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ON THE SPECTRUM AND THE SPECTRAL MAPPING THEOREM IN MAX ALGEBRA

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ABSTRACT. We give a new description of spectrum in max algebra of a given non-negative matrix A via local spectral radii and obtain a new block triangular form of A related to its Frobenius normal form. Related results for the usual spectrum of complex matrices and distinguished spectrum for non-negative matrices are also obtained.

As an application we provide a new proof of the spectral mapping theorem in max algebra and also generalize it to the setting of power series in max algebra.

Given a non-negative bounded infinite matrix A, we show that the Bonsall's cone spectral radius of a map $x \mapsto A \otimes x$, with respect to the cone l_+^∞, is included in its max algebra approximate point spectrum. Moreover, the spectral mapping theorem with respect to point and approximate point spectrum in max algebra is investigated. The corresponding results for more general max and max-plus type kernel operators and for tropical Bellman operators are obtained.

1. INTRODUCTION

The algebraic system max algebra and its isomorphic versions (max-plus algebra, tropical algebra) provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimization, mathematical physics, DNA analysis, ...(see e.g. [12], [6], [17], [7],[10], [33] and the references cited there). Max algebra's usefulness arises from a fact that these non-linear problems become linear when described in the max algebra language. Moreover, recently max algebra techniques were used to solve certain linear algebra problems (see e.g. [15], [30]). In particular, tropical

polynomial methods improved the accuracy of the numerical computation of the eigenvalues of a matrix polynomial (see e.g. [1], [2], [18], [19], [3], [11] and the references cited there).

The max algebra consists of the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab, where $a, b \geq 0$. Denote by Ω the set $\{1, 2, \ldots, n\}$ or $\mathbb{N}$. Let $A = [A_{ij}]$ be a non-negative matrix, i.e., $A_{ij} \geq 0$ for all $i, j \in \Omega$. A non-negative matrix A is called bounded if $\sup\{A_{ij} : i, j \in \Omega\} < \infty$.

Let $\mathbb{R}^{n \times n}$ ($\mathbb{C}^{n \times n}$) be the set of all $n \times n$ real (complex) matrices, $\mathbb{R}_+^{n \times n}$ the set of all $n \times n$ non-negative matrices and $M_+^{\infty \times \infty}$ the set of all non-negative bounded matrices in the case $\Omega = \mathbb{N}$. The operations between matrices and vectors in the max algebra are defined by analogy with the usual linear algebra. The product of non-negative bounded matrices A and B in the max algebra is denoted by $A \otimes B$, where $(A \otimes B)_{ij} = \sup_{k \in \Omega} A_{ik} B_{kj}$ and the sum $A \oplus B$ in the max algebra is defined by $(A \oplus B)_{ij} = \max\{A_{ij}, B_{ij}\}$. The notation $A_\otimes^2$ means $A \otimes A$, and $A_\otimes^k$ denotes the k-th max power of A. If $x = (x_i)_{i \in \Omega}$ is a non-negative bounded (i.e., $\|x\| = \sup_{i \in \Omega} x_i < \infty$) vector, then the notation $A \otimes x$ means $(A \otimes x)_i = \sup_{j \in \Omega} A_{ij} x_j$. The usual associative and distributive laws hold in this algebra.

The role of the spectral radius of $A \in \mathbb{R}_+^{n \times n}$ in max algebra is played by the maximum cycle geometric mean $r_\otimes(A)$, which is defined by

$$(1) \qquad r_\otimes(A) = \max\left\{ (A_{i_1 i_k} \cdots A_{i_3 i_2} A_{i_2 i_1})^{1/k} : k \in \mathbb{N} \text{ and } i_1, \ldots, i_k \in \{1, \ldots, n\} \right\}$$

and equal to

$$r_\otimes(A) = \max\left\{ (A_{i_1 i_k} \cdots A_{i_3 i_2} A_{i_2 i_1})^{1/k} : k \leq n \text{ and } i_1, \ldots, i_k \in \{1, \ldots, n\} \text{ mutually distinct} \right\}.$$

A digraph $\mathcal{G}(A) = (N(A), E(A))$ associated to $A \in \mathbb{R}_+^{n \times n}$ is defined by setting $N(A) = \{1, ..., n\}$ and letting $(i, j) \in E(A)$ whenever $A_{ij} > 0$. When this digraph contains at least one cycle, one distinguishes critical cycles, where the maximum in (1) is attained. A graph with just one node and no edges will be called trivial. A bit unusually, but in consistency with [12], [13], [20], [14], a matrix $A \in \mathbb{R}_+^{n \times n}$ is called irreducible if $\mathcal{G}(A)$ is trivial (A is 1×1 zero matrix) or strongly connected (for each $i, j \in N(A)$ there is a path in $\mathcal{G}(A)$ that starts in i and ends in j).

There are many different descriptions of the maximum cycle geometric mean $r_\otimes(A)$ (see e.g. [16], [12], [31], [32], [30] and the references cited there). It is known that $r_\otimes(A)$ is the largest max eigenvalue of A, i.e., $r_\otimes(A) = \max\{\lambda : \lambda \in \sigma_\otimes(A)\}$, where the spectrum in max algebra $\sigma_\otimes(A)$ is the set of all (max eigenvalues) $\lambda \geq 0$ for which there exists $x \in \mathbb{R}_+^n$, $x \neq 0$ with $A \otimes x = \lambda x$.

Moreover, if A is irreducible, then $r_\otimes(A)$ is the unique max eigenvalue and every max eigenvector is positive (see e.g. [7, Theorem 2], [12], [6], [9]). Also, the max version of the Gelfand formula holds for any $A \in \mathbb{R}_+^{n \times n}$, i.e.,

$$(2) \qquad r_\otimes(A) = \lim_{m \to \infty} \|A_\otimes^m\|^{1/m}$$

for an arbitrary vector norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ (see e.g. [31] and the references cited there).

As outlined e.g. in [27], due to (2) a natural generalization (and unification with the usual spectral radius) in infinite dimensions is the Bonsall's cone spectral radius, which we will recall in Section 4. An eigenproblem for max type kernel operators and its isomorphic versions (and an eigenproblem for more general maps) has already received a lot of attention (see e.g. [4], [27], [29], [21], [34] and the references cited there). The results can be applied in different contexts, for instance in optimal control problems (here the max eigenvectors correspond to stationary solutions of the dynamic programming equations and the max eigenvalues correspond to the maximal ergodic rewards per time unit), in the study of discrete event systems, in statistical mechanics, in the study of delay systems, ... (see e.g. [4], [27], [13] and the references cited there).

However, besides the study of the eigenproblem there seems to be a lack of more general functional analytic spectral theory in max algebra (and more generally in idempotent mathematics) in the literature, even though the need for it has already been explicitly requested by the idempotent community.

The paper is organized as follows. In Section 2 we give a new description of spectrum in max algebra of a given non-negative matrix A via local spectral radii (Theorem 2.6) and obtain a new block triangular description of A related to its Frobenius normal form. Consequently we provide a new proof of the spectral theorem in max algebra (Corollary 2.9). Related results for the usual spectrum of complex matrices and distinguished spectrum for non-negative matrices are also obtained (Proposition 2.11 and Theorem 2.13). In Section 3 we apply results of Section 2 to obtain a new proof of the spectral mapping theorem in max algebra (Theorem 3.3) and we also generalize it to the setting of power series in max algebra (Theorem 3.7) by applying the continuity properties of the spectrum in max algebra (Proposition 3.6).

Given a non-negative bounded infinite matrix A, we introduce in Section 4 the notion of the approximate point spectrum in max algebra. In particular, we show that the Bonsall's cone spectral radius of a map $x \mapsto A \otimes x$, with respect to the cone l_+^∞, is included in its max algebra approximate point spectrum (Theorem 4.2). Moreover, the spectral mapping theorem with respect to point and approximate point spectrum in max algebra is investigated. In particular, we prove that in both cases the spectral mapping theorem is valid for polynomials without an absolute term (Theorem 5.4 and Corollary 5.6). Also the corresponding results for more general max and max-plus type kernel operators and for tropical Bellman operators are obtained.

2. Spectrum in max algebra for $n \times n$ matrices

In this section we prove a new description of the spectrum $\sigma_\otimes(A)$ in max algebra and a corresponding block triangular decomposition of a given matrix $A \in \mathbb{R}_+^{n\times n}$.

For $x \in \mathbb{R}_+^n$ we define the local spectral radius in max algebra at x by

$$r_x(A) = \limsup_{k \to \infty} \|A_\otimes^k \otimes x\|^{1/k}$$

(we show later that in fact the limit $\lim_{k \to \infty} \|A_\otimes^k \otimes x\|^{1/k}$ always exists).

Let $e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^n$. Observe that $\|A_\otimes^k \otimes e_j\|$ is the largest entry of the jth column of the matrix $A_\otimes^k$, i.e.,

$$\|A_\otimes^k \otimes e_j\| = \max\{A_{j_k,j_{k-1}} \cdots A_{j_2,j_1} A_{j_1,j} : 1 \leq j, j_1, \cdots, j_k \leq n\}.$$

The following result describes $r_{e_j}(A)$ for all $j \in \{1, \ldots, n\}$.

Lemma 2.1. *Let $A \in \mathbb{R}_+^{n \times n}$, $j \in \{1, \ldots, n\}$. Then $r_{e_j}(A)$ is the maximum of all $t \geq 0$ with the following property (*):*

there exist $a \geq 0$, $b \geq 1$ and mutually distinct indices $i_0 := j, i_1, \ldots, i_a, i_{a+1}, \ldots, i_{a+b-1} \in \{1, \ldots, n\}$ such that

$$\prod_{s=0}^{a-1} A_{i_{s+1},i_s} \neq 0 \quad \text{and} \quad \prod_{s=a}^{a+b-1} A_{i_{s+1},i_s} = t^b,$$

where we set $i_{a+b} = i_a$.

