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Abstract

The aim of this paper is to proceed in the study of the system which will be specified below. The system concerns fluid flow in a simple hydraulic system consisting of a pipe with generator on one side and a valve or more complicated hydraulic elements, on the other end of the pipe. The purpose is a rigorous mathematical analysis of the corresponding linearized system.

Here, we analyze the linearized problem near the fixed steady state which already have been explicitely described. The theory of mixed linear partial differential systems and other tools are applied to derive as explicit form of the solution as possible.

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1 Introduction

In this paper we are going to analyze the mathematical model of a fluid flow of a real liquid in a pipe with in-flow and out-flow boundary conditions corresponding to hydraulic elements which are employed in the system to model the situation in a real hydraulic machinery. In next section we formulate the general problem which can be derived from physical balance laws applied in the system (see[1]).

Since the system is highly nonlinear and contains the features of general nonlinear hyperbolic systems which have not yet been managed even in much simpler cases, the necessary reasonably physical must be made to get at least partial picture about what is going on in the real machinery.

One step have already been made in [2] where we found some solutions in a closed form: stationary solutions, purely time-dependent solutions and a mixture of both (called combined solutions).

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Our ambition in the present paper is to analyze *linearized system* around a steady state or around special dynamical solutions.

We organize the paper as follows. In next section we formulate the general problem with some comments and present the particular results from [[2]] to have a base for the further formulation and analysis of the linearized problems. The section 2 is started with general scheme of linearization in a quite general setting to facilitate rather awckward linearization of the general system. Section 3 is devoted to the analysis of the linearized system, and finally, Section 3 brings for considerations for linearization arround a dynamical solution (in a closed form). In spite of restrictions we are forced to make, we believe that our approach is useful in the engineering practice, based on the authors' experience in application of similar mathematical approach to other problems.

2 Formulation of the problem

We consider the following equations for the two phase flow of real fluids in a pipe of the length l is possible to write in the form (see for instance [1]):

$$w_t + \rho_0^{-1} p_x + f(w) = 0, \tag{2.1}$$
$$p_t + \rho_0 c^2(p,\gamma) w_x = 0, \tag{2.2}$$
$$\gamma_t + w\gamma_x = g(\gamma, p), \quad x \in (0,l), \quad t \in (0,T),\ (T>0), \tag{2.3}$$
$$w(x,0) = w_0(x), \tag{2.4}$$
$$p(x,0) = p_0(x), \tag{2.5}$$
$$\gamma(x,0) = \gamma_0(x), \quad x \in [0,l], \tag{2.6}$$
$$C\big(p(0,t),\gamma(0,t)\big) + Q_V\big(p(0,t),H(t)\big) - S_0 w(0,t) + \varphi \dot{H}(t) = 0, \tag{2.7}$$
$$w(l,t) = h(t), \tag{2.8}$$
$$\ddot{H}(t) + \Phi\big(t, H(t), \dot{H}(t), p(0,t), p_t(0,t)\big) = 0, \quad t \in [0,T], \tag{2.9}$$
$$H(0) = H_0,\ \dot{H}(0) = H_1. \tag{2.10}$$

The quantities occurring in (2.1)-(2.10) have the following meaning:

$w = w(x,t)$	the velocity of the liquid in the point x and in the time t,
$p(x,t)$	the pressure,
$\gamma = \gamma(x,t)$	the mass of the freed air in the unit volume of the liquid,
ρ_0	the density of the liquid,
$c = c(p,\gamma)$	the sound velocity in the liquid and in the liquid containing the air, respectively (given function of $p,\ \gamma$),
$f = f(w)$	the coefficient of the resistance (the friction of the liquid on the wall of the duct),
$g(\gamma,p)$	$= \begin{cases} K_u\big((\bar\gamma-\gamma)/K_H - p\big), & \text{if} \quad (\bar\gamma-\gamma)/K_H \geq p, \\ K_r\big((\bar\gamma-\gamma)/K_H - p\big) & \text{if} \quad (\bar\gamma-\gamma)/K_H < p, \end{cases}$
$K_u,\ K_r$	the constants characterizing the proportionality of the velocity of loosening, and dissolution on the pressure gradient, respectively,
K_H	the coefficient of absorption,
$\bar\gamma$	the total mass of the air in the unit volume,
$w_0,\ p_0,\ \gamma_0$	the initial distribution of the velocity, the mass, and the pressure of the loosened air in unit volume, respectively,
$C = C(p,\gamma)$	the hydraulic capacity (the given function of $p,\ \gamma$),
H	the throw of the valve,
$Q_V = Q_V(p,H)$	the flow through the valve (the given function of $p,\ H$),
S_0	the cross-section of the duct,
φ	the acting facing of the valve,
h	the flow rate caused by the hydrogenerator at the end of the duct,
$H_0,\ H_1$	the initial position, and the velocity of the valve, respectively.

