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Abstract

We consider a parabolic-hyperbolic system of nonlinear partial differential equations modeling the motion of a chemically reacting mixture through porous medium. The existence of classical as well as weak solutions is established under several physically relevant choices of the constitutive equations and relevant boundary conditions

Key words: Chemically reacting flows, porous medium, parabolic-hyperbolic system

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1 Introduction

A simple model of the motion of a mixture of n chemically reacting fluids takes the form (see e.g. Giovangigli [10, Chapters 2,3]):

$$\partial_t(\varrho^i) + \mathrm{div}_x(\varrho^i \mathbf{v}) + \mathrm{div}_x \mathcal{F}^i = m_i \omega^i, \tag{1.1}$$

where ϱ^i is the mass density of the i-th species , $\mathbf{v}$ is the fluid bulk velocity of the mixture, m_i the molar mass of the i-th species, $\mathcal{F}^i$ the diffusive fluxes, and ω^i represent the molar production, typically given functions of $(\varrho^1, \dots, \varrho^n)$ and of the temperature. We also denote

$$\varrho = \sum_{i=1}^n \varrho^i,$$

the total density of the mixture and introduce the mass fractions

$$Y^i = \frac{\varrho^i}{\varrho}, \ i = 1, \dots, n.$$

Obviously,

$$Y^i \geq 0, \ \sum_{i=1}^n Y^i = 1. \tag{1.2}$$

We may sum up (1.1) to deduce the mass conservation (*equation of continuity*):

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}) = -\mathrm{div}_x \sum_{i=1}^n \mathcal{F}^i + \sum_{i=1}^n m_i \omega^i = 0, \tag{1.3}$$

where the last equality should be viewed as a natural constraint to be imposed on $\mathcal{F}^i$, ω^i enforced by the principle of mass conservation.

The diffusion fluxes are typically given through the empirical Fick's law:

$$\mathcal{F}^i = -d_i \nabla_x Y^i, \ d_i > 0, \ i = 1, \dots, n \tag{1.4}$$

If the motion takes place in the porous medium environment, we may close the system by imposing the standard hypothesis that the velocity $\mathbf{v}$ is given by the pressure gradient, more specifically

$$\mathbf{v} = -\nabla_x p + \varrho \mathbf{g}, \tag{1.5}$$

where $\mathbf{g}$ represents the gravitational force. For the one component compressible flow, the relation (1.5) has been rigorously identified as a homogenization limit of the compressible Navier-Stokes system, see Masmoudi [11]. The result has been extended to a more general class of pressure laws and also to the full Navier-Stokes-Fourier system in [8].

1.1 A parabolic-hyperbolic system

We consider the problem described above under the following simplifying assumptions:

- The diffusion coefficients d_i vanish for all $i = 1, 2, \ldots$.
- The process is isothermal, the temperature $T > 0$ is constant.
- The effect of the gravitational force is neglected, $\mathbf{g} = 0$.
- The production rates $\omega^i = \omega^i(\varrho^1, \ldots, \varrho^n)$ are given smooth functions of species densities. More specifically,

$$\omega^i = \mathcal{C}_i - \varrho^i \mathcal{D}_i, \tag{1.6}$$

where $\mathcal{C}_i \geq 0$, $\mathcal{D}_i \geq 0$,

$$\mathcal{C}_i = \sum_{j=1}^{m} \left[\nu^b_{i,j} K^f_j(T) \Pi^n_{l=1} \left(\frac{\varrho^l}{m_l} \right)^{\nu^f_{l,j}} + K^b_j(T) \Pi^n_{l=1} \left(\frac{\varrho^l}{m_l} \right)^{\nu^b_{l,j}} \right], \tag{1.7}$$

$$\mathcal{D}_i = \frac{1}{m_i} \left[\sum_{j=1, \nu^f_{i,j} \geq 1}^{m} \nu^f_{i,j} K^f_j(T) \left(\frac{\varrho^i}{m_i} \right)^{\nu^f_{i,j} - 1} \Pi^n_{l=1, l \neq i} \left(\frac{\varrho^l}{m_l} \right)^{\nu^f_{l,j}} \right. \tag{1.8}$$

$$\left. + \sum_{j=1, \nu^b_{i,j} \geq 1}^{m} \nu^b_{i,j} K^b_j(T) \left(\frac{\varrho^i}{m_i} \right)^{\nu^b_{i,j} - 1} \Pi^n_{l=1, l \neq i} \left(\frac{\varrho^l}{m_l} \right)^{\nu^b_{l,j}} \right],$$

where m is the number of chemical reactions, K^f_j, K^b_j are positive functions of the temperature, and $\nu^f_{i,j}$, $\nu^b_{i,j}$ are non-negative integers (stoichiometric coefficients), see [10, Section 6.4.6].