Proof. Let $t, i_0, \ldots, i_{a+b-1}$ satisfy (*). Then

$$r_{e_j}(A) = \limsup_{k \to \infty} \|A_\otimes^k \otimes e_j\|^{1/k} \geq \limsup_{m \to \infty} \|A_\otimes^{a+mb} \otimes e_j\|^{1/a+mb}$$

$$\geq \limsup_{m \to \infty} \left(\prod_{s=0}^{a-1} A_{i_{s+1},i_s} \left(\prod_{s=a}^{a+b-1} A_{i_{s+1},i_s} \right)^m \right)^{1/a+mb} = \limsup_{m \to \infty} \left(t^m \prod_{s=0}^{a-1} A_{i_{s+1},i_s} \right)^{1/a+mb} = t.$$

Conversely, let c be the maximum of all t with property (*). Then it is not difficult to show that $\|A_\otimes^k \otimes e_j\| \leq \|A\|^n \cdot c^{k-n}$ for all $k \in \mathbb{N}$. Hence $r_{e_j}(A) \leq c$ $\square$

The following result follows directly from Lemma 2.1 and definition (1) of $r_\otimes(A)$.

Corollary 2.2. *If $A \in \mathbb{R}_+^{n \times n}$, then*

$$r_\otimes(A) = \max_{j=1,\ldots,n} r_{e_j}(A).$$

Remark 2.3. Lemma 2.1 states that (using a terminology of e.g. [13], [12]) for each $j \in \{1, \ldots, n\}$, the radius $t = r_{e_j}(A)$ equals the maximum of cycle geometric means, such that the node j is accessible from one of the corresponding ($t-$critical) cycles. Now we could already deduce Theorem 2.6 bellow by applying the known spectral mapping theorem (see Corollary 2.9 bellow). However, we choose to prove Theorem 2.6 independently and thus we consequently provide a new proof of Corollary 2.9.

First we prove the following result.

Theorem 2.4. *Let $A \in \mathbb{R}_+^{n \times n}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $x \neq 0$. Then:*

(i) *the limit* $\lim_{k\to\infty} \|A_\otimes^k \otimes x\|^{1/k}$ *exists;*

(ii) $r_x(A) = \max\{r_{e_j}(A) : 1 \le j \le n, x_j \ne 0\}$.

Proof. We show first that $\lim_{k\to\infty} \|A_\otimes^k \otimes e_j\|^{1/k}$ exists for each $j = 1, \ldots, n$.

By Lemma 2.1 there exist $i_0 = j, i_1, \ldots, i_{a+b-1}$ satisfying property (*) with $t = r_{e_j}(A)$. Let $k \in \mathbb{N}$. Write $k = a + db + z$ with $d \in \mathbb{N}_0$ and $z < b$. Then

$$\|A_\otimes^k \otimes e_j\| \ge \prod_{s=0}^{a-1} A_{i_{s+1},i_s} \left(\prod_{s=a}^{a+b-1} A_{i_{s+1},i_s} \right)^d \prod_{s=a}^{a+z-1} A_{i_{s+1},i_s}$$

$$= \prod_{s=0}^{a-1} A_{i_{s+1},i_s} \cdot t^{db} \cdot \prod_{s=a}^{a+z-1} A_{i_{s+1},i_s}.$$

It follows that

$$\liminf_{k\to\infty} \|A_\otimes^k \otimes e_j\|^{1/k} \ge t = r_{e_j}(A)$$

and therefore $r_{e_j}(A) = \lim_{k\to\infty} \|A_\otimes^k \otimes e_j\|^{1/k}$ for all $j = 1, \ldots, n$.

Moreover, for each $k \in \mathbb{N}$, we have

$$\|A_\otimes^k \otimes x\| = \max\{x_j \|A_\otimes^k \otimes e_j\| : j = 1, \ldots, n\}.$$

So

$$\lim_{k\to\infty} \|A_\otimes^k \otimes x\|^{1/k} = \lim_{k\to\infty} \max\{\|A_\otimes^k \otimes e_j\|^{1/k} : x_j \ne 0\}$$

$$= \max\{\lim_{k\to\infty} \|A_\otimes^k \otimes e_j\|^{1/k} : x_j \ne 0\} = \max\{r_{e_j}(A) : 1 \le j \le n, x_j \ne 0\},$$

which completes the proof. $\square$

A set $C \subset \mathbb{R}_+^n$ is called *a max cone*, if $x \oplus y \in C$ and $\lambda x \in C$ for all $x, y \in C$ and $\lambda \ge 0$. For a set $S \subset \mathbb{R}_+^n$ we denote by $\bigvee S$ the max cone generated by S, i.e., $\bigvee S$ is the set of all $x \in \mathbb{R}_+^n$ for which there exist $k = k(x) \in \mathbb{N}$, $s_1, \ldots, s_k \in S$ and $\lambda_1, \ldots, \lambda_k \ge 0$ such that $x = \lambda_1 s_1 \oplus \cdots \oplus \lambda_k s_k$.

A max cone C is *invariant* for $A \in \mathbb{R}_+^{n \times n}$, if $A \otimes x \in C$ for all $x \in C$. We have the following result.

Corollary 2.5. *Let* $A \in \mathbb{R}_+^{n \times n}$, $c \ge 0$. *Then* $\{x \in \mathbb{R}_+^n : r_x(A) \le c\} = \bigvee\{e_j : r_{e_j}(A) \le c\}$ *is a max cone invariant for* A.

Proof. The equality follows from the previous theorem. Clearly $r_{A \otimes x}(A) = r_x(A)$ for all $x \in \mathbb{R}_+^n$, so the max cone is invariant for A. $\square$

Now we are in position to prove the following description of spectrum in max algebra.

Theorem 2.6. *If* $A \in \mathbb{R}_+^{n \times n}$, *then*

$$\sigma_\otimes(A) = \{t : \text{ there exists } j \in \{1, \ldots, n\}, t = r_{e_j}(A)\}.$$

Proof. If $t \in \sigma_\otimes(A)$, then there exists a nonzero $x \in \mathbb{R}_+^n$ such that $A \otimes x = tx$. So $r_x(A) = \lim_{k\to\infty} \|A_\otimes^k \otimes x\|^{1/k} = t$. By Theorem 2.4, there exists $j \in \{1, \ldots, n\}$ with $r_{e_j}(A) = t$.

Conversely, let $j \in \{1, \ldots, n\}$ and $c := r_{e_j}(A)$. Let $M = \bigvee\{e_s : r_{e_s}(A) \leq c\}$. By Corollary 2.5, the max cone M is invariant for A. For the restriction $A|_M$ of A to M we have by Corollary 2.2 that $r_\otimes(A|_M) = \max\{r_{e_s}(A|_M) : e_s \in M\} = c$. So there exists a max Perron-Frobenius eigenvector $x \in M \subset \mathbb{R}_+^n$ with $A \otimes x = cx$. $\qquad \square$

Remark 2.7. In [27] the equality

$$r_\otimes(A) = \max\{r_x(A) : x \in \mathbb{R}_+^n\}$$

is generalized to the setting of cone preserving maps in Banach spaces.

It follows from the proof of Theorem 2.6 that there exists a permutation matrix P such that the matrix $P^T A P$ equals

(3)
$$\begin{bmatrix} A_d & 0 & 0 & \ldots & 0 & 0 \\ * & A_{d-1} & 0 & \ldots & 0 & 0 \\ * & * & A_{d-2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \ldots & A_2 & 0 \\ * & * & * & \ldots & * & A_1 \end{bmatrix},$$

where $A_1, \cdots, A_d$ are square matrices, d is the cardinality of $\sigma_\otimes(A)$ and

$$r_\otimes(A_1) < r_\otimes(A_2) < \cdots < r_\otimes(A_d) = r_\otimes(A)$$

are max eigenvalues of a matrix A. Moreover, for each $i = 1, \ldots, d$ the set

$$M_i = \bigvee\{e_j : r_{e_j}(A) \leq r_\otimes(A_i)\}$$

is a max cone invariant for A such that $r_\otimes(A|_{M_i}) = r_\otimes(A_i)$.

In what follows we describe a correlation of our results with known results in terms of a Frobenius normal form of a matrix A (see e.g. [13], [12], [20], [14] and the references cited there). Each A_i for $i = 1, \ldots, d$ can be transformed by simultaneous permutations of the rows and columns to a Frobenius normal form (FNF) ([8], [12], [13], [20])

$$\begin{bmatrix} A_{l_i}^{[i]} & 0 & 0 & \ldots & 0 \\ * & A_{l_i-1}^{[i]} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ * & * & * & \ldots & A_1^{[i]} \end{bmatrix},$$

where $A_1^{[i]}, \ldots, A_{l_i}^{[i]}$ are irreducible square submatrices of A_i.

This gives a FNF of a matrix A denoted by

(4)
$$\begin{bmatrix} B_l & 0 & 0 & \dots & 0 \\ * & B_{l-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ * & * & * & \dots & B_1 \end{bmatrix}.$$

For example, the decomposition (3) of the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has only one block I, but its FNF has two blocks equal to [1]. In general, the diagonal blocks in FNF are determined uniquely (up to a simultaneous permutation of their rows and columns), however their order is not determined uniquely.