In what follows, we assume that all given functions are sufficiently smooth, and the solution will be sought smooth as well, i.e., continuously differentiable.

In what follows let us also assume for simplicity that instead of (2.9), (2.10) we have elementary condition

$$H(t) = H_0 = \text{const.} \tag{2.11}$$

The special solutions of our problem will be the functions $w,\ p,\ \gamma$ satisfying the equations (2.1), (2.2), and (2.3). The special feature of these solutions is that despite the fact that they do not satisfy all initial and boundary conditions, they express physical characteristics qualitatively analogous to those described by (2.1)-(2.10).

2.1 Stationary solution

Three functions $w = w(x),\ p = p(x),\ \gamma = \gamma(x)$ depending only on the length coordinate x of the pipe, and satisfying equations (2.1), (2.2), and (2.3) are called *stationary solution.* For this case, equations reduce to a simple system of three ordinary differential equations [2]

$$\rho_0^{-1}p' + f(w) = 0, \tag{2.1}$$
$$\rho_0 c^2(p,\gamma)w' = 0, \tag{2.2}$$
$$w\gamma' = g(\gamma, p), \quad x \in (0,l), \tag{2.3}$$

where $p' = dp/dx$ etc. Analogously as in [1] the function $c(p,\gamma)$ will be assumed in the form

$$c(p,\gamma) = \frac{c_1 p^2}{c_2 p^2 + \gamma + c_3}, \tag{2.4}$$

where $c_i > 0,\ i = 1,\ 2,\ 3$ are constants. The physical principles suggest the condition $c(p,\gamma) > 0$. Since the trivial solution with $p = 0$ is not interesting, the equation (2.2) gives us $w' = 0$ and from here we have

$$w = w_0 = \text{constant}. \tag{2.5}$$

Consequently, (2.1) implies

$$p(x) = p_0 - \rho_0 f(w_0)x, \tag{2.6}$$

and for the function γ we obtain the equation

$$\gamma' = \frac{1}{w_0} g\big(\gamma, p_0 - \rho_0 f(w_0)x\big). \tag{2.7}$$

The constants $w_0,\ p_0$ may be chosen arbitrarily. Also the integration of (2.7) gives an additional free integration constant γ_0. From various compatibility reasons we choose

$$w_0 = \lim_{t\to\infty} \frac{1}{t}\int_0^t h(s)\mathrm{d}s. \tag{2.8}$$

assuming that this limit exists.

The number p_0 in (2.6) is then determined from (2.7), i.e., from

$$Q_V(p_0, H_0) - S_0 w_0 = 0 \tag{2.9}$$

supposing that equation (2.9) is solvable with respect to p_0. It remains to determine the function γ from (2.7). This is substantially not difficult but a little bit lengthy computation. After doing that, the steady state problem is completely solved.

3 Oscillatory solutions

Oscillatory solutions are called such solutions of equations (2.1) to (2.3) which do not depend on the space variable x. Thus, $w = w(t),\ p = p(t),\ \gamma = \gamma(t)$. In this case, the system (2.1) to (2.3) may be write as follows:

$$\begin{aligned} \dot{w} + f(w) &= 0, && (3.1) \\ \dot{p} &= 0 && (3.2) \\ \dot{\gamma} &= g(\gamma, p), \quad t > 0. && (3.3) \end{aligned}$$

Equation (3.1) is solvable provided that

$$m = \inf_{w \in \mathbb{R}} f'(w) > -\infty. \tag{3.4}$$

Clearly, p is given by

$$p(t) = p_0 = \text{const.}, \tag{3.5}$$

and it remains to determine γ from the equation

$$\dot{\gamma} = g(\gamma, p_0). \tag{3.6}$$

which is a consequence of elementary existence theorem for ODE.