- The pressure of the mixture is given by the perfect gas law,

$$p = \sum_{i=1}^{n} \frac{1}{m_i} \varrho^i RT. \tag{1.9}$$

Remark 1.1 *It is interesting to note that* (1.9) *with* equal *molar masses* $m_i = m$ *is the only choice of the pressure compatible with the* Second Law of Thermodynamics *as soon as Fick's law is imposed, cf.* [9].

Our goal in the present paper is to discuss solvability and a proper choice of boundary conditions for system (1.1) under the simplifying conditions stated above. In Section 2, we study the case when the pressure p satisfies a parabolic equation of porous medium type independent of the species densities ϱ^i. The standard parabolic theory yields a regular pressure p that can be subsequently substituted in (1.1) to determine uniquely ϱ^i, $i = 1, \dots, n$ by the method of characteristics. Relevant boundary conditions are easy to discuss in this context.

In Section 3, we address the general situation when all equations in (1.1) are strongly coupled. The resulting system is of mixed parabolic-hyperbolic type. We derive *a priori* bounds and show weak sequential stability of the family of solutions. To this end, a variant of DiPerna, Lions [7] theory for the transport equation is used.

2 The case of "independent" pressure

We start with the simple situation of equal molar masses $m_i = m > 0$ for all $i = 1, \dots, n$. In accordance with (1.9) and (1.3),(1.5), we may sum up the equations (1.1) to obtain

$$\partial_t p - \mathrm{div}_x (p \nabla_x p) = 0. \tag{2.1}$$

Thus the pressure satisfies a parabolic type differential equation that may be solved separately and independently of the other quantities. Note that the same situation occurs in the absence of chemical reactions, meaning $\omega_i = 0$ for all $i = 1, \dots, n$. Most generally, we have (2.1) whenever

$$\sum_{j \in S_j} \omega_j = 0, \ m_j = m_{S_j} > 0 \text{ for all } j \in S_j, \ S_i \cap S_j = \emptyset \text{ if } i \neq j, \ \cup_j S_j = \{1, \dots, n\}. \tag{2.2}$$

2.1 Boundary value problem for the pressure equation

Equation (2.1) represents the standard porous medium equation studied frequently in the literature, see e.g. Di Benedetto [6]. Here, in addition, we avoid the "vacuum" problem by imposing positive initial and boundary conditions on p.

2.1.1 Mixed Neumann - Dirichlet boundary conditions

We suppose the boundary $\partial\Omega$ can be decomposed as

$$\partial\Omega = \Gamma_D \cup \Gamma_N, \ \Gamma_D, \ \Gamma_N \ \text{ smooth and compact with } \ \Gamma_D \cap \Gamma_N = \emptyset\,. \tag{2.3}$$

We impose the (non-homogeneous) Dirichlet boundary condition

$$p|_{\Gamma_D} = p_b - \ \text{ a positive constant,} \tag{2.4}$$

together with the (homogenous) Neumann boundary condition

$$\nabla_x p \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma_N \, . \tag{2.5}$$

As usual, $\mathbf{n}$ denotes the *outer* unit normal vector to the boundary $\partial\Omega$ of Ω.