The matrices $B_1, \dots, B_l$ from (4) correspond to the sets of nodes $N_1, \dots, N_l$ of the strongly connected components of the digraph $\mathcal{G}(A) = (N(A), E(A))$. Note that in (4) an edge from a node of N_μ to a node of N_ν in $\mathcal{G}(A)$ may exist only if $\mu \le \nu$.

The reduced graph denoted by $\mathcal{R}(A)$ is a digraph whose nodes correspond to N_μ for $\mu = 1, \dots, l$ and the set of edges is

$$\{(\mu, \nu) : \text{ there exist } k \in N_\mu \text{ and } j \in N_\nu \text{ such that } A_{kj} > 0\}.$$

By a class of A we mean a node μ (or also the corresponding set N_μ) of the reduced graph $\mathcal{R}(A)$. A class μ is trivial if B_μ is the 1×1 zero matrix. Class μ accesses class ν, denoted $\mu \to \nu$, if $\mu = \nu$ or if there exists a $\mu - \nu$ path in $\mathcal{R}(A)$ (a path that starts in μ and ends in ν). A node j of $\mathcal{G}(A)$ is accessed by a class μ, denoted by $\mu \to j$, if j belongs to a class ν such that $\mu \to \nu$.

The following result, that describes the max eigenvalues $r_{e_j}(A)$ via the access relation, follows from Lemma 2.1.

Corollary 2.8. *Let $A \in \mathbb{R}_+^{n \times n}$ and let $B_1, \dots, B_l$ be from (4). Then we have*

$$r_{e_j}(A) = \max\{r_\otimes(B_\mu) : \mu \to j\}.$$

for all $j = 1, \dots, n$.

For each $j = 1, \dots, n$ we have $r_{e_j}(A) = r_\otimes(B_\nu)$ for some class ν (but not vice versa in general). Now the following result (sometimes called the spectral theorem in max algebra ([13, Theorem 3.1], [12, Theorem 4.5.4], [20, Corollary 4.2(i)], [17], [10], [9]) follows from Corollary 2.8 and Theorem 2.5.

Corollary 2.9. *Let $A \in \mathbb{R}_+^{n \times n}$ and let $B_1, \dots, B_l$ be from (4). Then*

$$\sigma_\otimes(A) = \{r_\otimes(B_\nu) : r_\otimes(B_\nu) = \max\{r_\otimes(B_\mu) : \mu \to \nu\}\}.$$

If $r_\otimes(B_\nu) = \max\{r_\otimes(B_\mu) : \mu \to \nu\}$ is satisfied, then a class ν is called spectral. Thus $r_\otimes(B_\nu) \in \sigma_\otimes(A)$ if a class ν is spectral (but not necessarily vice versa as it is well known). The following example illustrates the results obtained above.

Example 2.10. Let

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then A is in the forms (3) and (4), where

$$A_3 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_1 = [1],$$

$$B_4 = [2], \quad B_3 = [3], \quad B_2 = A_2, \quad B_1 = A_1.$$

The spectral classes correspond to B_3, B_2, B_1 and $r_{e_1}(A) = r_{e_2}(A) = 3$, $r_{e_3}(A) = r_{e_4}(A) = 2$ and $r_{e_5}(A) = 1$.

To conclude this section we state some related linear algebra results for the spectrum of complex matrices and for the distinguished spectrum of non-negative matrices (see e.g. [20], [14]). First we describe the set $\{|\lambda| : \lambda \in \sigma(A)\}$ of a given matrix $A \in \mathbb{C}^{n \times n}$, where $\sigma(A)$ denotes the usual spectrum of A, via the usual local spectral radii. Recall that for $x \in \mathbb{C}^n$ the local spectral radius $\rho_x(A)$ at x is defined by

$$\rho_x(A) = \limsup_{k \to \infty} \|A^k x\|^{1/k},$$

where A^k denote the usual powers of A.

Proposition 2.11. *If $A \in \mathbb{C}^{n \times n}$, then*

$$\{|\lambda| : \lambda \in \sigma(A)\} = \{t : \text{ there exists } x \in \mathbb{C}^n, t = \rho_x(A)\}.$$

Proof. Since the inclusion $\subset$ is obvious, it remains to prove the reverse inclusion. For a given $x \in \mathbb{C}^n$ let us denote $\rho_x(A) = t$. By M let us denote the linear span of the set $\{y : \rho_y(A) \leq t\}$. Then M is an invariant subspace and the usual spectral radius $\rho(A|_M) = t$. Thus there exists $\lambda \in \sigma(A)$ such that $|\lambda| = t$. This completes the proof. $\square$

Proposition 2.11 implies the following known result (see e.g. [27]).

Corollary 2.12. *If $A \in \mathbb{C}^{n \times n}$, then the usual spectral radius*

$$\rho(A) = \max\{\rho_x(A) : x \in \mathbb{C}^n\}.$$

Recall that $\lambda \in \sigma(A)$ of $A \in M_+^{n \times n}$ is called a distinguished eigenvalue if there exists $x \in \mathbb{R}_+^n$, $x \neq 0$, such that $Ax = \lambda x$. Let us denote the set of all distinguished eigenvalues of A by $\sigma_D(A)$, which is a non-empty set since $\rho(A) \in \sigma_D(A)$. The following result is an

analogue of Theorem 2.6 in nonnegative linear algebra (linear algebra over a semiring of non-negative reals equipped with the usual sum and product).

Theorem 2.13. *If $A \in \mathbb{R}_+^{n \times n}$, then*

$$\sigma_D(A) = \{t : \text{ there exists } j \in \{1, \dots, n\}, t = \rho_{e_j}(A)\}.$$

Proof. $\subset$: If $Ax = \lambda x$ for some $x \geq 0$, $x \neq 0$, then $\lambda = \rho_x(A)$. It is easy to see that $\rho_x(A) = \max\{\rho_{e_j}(A) : x_j \neq 0\}$, which proves the inclusion $\subset$.

$\supset$: Let $t = \rho_{e_j}(A)$ for some $j = 1, \cdots, n$. By M let us denote the subcone generated by the set $\{e_i : \rho_{e_i}(A) \leq t\}$. Then $AM \subset M$ and $\rho(A|_M) = t$. Since $\rho(A|_M) = \max\{\rho_x(A|_M) : x \in M\}$, there exists $x \in M$, $x \neq 0$, such that $Ax = tx$. This completes the proof. $\square$

Remark 2.14. From the proof of Theorem 2.13 we can obtain a block triangular decomposition of $A \in \mathbb{R}_+^{n \times n}$, which is an analogue of (3) and consequently a FNF of A in a similar manner as above. We omit the details. There are several similarities, but also some differences, in the description of $\sigma_\otimes(A)$ and $\sigma_D(A)$ via an access relation (we refer the reader to [20], [14], [13], [12]).

3. SPECTRAL MAPPING THEOREM IN MAX ALGEBRA FOR $n \times n$ MATRICES

The spectral mapping theorem for polynomials (see (5) bellow) in max algebra was established in [20, Theorem 3.6]. Applying Theorem 2.6 we give a different proof of this result (Theorem 3.3). Moreover, we extend it to power series in max algebra (Theorem 3.7).

Let $\mathcal{P}_+$ be the set of all polynomials with non-negative coefficients,

$$\mathcal{P}_+ = \{p = \sum_{j=0}^{\deg p} \alpha_j z^j : \alpha_j \geq 0, j = 0, \dots \deg p\}.$$

For $p, q \in \mathcal{P}_+$, $p = \sum_{j=0}^{\deg p} \alpha_j z^j$, $q = \sum_{j=0}^{\deg q} \beta_j z^j$ define $p \oplus q, p \otimes q \in \mathcal{P}_+$ by

$$p \oplus q = \sum_{j=0}^{\max\{\deg p, \deg q\}} \max\{\alpha_j, \beta_j\} z^j,$$

$$p \otimes q = \sum_{j=0}^{\deg p + \deg q} \max\{\alpha_i \beta_{j-i} : 0 \leq i \leq j\} z^j.$$

These algebraic operations on $\mathcal{P}_+$ satisfy the familiar laws. The proof of the following result is elementary and we omit it.

Proposition 3.1. *Let $p, q, h \in \mathcal{P}_+$. Then we have*

   (i) $p \oplus p = p$,
  (ii) $p \oplus q = q \oplus p$,
 (iii) $(p \oplus q) \oplus h = p \oplus (q \oplus h)$,

(iv) $p \otimes q = q \otimes p$,

(v) $(p \otimes q) \otimes h = p \otimes (q \otimes h)$,

(vi) $p \otimes (q \oplus h) = p \otimes q \oplus p \otimes h$.

Let $p \in \mathcal{P}_+$, $p = \sum_{j=0}^{\deg p} \alpha_j z^j$ and $t \geq 0$. Write $p_\otimes(t) = \max\{\alpha_j t^j : 0 \leq j \leq \deg p\}$. For $A \in \mathbb{R}_+^{n \times n}$ define

$$p_\otimes(A) = \bigoplus_{j=0}^{\deg p} \alpha_j A_\otimes^j.$$

The proof of the following fact is again straightforward and it is omitted.

Proposition 3.2. *Let $p, q \in \mathcal{P}_+$ and $A \in \mathbb{R}_+^{n \times n}$. Then*

(i) $(p \oplus q)_\otimes(A) = p_\otimes(A) \oplus q_\otimes(A)$,

(ii) $(p \otimes q)_\otimes(A) = p_\otimes(A) \otimes q_\otimes(A)$.

