array}{c} \tilde{w}_{xxt}^{(n)} \to \varepsilon_t = w_{xxt}\,,\ \tilde{u}_{xx}^{(n)} \to u_{xx} \ \text{ strongly in } L^2(\Omega_T), \\ \bar{\theta}_t^{(n)} \to \theta_t\,,\ \tilde{\theta}_{xx}^{(n)} \to \theta_{xx} \ \text{ weakly in } L^2(\Omega_T), \quad \bar{\theta}^{(n)} \to \theta \ \text{ strongly in } L^\infty(\Omega_T). \end{array}\right\} \tag{6.23}$$

Finally, for all $n, l \in \mathbb{N}$ we have by Proposition 1.2 (ii)

$$\begin{aligned} |\bar{\chi}^{(n)} - \bar{\chi}^{(l)}|(x,t) &\le \int_0^t |\bar{\chi}_t^{(n)} - \bar{\chi}_t^{(l)}|(x,\tau)\,\mathrm{d}\tau \qquad (6.24)\\ &\le C\int_0^t |\bar{\theta}^{(n)} - \bar{\theta}^{(l)}|(x,\tau)\,\mathrm{d}\tau + |\bar{\chi}^{(n)} - \bar{\chi}^{(l)}|(x,0). \end{aligned}$$

Both sequences $\bar{\theta}^{(n)}$ and $\bar{\chi}^{(n)}(\cdot,0)$ are uniformly convergent by (6.23) and Hypothesis 2.1 (v). It follows that $\bar{\chi}^{(n)}$ and $\bar{\chi}_t^{(n)}$ are Cauchy sequences in $L^\infty(\Omega_T)$ and in $L^\infty(0,1;L^1(0,T))$, respectively. Moreover we have for all $x \in \Omega$ by Proposition 1.2 (ii) and by (6.15)–(6.18) that

$$\begin{aligned} &\int_0^t \left|\bar{m}_t^{(n)} - \bar{m}_t^{(l)}\right|(x,\tau)\,\mathrm{d}\tau \le C\int_0^t |\bar{\chi}_t^{(n)} - \bar{\chi}_t^{(l)}|(x,\tau)\,\mathrm{d}\tau \\ &+ \int_0^t \int_0^1 \int_0^\infty \Big|\lambda^{(n)}(x,y)\varphi(\bar{m}^{(n)},r)\delta^{(n)}(y,t,r) \qquad (6.25)\\ &-\lambda^{(l)}(x,y)\varphi(\bar{m}^{(l)},r)\delta^{(l)}(y,t,r)\Big|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}\tau, \end{aligned}$$

where we denote

$$\delta^{(n)} = \delta^{(n)}(y,t,r) = \mathfrak{s}_r[\bar{\varepsilon}^{(n)}](\bar{\varepsilon}^{(n)} - \mathfrak{s}_r[\bar{\varepsilon}^{(n)}])_t(y,t) = r|\mathfrak{p}_r[\bar{\varepsilon}^{(n)}]_t(y,t)|.$$

By Proposition 1.2 (ii) we have (note that $\big||a| - |b|\big| \le |a-b|$ for $a, b \in \mathbb{R}$)

$$\int_0^t |\delta^{(n)} - \delta^{(l)}|(y,\tau)\,\mathrm{d}\tau \le r\int_0^t |\bar{\varepsilon}_t^{(n)} - \bar{\varepsilon}_t^{(l)}|(y,\tau)\,\mathrm{d}\tau,$$

hence, by Hypothesis 2.1 (i), (iii),

$$\begin{aligned}&\int_0^t\int_0^1\int_0^\infty \lambda^{(n)}(x,y)\varphi(\bar m^{(n)},r)|\delta^{(n)}-\delta^{(l)}|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}\tau \\ \le\ & C\int_0^t\int_0^1 |\bar\varepsilon_t^{(n)}-\bar\varepsilon_t^{(l)}|(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau.\end{aligned} \tag{6.26}$$

Similarly, by Hypothesis 2.1 (i),

$$\begin{aligned}&\int_0^t\int_0^1\int_0^\infty \delta^{(l)}\lambda^{(n)}(x,y)|\varphi(\bar m^{(n)},r)-\varphi(\bar m^{(l)},r)|\,\mathrm{d}r\,\mathrm{d}y\,\mathrm{d}\tau \\ &\qquad\le C\int_0^t\left(\int_0^1|\bar\varepsilon_t^{(l)}(y,\tau)|\,\mathrm{d}y\right)\max_{x\in\Omega}|m^{(n)}(x,\tau)-m^{(l)}(x,\tau)|\,\mathrm{d}\tau.\end{aligned} \tag{6.27}$$