Remark 2.1 *This choice of boundary conditions corresponds to the presence of a "well" in the container Ω on the boundary of which a (constant) pressure is maintained, with the rest of $\partial\Omega$ being an impermeable wall.*

In order to deal with a well-posed problem, we prescribe the initial pressure distribution

$$p(0,x) = p_0(x) \text{ in } \Omega. \tag{2.6}$$

(i) Consider the situation

$$p_0(x) \geq p_b > 0 \text{ for all } x \in \Omega, \ p_0 \in W^{2,\infty}(\Omega), \ p_0 \not\equiv p_b.$$

By virtue of the standard parabolic theory, problem (2.1), (2.3) – (2.6) admits a unique solution

$$p(t,x) \geq p_b \ \text{ for any } \ (t,x) \in (0,\tau) \times \Omega \, .$$

Moreover, the solution is smooth in the *open* set $(0,\tau) \times \Omega$ and, by virtue of the strong maximum principle (Hopf's boundary point lemma),

$$\nabla_x p \cdot \mathbf{n} < 0 \ \text{ on } \ \Gamma_D \, . \tag{2.7}$$

Now, equation (1.1) reduces to the transport problem

$$\partial_t(\varrho^i) - \mathrm{div}_x(\varrho^i \nabla_x p) = m_i \omega^i(\varrho^1, \ldots, \varrho^n), \ i = 1, \ldots n, \tag{2.8}$$

with a given (regular) velocity field $\mathbf{v} = -\nabla_x p$. Keeping (2.5), (2.7) in mind, equation (2.8) admits a unique solution for any initial data

$$\varrho^i(0,\cdot) = \varrho_0^i \, , \qquad \text{in } \ \Omega \tag{2.9}$$

satisfying the obvious compatibility condition

$$\sum_{i=1}^n \frac{1}{m_i} \varrho_0^i RT = p_0 \qquad \text{in } \ \Omega \, . \tag{2.10}$$

(ii) Now, we examine the complementary situation

$$0 < p_0(x) \leq p_b \quad \text{for all } x \in \overline{\Omega},\ p_0 \in W^{2,\infty}(\Omega),\ p_0 \not\equiv p_b\,.$$

It is easy to check, by means of the same arguments as above, that

$$\nabla_x p \cdot \mathbf{n} > 0 \text{ on } \Gamma_D. \tag{2.11}$$

Consequently, for the transport problem (2.8), (2.9) to be uniquely solvable, we have to prescribe the boundary conditions

$$\varrho^i|_{\Gamma_D} = \varrho^i_b,\ i = i, \ldots, n,$$

with the compatibility condition

$$\sum_{i=1}^{n} \frac{1}{m_i} \varrho^i_b RT = p_b.$$

(iii) In general, the sign of the normal component of the velocity $-\nabla_x p \cdot \mathbf{n}$ on Γ_D is determined by the pressure. In particular, the relevant boundary conditions for ϱ^i must be prescribed a posteriori, after having solved problem (2.1), (2.3) – (2.6).

2.2 Other boundary conditions

More general boundary conditions can be handled in a similar fashion. One should always keep in mind that the boundary conditions for the species densities ϱ^i_b must be determined after having identified the sign of $\nabla_x p \cdot \mathbf{n}$ together with p on $\partial\Omega$.

3 General system

We focus on the general case in which the equations for the pressure and the species densities are coupled. It turns out that it is more convenient to consider p, together with the mass fractions Y^i, as independent variables. Accordingly, the resulting system of equations reads:

$$\partial_t p - \mathrm{div}_x (p \nabla_x p) = RT \sum_{i=1}^{n} \omega^i, \tag{3.1}$$

$$\partial_t Y^i - \nabla_x p \cdot \nabla_x Y^i = \frac{m_i}{\varrho} \omega^i,\ i = 1, \ldots, n. \tag{3.2}$$

Recalling the pressure-density relation

$$\varrho = p \left(\sum_{i=1}^{n} \frac{1}{m_i} Y^i RT \right)^{-1},\ \varrho^i = Y^i \varrho, \tag{3.3}$$

and using the specific form of ω^i stated in (1.6) – (1.8), we view the right-hand sides of the above equations as functions of p and $Y^1, \dots, Y^n$.

System (3.1) – (3.3) is nonlinear of parabolic-hyperbolic type. To avoid unnecessary technicalities, we impose the homogeneous Neumann boundary conditions for the pressure,

$$\nabla_x p \cdot \mathbf{n}|_{\partial\Omega} = 0\,. \tag{3.4}$$

Accordingly, only the *initial conditions* for Y^i are necessary to make the problem, at least formally, well-posed.