Thus the mapping $p \mapsto p_\otimes(A)$ defines naturally a polynomial functional calculus for matrices $A \in \mathbb{R}_+^{n \times n}$.

Let $A \in \mathbb{R}_+^{n \times n}$ and $p \in \mathcal{P}_+$. It follows from [20, Theorem 3.6] that

$$(5) \qquad\qquad\qquad \sigma_\otimes(p_\otimes(A)) = p_\otimes(\sigma_\otimes(A)).$$

In [20] the equality (5) was deduced from the existence theorem of common max eigenvectors for commutative matrices in max algebra ([20, Theorem 3.5], [12]) applied to matrices A and $p_\otimes(A)$.

Applying the results of the previous section we give a different proof of (5).

Theorem 3.3. *Let $A \in \mathbb{R}_+^{n \times n}$ and $p \in \mathcal{P}_+$. Then*

$$(6) \qquad\qquad\qquad r_\otimes(p_\otimes(A)) = p_\otimes(r_\otimes(A))$$

and

$$(7) \qquad\qquad\qquad \sigma_\otimes(p_\otimes(A)) = p_\otimes(\sigma_\otimes(A)).$$

Proof. Let $t \in \sigma_\otimes(A)$. There exists $x \in \mathbb{R}_+^n$, $x \neq 0$ with $A \otimes x = tx$. So $A_\otimes^j \otimes x = t^j x$ for all j, and so $p_\otimes(A) \otimes x = p_\otimes(t) \otimes x$. It follows $p_\otimes(\sigma_\otimes(A)) \subset \sigma_\otimes(p_\otimes(A))$ and so $p_\otimes(r_\otimes(A)) \leq r_\otimes(p_\otimes(A))$.

Let us denote $b = r_\otimes(p_\otimes(A))$. To prove (6) it remains to show that $p_\otimes(r_\otimes(A)) \geq b$ if $b > 0$. Let us consider a critical cycle for $p_\otimes(A)$, i.e., let $i_0, i_1, \ldots, i_{k-1}, i_k = i_0 \in \{1, \ldots, n\}$ be such that

$$\prod_{s=0}^{k-1} (p_\otimes(A))_{i_{s+1}, i_s} = b^k.$$

For $s = 0, \ldots, k-1$ we have

$$(p_\otimes(A))_{i_{s+1}, i_s} = \max\{(\alpha_j A_\otimes^j)_{i_{s+1}, i_s} : 0 \leq j \leq \deg p\}.$$

Choose $j(s) \in \{0, 1, \ldots, \deg p\}$ such that

$$(p_{\otimes}(A))_{i_{s+1},i_s} = (\alpha_{j(s)} A_{\otimes}^{j(s)})_{i_{s+1},i_s}.$$

Since $i_{s+1} \neq i_s$ we have $j(s) \neq 0$. Choose $i_{s,0} = i_s, i_{s,1}, \ldots, i_{s,j(s)} = i_{s+1}$ such that

$$(A_{\otimes}^{j(s)})_{i_{s+1},i_s} = \prod_{m=0}^{j(s)-1} A_{i_{s,m+1},i_{s,m}}.$$

Consider the cycle

$$i_0 = i_{0,0}, i_{0,1}, \ldots, i_{0,j(0)-1}, i_{0,j(0)} = i_1 = i_{1,0}, i_{1,1} \ldots, i_{1,j(1)} = i_2, i_{2,1}, \ldots, i_k = i_0$$

of length $\sum_{s=0}^{k-1} j(s)$. We have

$$r_{\otimes}(A)^{\sum_{s=0}^{k-1} j(s)} \geq \prod_{s=0}^{k-1} \prod_{m=0}^{j(s)-1} A_{i_{s,m+1},i_{s,m}} = \prod_{s=0}^{k-1} (A_{\otimes}^{j(s)})_{i_{s+1},i_s}$$

$$= \prod_{s=0}^{k-1} \frac{(p_{\otimes}(A))_{i_{s+1},i_s}}{\alpha_{j(s)}} = \frac{b^k}{\prod_{s=0}^{k-1} \alpha_{j(s)}}.$$

Hence

$$\prod_{s=0}^{k-1} \left(\alpha_{j(s)} r_{\otimes}(A)^{j(s)} \right) \geq b^k$$

and there exists s, $0 \leq s \leq k-1$ with $\alpha_{j(s)} r_{\otimes}(A)^{j(s)} \geq b$. So $p_{\otimes}(r_{\otimes}(A)) \geq b$. This proves (6).

It remains to prove that $\sigma_{\otimes}(p_{\otimes}(A)) \subset p_{\otimes}(\sigma_{\otimes}(A))$. Suppose on the contrary that there exists $s \in \sigma_{\otimes}(p_{\otimes}(A)) \setminus p_{\otimes}(\sigma_{\otimes}(A))$. We will show that this implies $r_{e_j}(p_{\otimes}(A)) \neq s$ for all $j \in \{1, \ldots, n\}$, which contradicts Theorem 2.6.

Let $L = \{j \in \{1, \ldots, n\} : p_{\otimes}(r_{e_j}(A)) < s\}$ and $X_L = \bigvee \{e_j : j \in L\}$. For $j \notin L$ we have $p_{\otimes}(r_{e_j}(A)) > s$ by Theorem 2.6. Therefore

$$r_{e_j}(p_{\otimes}(A)) \geq \max\{r_{e_j}(\alpha_m A_{\otimes}^m) : 0 \leq m \leq \deg p\}$$

$$\geq \max\{\alpha_m (r_{e_j}(A))^m : 0 \leq m \leq \deg p\} = p_{\otimes}(r_{e_j}(A)) > s.$$

On the other hand, X_L is invariant for A. So for $j \in L$,

$$r_{e_j}(p_{\otimes}(A)) \leq r_{\otimes}(p_{\otimes}(A|X_L)) = p_{\otimes}(r_{\otimes}(A|X_L)) < s$$

by (6) and Theorem 2.6. So $r_{e_j}(p_{\otimes}(A)) \neq s$ for all $j \in \{1, \ldots, n\}$, and so $s \notin \sigma_{\otimes}(p_{\otimes}(A))$ by Theorem 2.6. This contradiction completes the proof. $\square$

Remark 3.4. Alternatively, the inequality $p_{\otimes}(r_{\otimes}(A)) \geq r_{\otimes}(p_{\otimes}(A))$ can be proved also in the following way.

It follows from [12, Theorems 5.3.4 and 5.3.2] that

$$(8) \qquad r_{\otimes}(A) = \max\{\mathrm{maper}(B)^{1/k} : B \in P_k(A), k = 1, \ldots, n\},$$

where $P_k(A)$ is the set of all principal submatrices of A of order k. The permanent maper(B) in max algebra is defined by

$$\text{maper}(B) = \max_{\sigma \in P_k} B_{1\sigma(1)} \cdots B_{1\sigma(k)},$$

where P_k denotes the set of all permutations of the set $\{1, 2, \ldots, k\}$.

By (8) there exist $k \in \{1, \ldots, n\}$, $\sigma \in P_k$ and $j(r) \in \{1, \ldots \deg p\}$ for each $r \in \{1, \ldots, k\}$ such that

$$r_\otimes(p_\otimes(A))^k = \alpha_{j(1)} \cdots \alpha_{j(k)} \prod_{r=1}^{k} (A_\otimes^{j(r)})_{r\sigma(r)}.$$

Since σ is the product of cyclic permutations, it follows that

$$r_\otimes(p_\otimes(A))^k \leq \alpha_{j(1)} \cdots \alpha_{j(k)} r_\otimes(A)^{j(1)+\cdots+j(k)}.$$

Thus there exist $j(r) \in \{1, \ldots \deg p\}$ such that

$$r_\otimes(p_\otimes(A)) \leq \alpha_{j(r)} r_\otimes(A)^{j(r)} \leq p_\otimes(r_\otimes(A)),$$

which reproves the desired inequality.

A reformulation of the equality (6) therefore asserts that

$$\max_{B \in P_k(p_\otimes(A)),\ k=1,\ldots,n} \text{maper}(B)^{1/k} = p_\otimes \left(\max_{C \in P_l(A),\ l=1,\ldots,n} \text{maper}(C)^{1/l} \right).$$

In what follows we generalize Theorem 3.3 by considering power series instead of polynomials.

Let $\mathcal{A}_+ = \{f = \sum_{j=0}^{\infty} \alpha_j z^j : \alpha_j \geq 0, j = 0, 1, \ldots \}$.

For $f, g \in \mathcal{A}_+$, $f = \sum_{j=0}^{\infty} \alpha z^j$, $g = \sum_{j=0}^{\infty} \beta_j z^j$ define

$$f \oplus g = \sum_{j=0}^{\infty} \max\{\alpha_j, \beta_j\} z^j,$$

$$f \otimes g = \sum_{j=0}^{\infty} \max\{\alpha_i \beta_{j-i} : 0 \leq i \leq j\} z^j.$$

These operations satisfy the same rules as polynomials, i.e., an analogue of Proposition 3.1 for a set $\mathcal{A}_+$ holds.

For $f \in \mathcal{A}_+$, $f = \sum_{j=0}^{\infty} \alpha_j z^j$ write $R_f = \liminf_{j \to \infty} \alpha_j^{-1/j}$. For $0 \leq t < R_f$ write $f_\otimes(t) = \sup\{\alpha_j t^j : j = 0, 1, \ldots \}$ (note that for $t < R_f$ we have $\sup_j \alpha_j t^j < \infty$).