Finally, we have the pointwise bound

$$|\lambda^{(n)}(x,y)-\lambda^{(l)}(x,y)|\le\frac{4\Lambda}{\min\{n,l\}}\,. \tag{6.28}$$

Combining (6.24)–(6.28) gives the inequality

$$\begin{aligned}&\max_{x\in\Omega}|m^{(n)}-m^{(l)}|(x,t)\\ \le\ &\max_{x\in\Omega}\int_0^t\left|\bar m_t^{(n)}-\bar m_t^{(l)}\right|(x,\tau)\,\mathrm{d}\tau\\ \le\ & q_{nl}+C\int_0^t\left(\int_0^1|\bar\varepsilon_t^{(l)}(y,\tau)|\,\mathrm{d}y\right)\max_{x\in\Omega}|m^{(n)}-m^{(l)}|(x,\tau)\,\mathrm{d}\tau,\end{aligned} \tag{6.29}$$

with

$$q_{nl}=C\left(\frac{1}{\min\{n,l\}}+|\bar\chi^{(n)}(\cdot,0)-\bar\chi^{(l)}(\cdot,0)|_1+\|\bar\theta^{(n)}-\bar\theta^{(l)}\|_\infty+\|\bar\varepsilon_t^{(n)}-\bar\varepsilon_t^{(l)}\|_1\right).$$

Inequality (6.29) can be interpreted as an inequality of the form

$$q(t)\le q_{nl}+\int_0^t s^{(l)}(\tau)q(\tau)\,\mathrm{d}\tau,$$

with $q(t)=\max_{x\in\Omega}|\bar m^{(n)}-\bar m^{(l)}|(x,t)$, $s^{(l)}(t)=C\int_0^1|\bar\varepsilon_t^{(l)}(y,t)|\,\mathrm{d}y$, with $s^{(l)}$ uniformly bounded in $L^1(0,T)$. We obtain using Gronwall's lemma that

$$q(t)\le q_{nl}\mathrm{e}^{\int_0^t s^{(l)}(\tau)\,\mathrm{d}\tau}\le Cq_{nl}.$$

The convergences (6.23) imply that q_{nl} is small if n,l are large. Hence, $\bar m^{(n)}$ is a Cauchy sequence, so that

$$\bar m^{(n)}\to m \ \text{ strongly in } L^\infty(\Omega_T),$$

and, by (6.29),

$$\bar m_t^{(n)}\to m_t \ \text{ strongly in } L^\infty(0,1;L^1(0,T)).$$

Furthermore, by virtue of (5.41), $\bar m_t^{(n)}$ and $\bar\chi_t^{(n)}$ are uniformly bounded in $L^\infty(\Omega_T)$, hence $\bar m_t^{(n)}\to m_t$, $\bar\chi_t^{(n)}\to\chi_t$ in $L^\infty(\Omega_T)$ weakly star. Using the convergences (6.21), (6.22), and Proposition 1.2, we conclude that $\bar D^{(n)}(x,\cdot)$, $\bar K^{(n)}(x,\cdot)$ converge for all $x\in[0,1]$ to $D[m,\varepsilon](x,\cdot)$, $\mathcal{K}[m,\varepsilon](x,\cdot)$, respectively, strongly in $L^2(0,T)$. By the Lebesgue Dominated Convergence Theorem, we can pass in $L^2(\Omega_T)$ to the limit in (6.10)–(6.19) and conclude that (u,w,θ,m,χ) is a strong solution to (2.1)–(2.4) with the regularity indicated in Theorem 2.2 and satisfying the initial conditions (2.6).

It remains to check that the boundary conditions (2.7) hold. We have $w_n(t) = 0$, hence

$$\begin{aligned} |\tilde{w}^{(n)}(1,t)| &= \left|2w_n(t) - \frac{3}{2}w_{n-1}(t) + \frac{1}{2}w_{n-2}(t)\right| \\ &= \left|w_n(t) - w_{n-1}(t) - \frac{1}{2}(w_{n-1}(t) - w_{n-2}(t))\right| \\ &\le 2\Big(\sum_{k=1}^{n} |w_k - w_{k-1}|^2(t)\Big)^{1/2} \le \frac{C}{\sqrt{n}}, \end{aligned}$$

and similarly for $w(0,t), u(1,t), u(0,t)$. To complete the existence proof, we only have to verify the homogeneous Neumann boundary condition for θ. In other words, we have to check that for every $\tilde{\psi} \in C^1(\overline{\Omega}_T)$ we have

$$\int_0^T \int_0^1 (\theta_x \tilde{\psi}_x + \theta_{xx}\tilde{\psi})(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{6.30}$$

A straightforward computation yields

$$\int_0^T \int_0^1 (\tilde{\theta}^{(n)}_x \tilde{\psi}_x + \tilde{\theta}^{(n)}_{xx}\tilde{\psi})(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \tilde{\psi}(1,t) n(\theta_{n-1} - \theta_{n-2})(t)\,\mathrm{d}t\,. \tag{6.31}$$