3.1 A priori estimates

We start by deriving suitable *a priori* estimates for (smooth) solutions of problem (3.1), (3.2), (3.4).

3.1.1 Uniform bounds on the pressure

Uniform bounds on the pressure are usually derived by application of some form of the maximum principle. A short inspection of the pressure equation (3.1) and the structure (1.6) of the functions ω^i reveals that

$$\sum_{i=1}^n \omega_i = \sum_{i=1}^n \mathcal{C}_i - \varrho^i \mathcal{D}_i \lesssim \sum_{j=1}^n \left(p^{\sum_{l=1}^m \nu_{l,j}^f} + p^{\sum_{l=1}^m \nu_{l,j}^b} \right).$$

Consequently, in view of the standard maximum principle estimates, we get a uniform bound

$$0 \le p(t,x) \le \overline{p} \text{ on the time interval } (0,\tau), \tag{3.5}$$

where $\tau > 0$ depends, in general, on $\|p(0,\cdot)\|_{L^\infty(\Omega)}$. Moreover, the estimate is uniform, meaning extendable to any positive τ if at least one of the following situations occurs:

- $$\sum_{i=1}^n \mathcal{C}_i - \varrho^i \mathcal{D}_i \lesssim (p+1),$$

  for specific examples see [10, Section 3.2.3];

- $$\|p(0,\cdot)\|_{L^\infty(\Omega)} \text{ is sufficiently small,}$$

  where "small" means in terms of τ and the structural constants appearing in (1.7), (1.8).

Accordingly, in the remaining part of this section, we assume the validity of the bound (3.5). Note that, in view of the structure of ω^i stated in (1.6), relation (3.5) implies that

$$p(t,\cdot) \geq \underline{p} > 0 \text{ for any } t \in (0,\tau) \text{ as soon as } \inf_{x\in\Omega} p(0,x) > 0, \tag{3.6}$$

where the lower bound $\underline{p}$ may depend on τ.

3.1.2 Maximal regularity estimates

In view of (3.5), (3.6) we may use the maximal regularity estimates for (non-degenerate) parabolic equations, see Denk, Hieber, and Pruess [4] or Ashyralyev and Sobolevskii [5], to deduce the bounds

$$\partial_t p,\ \nabla_x^l p\,;\ l = 0,1,2\,,\ \ \text{bounded in}\ \ L^q((0,\tau)\times\Omega)\ \ \text{for any finite}\ \ 1 < q < \infty\,. \tag{3.7}$$

Unfortunately, the bounds (3.7) are still not sufficient for the transport equations (3.2) to be well-posed. The available DiPerna, Lions theory [7] (see also Ambrosio [2], Crippa and De Lellis [3]) require that, at least,

$$\mathrm{div}_x \nabla_x p = \Delta_x p \in L^1(0,\tau; L^\infty(\Omega))\,. \tag{3.8}$$

In order to guarantee (3.8), higher order regularity estimates are needed that will be established in the next section.

3.1.3 Higher order regularity

Taking the time derivative of (3.1) with respect to t and denoting $P = \partial_t p$, we obtain

$$\partial_t P - \mathrm{div}_x(p\nabla_x P) = \mathrm{div}_x(\partial_t p \nabla_x p) + RT\sum_{i=1}^n \partial_t \omega^i. \tag{3.9}$$

To evaluate $\partial_t \omega^i$ we realize that, thanks to (1.6 – 1.8),

$$\omega_i = \sum_{k=1}^{k_i} \varrho^k G_{k,i}(\varrho^1,\dots,\varrho^n),\ i = 1,\dots,n,$$

where $G_{k,i}$ are continuously differentiable functions. Using (2.8) we compute

$$\partial_t \left(\varrho^k G_{k,i}(\varrho^1,\dots,\varrho^n)\right) = \partial_t \varrho^k G_{k,i}(\varrho^1,\dots,\varrho^n) + \varrho^k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} \partial_t \varrho^j$$

$$= \mathrm{div}_x(\varrho^k \nabla_x p) G_{k,i}(\varrho^1,\dots,\varrho^n) + \varrho^k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} \mathrm{div}_x(\varrho^j \nabla_x p)$$

$$+ m_i \omega_i G_{k,i}(\varrho^1,\dots,\varrho^n) + \varrho^k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} m_j \omega_j .$$

