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# Perturbation of *m*-isometries by nilpotent operators


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# PERTURBATION OF $m$-ISOMETRIES BY NILPOTENT OPERATORS

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Abstract. We prove that if $T$ is an $m$-isometry on a Hilbert space and $Q$ an $n$-nilpotent operator commuting with $T$, then $T+Q$ is a $(2n+m-2)$-isometry. Moreover, we show that a similar result for $(m,q)$-isometries on Banach spaces is not true.

## 1. Introduction

The notion of $m$-isometric operators on Hilbert spaces was introduced by Agler [1]. See also [14], [6], [4] and [5]. Recently Sid Ahmed [15] has defined $m$-isometries on Banach spaces, Bayart [7] introduced $(m,q)$-isometries on Banach spaces, and $(m,q)$-isometries on metric spaces were considered in [8]. Moreover, Hoffman, Mackey and Searcóid [13] have studied the role of the second parameter $q$. Recall the main definitions.

A map $T: E \longrightarrow E$ ($m \geq 1$ integer and $q>0$ real), defined on a metric space $E$ with distance $d$, is called an $(m,q)$-*isometry* if, for all $x,y \in E$,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} d(T^k x, T^k y)^q = 0 \; . \tag{1.1}$$

We say that $T$ is a *strict* $(m,q)$-*isometry* if either $m=1$ or $T$ is an $(m,q)$-isometry with $m>1$, but is not an $(m-1,q)$-isometry. Note that $(1,q)$-isometries are isometries.

The above notion of an $(m,q)$-isometry can be adapted to Banach spaces in the following way: a bounded linear operator $T: X \longrightarrow X$, where $X$ is a Banach space with norm $\|\cdot\|$, is an $(m,q)$-isometry if and only if, for all $x \in X$,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} \|T^k x\|^q = 0 \; . \tag{1.2}$$

In the setting of Hilbert spaces, the case $q=2$ can be expressed in a special way. Agler [1] gives the following definition: a linear bounded operator $T: H \longrightarrow H$ acting on a Hilbert

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space $H$ is an $(m,2)$-isometry if

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k = 0\ . \tag{1.3}$$

$(m,2)$-isometries on Hilbert spaces will be called for short $m$-isometries.

The paper is organized as follows. In the next section we collect some results about applications of arithmetic progressions to $m$-isometric operators.

In section 3 we prove that, in the setting of Hilbert spaces, if $T$ is an $m$-isometry, $Q$ is an $n$-nilpotent operator and they commute, then $T+Q$ is a $(2n+m-2)$-isometry. This is a partial generalization of the following result obtained in [9, Theorem 2.2]: if $T$ is an isometry and $Q$ is a nilpotent operator of order $n$ commuting with $T$, then $T+Q$ is a strict $(2n-1)$-isometry.

In the last section we give some examples of operators on Banach spaces which are of the form identity plus nilpotent, but they are not $(m,q)$-isometries, for any positive integer $m$ and any positive real number $q$.

**Notation.** Throughout this paper $H$ denotes a Hilbert space and $B(H)$ the algebra of all linear bounded operators on $H$. Given $T\in B(H)$, $T^*$ denotes its adjoint. Moreover, $m\geq 1$ is an integer and $q>0$ a real number.

## 2. Preliminaries: arithmetic progressions and $(m,q)$-isometries

In this section we give some basic properties of $m$-isometries. We need some preliminaries about arithmetic progressions and their applications to $m$-isometries. In [10], some results about this topic are recollected.

Let $G$ be a group, not necessarily commutative, and denote its operation by $+$. Given a sequence $a=(a_n)_{n\geq 0}$ in $G$, the difference sequence $Da=(Da)_{n\geq 0}$ is defined by $(Da)_n := a_{n+1}-a_n$. The powers of $D$ are defined recursively by $D^0a := a$, $D^{k+1}a = D(D^ka)$. It is easy to show that

$$(D^ka)_n = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}a_{i+n}\ ,$$

for all $k\geq 0$ and $n\geq 0$ integers.