We have

$$\begin{aligned} \int_0^T n^2(\theta_{n-1} - \theta_{n-2})^2(t)\,\mathrm{d}t &= \int_0^T n^2(\theta_n - 2\theta_{n-1} + \theta_{n-2})^2(t)\,\mathrm{d}t \\ &\le \int_0^T n^2 \sum_{k=1}^{n-1} (\theta_{k+1} - 2\theta_k + \theta_{k-1})^2(t)\,\mathrm{d}t \le \frac{C}{n}\,, \end{aligned}$$

where we used (5.42). Hence,

$$\lim_{n\to\infty} \int_0^T \int_0^1 (\tilde{\theta}^{(n)}_x \tilde{\psi}_x + \tilde{\theta}^{(n)}_{xx}\tilde{\psi})(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0$$

and (6.30) follows.

6.2. **Uniqueness.** Let (u, w, θ, χ, m), $(\tilde{u}, \tilde{w}, \tilde{\theta}, \tilde{\chi}, \tilde{m})$ be two solutions of (2.1)–(2.7), with the regularity as in Theorem 2.2, and with the same initial conditions and the same right hand sides. We integrate the difference of (2.3) for θ and $\tilde{\theta}$ in time, and estimate the terms on the right hand side as follows:

$$\begin{aligned} &\int_0^t |D[m, w_{xx}] - D[\tilde{m}, \tilde{w}_{xx}]|(x,\tau)\,\mathrm{d}\tau \\ \le\ & C\left(\int_0^t (|m - \tilde{m}|\,|w_{xxt}| + |w_{xxt} - \tilde{w}_{xxt}|)(x,\tau)\,\mathrm{d}\tau\right), \end{aligned} \tag{6.32}$$

where we have used Hypothesis 2.1 (i), (iv) and Proposition 1.2 (ii). Furthermore,

$$\begin{aligned} &\int_0^t |\theta w_{xxt} - \tilde{\theta}\tilde{w}_{xxt}|\,\mathrm{d}\tau \\ \le\ & \|\tilde{\theta}\|_\infty \int_0^t |w_{xxt} - \tilde{w}_{xxt}|\,\mathrm{d}\tau + \left(\int_0^t |w_{xxt}|^2\,\mathrm{d}\tau\right)^{1/2} \left(\int_0^t |\theta - \tilde{\theta}|^2\,\mathrm{d}\tau\right)^{1/2}. \end{aligned} \tag{6.33}$$

We have by (4.48) that

$$\max_{x\in[0,1]} \int_0^t |w_{xxt}|^2(x,\tau)\,\mathrm{d}\tau \le C(\|w_{xxt}\|_2^2 + \|w_{xxxt}\|_2^2) \le C,$$

hence

$$\begin{aligned} & \int_0^t |\theta w_{xxt} - \tilde\theta \tilde w_{xxt}|(x,\tau)\,\mathrm{d}\tau \\ \le\ & C\left(\int_0^t |w_{xxt} - \tilde w_{xxt}|(x,\tau)\,\mathrm{d}\tau + \left(\int_0^t |\theta - \tilde\theta|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2}\right). \end{aligned} \tag{6.34}$$

Similarly,

$$\int_0^t |w_{xxt}^2 - \tilde w_{xxt}^2|(x,\tau)\,\mathrm{d}\tau \le C\left(\int_0^t |w_{xxt} - \tilde w_{xxt}|^2(x,\tau)\,\mathrm{d}\tau\right)^{1/2}. \tag{6.35}$$

The fatigue term is estimated as

$$\begin{aligned} & \int_0^t |m_t\mathcal{K}[m,w_{xx}] - \tilde m_t\mathcal{K}[\tilde m,\tilde w_{xx}]|\,\mathrm{d}\tau \\ \le\ & C\int_0^t (|m_t - \tilde m_t| + |m_t|(|m-\tilde m| + |w_{xxt} - \tilde w_{xxt}|))\,\mathrm{d}\tau, \end{aligned}$$

where $|m_t(x,t)| \le C$ by Theorem 2.2, and

$$\begin{aligned} & \int_0^t |m_t - \tilde m_t|(x,\tau)\,\mathrm{d}\tau \\ \le\ & C\int_0^t \left(|\theta-\tilde\theta|(x,\tau) + \int_0^1 (|m-\tilde m|\,|w_{xxt}| + |w_{xxt} - \tilde w_{xxt}|)(y,\tau)\,\mathrm{d}y\right)\mathrm{d}\tau \end{aligned} \tag{6.36}$$