Furthermore,

$$\mathrm{div}_x(\varrho^k \nabla_x p) G_{k,i}(\varrho^1,\dots,\varrho^n) + \varrho^k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} \mathrm{div}_x(\varrho^j \nabla_x p)$$

$$= \mathrm{div}_x \left[\varrho^k \nabla_x p G_{k,i}(\varrho^1,\dots,\varrho^n) \right] - \varrho_k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} \nabla_x p \cdot \nabla_x \varrho^j$$

$$+ \varrho_k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} \nabla_x p \cdot \nabla_x \varrho^j + \varrho^k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} \varrho^j \Delta_x p$$

$$= \mathrm{div}_x \left[\varrho^k \nabla_x p G_{k,i}(\varrho^1,\dots,\varrho^n) \right] + \varrho^k \sum_{j=1}^n \frac{G_{k,i}(\varrho^1,\dots,\varrho^n)}{\partial \varrho^j} \varrho^j \Delta_x p.$$

Summing up the previous observations and going back to (3.9) we infer that

$$\partial_t P - \mathrm{div}_x(p \nabla_x P) = \mathrm{div}_x(\mathbf{F}) + G\,,$$

with

$$\mathbf{F},\ G \quad \text{bounded in } \ L^q((0,\tau)\times\Omega) \ \text{ for any finite } \ 1<q<\infty,\ \mathbf{F}\cdot\mathbf{n}|_{\partial\Omega} = 0\,.$$

Thus, applying the (weak) maximal regularity theory for parabolic equations (see Amann [1]), we conclude that

$$\partial_t p = P \quad \text{ is bounded in } \ L^q(0,\tau; W^{1,q}(\Omega)) \ \text{ for any } \ 1<q<\infty\,. \tag{3.10}$$

Note that this step requires higher regularity of the initial data (at $t=0$), specifically,

$$\partial_t p(0,\,\cdot\,) = P(0,\,\cdot\,) \in B^{1-(2/q);q,q}(\Omega)\,,$$

see Amann [1, Theorem 2.1]. This kind of initial regularity hypothesis is not unusual for a parabolic problem.

Finally, returning to (3.1), we obtain the desired conclusion

$$\nabla_x \mathrm{div}_x p = \Delta_x p \in L^q(0,\tau; L^\infty(\Omega)) \quad \text{ for any } \ 1<q<\infty\,. \tag{3.11}$$

3.2 Weak sequential stability

Our goal is to establish the following result:

Theorem 3.1 *Let $\{p_\varepsilon\}_{\varepsilon>0}$, $\{Y^i_\varepsilon\}_{\varepsilon>0}$; $i=1,\dots,n$, be a family of (smooth) solutions of problem* (3.1), (3.2) *such that:*

$$p_\varepsilon \to p,\ \nabla_x p_\varepsilon \to \nabla_x p \text{ in } C([0,\tau]\times\Omega),\ \Delta_x p \to \Delta_x p \text{ weakly-(*) in } L^q(0,\tau;L^\infty(\Omega)),\ 1<q<\infty, \tag{3.12}$$

$$Y^i_\varepsilon \to Y^i \text{ weakly-(*) in } L^\infty((0,\tau)\times\Omega), \tag{3.13}$$

$$Y^i_\varepsilon(0,\cdot) \to Y^i_0 \text{ in } L^1(\Omega). \tag{3.14}$$

Then

$$Y^i_\varepsilon \to Y^i \text{ a.e. in } (0,\tau)\times\Omega\,, \tag{3.15}$$

where p and $Y^1,\dots,Y^n$ satisfy (3.2)*, specifically,*

$$\partial_t Y^i - \mathrm{div}_x(Y^i\nabla_x p) + Y^i\Delta_x p = \frac{1}{p}\omega_i(p,Y^1,\dots,Y^n)\sum_{j=1}^n \frac{m_i}{m_j}Y^j RT,\ i=1,\dots,n. \tag{3.16}$$

The rest of the paper is devoted to the proof of Theorem 3.1. We use the approach proposed in the seminal paper by DiPerna and Lions [7].