We are free to prescribe the initial conditions

$$w(0) = w_0, \quad \gamma(0) = \gamma_0. \tag{3.7}$$

with given constants w_0, γ_0 which is a little bit far from the required generality.

For example, for the case

$$f(w) = k|w|w. \tag{3.8}$$

we obtain

$$w(t) = \frac{w_0}{1 + k|w_0|t}. \tag{3.9}$$

3.1 Combined solutions

By *combined solution* we mean such solution of equations (2.1) to (2.3) which is neither stationary - nor oscillatory, and for which at least one of the functions $w,\ p,\ \gamma$ depends only on x or t.

In [2] we considered "Anzatz" $w = w(t),\ p = p(x)$. For this case the equations (2.1), (2.2), (2.3) have the form

$$\dot{w} + \frac{1}{\rho_0} p' + f(w) = 0, \tag{3.1}$$

$$\gamma_t + w\gamma_x = g(\gamma, p). \tag{3.2}$$

Then we obtain

$$p'(x) = \text{const.} \tag{3.3}$$

Instead of (2.7), (2.8), we chose for p the boundary conditions

$$p(0) = p_0, \quad p(l) = p_1. \tag{3.4}$$

Then

$$p(x) = p_0 + \frac{p_1 - p_0}{l} x, \tag{3.5}$$

as it follows from (3.3). Consequently, (3.1) gives us

$$\dot{w} + f(w) = \frac{p_1 - p_0}{\rho_0 l} \tag{3.6}$$

for the function w. If we suppose (3.4) and supply the initial condition

$$w(0) = w_0 \tag{3.7}$$

we know that there exists a global solution $w(t)$ of the problem (3.6), (3.7). If we know such solution we can find out the function γ from the equation

$$\gamma_t + w(t)\gamma_x = g\big(\gamma, p_0 + \frac{p_1 - p_0}{l} x\big) \tag{3.8}$$

by the method of characteristics. We prescribe the initial condition

$$\gamma(x, 0) = \gamma_0(x). \tag{3.9}$$

In coincidence with the theory of hyperbolic equation we prescribe boundery conditions accordingly. For example, if $w(t) > 0$, we must prescribe

$$\gamma(0, t) = \gamma^0(t), \tag{3.10}$$

If $w(t) < 0$ it is necessary to prescribe the out-flow condition

$$\gamma(l, t) = \gamma^1(t). \tag{3.11}$$

In [2] we have completely resolved the case

$$\begin{aligned} p_1 > p_0,\ w < 0,\ \frac{p_0 - p_1}{\rho_0 l} - f(w_0) < 0, \\ f(-\xi) = -f(\xi),\ f'(\xi) \geq 0, \quad \xi \in \mathbb{R}. \end{aligned} \tag{3.12}$$

Physical interpretation of these assumptions is given.

Then we determine γ by the method of characteristics. After a tedious but principially clear discussion of a greater number of cases, and under the assumption

$$\gamma_0\Big(x - \int_0^t w(s)\mathrm{d}s\Big) < \bar{\gamma} - K_H\Big(p_0 + \frac{p_1 - p_0}{l}\big(x + \int_t^0 w(s)\mathrm{d}s\big)\Big), \tag{3.13}$$

we get

$$\begin{aligned}
\gamma(x,t) &= \phi(t) = \varphi(t; x, t) - y(t) \\
&= \bar{\gamma} - K_H\Big(p_0 + \frac{p_1 - p_0}{x}\Big) \\
&\quad - \exp\Big(-\frac{K_u}{K_H}t\Big)\Big(\bar{\gamma} - K_H\Big(p_0 + \frac{p_1 - p_0}{l}\Big(x + \int_t^0 w(s)\,\mathrm{d}s\Big)\Big) \\
&\quad - \gamma_0\Big(x - \int_0^t w(s)\,\mathrm{d}s\Big)\Big) - K_H\frac{p_0 - p_1}{l}\int_0^t \exp\Big(-\frac{K_u}{K_H}(t-s)\Big)w(s)\mathrm{d}s.
\end{aligned} \tag{3.14}$$

So, we are able to compute $\gamma(x,t)$ in terms of $w(t)$.