A sequence $a$ in a group $G$ is called an *arithmetic progression* of order $h=0,1,2...$, if $D^{h+1}a=0$. Equivalently,

$$\sum_{i=0}^{h+1}(-1)^{h+1-i}\binom{h+1}{i}a_{i+j} = 0 \tag{2.4}$$

for $j = 0, 1, 2....$ It is well known that the sequence $a$ in $G$ is an arithmetic progression of order $h$ if and only if there exists a polynomial $p(n)$ in $n$, with coefficients in $G$ and of degree less or equal to $h$, such that $p(n) = a_n$, for every $n = 0, 1, 2...$; that is, there are $\gamma_h, \gamma_{h-1}, ..., \gamma_1, \gamma_0 \in G$, which depend only on $a$, such that, for every $n = 0, 1, 2...$,

$$a_n = p(n) = \sum_{i=0}^{h} \gamma_i n^i \ . \tag{2.5}$$

We say that the sequence $a$ is an *arithmetic progression of strict order* $h = 0, 1, 2...$, if $h = 0$ or if it is of order $h > 0$, but is not of order $h - 1$; that is, the polynomial $p$ of (2.5) has degree $h$.

Moreover, a sequence $a$ in a group $G$ is an arithmetic progression of order $h$ if and only if, for all $n \geq 0$,

$$a_n = \sum_{k=0}^{h} (-1)^{h-k} \frac{n(n-1)\cdots\overbrace{(n-k)}\cdots(n-h)}{k!(h-k)!} a_k \ ; \tag{2.6}$$

that is,

$$a_n = \sum_{k=0}^{h} (-1)^{h-k} \binom{n}{k} \binom{n-k-1}{h-k} a_k \ . \tag{2.7}$$

Now we give a basic result about $m$-isometries.

**Theorem 2.1.** *Let $H$ be a Hilbert space. An operator $T \in B(H)$ is a strict $m$-isometry if and only if there are $A_{m-1} \neq 0, A_{m-2}, ..., A_1, A_0$ in $B(H)$, which depend only on $T$, such that, for every $n = 0, 1, 2...$,*

$$T^{*n}T^n = \sum_{i=0}^{m-1} A_i n^i \ ; \tag{2.8}$$

*that is, the sequence $(T^{*n}T^n)_{n\geq 0}$ is an arithmetic progression of strict order $m-1$ in $B(H)$.*

*Proof.* If $T \in B(H)$ is a strict $m$-isometry, then it satisfies equation (1.3). Hence, for each integer $i \geq 0$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*i}T^{*k}T^kT^i = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*k+i}T^{k+i} = 0 \ , \tag{2.9}$$

but

$$\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} T^{*k}T^k \neq 0 \ . \tag{2.10}$$

By (2.4), the operator sequence $(T^{*n}T^n)_{n\geq 0}$ is an arithmetic progression of strict order $m - 1$. Therefore, from (2.5) we obtain that there is a polynomial $p(n)$, of degree $m - 1$ in

$n$, with coefficients in $B(H)$ satisfying $p(n) = T^{*n}T^n$; that is, there are operators $A_{m-1} \neq 0, A_{m-2}, ..., A_1, A_0$ in $B(H)$, such that, for every $n = 0, 1, 2...$,

$$T^{*n}T^n = A_{m-1}n^{m-1} + A_{m-2}n^{m-2} + \cdots + A_1 n + A_0 \ .$$

Conversely, if $(T^{*n}T^n)_{n\geq 0}$ is an arithmetic progression of strict order $m-1$, then the equations (2.9) and (2.10) hold. Taking $i = 0$ we obtain (1.3), so $T$ is a strict $m$-isometry. $\square$

Now we recall an elementary property of $(m, q)$-isometries on metric spaces which will be used in next sections:

**Proposition 2.1.** [8, Proposition 3.11] *Let $E$ be a metric space and $T : E \longrightarrow E$ be an $(m, q)$-isometry. If $T$ is an invertible strict $(m, q)$-isometry, then $m$ is odd.*

## 3. $m$-isometry plus $n$-nilpotent

Recall that an operator $Q \in B(H)$ is *nilpotent of order $n$* ($n \geq 1$ integer), or *$n$-nilpotent*, if $Q^n = 0$ and $Q^{n-1} \neq 0$.