Let $A \in \mathbb{R}_+^{n \times n}$, $f = \sum_{j=0}^{\infty} \alpha_j z^j \in \mathcal{A}_+$, $r_\otimes(A) < R_f$. Define

$$f_\otimes(A) = \bigoplus_{j=0}^{\infty} \alpha_j A_\otimes^j.$$

Since $r_\otimes(A) = \lim_{j \to \infty} \|A_\otimes^j\|^{1/j}$ this definition makes sense and $f \mapsto f_\otimes(A)$ defines an analytic functional calculus for $A \in \mathbb{R}_+^{n \times n}$ with properties analogous to the polynomial functional calculus (an analogue of Proposition 3.2 for a set $\mathcal{A}_+$ holds.)

The spectral mapping theorem remains valid for power series as we will prove bellow in Theorem 3.7.

For $f \in \mathcal{A}_+$, $f = \sum_{j=0}^{\infty} \alpha_j z^j$ and $k \in \mathbb{N}$ denote by S_k the k-th partial sum $S_k = \sum_{j=0}^{k} \alpha_j z^k$. Clearly for $A \in \mathbb{R}_+^{n \times n}$ with $r_\otimes(A) < R_f$ we have $f_\otimes(A) = \lim_{k \to \infty} S_{k\otimes}(A)$. Moreover, the sequence of partial sums is non-decreasing, $S_{0\otimes}(A) \leq S_{1\otimes}(A) \leq \cdots$.

It is known that the spectral radius in max algebra $A \mapsto r_\otimes(A)$ is continuous in $\mathbb{R}_+^{n \times n}$ (see e.g. [30], [33], [26], [22]). However, as the following example shows the spectrum in max algebra $A \mapsto \sigma_\otimes(A)$ is in general not continuous.

Example 3.5. Let $A_k = \begin{bmatrix} 1 & 0 \\ k^{-1} & 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $\sigma_\otimes(A_k) = \{2\}$ for all $k \in \mathbb{N}$, $\|A_k - A\| \to 0$ as $k \to \infty$ and $\sigma_\otimes(A) = \{1, 2\}$.

The next result summarizes the continuity properties of the spectrum in max algebra.

Proposition 3.6. *(i) The spectrum in max algebra $A \mapsto \sigma_\otimes(A)$ is upper semi-continuous in $\mathbb{R}_+^{n \times n}$.*
(ii) If $A, A_k \in \mathbb{R}_+^{n \times n}$, $\|A_k - A\| \to 0$ as $k \to \infty$ and $A_1 \leq A_2 \leq \cdots$ then $\sigma_\otimes(A_k) \to \sigma_\otimes(A)$.

Proof. (i) Let $A, A_k \in \mathbb{R}_+^{n \times n}$ such that $\|A_k - A\| \to 0$ as $k \to \infty$. Let $\lambda_k \in \sigma_\otimes(A_k)$, $\lambda_k \to \lambda$. For each $k \in \mathbb{N}$ there exists an eigenvector $x_k \in \mathbb{R}_+^n$ such that $\|x_k\| = 1$ and $A_k \otimes x_k = \lambda_k x_k$. By a compactness argument there exists a convergent subsequence of the sequence (x_k). Clearly its limit x satisfies $\|x\| = 1$ and $A \otimes x = \lambda x$, which proves (i).

(ii) Let $A, A_k \in \mathbb{R}_+^{n \times n}$ such that $\|A_k - A\| \to 0$ as $k \to \infty$ and $A_1 \leq A_2 \leq \cdots$. Let $\lambda \in \sigma_\otimes(A)$. By Theorem 2.6 there exists $j \in \{1, \ldots, n\}$ such that $\lambda = r_{e_j}(A)$. Clearly there exists k_0 such that for all $k \geq k_0$ and $i, l \in \{1, \ldots, n\}$,

$$(A_k)_{i,l} \neq 0 \Leftrightarrow A_{i,l} \neq 0.$$

Then $\lambda_k = r_{e_j}(A_k) \in \sigma_\otimes(A_k)$ and $\lambda_k = r_{e_j}(A_k) \to r_{e_j}(A) = \lambda$ as $k \to \infty$, which completes the proof. $\qquad\square$

Now the spectral mapping theorem for power series in max algebra follows obviously from Theorem 3.3 and Proposition 3.6(ii).

Theorem 3.7. *Let $f \in \mathcal{A}_+$ and $A \in \mathbb{R}_+^{n \times n}$ such that $r_\otimes(A) < R_f$. Then $\sigma_\otimes(f_\otimes(A)) = f_\otimes(\sigma_\otimes(A))$ and so $r_\otimes(f_\otimes(A)) = f_\otimes(r_\otimes(A))$.*

4. Approximate point spectrum in max algebra and Bonsall's cone spectral radius

Recall that by $M_+^{\infty \times \infty}$ we denote the set of all non-negative bounded matrices (i.e., $\|A\| = \sup_{i,j \in \mathbb{N}} A_{ij} < \infty$ for all $A \in M_+^{\infty \times \infty}$). Similarly l_+^∞ denotes a cone (and also a max cone) of non-negative bounded sequences. For $x, y \in l_+^\infty$ we write $\|x - y\| = \sup_{i \in \mathbb{N}} |x_i - y_i|$.

Let $A \in M_+^{\infty \times \infty}$. It is not hard to see that $\|A \otimes x - A \otimes y\| \le \|A\| \cdot \|x - y\|$ for all $x, y \in l_+^\infty$. A map $g_A : l_+^\infty \to l_+^\infty$, $g_A : x \mapsto A \otimes x$ is therefore a well defined continuous max linear map. We also have

$$\|A\| = \sup\{\|A \otimes x\| : \|x\| \le 1, x \in l_+^\infty\} = \sup\left\{\frac{\|A \otimes x\|}{\|x\|} : x \ne 0, x \in l_+^\infty\right\} = \sup_{j \in \mathbb{N}} \|A \otimes e_j\|.$$

Since the map $g_A : l_+^\infty \to l_+^\infty$ is monotone, positively homogeneous and continuous, the spectral radius $r_\otimes(A)$ of $A \in M_+^{\infty \times \infty}$ in max algebra

$$r_\otimes(A) := \lim_{k \to \infty} \|A_\otimes^k\|^{1/k} = \inf_{k \in \mathbb{N}} \|A_\otimes^k\|^{1/k}$$

by definition equals the Bonsall's cone spectral radius $r_{l_+^\infty}(g_A)$ of the map g_A with respect to the cone l_+^∞ (see e.g. [27], [5], [28], [22]).

Let $A \in M_+^{\infty \times \infty}$ and $x \in l_+^\infty$. As in the finite-dimensional case we define the local spectral radius at x by $r_x(A) = \limsup_{k \to \infty} \|A_\otimes^k \otimes x\|^{1/k}$ (in general the limit does not exist).

Clearly $r_x(A) \le r_\otimes(A)$ for all $x \in l_+^\infty$. Moreover, $r_y(A) = r_\otimes(A)$ for $y = (1, 1, \dots)$.

Let $\{e_j : j \in \mathbb{N}\}$ be the standard basis vectors. The following example shows that in general $\sup r_{e_j}(A) \ne r_\otimes(A)$, an so Theorem 2.6 for infinite matrices is not true.

Example 4.1. Consider the direct sum $A = \bigoplus_{n=1}^\infty S_n$, where S_n is the $n-$dimensional left shift, i.e., $A \otimes e_1 = 0$ and $A \otimes e_j = e_{j-1}$ for all $j \ge 2$. Then $r_{e_j}(A) = 0$ for each basis element e_j, but $r_\otimes(A) = 1$.

The point spectrum in max algebra of $A \in M_+^{\infty \times \infty}$ is the set of all $t \ge 0$ for which there exists $x \in \mathbb{R}_+^\infty$, $\|x\| = 1$ with $A \otimes x = tx$.

The approximate point spectrum is the set of all $t \ge 0$ for which there exists a sequence $(x_k) \subset l_+^\infty$ of unit vectors such that

$$\lim_{k \to \infty} \|A \otimes x_k - tx_k\| = 0.$$

We denote the point and the approximate point spectrum in max algebra simply by $\sigma_p(A)$ and $\sigma_{ap}(A)$, respectively. Obviously $\sigma_p(A) \subset \sigma_{ap}(A)$. Also $\sigma_{ap}(A)$ is always closed and nonempty, which follows from Theorem 4.2. It is proved in [27] that under certain compactness assumptions we have $r_\otimes(A) = \max\{t : t \in \sigma_p(A)\}$. Next we show that we always have $r_\otimes(A) = \max\{t : t \in \sigma_{ap}(A)\}$.

Theorem 4.2. Let $A \in M_+^{\infty \times \infty}$. Let $\sup\{r_{e_j}(A) : j \in \mathbb{N}\} \le t \le r_\otimes(A)$. Then $t \in \sigma_{ap}(A)$.

In particular, $r_\otimes(A) \in \sigma_{ap}(A)$. Moreover, $r_\otimes(A) = \max\{t : t \in \sigma_{ap}(A)\}$.

Proof. First we prove that $r_\otimes(A) \ge t$ for all $t \in \sigma_{ap}(A)$. If $t \in \sigma_{ap}(A)$, then there exists a sequence (x_k) of unit vectors such that $\lim_{k \to \infty} \|A \otimes x_k - tx_k\| = 0$. By induction it follows that $\lim_{k \to \infty} \|A_\otimes^j \otimes x_k - t^j x_k\| = 0$ for all $j \in \mathbb{N}$. Indeed,

$$\|A_\otimes^j \otimes x_k - t^j x_k\| \le \|A_\otimes^j \otimes x_k - A_\otimes^{j-1} \otimes tx_k\| + \|A_\otimes^{j-1} \otimes tx_k - t^j x_k\|$$

$$\le \|A_\otimes^{j-1}\| \cdot \|A \otimes x_k - tx_k\| + t\|A_\otimes^{j-1} \otimes x_k - t^{j-1}x_k\| \to 0$$

as $k \to \infty$ by the induction assumption.