On the other hand, for quite frequent physically justified choice $f(w) = k|w|w$ we are able to solve the problem explicitly. The result is

$$w(t) = -\frac{\alpha_0}{k}\left(1 - \Big(1 - \frac{k^2}{\alpha_0^2}w_0^2\Big)\exp\Big(-\frac{\alpha_0 t}{2}\Big)\right)^{\frac{1}{2}},$$

where

$$\alpha_0 = \Big(\frac{k(p_1 - p_0)}{\rho_0 l}\Big)^{\frac{1}{2}}.$$

Now it is possible to express the whole combined solution via quadratures.

Finally, if we assume $f \equiv 0$, $w = w_1 x + w_0$, $p = p(t)$, and $\gamma = \gamma(t)$ where w_0, w_1 are constants, then the system reduces to two ODE's, namely,

$$\begin{aligned}
\dot{p}(t) &= \rho_0 c^2(p(t), \gamma(t)), \\
\dot{\gamma}(t) &= g(p(t), \gamma(t)), \quad t > 0,
\end{aligned}$$

with the initial conditions

$$\begin{aligned}
p(0) &= p_0 = \text{const.} \\
\gamma(0) &= \gamma_0 = \text{const.}
\end{aligned}$$

3.2 Linearized system

Let us start with the general description of the linearization principle which we use for the system (2.1)-(2.9). Let us consider the model equation (system)

$$u_t + F(u)_x = Z(u),\ u(t) \in B_1,\ Z(u) \in B_2, \tag{3.15}$$

where B_j are Banach spaces of functions dependent of x and F is a sufficiently smooth mapping from B_1 into B_2. Assume that we have a steady state u_s satisfying the equation

$$F(u_s)_x = Z(u_s). \tag{3.16}$$

By subtraction we get for $\overline{u} := u - u_s$ the equation

$$\overline{u}_t + F(u_s + \overline{u})_x - F(u_s)_x = Z(u) - Z(u_s). \tag{3.17}$$

Now we have

$$F(u_s + \overline{u})_x - F(u_s)_x = (F'(u_s)\overline{u})_x + Z'(u_s)\overline{u} + \text{ higher order terms.} \tag{3.18}$$

Here $F'(u_s)$ and $Z'(u_s)$ denote the Fréchet derivatives of F and Z at the point u_s, respectively. By the *linearized equation* at the point u_s corresponding to equation (2.1) we call the equation

$$\overline{u}_t + (F'(u_s)\overline{u})_x = Z'(u_s)\overline{u}. \tag{3.19}$$

Applying the linearization procedure to the system (2.1)-(2.9) and putting $u = (w, p, \gamma)^{\mathrm{T}}$, $\overline{w} = w - w_s$, $\overline{p} = p - p_s$, $\overline{\gamma} = \gamma - \gamma_s$, and $Z(u) = (0, 0, g(p, \gamma))$, we obtain the following system:

$$\begin{aligned} \overline{w}_t + \rho_0^{-1}\overline{p}_x + f'(w_s)\overline{w} &= 0 \\ \overline{p}_t + \rho_0 w_{sx}\frac{\partial c^2}{\partial p}(p_s,\gamma_s)\overline{p} + \rho_0 w_{sx}\frac{\partial c^2}{\partial \gamma}(p_s,\gamma_s)\overline{\gamma} + \rho_0 c^2(p_s,\gamma_s)\overline{w}_x &= 0 \\ \overline{\gamma}_t + \gamma_{sx}\overline{w} + w_s\overline{\gamma}_x = \frac{\partial g}{\partial \gamma}(\gamma_s,p_s)\overline{\gamma} + \frac{\partial g}{\partial p}(\gamma_s,p_s)\overline{p}&. \end{aligned} \tag{3.20}$$

