



ACADEMY of SCIENCES of the CZECH REPUBLIC

INSTITUTE of MATHEMATICS

**Olsen's problem and essentially power
bounded operators**

Vladimír Müller

Preprint No. 8-2014

PRAHA 2014

OLSEN'S PROBLEM AND ESSENTIALLY POWER BOUNDED OPERATORS

V. MÜLLER

ABSTRACT. Let T be a non-Riesz operator on an infinite-dimensional Hilbert space. Then there exists a compact operator K such that $\|(T + K)^n\| = \|T^n\|_e$ for all n . In particular, every essentially power bounded operator is a compact perturbation of a power bounded operator.

1. INTRODUCTION

Let H be an infinite-dimensional Hilbert space. Denote by $B(H)$, $\mathcal{K}(H)$ and $\mathcal{F}(H)$ the set of all bounded, compact and finite-rank operators on H , respectively.

For $T \in B(H)$ denote by $\|T\|_e$ the essential norm of T , i.e., $\|T\|_e = \inf\{\|T + K\| : K \in \mathcal{K}(H)\}$. Let $\sigma_e(T)$ and $r_e(T)$ denote the essential spectrum and essential spectral radius, respectively, i.e., the spectrum and the spectral radius of the class $T + \mathcal{K}(H)$ in the Calkin algebra $B(H)/\mathcal{K}(H)$. Recall that $T \in B(H)$ is called Riesz if $r_e(T) = 0$.

For a subspace $M \subset H$ denote by P_M the orthogonal projection onto M .

Properties of an operator $T \in B(H)$ can be frequently improved by a suitable compact perturbation. By [W], any Riesz operator $T \in B(H)$ can be written as a sum $T = Q + K$ where K is a compact and Q quasinilpotent operator. More generally, by [S], for any $T \in B(H)$ there exists $K \in \mathcal{K}(H)$ such that $\sigma(T + K)$ is equal to the Weyl spectrum of T ,

$$\sigma_W(T) = \bigcap \{\sigma(T + K) : K \in \mathcal{K}(H)\}.$$

By [CSSW], for any $T \in B(H)$ there exists a compact operator $K \in \mathcal{K}(H)$ such that the closure of the numerical range $\overline{W(T + K)}$ is equal to the essential numerical range $W_e(T)$.

2000 *Mathematics Subject Classification.* 47A30, 47A55.

Key words and phrases. Olsen's problem, essentially power bounded operators. Supported by grants 14-07880S of GA CR and RVO: 67985840 .

By [O1], if $T \in B(H)$ is polynomially compact, i.e., $p(T) \in \mathcal{K}(H)$ for some polynomial p , then there exists $K \in \mathcal{K}(H)$ such that $T + K$ is algebraic, $p(T + K) = 0$ (with the same polynomial p).

The following problem was raised by C. Olsen [O2], cf. also [O1], [OP]:

Problem. Let $T \in B(H)$. Does there exist a compact operator $K \in \mathcal{K}(H)$ such that $\|p(T + K)\| = \|p(T)\|_e$ for all polynomials p ? Less ambitiously, if T and p are both given, is there a compact operator K_p such that $\|p(T + K_p)\| = \|p(T)\|_e$?

Even the more modest formulation of the problem is surprisingly difficult. A positive answer was given in [O2] for the polynomials $p(z) = z, z^2, z^3$ and in [CLSW] for all linear polynomials. In [M], a positive answer was given for any power $p(z) = z^n$.

In the present paper we refine the estimates of [M] and show that for any non-Riesz operator $T \in B(H)$ there exists a compact operator $K \in \mathcal{K}(H)$ such that $\|(T + K)^n\| = \|T^n\|_e$ for all n simultaneously.

In particular, this implies that for any essentially power bounded operator $T \in B(H)$, (i.e., $\sup_n \|T^n\|_e < \infty$) there exists a compact operator K such that $T + K$ is power bounded, $\sup_n \|(T + K)^n\| < \infty$.

2. MAIN RESULT

We need several lemmas that were proved in [CLSW] and [M].

Proposition 1. (see [M], Proposition 6) Let H be a separable infinite-dimensional Hilbert space, let (e_1, e_2, \dots) be an orthonormal basis in H . Let $S \in B(H)$. Then

$$\|S\|_e = \lim_{k \rightarrow \infty} \|P_{H_k^\perp} S P_{H_k^\perp}\|,$$

where $H_k = \bigvee_{j=1}^k e_j$ ($k \in \mathbb{N}$).