It follows $\|A_\otimes^j\| \geq \lim_{k\to\infty} \|A_\otimes^j \otimes x_k\| = t^j$, and so $r_\otimes(A) \geq t$.

Let $\sup\{r_{e_j}(A) : j \in \mathbb{N}\} \leq t \leq r_\otimes(A)$ and $\varepsilon > 0$. We will construct $u \in l_+^\infty$, $\|u\| = 1$ such that $\|A \otimes u - tu\| \leq \varepsilon$.

If $t = 0$, then $r_{e_j}(A) = \lim_{k\to\infty} \|A_\otimes^k \otimes e_j\|^{1/k} = 0$ for all $j \in \mathbb{N}$. Let $\varepsilon > 0$. Then there exists $n \geq 0$ such that $A_\otimes^n \otimes e_1 \neq 0$ and $\|A_\otimes^{n+1} \otimes e_1\| < \varepsilon \|A_\otimes^n \otimes e_1\|$. So it is sufficient to take $u = \frac{A_\otimes^n \otimes e_1}{\|A_\otimes^n \otimes e_1\|}$. Then $\|u\| = 1$ and $\|A \otimes u\| < \varepsilon$. Hence $0 \in \sigma_{ap}(A)$.

So without loss of generality we may assume that $t = 1$. Clearly $\|A_\otimes^j\| \geq r_\otimes(A)^j \geq 1$ for all j. Let $K = \|A\|$. Let $\varepsilon > 0$. Choose $m_0 \in \mathbb{N}$ such that $(1 + \varepsilon)^{m_0} > \frac{2}{\varepsilon}$ and $n_0 \in \mathbb{N}$ such that $(1 + \varepsilon)^{n_0} > 2K^{m_0}(1 + \varepsilon)^{m_0}$. Let $i_0 \in \mathbb{N}$ and $n \geq n_0$ satisfy $\|A_\otimes^n \otimes e_{i_0}\| > 1/2$. For $k \geq 0$ set $\alpha_k = \|A_\otimes^k \otimes e_{i_0}\|$.

Claim. There exist numbers $\beta_k \geq \alpha_k$ and $m \geq m_0$ such that $\beta_m = \alpha_m$, $\beta_0 > \varepsilon^{-1}$ and
$$|\beta_k^{-1} - \beta_{k+1}^{-1}| \leq \frac{\varepsilon}{\max\{\beta_k, \beta_{k+1}\}}$$
for all $k \geq 0$.

Proof of the Claim. Set $\gamma_k = \alpha_n(1+\varepsilon)^{|k-n|}$. Then $\gamma_n = \alpha_n$. Let m satisfy $\frac{\alpha_m}{\gamma_m} = \max_k\{\frac{\alpha_k}{\gamma_k}\}$ (such an m exists since $\lim_{k\to\infty} \frac{\alpha_k}{\gamma_k} = \frac{1}{\alpha_n}\lim_{k\to\infty}\frac{\alpha_k}{(1+\varepsilon)^{k-n}} = 0$).

In particular we have $\frac{\alpha_m}{\gamma_m} \geq \frac{\alpha_n}{\gamma_n} = 1$, so $\alpha_m \geq \alpha_n > 1/2$.

Set $\beta_k = \alpha_m(1 + \varepsilon)^{|k-m|}$. Clearly $\beta_m = \alpha_m$. For each $k \geq 0$ we have $\{\beta_k, \beta_{k+1}\} = \{\alpha_m(1+\varepsilon)^r, \alpha_m(1+\varepsilon)^{r+1}\}$ for some r ($r = \min\{|k-m|, |k+1-m|\}$) and so
$$|\beta_k^{-1} - \beta_{k+1}^{-1}| = \frac{1}{\alpha_m(1+\varepsilon)^r}\left|1 - \frac{1}{1+\varepsilon}\right| = \frac{\varepsilon}{\alpha_m(1+\varepsilon)^{r+1}} = \frac{\varepsilon}{\max\{\beta_k, \beta_{k+1}\}}.$$
We show that $m \geq m_0$. Suppose the contrary. We have $\alpha_m \leq K^m$ and $\gamma_m = \alpha_n(1+\varepsilon)^{n-m}$. So $\frac{\alpha_m}{\gamma_m} \leq \frac{K^m}{\alpha_n(1+\varepsilon)^{n-m}} < \frac{2K^m(1+\varepsilon)^{m_0}}{(1+\varepsilon)^{n_0}} < 1$, a contradiction. So $m \geq m_0$ and $\beta_0 = \alpha_m(1+\varepsilon)^m > \frac{(1+\varepsilon)^m}{2} > \frac{1}{\varepsilon}$.

Finally, for each k we have $\frac{\alpha_k}{\gamma_k} \leq \frac{\alpha_m}{\gamma_m}$. So
$$\alpha_k \leq \frac{\alpha_m \gamma_k}{\gamma_m} = \alpha_m(1+\varepsilon)^{|k-n|-|m-n|} \leq \alpha_m(1+\varepsilon)^{|k-m|} = \beta_k.$$

Continuation of the proof of Theorem 4.2. Let $u = \bigoplus_{k=0}^\infty \frac{A_\otimes^k \otimes e_{i_0}}{\beta_k}$. Since $\beta_k \leq \alpha_k$ for all k and $\beta_m = \alpha_m$, we have $\|u\| = 1$.

We show $\|A \otimes u - u\| \leq \varepsilon$. Fix $j \in \mathbb{N}$. We have
$$u_j = \max_{k\geq 0, i_1,\dots,i_{k-1}\in\mathbb{N}}\left\{\frac{1}{\beta_k}A_{j,i_{k-1}}A_{i_{k-1},i_{k-2}}\cdots A_{i_1,i_0}\right\}$$
and
$$(A \otimes u)_j = \max_{k\geq 1, i_1,\dots,i_{k-1}\in\mathbb{N}}\left\{\frac{1}{\beta_{k-1}}A_{j,i_{k-1}}A_{i_{k-1},i_{k-2}}\cdots A_{i_1,i_0}\right\}.$$
For $k \geq 1$, $i_1, \dots, i_{k-1} \in \mathbb{N}$ we have
$$|\beta_k^{-1} - \beta_{k-1}^{-1}|A_{j,i_{k-1}}A_{i_{k-1},i_{k-2}}\cdots A_{i_1,i_0} \leq \frac{\varepsilon\alpha_k}{\max\{\beta_k, \beta_{k+1}\}} \leq \varepsilon.$$

For $k = 0$ we have $\left\|\frac{e_{i_0}}{\beta_0}\right\| = \beta_0^{-1} < \varepsilon$. So

$$\|A \otimes u - u\| = \max_j (A \otimes u - u)_j \leq \varepsilon.$$

Hence $1 \in \sigma_{ap}(A)$. $\square$

Remark 4.3. In the proof of Theorem 4.2 we can clearly in the definition of u take a finite max sum, so we can take u with a finite support. Therefore we can conclude for example that $u \in c_0$.

Note that for a left shift A from Example 4.1 we have $\sigma_p(A) = \sigma_{ap}(A) = [0, 1]$. On the other hand, for its restriction $A|_{c_0}$ to c_0 we have $\sigma_p(A|_{c_0}) = [0, 1)$ and $\sigma_{ap}(A|_{c_0}) = [0, 1]$.

We conclude this section with some remarks on Corollary 2.5 and Theorem 2.6 in the infinite setting. A max cone $C \subset l_+^\infty$ is called $\sigma-$complete, if $\oplus_{n \in \mathbb{N}} x_n \in C$ for all $x_n \in C$. For a set $S \subset l_+^\infty$ we denote by $\bigvee_\sigma S$ the $\sigma-$complete max cone generated by S, i.e., $\bigvee_\sigma S$ is the set of all $x \in l_+^\infty$ for which there exist $s_1, s_2, \ldots \in S$ and $\alpha_1, \alpha_2, \ldots \geq 0$ such that $x = \oplus_{n \in \mathbb{N}} \alpha_n s_n$.

The proof of the following result is straightforward and it is omitted.

Proposition 4.4. Let $A \in M_+^{\infty \times \infty}$, $c \geq 0$. Then $\bigvee_\sigma \{e_j : r_{e_j}(A) \leq c\}$ is a $\sigma-$complete max cone invariant for A.

Corollary 4.5. If $A \in M_+^{\infty \times \infty}$, then

$$\{r_{e_j}(A) : j \in \mathbb{N}\} \subset \sigma_{ap}(A).$$

Proof. Let $t = r_{e_j}(A)$ for some $j \in \mathbb{N}$. Let $M = \bigvee_\sigma \{e_s : r_{e_s}(A) \leq t\}$. By Proposition 4.4 the set M is a $\sigma-$complete max cone invariant for A. For the restriction $A|_M$ of A to M we have by Theorem 4.2 that $\sup\{r_{e_i}(A) : e_i \in M\} \leq t \leq r_\otimes(A|_M)$ implies $t \in \sigma_{ap}(A)$, which completes the proof. $\square$

5. Spectral mapping theorem in the infinite case

Let $q \in \mathcal{P}_+$, $q = \sum_{j=0}^{\deg q} \alpha_j z^j$ be a polynomial with nonnegative coefficients. We write $q_\otimes(A) = \bigoplus_{j=0}^{\deg q} \alpha_j A_\otimes^j$. For $t \geq 0$ write $q_\otimes(t) = \max\{\alpha_j t^j : 0 \leq j \leq \deg q\}$.