In any finite dimensional Hilbert space $H$, strict $m$-isometries can be characterized in a very simple way: a linear operator $T \in B(H)$ is a strict $m$-isometry if and only if $m$ is odd and $T = A + Q$, where $A$ and $Q$ are commuting operators on $H$, $A$ is unitary and $Q$ a nilpotent operator of order $\frac{m+1}{2}$, ([2, page 134] & [9, Theorem 2.7]).

It was proved in [9, Theorem 2.2] that if $A \in B(H)$ is an isometry and $Q \in B(H)$ is an $n$-nilpotent operator such that $TQ = QT$, then $T + Q$ is a strict $(2n-1)$-isometry. Now we obtain a partial generalization of this result: if $T \in B(H)$ is an $m$-isometry and $Q \in B(H)$ is an $n$-nilpotent operator commuting with $T$, then $T+Q$ is a $(2n+m-2)$-isometry. However, $T+Q$ is not necessarily a strict $(2n+m-2)$-isometry. For example, if $T$ is an isometry and $Q$ any $n$-nilpotent operator ($n > 1$) such that $TQ = QT$, then $T = T + Q + (-Q)$ is not a strict $(4n-3)$-isometry.

**Theorem 3.1.** *Let $H$ be a Hilbert space. Let $T \in B(H)$ be an $m$-isometry and $Q \in B(H)$ an $n$-nilpotent operator ($n \geq 1$ integer) such that $TQ = QT$. Then $T + Q$ is $(2n + m - 2)$-isometry.*

*Proof.* Fix an integer $k \geq 0$ and denote by $h := \min\{k, n-1\}$. Then we have

$$(T+Q)^{*k}(T+Q)^k = \left(\sum_{i=0}^{h} \binom{k}{i} Q^{*i} T^{*k-i}\right)\left(\sum_{j=0}^{h} \binom{k}{j} T^{k-j} Q^j\right) =$$

$$= \sum_{i,j=0}^{h} \binom{k}{i}\binom{k}{j} Q^{*i} T^{*k-i} T^{k-j} Q^j =$$

$$= \sum_{0\leq i<j\leq h} \binom{k}{i}\binom{k}{j} Q^{*i} T^{*j-i} T^{*k-j} T^{k-j} Q^j + \sum_{0\leq j\leq i\leq h} \binom{k}{i}\binom{k}{j} Q^{*i} T^{*k-i} T^{k-i} T^{i-j} Q^j \ .$$

From (2.8) we obtain, for certain $A_{m-1}, ...., A_0 \in B(H)$,

$$(T+Q)^{*k}(T+Q)^k = \sum_{0\leq i<j\leq h} \binom{k}{i}\binom{k}{j} Q^{*i} T^{*j-i} \left(\sum_{r=0}^{m-1} A_r (k-j)^r\right) Q^j +$$

$$+ \sum_{0\leq j\leq i\leq h} \binom{k}{i}\binom{k}{j} Q^{*i} \left(\sum_{r=0}^{m-1} A_r (k-i)^r\right) T^{i-j} Q^j \ .$$

Write

$$B_{r,i,j} := Q^{*i} T^{*j-i} A_r Q^j \in B(H) \ , \quad C_{r,i,j} := Q^{*i} A_r T^{i-j} Q^j \in B(H) \ ,$$

$$q_{r,i,j} := \binom{k}{i}\binom{k}{j}(k-j)^r \ , \quad p_{r,i,j} := \binom{k}{i}\binom{k}{j}(k-i)^r \ .$$