The next lemma is a slight modification of [M], Lemma 8, cf. also [CLSW], Lemma 5. For the sake of convenience we give a proof here.

Lemma 2. Let H be a separable infinite-dimensional Hilbert space, $S_1, \dots, S_n \in B(H)$. Let $F_0 \subset H$ be a finite-dimensional subspace. Then there exist mutually orthogonal finite-dimensional subspaces $F_k \subset H$ such that $H = \bigoplus_{k=0}^{\infty} F_k$ and $P_{F_r} S_j P_{F_s} = 0$ for all $r, s \in \mathbb{N}$, $|r - s| \geq 2$ and $j = 1, \dots, n$ (i.e., the operators S_1, \dots, S_n are simultaneously block 3-diagonal).

Proof. Let (e_1, e_2, \dots) be an orthonormal basis in H .

We construct the subspaces F_k inductively. If $k \geq 1$ and the subspaces F_0, \dots, F_{k-1} have already been constructed, then set

$$G_k = \bigvee \{F_0, \dots, F_{k-1}, S_j F_{k-1}, S_j^* F_{k-1} \quad (j = 1, \dots, n), e_k\}$$

and $F_k = G_k \ominus (F_0 \oplus \dots \oplus F_{k-1})$. Then $\dim F_k < \infty$, $S_j F_{k-1} \subset F_0 \oplus \dots \oplus F_k$ and $S_j^* F_{k-1} \subset F_0 \oplus \dots \oplus F_k$ for all $j = 1, \dots, n$.

If we continue this construction for all $k \in \mathbb{N}$ then we get the required decomposition. Note that $\bigoplus_{k=0}^{\infty} F_k = H$ since $e_k \in F_0 \oplus \dots \oplus F_k$ for each k .

Let $r, s \geq 0$, $|r - s| \geq 2$. If $r > s$ then $P_{F_r} S_j P_{F_s} = 0$. If $r < s$ then $P_{F_s} S_j^* P_{F_r} = 0$, and so $P_{F_r} S_j P_{F_s} = 0$. \square

Lemma 3. (see [M], Lemma 9, cf. also [CLSW], Lemma 6) Let $S \in B(H)$ be block 3-diagonal, i.e., there are mutually orthogonal finite-dimensional subspaces F_j such that $H = \bigoplus_{j=0}^{\infty} F_j$ and $P_{F_r} S P_{F_s} = 0$ whenever $|r - s| \geq 2$. Denote by Q_k the orthogonal projection onto $\bigoplus_{j=k+1}^{\infty} F_j$.

Let $l, d \in \mathbb{N}$, $k = l + 2d$ and let $V \in B(H)$ satisfy $V = Q_k V Q_k$. Then $\|S + V\| \leq \max\{\|S\|, \|Q_l(S + V)Q_l\|\} + \frac{\|S\|}{\sqrt{d}}$.

The next result is a modification of [M], Theorem 16.

Theorem 4. Let H be an infinite-dimensional separable Hilbert space, $S \in B(H)$, $m \in \mathbb{N}$, $\|S^m\|_e \neq 0$, $0 \leq r < r' < 1$. Let $T \in B(H)$ satisfy $\|T^j\| < \|S^j\|_e$ ($j = 1, \dots, m$) and $T - rS \in \mathcal{F}(H)$. Then there exists $T' \in B(H)$ such that

$$\begin{aligned} \|T - T'\| &\leq (r' - r)\|S\|, \\ T' - r'S &\in \mathcal{F}(H) \end{aligned}$$

and

$$\|T'^j\| < \|S^j\|_e \quad (j = 1, \dots, m).$$

Proof. Without loss of generality we may assume that $\|S\| = 1$.

Let $F_0 = R(T - rS) \vee R(T^* - rS^*)$. By Lemma 2, there exist mutually orthogonal finite-dimensional subspaces F_1, F_2, \dots such that $H = \bigoplus_{i=0}^{\infty} F_i$ and S, S^2, \dots, S^m are simultaneously block 3-diagonal with respect to this decomposition. Denote by Q_l the orthogonal projection onto $\bigoplus_{i=l+1}^{\infty} F_i$. We have $Q_0(T - rS)Q_0 = 0$.