Both the point and approximate point spectrum satisfy the spectral mapping property (for polynomials without the absolute term; see Theorem 5.4 bellow). One inclusion is simple as the next lemma shows.

Lemma 5.1. Let $A \in M_+^{\infty \times \infty}$, let $q \in \mathcal{P}_+$, $q = \sum_{j=0}^{\deg q} \alpha_j z^j$ be a polynomial with nonnegative coefficients. Then $q_\otimes(\sigma_p(A)) \subset \sigma_p(q_\otimes(A))$ and $q_\otimes(\sigma_{ap}(A)) \subset \sigma_{ap}(q_\otimes(A))$.

Proof. Let $t \in \sigma_p(A)$, let x be a nonzero vector satisfying $A \otimes x = tx$. Then $A_\otimes^j \otimes x = t^j x$ for all j, and so $q_\otimes(A) \otimes x = q_\otimes(t)x$. Hence $q_\otimes(t) \in \sigma_p(q_\otimes(A))$.

Let $t \in \sigma_{ap}(A)$. Then there exists a sequence $(x_k) \subset l_+^\infty$ of unit vectors with $\lim_{k\to\infty} \|A \otimes x_k - tx_k\| = 0$. Then for each $j = 0, 1, \ldots, \deg q$ we have

$$\|\alpha_j A_\otimes^j \otimes x_k - \alpha_j t^j x_k\| \to 0,$$

and so

$$\left\| q_\otimes(A) \otimes x_k - q_\otimes(t)x_k \right\| = \max\left\{ \|\alpha_j A_\otimes^j \otimes x_k - \alpha_j t^j x_k\| : j = 0, 1, \ldots, \deg q \right\} \to 0.$$

So $q_\otimes(\sigma_{ap}(A)) \subset \sigma_{ap}(q_\otimes(A))$. $\qquad\square$

For the opposite inequality (for polynomials without the absolute term) we need the following lemma.

Lemma 5.2. *Let $A \in M_+^{\infty\times\infty}$, $q \in \mathcal{P}_+$, $q = \sum_{j=1}^{\deg q} \alpha_j z^j$. Suppose that $q_\otimes(1) = 1$, i.e., $\alpha_j \le 1$ for all j and there exists $m, 1 \le m \le \deg q$ with $\alpha_m = 1$. Let $\delta > 0$ and $x \in l_+^\infty$ satisfy $\|x\| = 1$ and $\|q_\otimes(A) \otimes x - x\| \le \delta$. Then*

$$\|A_\otimes^m \otimes x - x\| \le 3\delta \deg q \cdot \max\{1, \|q_\otimes(A)\|^{2\deg q}\}.$$

Proof. Write $B = q_\otimes(A)$, $n = \deg q$ and $K = \max\{1, \|B\|^n\}$.

For every r we have $(A_\otimes^m \otimes x - x)_r \le (q_\otimes(A) \otimes x - x)_r \le \delta$. So it is sufficient to show $(x - A_\otimes^m \otimes x)_r \le 3n\delta K^2$ for all r.

Fix $r = r_0$. Find $r_1, \ldots, r_m$ such that

$$x_{r_s} - B_{r_s,r_{s+1}} \cdot x_{r_{s+1}} \le \delta \qquad (s = 0, 1, \ldots, m-1).$$

For $s = 0, \ldots, m-1$ there exists j_s, $1 \le j_s \le n$ such that

$$B_{r_s,r_{s+1}} = \alpha_{j_s}(A_\otimes^{j_s})_{r_s,r_{s+1}}.$$

So there exist k, k', $0 \le k < k' \le m$ such that

$$\sum_{s=0}^{k-1} j_s = \sum_{s=0}^{k'-1} j_s \qquad (\mathrm{mod}\ m),$$

i.e.,

$$\sum_{s=k}^{k'-1} j_s = ma$$

for some $a \in \mathbb{N}$, $a = m^{-1}\sum_{s=k}^{k'-1} j_s \le nm^{-1}(k'-k) \le n$.

We have

$$x_{r_0} - x_{r_k} \cdot (B_\otimes^k)_{r_0,r_k} \le x_{r_0} - x_{r_k} \cdot B_{r_0,r_1} \cdot B_{r_1,r_2} \cdots B_{r_{k-1},r_k}$$

$$\le (x_{r_0} - x_{r_1} \cdot B_{r_0,r_1}) + B_{r_0,r_1}(x_{r_1} - x_{r_2}B_{r_1,r_2}) + \cdots$$

$$\cdots + B_{r_0,r_1}\cdots B_{r_{k-2},r_{k-1}}(x_{r_{k-1}} - x_{r_k}B_{r_{k-1},r_k}) \le kK\delta.$$

Similarly,

$$x_{r_k} - (B_\otimes^{k'-k})_{r_k,r_{k'}}x_{r_{k'}} \le x_{r_k} - x_{r_{k'}} \cdot B_{r_k,r_{k+1}} \cdot B_{r_{k+1},r_{k+2}}\cdots B_{r_{k'-1},r_{k'}} \le (k'-k)K\delta.$$

So

$$x_{r_k} - (A_\otimes^{ma})_{r_k,r_{k'}} x_{r_{k'}} \leq x_{r_k} - \alpha_{j_k} \cdots \alpha_{j_{k'-1}} (A_\otimes^{ma})_{r_k,r_{k'}} x_{r_{k'}}$$

$$\leq x_{r_k} - \alpha_{j_k} \cdots \alpha_{j_{k'-1}} \cdot (A_\otimes^{j_k})_{r_k,r_{k+1}} \cdots (A_\otimes^{j_{k'-1}})_{r_{k'-1},r_{k'}} x_{r_{k'}}$$

$$= x_{r_k} - B_{r_k,r_{k+1}} \cdots B_{r_{k'-1},r_{k'}} x_{r_{k'}} \leq (k'-k)K\delta.$$

Thus

$$x_{r_0} - (B_\otimes^k \otimes A_\otimes^{ma} \otimes x)_{r_0} \leq \Big(x_{r_0} - (B_\otimes^k)_{r_0,r_k} x_{r_k} \Big) + (B_\otimes^k)_{r_0,r_k} \Big(x_{r_k} - (A_\otimes^{ma})_{r_k,r_{k'}} \Big) x_{r_{k'}}$$

$$\leq kK\delta + K^2(k'-k)\delta \leq nK^2\delta.$$

Hence

$$x_{r_0} - (A_\otimes^{ma} \otimes x)_{r_0} \leq \Big(x_{r_0} - (A^{\otimes ma} \otimes B_\otimes^k \otimes x)_{r_0} \Big) + \Big(A_\otimes^{ma} \otimes B_\otimes^k \otimes x - A_\otimes^{ma} \otimes x) \Big)_{r_0}$$

$$\leq nK^2\delta + \|A_\otimes^{ma}\| \Big(\|B_\otimes^k \otimes x - B_\otimes^{k-1} \otimes x\| + \|B_\otimes^{k-1} \otimes x - B_\otimes^{k-2} \otimes x\| + \cdots + \|B \otimes x - x\| \Big)$$

$$\leq nK^2\delta + K^2 k\delta \leq 2nK^2\delta.$$

Thus there exist $s_0 = r_0, s_1, \ldots, s_a$ such that

$$x_{s_0} - (A_\otimes^m)_{s_0,s_1} (A_\otimes^m)_{s_1,s_2} \cdots (A_\otimes^m)_{s_{a-1},s_a} x_{s_a} \leq 2nK^2\delta$$

and

$$x_{r_0} - (A_\otimes^m \otimes x)_{r_0} \leq x_{s_0} - (A_\otimes^m)_{s_0,s_1} x_{s_1}$$

$$\leq \Big(x_{s_0} - (A_\otimes^m)_{s_0,s_1} (A_\otimes^m)_{s_1,s_2} \cdots (A_\otimes^m)_{s_{a-1},s_a} x_{s_a} \Big)$$

$$+ (A_\otimes^m)_{s_0,s_1} \cdots (A_\otimes^m)_{s_{a-2},s_{a-1}} \Big((A_\otimes^m)_{s_{a-1},s_a} x_{s_a} - x_{s_{a-1}} \Big) + \cdots$$

$$\cdots + (A^{\otimes m})_{s_0,s_1} \Big((A_\otimes^m)_{s_1,s_2} x_{s_2} - x_{s_1} \Big) \leq 2nK^2\delta + aK\delta \leq 3nK^2\delta.$$

Since r_0 was arbitrary, we have $\|A_\otimes^m \otimes x\| \leq 3nK^2\delta$. $\qquad\square$

Corollary 5.3. *Let $A \in M_+^{\infty \times \infty}$, $q \in \mathcal{P}_+$, $q = \sum_{j=1}^{\deg q} \alpha_j z^j$, $q_\otimes(1) = 1$. Then:*

(i) *if $1 \in \sigma_{ap}(q_\otimes(A))$ then $1 \in \sigma_{ap}(A)$;*

(ii) *if $1 \in \sigma_p(q_\otimes(A))$ then $1 \in \sigma_p(A)$.*

Proof. (i) Let $1 \leq m \leq \deg q$ such that $\alpha_m = 1$ and let $1 \in \sigma_{ap}(q_\otimes(A))$. Let $\varepsilon > 0$. By Lemma 5.2 there exists a unit vector $x \in l_+^\infty$ such that $\|A_\otimes^m \otimes x - x\| < \varepsilon$. Let $y = \bigoplus_{j=0}^{m-1} A_\otimes^j \otimes x$. Then $\|y\| \geq \|x\| = 1$ and

$$\|A \otimes y - y\| = \Big\| \bigoplus_{j=1}^m A_\otimes^j \otimes x - \bigoplus_{j=0}^{m-1} A_\otimes^j \otimes x \Big\| \leq \|A_\otimes^m \otimes x - x\| < \varepsilon.$$

So $1 \in \sigma_{ap}(A)$, which proves (i).