Note that $\binom{k}{i}$ and $\binom{k}{j}$ are real polynomials in $k$ of degree less or equal to $h \leq n-1$, and $(k-j)^r$ and $(k-i)^r$ have degree $r \leq m-1$. Hence $q_{r,i,j}$ and $p_{r,i,j}$ are real polynomials of degree less or equal to $m-1+2(n-1) = 2n+m-3$. Consequently we can write

$$(T+Q)^{*k}(T+Q)^k = \sum_{r=0}^{m-1} \sum_{0\leq i<j\leq h} B_{r,i,j} q_{r,i,j} + \sum_{r=0}^{m-1} \sum_{0\leq j\leq i\leq h} C_{r,i,j} p_{r,i,j} \ ,$$

which is a polynomial in $k$, of degree less or equal to $2n+m-3$ with coefficients in $B(H)$. By Theorem 2.1, the operator $T+Q$ is an $(2n+m-2)$-isometry. □

For isometries it is possible to say more [9, Theorem 2.2].

**Theorem 3.2.** *Let $H$ be a Hilbert space. Let $T \in B(H)$ be an isometry and $Q \in B(H)$ be an $n$-nilpotent operator ($n \geq 1$ integer) such that $TQ = QT$. Then $T+Q$ is a strict $(2n-1)$-isometry.*

*Proof.* By Theorem 3.1 we obtain that $T+Q$ is a $(2n-1)$-isometry. Note that as $T$ is an isometry we have $T^{*k}T^k = I$, for every positive integer $k$.

As in the proof of Theorem 3.1, for any integer $k \geq 0$, we have that

$$(T+Q)^{*k}(T+Q)^k = \sum_{i,j=0}^{h} \binom{k}{i}\binom{k}{j} Q^{*i}T^{*h-i}T^{h-j}Q^j =$$

$$= \sum_{0\leq i<j\leq h} \binom{k}{i}\binom{k}{j} Q^{*i}T^{*j-i}Q^j + \sum_{0\leq j\leq i\leq h} \binom{k}{i}\binom{k}{j} Q^{*i}T^{i-j}Q^j \ ,$$

where $h := \min\{k, n-1\}$.

The coefficient of the summand that appears at $k^{2n-1}$ is equal to

$$\binom{k}{n-1}^2 Q^{*n-1}Q^{n-1} \ ,$$

which is null if and only if $Q^{*n-1}Q^{n-1} = 0$; that is, if and only if $Q^{n-1} = 0$. Therefore, if $Q$ is nilpotent of order $n$, then $(T+Q)^{*k}(T+Q)^k$ can be written as a polynomial in $k$, of degree $2n-1$ and coefficients in $B(H)$. Consequently $T+Q$ is a strict $(2n-1)$-isometry. □

Now we obtain the following corollary of Theorem 3.2.

**Corollary 3.1.** *Let $H$ be a Hilbert space. Let $Q \in B(H)$ be an $n$-nilpotent operator ($n \geq 1$ integer). Then $I+Q$ is a strict $(2n-1)$-isometry.*

Recall that an operator $T \in B(H)$ is $N$*-supercyclic* ($N \geq 1$ integer) if there exists a subspace $F \subset H$ of dimension $N$ such that its orbit $\{T^n x : n \geq 0, x \in F\}$ is dense in $H$. Moreover, $T$ is called *supercyclic* if it is 1-supercyclic. See [12] and [11].

Bayart [7, Theorem 3.3] proved that on an infinite dimensional Banach space an $(m,q)$-isometry is never $N$-supercyclic, for any $N \geq 1$. In the setting of Banach spaces, Yarmahmoodi, Hedayatian and Yousefi [16, Theorem 2.2] showed that any sum of an isometry and a commuting nilpotent operator is never supercyclic. For Hilbert space operators we extend the result [16, Theorem 2.2] to $m$-isometries plus commuting nilpotent operators.

**Corollary 3.2.** *Let $H$ be an infinite dimensional Hilbert space. If $T \in B(H)$ is an $m$-isometry that commutes with a nilpotent operator $Q$, then $T+Q$ is never $N$-supercyclic for any $N$.*

## 4. Some examples in the setting of Banach spaces

Theorem 3.2 is not true for finite-dimensional Banach spaces even for $m = 1$.