Choose ε such that

$$0 < \varepsilon < \frac{1}{3} \min\{\|S^j\|_e - \|T^j\|, (1 - r')\|S^j\|_e \quad (j = 1, \dots, m)\}.$$

Choose $n \in \mathbb{N}$ such that

$$\frac{(2^m - 1)(r' - r)}{n} < \varepsilon$$

and $d \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{d}} < \frac{\varepsilon}{n}.$$

By Proposition 1, there exists k_0 such that $\|Q_l S^j Q_l\| < \|S^j\|_e + \varepsilon$ for all $l \geq k_0$ and $j = 1, \dots, m$.

For $t = 0, \dots, n$ let $s_t = r + \frac{t(r' - r)}{n}$. So $s_0 = r$ and $s_n = r'$. Choose numbers $k_1, \dots, k_n \in \mathbb{N}$ such that $k_{t+1} > k_t + 2d + 1$.

Define inductively operators S_0, \dots, S_n by $S_0 = T$ and

$$S_{t+1} = S_t + (s_{t+1} - s_t)Q_{k_{t+1}}S Q_{k_{t+1}}.$$

Let $T' = S_n$.

Clearly $\|S_{t+1} - S_t\| \leq (s_{t+1} - s_t)\|S\| \leq \frac{r' - r}{n}$. So

$$\|T' - T\| \leq \|S_n - S_{n-1}\| + \dots + \|S_1 - S_0\| \leq r' - r.$$

For each t we have $Q_{k_t}(S_t - s_t S)Q_{k_t} = 0$. In particular, $Q_{k_n}(T' - r' S)Q_{k_n} = 0$, and so $T' - r' S \in \mathcal{F}(H)$.

We prove by induction on t that

$$\|S_t^j\| < \|S^j\|_e - \frac{\varepsilon(n-t)}{n} \quad (j = 1, \dots, m). \quad (1)$$

For $t = 0$ this follows by the definition of ε . Suppose that (1) is true for some t , $0 \leq t \leq n - 1$. Fix $j \in \{1, \dots, m\}$. By Lemma 3,

$$\|S_{t+1}^j\| \leq \max\{\|S_t^j\|, \|Q_{k_{t+1}-2d}S_{t+1}^jQ_{k_{t+1}-2d}\|\} + \frac{\|S\|}{\sqrt{d}},$$

where $\|S_t^j\| < \|S^j\|_e - \frac{\varepsilon(n-t)}{n}$ by the induction assumption and $\frac{\|S\|}{\sqrt{d}} < \frac{\varepsilon}{n}$. So it is sufficient to show that

$$\|Q_{k_{t+1}-2d}S_{t+1}^jQ_{k_{t+1}-2d}\| < \|S^j\|_e - \frac{\varepsilon(n-t)}{n}.$$

Write $V = S_{t+1} - S_t = (s_{t+1} - s_t)Q_{k_{t+1}}S^jQ_{k_{t+1}}$. Clearly $\|V\| \leq (s_{t+1} - s_t)\|S\| = \frac{r' - r}{n}$. Since $S_{t+1}^j - S_t^j = (S_t + V)^j - S_t^j$ can be expressed as a sum of $2^j - 1$ products, each of them containing V , we have

$$\|S_{t+1}^j - S_t^j\| \leq (2^j - 1)\frac{r' - r}{n} \leq \frac{(2^m - 1)(r' - r)}{n} < \varepsilon.$$

For $x \in \bigoplus_{i=k_{t+1}-2d+1}^{\infty} F_i$ we have $Sx, S^2x, \dots, S^jx \in \bigoplus_{i=k_{t+1}-2d}^{\infty} F_i \subset \bigoplus_{i=k_t+2}^{\infty} F_i$. Moreover, S_t behaves on $\bigoplus_{i=k_t+2}^{\infty} F_i$ as $s_t S$. Thus $S_t^j x =$

$s_t^j S^j x$. Hence

$$\begin{aligned} \|Q_{k_{t+1}-2d} S_{t+1}^j Q_{k_{t+1}-2d}\| &\leq \|Q_{k_{t+1}-2d} S_t^j Q_{k_{t+1}-2d}\| + \varepsilon \\ &= s_t^j \|Q_{k_{t+1}-2d} S^j Q_{k_{t+1}-2d}\| + \varepsilon \leq s_t (\|S^j\|_e + \varepsilon) + \varepsilon \\ &\leq s_t \|S^j\|_e + 2\varepsilon \leq \|S^j\|_e + 2\varepsilon - (1 - r') \|S^j\|_e \leq \|S^j\|_e - \varepsilon. \end{aligned}$$

This proves (1).