3.2.1 Existence for the limit problem

We show that the limit problem (3.16) admits a weak solution $Y^1,\dots,Y^n$ such that

$$Y^i \ge 0 \text{ for any } i=1,\dots,n,\ \sum_{i=1}^n Y^i = 1\,,$$

provided the initial data satisfy

$$Y^i_0 \ge 0\,,\quad \sum_{i=1}^n Y^i_0 = 1\,.$$

Step 1

We approximate the pressure p by a family of smooth functions $\{p_\delta\}_{\delta>0}$,

$$p_\delta \to p,\ \nabla_x p_\delta \to \nabla_x p \text{ uniformly in } [0,\tau]\times\overline{\Omega},$$

$$\Delta_x p_\delta \to \Delta_x p \text{ a.e. in } (0,\tau)\times\Omega\,,\quad \|\Delta_x p_\delta\|_{L^q(0,\tau;L^\infty(\Omega))} \lesssim 1 \quad \text{for any } 1<q<\infty\,.$$

as $\delta \to 0$. Using the standard method of characteristics, we find a unique solution $Y_\delta^1, \dots, Y_\delta^n$ emanating from the initial data $Y_0^1, \dots, Y_0^n$.

Thanks to hypothesis (1.6),

$$Y_\delta^i \geq 0 \quad \text{for all } \ i = 1, \dots, n\,,$$

and, by virtue of (1.3),

$$\sum_{i=1}^n Y_\delta^i = 1\,.$$

Consequently, passing to a suitable subsequence if necessary, we may assume that

$$Y_\delta^i \to Y^i \text{ weakly-}(*) \text{ in } L^\infty((0,\tau) \times \Omega) \cap C_{\text{weak}}([0,\tau]; L^1(\Omega)) \text{ as } \delta \to 0,$$

where

$$\partial_t Y^i - \mathrm{div}_x (Y^i \nabla_x p) + Y^i \Delta_x p = \frac{1}{p} \overline{\omega_i(p, Y^1, \dots, Y^n) \sum_{j=1}^n \frac{m_i}{m_j} Y^j} RT, \ i = 1, \dots, n\,. \tag{3.17}$$

$$Y^i(0, \cdot) = Y_0^i\,. \tag{3.18}$$

Here and hereafter, the upper bar denotes a weak limit of compositions of smooth functions applied to weakly convergent sequences.

Step 2

In order to complete the proof, we have to show strong convergence

$$Y_\delta^i \to Y^i \text{ a.a. in } (0,\tau) \times \Omega \text{ as } \delta \to 0. \tag{3.19}$$

To this end, we write down a renormalized formulation of the $\delta-$problem in the form:

$$\partial_t |Y_\delta|^2 - \mathrm{div}_x (|Y_\delta|^2 \nabla_x p_\delta) + |Y_\delta|^2 \Delta_x p_\delta = \frac{2RT}{p_\delta} \sum_{i,j=1}^n \frac{m_i}{m_j} \omega_i(p_\delta, Y_\delta^1, \dots, Y_\delta^n) Y_\delta^i Y_\delta^j.$$

Letting $\delta \to 0$ we obtain

$$\partial_t \overline{|Y|^2} - \mathrm{div}_x (\overline{|Y|^2} \nabla_x p) + \overline{|Y|^2} \Delta_x p = \frac{2RT}{p} \sum_{i,j=1}^n \frac{m_i}{m_j} \overline{\omega_i(p, Y^1, \dots, Y^n) Y^i Y^j}. \tag{3.20}$$

Now, applying the regularization procedure of DiPerna and Lions [7] to (3.17) we deduce that

$$\partial_t |Y|^2 - \mathrm{div}_x (|Y|^2 \nabla_x p) + |Y|^2 \Delta_x p = \frac{2RT}{p} \sum_{i,j=1}^n \frac{m_i}{m_j} \overline{\omega_i(p, Y^1, \dots, Y^n) Y^j} Y^i. \tag{3.21}$$