Denote by $\ell_p^d := (\mathbb{C}^d, \|\cdot\|_p)$.

**Example 4.1.** *Let $Q : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be defined by $Q(x, y) := (y, 0)$, hence $Q$ is a 2-nilpotent operator. The following assertions hold:*

*(1) $I + Q$ is not a $(3, p)$-isometry on $\ell_p^2$ for any $1 \leq p < \infty$ and $p \neq 2$.*

*(2) $I + Q$ is not a $(3, p)$-isometry on $\ell_\infty^2$ for any $p > 0$.*

*(3) $I + Q$ is a strict $(2k + 1, 2k)$-isometry on $(\mathbb{C}^2, \|.\|_{2k})$ for any $k = 1,\ 2,\ 3,\ \ldots$*

*Proof.* For $(x, y) \in \mathbb{C}^2$ we have

$$(I + Q)(x, y) = (x + y, y)\ ,\quad (I + Q)^2(x, y) = (x + 2y, y)\ ,\quad (I + Q)^3(x, y) = (x + 3y, y)\ .$$

Write

$$A(x, y; p, q) := \|(I + Q)^3(x, y)\|_p^q - 3\|(I + Q)^2(x, y)\|_p^q + 3\|(I + Q)(x, y)\|_p^q - \|(x, y)\|_p^q.$$

(1) We consider two cases, $1 < p < \infty$ and $p = 1$.

(a) *Case* $1 < p < \infty$. For $x = 0$, $y = 1$ and $q = p$, we have

$$A(0, 1; p, p) = 3^p + 1 - 3 \cdot 2^p - 3 + 6 - 1 = 3^p - 3 \cdot 2^p + 3\ .$$

So $A(0, 1; p, p) = 0$ if and only if $3^{p-1} + 1 = 2^p$, which is true only when $p = 2$ or $p = 1$ since the function $f(t) = 3^{t-1} + 1 - 2^t$ is null only for $t = 1$ and $t = 2$.

Consequently $I + Q$ is not a $(3, p)$-isometry on $\ell_p^2$ if $p \neq 2$ and $1 < p < \infty$.

(b) *Case* $p = 1$. In order to prove that $I + Q$ is not a $(3, 1)$-isometry on $\ell_1^2$, we take the vector $(1, -1)$ and obtain that

$$A(1, -1; 1, 1) = \|(I+Q)^3(1, -1)\|_1 - 3\|(I+Q)^2(1, -1)\|_1 + 3\|(I+Q)(1, -1)\|_1 - \|(1, -1)\|_1 \neq 0.$$

(2) For $(x, y) \in \mathbb{C}^2$ we have

$$A(x, y; \infty, p) := \|(I + Q)^3(x, y)\|_\infty^p - 3\|(I + Q)^2(x, y)\|_\infty^p + 3\|(I + Q)(x, y)\|_\infty^p - \|(x, y)\|_\infty^p =$$

$$= \max\{|x + 3y|, |y|\}^p - 3 \max\{|x + 2y|, |y|\}^p + 3 \max\{|x + y|, |y|\}^p - \max\{|x|, |y|\}^p\ .$$

In particular, for $x := (1, -1)$,

$$A(1, -1; \infty, p) = 2^p - 1 \neq 0.$$

Therefore $I + Q$ is not a $(3, p)$-isometry on $\ell^2_\infty$ for any $p > 0$.