In particular, for $t = n$ we get $\|T'^j\| < \|S^j\|_e$ ($j = 1, \dots, m$). \square

Let (c_j) be a submultiplicative sequence of positive numbers, i.e., $c_{i+j} \leq c_i c_j$ for all $i, j \in \mathbb{N}$. It is well known that the limit $\lim_{j \rightarrow \infty} c_j^{1/j}$ exists and it is equal to $\inf_j c_j^{1/j}$.

We need the following lemma.

Lemma 5. Let (c_j) be a submultiplicative sequence of positive numbers such that $\lim_{j \rightarrow \infty} c_j^{1/j} = 1$. Let $0 < r < r' < 1$. Then there exists $k \in \mathbb{N}$ with the following property: if (d_j) is a submultiplicative sequence satisfying

$$0 \leq d_j \leq c_j \quad (j = 1, 2, \dots)$$

and

$$d_j < r^j c_j \quad (1 \leq j \leq k)$$

then

$$d_j < r'^j c_j \quad (j = 1, 2, \dots).$$

Proof. Choose r'' , $r < r'' < r'$. Find k_0 such that $c_{k_0} < \left(\frac{r''}{r}\right)^{k_0}$.

Let $M = \max\{1, c_1, \dots, c_{k_0-1}\}$ and $L = \frac{M}{r''^{k_0}}$. Then there exists $k \geq k_0$ such that $L r''^k < r'^k$.

Let (d_j) be a submultiplicative sequence satisfying $d_j \leq c_j$ ($j \in \mathbb{N}$) and $d_j < r^j c_j$ ($1 \leq j \leq k$). For $1 \leq j \leq k$ we have $d_j < r^j c_j < r'^j c_j$.

Let $j > k$. Then $j = sk_0 + z$ for some $s \in \mathbb{N}$ and $z \in \{0, 1, \dots, k_0 - 1\}$. Then

$$d_j \leq d_{k_0}^s \cdot d_z < (r^{k_0} c_{k_0})^s \cdot c_z < M \cdot r''^{k_0 s} = \frac{M r''^j}{r''^z} \leq L r''^j < r'^j \leq r'^j c_j.$$

\square

Theorem 6. Let H be an infinite-dimensional Hilbert space. Let $S \in B(H)$ satisfy $r_e(S) \neq 0$. Then there exists a compact operator $K \in B(H)$ such that

$$\|(S + K)^j\| = \|S^j\|_e$$

for all $j = 1, 2, \dots$

Proof. Without loss of generality we may assume that H is separable. Indeed, there exists a decomposition $H = \bigoplus_{\nu \in J} H_\nu$ such that all the subspaces H_ν are separable and reducing for S . Write $S_\nu = P_{H_\nu} S P_{H_\nu}$.

For all $n \in \mathbb{N}$ and $\varepsilon > 0$ there are only finitely many $\nu \in J$ such that $\|S_\nu^n\| > \|S^n\|_e + \varepsilon$. So there are only countable many $\nu \in J$ such that $\|S_\nu^n\| > \|S^n\|_e$. Hence there exists a countable subset $J_0 \subset J$ such that $\|S_\nu^n\| \leq \|S^n\|_e$ for all $\nu \notin J_0$ and $n \in \mathbb{N}$. Let $H_0 = \bigoplus_{\nu \in J_0} H_\nu$. Then H_0 is a separable subspace reducing for T and $\|P_{H_0} S^n P_{H_0}\| \leq \|S^n\|_e$ for all $n \in \mathbb{N}$. So we may consider only the operator $P_{H_0} S P_{H_0}$.

Without loss of generality we may assume that $r_e(S) = 1$. Fix a sequence (r_n) such that $0 = r_0 < r_1 < \dots < 1$ and $\lim_{n \rightarrow \infty} r_n = 1$.