by Proposition 1.2 (ii), together with (1.14), (1.16), and (6.32). We have

$$|m-\tilde m|(x,t) \le \int_0^t |m_t - \tilde m_t|(x,\tau)\,\mathrm{d}\tau$$

for almost every x, and

$$\int_0^t\int_0^1 |m-\tilde m|\,|w_{xxt}|\,\mathrm{d}y\,\mathrm{d}\tau \le C\int_0^t \left(\int_0^1 |m-\tilde m|^2\,\mathrm{d}y\right)^{1/2}\mathrm{d}\tau$$

by (4.48). Hence,

$$\begin{aligned} |m-\tilde m|^2(x,t) \le C\Bigg(&\int_0^t |\theta-\tilde\theta|^2(x,\tau)\,\mathrm{d}\tau \\ &+ \int_0^t\int_0^1 (|m-\tilde m|^2 + |w_{xxt} - \tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\Bigg). \end{aligned}$$

Integrating in space and using Gronwall's argument, we obtain from (6.36) that

$$\begin{aligned} \int_0^t |m_t - \tilde m_t|(x,\tau)\,\mathrm{d}\tau \le C\Bigg(&\int_0^t |\theta-\tilde\theta|(x,\tau)\,\mathrm{d}\tau \\ &+ \left(\int_0^t\int_0^1 (|\theta-\tilde\theta|^2 + |w_{xxt} - \tilde w_{xxt}|^2)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau\right)^{1/2}\Bigg). \end{aligned} \tag{6.37}$$

Finally,

$$
\begin{aligned}
|\chi(x,t)-\tilde\chi(x,t)| &\le C\int_0^t |\theta-\tilde\theta|(x,\tau)\,\mathrm{d}\tau,\ \int_0^t |g(\theta,x,\tau)-g(\tilde\theta,x,\tau)|\,\mathrm{d}\tau\\
&\le C\int_0^t |\theta-\tilde\theta|(x,\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

We now test the inequality obtained by integrating in time from 0 to t the difference of (2.3) for θ and $\tilde\theta$ by $\theta(x,t)-\tilde\theta(x,t)$, and integrate in x. Taking into account the above estimates, we finally obtain

$$
\begin{aligned}
&\int_0^1 |\theta-\tilde\theta|^2(x,t)\,\mathrm{d}x + \frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\int_0^t(\theta_x-\tilde\theta_x)(x,\tau)\,\mathrm{d}\tau\Big)^2\mathrm{d}x\\
&\qquad\le C\int_0^t\int_0^1\big(|w_{xxt}-\tilde w_{xxt}|^2+|\theta-\tilde\theta|^2\big)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\,. \qquad (6.38)
\end{aligned}
$$

In the next step, we test the difference of the time derivatives of (2.2) for w and $\tilde w$ by $w_t-\tilde w_t$, the difference of (2.1) for u and $\tilde u$ by $w_{xxt}-\tilde w_{xxt}$, and sum up. Arguing as above, we obtain, using Proposition 1.2 and Hypothesis 2.1 (i) to estimate the difference $P[m,w_{xx}]-P[\tilde m,\tilde w_{xx}]$, that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\big((w_t-\tilde w_t)^2+(w_{xt}-\tilde w_{xt})^2\\
&+(w_{xx}-\tilde w_{xx})^2\big)(x,t)\,\mathrm{d}x+\int_0^1|w_{xxt}-\tilde w_{xxt}|^2(x,t)\,\mathrm{d}x\\
&\quad\le C\Big(\int_0^1|\theta-\tilde\theta|^2(x,t)\,\mathrm{d}x+\int_0^t\int_0^1|w_{xxt}-\tilde w_{xxt}|^2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\Big). \qquad (6.39)
\end{aligned}
$$

We now multiply (6.38) by $2C$ and add the result to (6.39) to obtain

$$
\begin{aligned}
&\int_0^1\big(|w_{xxt}-\tilde w_{xxt}|^2+C|\theta-\tilde\theta|^2\big)(x,t)\,\mathrm{d}x+\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(2C\Big(\int_0^t(\theta_x-\tilde\theta_x)(x,\tau)\,\mathrm{d}\tau\Big)^2\\
&\quad+(w_t-\tilde w_t)^2+(w_{xt}-\tilde w_{xt})^2+(w_{xx}-\tilde w_{xx})^2\Big)(x,t)\,\mathrm{d}x\\
&\le (C+2C^2)\int_0^t\int_0^1\big(|w_{xxt}-\tilde w_{xxt}|^2+|\theta-\tilde\theta|^2\big)(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\,.
\end{aligned}
$$

Gronwall's argument now yields that $w=\tilde w$, $\theta=\tilde\theta$, and the proof of Theorem 2.2 is complete.

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Received December 2013; 1st revision April 2014; 2nd revision April 2014.

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