The proof of (ii) is similar. $\qquad\square$

Now we can prove the spectral mapping theorem for the point spectrum and for the approximate point spectrum (for polynomials without the absolute term).

Theorem 5.4. *Let $A \in M_+^{\infty \times \infty}$, $q \in \mathcal{P}_+$, $q = \sum_{j=1}^{\deg q} \alpha_j z^j$, $q \neq 0$. Then:*

(i) $\sigma_{ap}(q_\otimes(A)) = q_\otimes(\sigma_{ap}(A))$;

(ii) $\sigma_p(q_\otimes(A)) = q_\otimes(\sigma_p(A))$.

Proof. (i) The inclusion $\sigma_{ap}(q_\otimes(A)) \supset q_\otimes(\sigma_{ap}(A)$ was proved in Lemma 5.1.

Conversely, let $s \in \sigma_{ap}(q_\otimes(A))$ and let $t \geq 0$ satisfy $s = q_\otimes(t)$ (note that the function $t \mapsto q_\otimes(t)$ is injective, continuous, $q_\otimes(0) = 0$ and $\lim_{t\to\infty} q_\otimes(t) = \infty$). We show that $t \in \sigma_{ap}(A)$.

If $t = 0 = s$, then we have $0 \in \sigma_{ap}(q_\otimes(A))$. Since $q \neq 0$, there exists $m \in \{1, \ldots, \deg q\}$ such that $\alpha_m \neq 0$. Since $\alpha_m A_\otimes^m \leq q_\otimes(A)$, we have $0 \in \sigma_{ap}(A_\otimes^m)$. For each $\varepsilon > 0$ there exists a unit vector x with $\|A_\otimes^m \otimes x\| < \varepsilon^m$. So there exists j, $0 \leq j \leq m - 1$ with $\|A_\otimes^{j+1} \otimes x\| < \varepsilon\|A_\otimes^j \otimes x\|$ and $y := \frac{A_\otimes^j \otimes x}{\|A_\otimes^j \otimes x\|}$ that satisfies $\|y\| = 1$ and $\|A \otimes y\| < \varepsilon$.

So we may assume that $t > 0$ and $s > 0$. Set $A' = A/t$ and $q' = \sum_{j=1}^{\deg q} \frac{\alpha_j t^j}{s} z^j$. Then

$$q'_\otimes(A') = \bigoplus_{j=1}^{\deg q} \frac{\alpha_j t^j}{s} \frac{A_\otimes^j}{t^j} = \frac{q_\otimes(A)}{s}, \ 1 \in \sigma_{ap}(q'_\otimes(A'))$$ and $q'_\otimes(1) = 1$. By Corollary 5.3 it follows that $1 \in \sigma_{ap}(A')$ and so $t \in \sigma_{ap}(A)$, which proves (i).

The proof of (ii) is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Given $A \in M_+^{\infty \times \infty}$ and $\alpha \geq 0$ it is easy to see that $\sigma_{ap}(\alpha I \oplus A) \subset [\alpha, \infty)$ and $\sigma_p(\alpha I \oplus A) \subset [\alpha, \infty)$. Moreover, we have the following result.

Proposition 5.5. *Let $A \in M_+^{\infty \times \infty}$, $\alpha \geq 0$, $q(z) = \alpha + z$. Then*

(9) $$\sigma_{ap}(q_\otimes(A)) \cap (\alpha, \infty) = q_\otimes(\sigma_{ap}(A)) \cap (\alpha, \infty)$$

and

$$\sigma_p(q_\otimes(A)) \cap (\alpha, \infty) = q_\otimes(\sigma_p(A)) \cap (\alpha, \infty)$$

Proof. $\supset$: The inclusions $q_\otimes(\sigma_{ap}(A)) \subset \sigma_{ap}(q_\otimes(A))$ and $q_\otimes(\sigma_p(A)) \subset \sigma_p(q_\otimes(A))$ were proved in Lemma 5.1.

$\subset$: Write $B = q_\otimes(A) = \alpha I \oplus A$ and let $s \in \sigma_{ap}(B)$, $s > \alpha$. Let $0 < \varepsilon < s - \alpha$. Take $x \in l_+^\infty$, $\|x\| = 1$ such that $\|B \otimes x - sx\| < \varepsilon$.

For each r, $(A \otimes x - sx)_r \leq (B \otimes x - sx)_r < \varepsilon$. We show that $(sx - A \otimes x)_r < \varepsilon$. We distinguish two cases:

Suppose first that $x_r > \frac{\varepsilon}{s-\alpha}$. Then there exists r' with

$$sx_r - B_{r,r'} \cdot x_{r'} < \varepsilon,$$

so either $sx_r - A_{r,r'}x_{r'} < \varepsilon$ or $r' = r$ and $sx_r - \alpha x_r < \varepsilon$. Since the second case is not possible, we have

$$(sx - A \otimes x)_r \leq sx_r - A_{r,r'}x_{r'} < \varepsilon.$$

If $x_r \leq \frac{\varepsilon}{s-\alpha}$ then

$$(sx - A \otimes x)_r \leq sx_r \leq \frac{s\varepsilon}{s - \alpha}.$$

So
$$\|A \otimes x - sx\| = \sup_r |(A \otimes x - sx)_r| \le \max\left\{\varepsilon, \frac{s\varepsilon}{s-\alpha}\right\} = \frac{s\varepsilon}{s-\alpha}.$$
Letting $\varepsilon \to 0$ we get $s \in \sigma_{ap}(A)$, which proves (9).

Similarly, if $s > \alpha$, $s \in \sigma_p(B)$ and $B \otimes x = sx$ for some nonzero $x \in l_+^\infty$, then it is easy to see that $A \otimes x = sx$ and $s \in \sigma_p(A)$. $\qquad\square$

The following result follows from Lemma 5.1, Theorem 5.4 and Proposition 5.5.

Corollary 5.6. *Let $A \in M_+^{\infty \times \infty}$, $q \in \mathcal{P}_+$, $q = \sum_{j=0}^{\deg q} \alpha_j z^j$. Then*

(10)
$$q_\otimes(\sigma_p(A)) \subset \sigma_p(q_\otimes(A)) \subset q_\otimes(\sigma_p(A)) \cup \{\alpha_0\}$$

and

$$q_\otimes(\sigma_{ap}(A)) \subset \sigma_{ap}(q_\otimes(A)) \subset q_\otimes(\sigma_{ap}(A)) \cup \{\alpha_0\}.$$

Remark 5.7. In general for the point spectrum nothing better than (10) can be said, for example consider the right shift $A_{i+1,i} = 1 \quad (i \ge 1)$, $A_{i,j} = 0$ otherwise, with $\sigma_p(A) = \emptyset$ and the polynomial $p(z) = z + 1$. Then $x = (1, 1, \dots)$ satisfies $(A \oplus I)x = x$. Moreover, $\sigma_p(I \oplus A) = \{1\}$ and the corresponding max eigencone consists of $x = (x_i)_{i \in \mathbb{N}} \in l_+^\infty$ such that $x_i \le x_{i+1}$ for all $i \in \mathbb{N}$.

We do not know if this may happen for the approximate point spectrum.

We conclude the paper with some remarks on the generalizations of results from Sections 4 and 5. The matrices in these sections may be of any size, not necessarily countable. More precisely, these results are true for max type kernel operators
$$(Af)(x) = \sup_{y \in \Omega} a(x, y) f(y).$$
Here Ω is a non-empty set and the kernel $a : \Omega \times \Omega \to [0, \infty)$ is bounded (i.e., $\|A\|_\infty = \sup_{x,y \in \Omega} a(x, y) < \infty$). Thus A acts on the max cone $l_+^\infty(\Omega)$ of bounded functions $f : \Omega \to [0, \infty)$ (i.e., $\|f\|_\infty = \sup_{x \in \Omega} f(x) < \infty$). The proofs are similar as above and we omit the details (for example the standard vectors e_n are replaced by the functions $e_y = \chi_{\{y\}}$).

It is well known that max algebra is an idempotent semifield isomorphic to max + algebra $\mathbb{R}_{max}$ (the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$) and to min + algebra or tropical algebra $\mathbb{R}_{min}$ ($\mathbb{R} \cup \{\infty\}$ equipped with the operations $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$) via maps $x \mapsto \log x$ and $x \mapsto -\log x$, respectively.

The results above can thus be reformulated and applied also to these settings, i.e., to max-plus type operators
$$(Bg)(x) = \sup_{y \in \Omega}(b(x, y) + g(y))$$
on an idempotent semimodule of bounded (from above) functions $g : \Omega \to \mathbb{R}_{max}$ (see e.g. [12], [27], [23] and the references cited there) and its tropical versions known also

as Bellman operators (which arise in numerous applications to optimal control problems, discrete mathematics, turnpike theory, mathematical economics, games and controlled Markov processes, the theory of generalized solutions of the Hamilton-Jacobi-Bellman differential equations, the theory of continuously observed and controlled quantum systems, ... - see e.g. [21], [25], [24] and the references cited there).

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