Consider the submultiplicative sequence $c_n = \|S^n\|_e$. Then $1 = r_e(S) = \lim_{j \rightarrow \infty} \|S^j\|_e^{1/j} = \lim_{j \rightarrow \infty} c_j^{1/j}$.

For $n = 1, 2, \dots$, let k_n be the number constructed in Lemma 5 for the sequence (c_j) and numbers r_n, r_{n+1} . Thus, if (d_j) is a submultiplicative sequence satisfying $d_j \leq c_j$ for all j and $d_j < r_n^j c_j$ ($1 \leq j \leq k_n$) then $d_j < r_{n+1}^j c_j$ for all $j \in \mathbb{N}$.

Construct inductively a sequence (T_n) of operators such that

$$\|T_n^j\| < r_n^j \|S^j\|_e \quad (j = 1, \dots, k_n),$$

$$\|T_{n+1} - T_n\| \leq (r_{n+1} - r_n) \|S\|$$

and

$$T_n - r_{n-1} S \in \mathcal{F}(H).$$

Set $T_1 = 0$. Suppose that $n \geq 1$ and $T_1, \dots, T_n \in B(H)$ with the above properties have already been constructed. Since $\|T_n^j\| < r_n^j \|S^j\|_e$ ($1 \leq j \leq k_n$), we have $\|T_n^j\| < r_{n+1}^j \|S^j\|_e$ for all $j \in \mathbb{N}$.

By Theorem 4 for the operator $r_{n+1} S$ and the pair of numbers $\frac{r_{n-1}}{r_{n+1}}, \frac{r_n}{r_{n+1}}$, there exists $T_{n+1} \in B(H)$ such that

$$\|T_{n+1} - T_n\| \leq (r_n - r_{n-1}) \|S\|,$$

$$T_{n+1} - r_n S \in \mathcal{F}(H)$$

and

$$\|T_{n+1}^j\| < r_{n+1}^j \|S^j\|_e \quad (1 \leq j \leq k_{n+1}).$$

Let (T_n) be the sequence of operators satisfying the above properties. Clearly (T_n) is a Cauchy sequence. Let T be the norm-limit of the sequence (T_n) and $K = T - S$. For each $j \in \mathbb{N}$ we have

$$\|T^j\| = \lim_{n \rightarrow \infty} \|T_n^j\| \leq \lim_{n \rightarrow \infty} r_n^j \|S^j\|_e \leq \|S^j\|_e.$$

Moreover,

$$K = \lim_{n \rightarrow \infty} T_n - S = \lim_{n \rightarrow \infty} (T_n - r_{n-1}S),$$

where $T_n - r_{n-1}S \in \mathcal{F}(H)$ for all n . So K is a compact operator and $\|T^j\| = \|S^j\|_e$ for all j . \square

Problem 7. Is the statement of Theorem 6 true for Riesz operators?

Recall that $T \in B(H)$ is called power bounded if $\sup_n \|T^n\| < \infty$. An operator $S \in B(H)$ is called essentially power bounded if $\sup_n \|S^n\|_e < \infty$.

Essentially power bounded operators may serve as a source of interesting examples of Hilbert space operators. Note that for example the Read-type operator constructed by S. Grivaux and M. Roginskaya [GR] is of this class.

Theorem 6 has the following simple consequence.

Corollary 8. An operator $S \in B(H)$ is essentially power bounded if and only if $S = T + K$ for some power bounded operator $T \in B(H)$ and a compact operator $K \in B(H)$.

Proof. If $\sup_n \|T^n\| < \infty$ and $K \in B(H)$ is compact, then $\|(T + K)^n\|_e \leq \|T^n\|$ for all n , and so $T + K$ is essentially power bounded.

Let $S \in B(H)$ be essentially power bounded. If $r_e(S) \neq 0$ then $S = T + K$ for some compact operator K and an operator T satisfying $\|T^n\| = \|S^n\|_e$ for all $n \in \mathbb{N}$. So T is power bounded.

If $r_e(s) = 0$, then $S = T + K$ for some $K \in \mathcal{K}(H)$ and a quasinilpotent $T \in B(H)$ by the West decomposition. Clearly then T is power bounded. \square

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MATHEMATICAL INSTITUTE, CZECH ACADEMY OF SCIENCES, ŽITNA 25, 11567
PRAGUE 1, CZECH REPUBLIC
E-mail address: muller@math.cas.cz