Step 3
Finally, we integrate the difference of (3.20), (3.21) over Ω:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\overline{|Y|^2} - |Y|^2\right)\ \mathrm{d}x = -\int_\Omega \Delta_x p \left(\overline{|Y|^2} - |Y|^2\right)\ \mathrm{d}x$$

$$+\int_\Omega \frac{2RT}{p}\sum_{i,j=1}^n \frac{m_i}{m_j}\left[\overline{\omega_i(p,Y^1,\dots,Y^n)Y^iY^j} - \overline{\omega_i(p,Y^1,\dots,Y^n)Y^j}Y^i\right]\ \mathrm{d}x,$$

where

$$\int_\Omega \left[\overline{\omega_i(p,Y^1,\dots,Y^n)Y^iY^j} - \overline{\omega_i(p,Y^1,\dots,Y^n)Y^j}Y^i\right]\ \mathrm{d}x$$

$$= \lim_{\delta\to 0}\int_\Omega \left[\omega_i(p_\delta,Y^1_\delta,\dots,Y^n_\delta)Y^i_\delta - \omega_i(p_\delta,Y^1,\dots,Y^n)Y^i\right](Y^j_\delta - Y^j)\ \mathrm{d}x$$

$$\lesssim \lim_{\delta\to 0}\int_\Omega |Y_\delta - Y|^2\ \mathrm{d}x = \int_\Omega \left(\overline{|Y|^2} - |Y|^2\right)\ \mathrm{d}x.$$

Thus, applying Gronwall's lemma and using the fact that the initial values converge strongly, we conclude

$$\overline{|Y|^2} = |Y|^2$$

yielding (3.19).

3.2.2 Compactness

Our ultimate goal is to show (3.15), (3.16). As Y_ε are smooth, we may rewrite (3.16) as

$$\partial_t Y^i_\varepsilon - \nabla_x Y^i_\varepsilon \cdot \nabla_x p_\varepsilon = \frac{RT}{p_\varepsilon}\sum_{j=1}^n \frac{m_i}{m_j}\omega_i(p_\varepsilon, Y^1_\varepsilon,\dots,Y^n_\varepsilon)Y^j_\varepsilon,\ i = 1,\dots,n. \tag{3.22}$$

At this stage, we employ once more the regularization procedure of DiPerna, Lions [7] to equation (3.16):

$$\partial_t Y^i_r - \nabla_x Y^i_r \nabla_x p = \frac{RT}{p}\sum_{j=1}^n \frac{m_i}{m_j}\omega_i(p, Y^1_r,\dots,Y^n_r)Y^j_r + e_r,\ i = 1,\dots,n, \tag{3.23}$$

where

$$e_r \to 0 \text{ in } L^1((0,\tau)\times\Omega) \text{ as } r \to 0.$$

Similarly to the above, we subtract (3.22), (3.23), multiply the resulting expression by $Y^i_\varepsilon - Y^i_r$, and integrate over Ω obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |Y_\varepsilon - Y_r|^2 \ \mathrm{d}x + \int_\Omega \Delta_x p_\varepsilon |Y_\varepsilon - Y_r|^2 \ \mathrm{d}x = \int_\Omega \sum_{i=1}^n (\nabla_x p_\varepsilon - \nabla_x p)\cdot \nabla_x Y^i_r (Y^i_\varepsilon - Y^i_r) \ \mathrm{d}x$$

$$= \int_\Omega \frac{RT}{p_\varepsilon} \sum_{i,j=1}^n \frac{m_i}{m_j} \left[\omega_i(p_\varepsilon, Y^1_\varepsilon, \dots, Y^n_\varepsilon) Y^j_\varepsilon - \omega_i(p_\varepsilon, Y^1_r, \dots, Y^n_r) Y^j_r\right] \ \mathrm{d}x + e_\varepsilon(r) + e_r,$$

where

$$e_\varepsilon(r) \to 0 \text{ in } L^1((0,\tau)\times\Omega) \text{ as } \varepsilon \to 0 \text{ for any fixed } r.$$

Finally, letting first $\varepsilon \to 0$, then $r \to 0$, and realizing that

$$Y^i_r \to Y^i \text{ in } C([0,\tau]; L^2(\Omega)),$$

we get the desired conclusion (3.15), (3.16).

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