(3) First we prove by induction on $k$ that $I + Q$ is a $(2k+1, 2k)$-isometry on $\ell^2_{2k}$ for any $k = 1,\ 2,\ 3 \ldots$ Note that, for $(x, y) \in \mathbb{C}^2$,

$$(I + Q)^s(x, y) = (x + sy, y) \qquad (s = 0, 1, 2 \ldots)\ .$$

By Corollary 3.1, the operator $I+Q$ is a strict (3,2)-isometry on $\ell^2_2$. Hence $I+Q$ is a strict $(2k+1, 2k)$-isometry on $\ell^2_2$ for all $k = 1,\ 2,\ 3\ \ldots$ [13, Corollary 4.6]. Thus for $(x, y) \in \mathbb{C}^2$,

$$\sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} (|x + sy|^2 + |y|^2)^k = 0\ . \tag{4.11}$$

Suppose that $I + Q$ is a $(2i - 1, 2i - 2)$-isometry on $\ell^2_{2i-2}$ for every $i = 2, 3, \ldots, k$. Hence $I + Q$ is also a $(2k + 1, 2i - 2)$-isometry on $\ell^2_{2i-2}$. Then, for $(x, y) \in \mathbb{C}^2$,

$$\sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} (|x + sy|^{2i-2} + |y|^{2i-2}) = 0, \quad (2 \le i \le k)\ .$$

Therefore

$$\sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} |x + sy|^{2i-2} = 0, \quad (2 \le i \le k)\ . \tag{4.12}$$

Taking into account the equality (4.12) we can write (4.11) in the following way:

$$\begin{aligned} 0 &= \sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} \sum_{i=0}^{k} \binom{k}{i} |x + sy|^{2i} |y|^{2(k-i)} \\ &= \sum_{i=0}^{k-1} \binom{k}{i} |y|^{2(k-i)} \sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} |x + sy|^{2i} \\ &\quad + \sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} |x + sy|^{2k} \\ &= \sum_{s=0}^{2k+1} (-1)^{2k+1-s} \binom{2k+1}{s} (|x + sy|^{2k} + |y|^{2k})\ . \end{aligned}$$

Therefore $I + Q$ is a $(2k + 1, 2k)$-isometry on $\ell^2_{2k}$.

Now we prove that $I+Q$ is a strict $(2k+1,2k)$-isometry on $\ell^2_{2k}$. Suppose on the contrary that $I+Q$ is a $(2k,2k)$-isometry on $\ell^2_{2k}$. Then

$$\sum_{s=0}^{2k-1}(-1)^{2k-1-s}\binom{2k-1}{s}(|x+sy|^{2k}+|y|^{2k})=0$$

for all $(x,y)\in\mathbb{C}^2$. So

$$\sum_{s=0}^{2k-1}(-1)^{2k-1-s}\binom{2k-1}{s}|x+sy|^{2k}=0 \tag{4.13}$$

for all $(x,y)\in\mathbb{C}^2$. In particular, for $y=1$ and $x=0,1,2,\dots$ we have

$$\sum_{s=0}^{2k-1}(-1)^{2k-1-s}\binom{2k-1}{s}(x+s)^{2k}=0\ . \tag{4.14}$$

So $(s^{2k})_{s=0}^{\infty}$ is an arithmetic progression of order $2k-2$, which is a contradiction with (2.5).

□

**Remark 4.2.** Notice that in any Hilbert space of dimension $n$, there are strict $m$-isometries only for any $m\le 2n-1$. However, as the above example shows, there are strict $(2k+1,2k)$-isometries for any integer $k$ in a Banach space of dimension 2.

The following example gives an operator of the form $I+Q$ with $Q$ a nilpotent operator such that $I+Q$ is not an $(m,q)$-isometry for any integer $m$ and any $q>0$.

**Example 4.3.** *Let $X$ be the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ such that vanish at 1 endowed with sup-norm. Define $Q:X\longrightarrow X$ by*

$$(Qf)(t):=\begin{cases} f(t+\frac{1}{2}) & \text{if } 0\le t\le \frac{1}{2}\\ 0 & \text{if } \frac{1}{2}<t\le 1\ .\end{cases}$$

*Then $Q\in B(X)$ is 2-nilpotent operator. Moreover, $I+Q$ is not an $(m,q)$-isometry for any $m=1,2,3...$ and any $q>0$.*

*Proof.* It is clear that $I+Q$ is not an isometry since the function $f\in X$ given by

$$f(t):=\begin{cases} 1 & \text{if } 0\le t\le \frac{1}{2}\\ -2t+2 & \text{if } \frac{1}{2}<t\le 1\end{cases}$$

satisfies $\|f\|=1$ and $\|(I+Q)f\|=2$.

For $m = 2, 3, 4...$ consider the function $f_m \in X$ defined by

$$f_m(t) := \begin{cases} -4t + 1 & \text{if } 0 \le t \le \frac{1}{4} \\ 0 & \text{if } \frac{1}{4} < t \le \frac{1}{2} \\ \frac{-4}{m-1}t + \frac{2}{m-1} & \text{if } \frac{1}{2} < t \le \frac{3}{4} \\ \frac{4}{m-1}t - \frac{4}{m-1} & \text{if } \frac{3}{4} < t \le 1 \,. \end{cases}$$

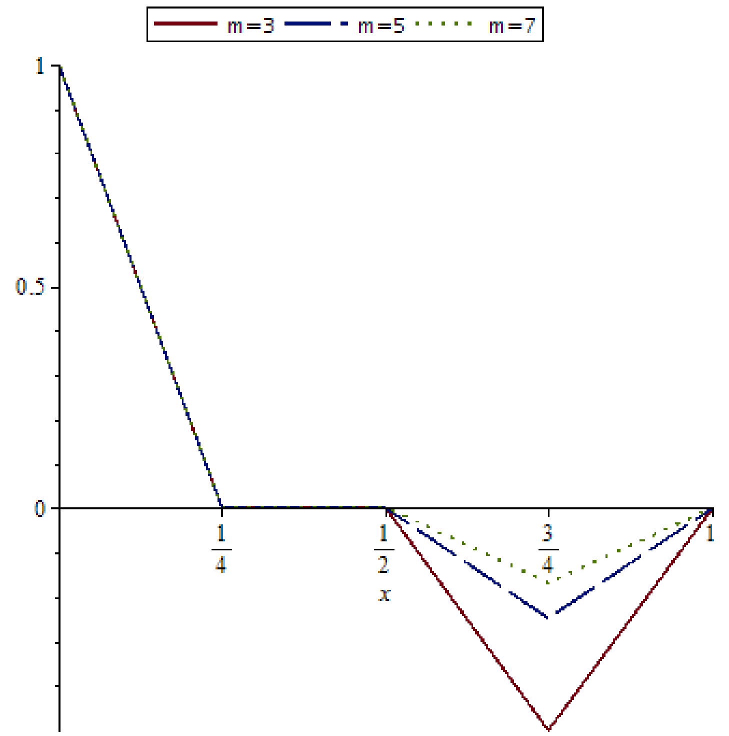


FIGURE 1. Graphics of functions $f_3$, $f_5$ and $f_7$

Note that $f_m(\frac{3}{4}) = \frac{1}{1-m} = \min_{0 \le t \le 1} f_m(t)$.

Fix $q > 0$. For $k = 0, 1, 2...$ we have

$$\|(I+Q)^k f_m\|^q = \|(I+kQ)f_m\|^q = \sup_{0\le t\le 1} |f_m(t) + k(Qf_m)(t)|^q \,.$$

If $0 \le k \le m-1$, then

$$\|(I+Q)^k f_m\|^q = |f_m(0) + kf_m(1/2)|^q = 1 \,,$$

since $k\frac{1}{m-1} \le 1$. But as $m\frac{1}{m-1} > 1$ we obtain

$$\|(I+Q)^m f_m\|^q = |f_m(1/4) + mf_m(3/4)|^q = \left(\frac{m}{m-1}\right)^q > 1 \,.$$

Consequently

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|(I+Q)^k f_m\|^q = \sum_{k=0}^{m-1}(-1)^{m-\ell}\binom{m}{k}+\|(I+Q)^m f_m\|^q = -1+\left(\frac{m}{m-1}\right)^q \neq 0 \ .$$

Therefore $I+Q$ is not an $(m,q)$-isometry for any $m=1,2,3...$ and any $q>0$. $\square$

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