The system (3.20) is a first order linear system of the type

$$u_t + A(x)u_x + B(x)u = G, \tag{3.21}$$

where $u = u(x,t) = (\overline{w}(x,t), \overline{p}(x,t), \overline{\gamma}(x,t))^{\mathrm{T}}$ is an unknown threedimensional vector, and $A(x)$ and $B(x)$ are given 3×3 matrices and G a given threedimensional vector, namely,

$$A(x) = \begin{pmatrix} 0 & \rho_0^{-1} & 0 \\ \rho_0 c^2(p_s,\gamma_s) & 0 & 0 \\ 0 & 0 & w_s \end{pmatrix}, \tag{3.22}$$

$$B(x) = \begin{pmatrix} f'(w_s) & 0 & 0 \\ 0 & \frac{\partial c^2}{\partial p}(p_s,\gamma_s) & \rho_0 w_{sx}\frac{\partial c^2}{\partial \gamma}(p_s,\gamma_s) \\ \gamma_{sx} & -\frac{\partial g}{\partial p}(\gamma_s,p_s) & -\frac{\partial g}{\partial \gamma}(\gamma_s,p_s) \end{pmatrix}, \tag{3.23}$$

$$G(x) = (0, 0, g(p_s(x), \gamma_s(x)))^{\mathrm{T}}. \tag{3.24}$$

Comming out from the theory of first order partial differential equations, we are interested in the properties of the matrix $A(x)$. Solving equation

$$\det(\lambda I - A(x)) = 0,$$

where I is the identical matrix, we get eigenvalues of the matrix $A(x)$:

$$\lambda_1(x) = c(p_s(x), \gamma_s(x)), \quad \lambda_2(x) = -c(p_s(x), \gamma_s(x)), \quad \lambda_3(x) = w_s(x). \tag{3.25}$$

Let us impose initial conditions to the system (3.20):

$$\overline{w}(x,0) = \overline{w}_0(x), \quad \overline{p}(x,0) = \overline{p}_0(x), \quad \overline{\gamma}(x,0) = \overline{\gamma}(x), \quad x \in [0,l]. \tag{3.26}$$

It remains to input linearized boundary condition corresponding to the condition (2.7), which in this particular case is of the form

$$C(p(0,t), \gamma(0,t)) + Q_V(p(0,t), H_0) - S_0 w(0,t) = 0. \tag{3.27}$$

By an easy manipulation in the spirit of preceeding linearization proceedures we arrive at the following *linearized boundary condition* derived from (3.27):

$$\begin{aligned} &\left(\frac{\partial C}{\partial p}(p_s(0), \gamma_s(0)) + \frac{\partial Q_v(p_s(0), H_0)}{\partial p}\right)(p(0,t) - p_s(0)) + \\ &\quad \left(\frac{\partial C}{\partial \gamma}(p_s(0), \gamma_0(0)) - \frac{\partial C}{\partial \gamma}(p_s(0), \gamma_0(0)\right)(\gamma(0,t) - \gamma_0(0)) \\ &-S_0(w(0,t) - w_s(0)) = S_0 w_s(0) - C(p_s(0), \gamma_s(0)) - Q_V(p_s(0), \gamma_s(0)). \end{aligned} \tag{3.28}$$

In terms of $\overline{w}, \overline{p}, \overline{\gamma}$ condition (3.28) reads

$$\begin{aligned} &\left(\frac{\partial C}{\partial p}(p_s(0), \gamma_s(0)) + \frac{\partial Q_v(p_s(0), H_0)}{\partial p}\right)\overline{p}(0,t) \\ &+\left(\frac{\partial C}{\partial \gamma}(p_s(0), \gamma_0(0)) - \frac{\partial C}{\partial \gamma}(p_s(0), \gamma_0(0)\right)\overline{\gamma}(0,t) \\ &-S_0\overline{w}(0,t) = S_0 w_s(0) - C(p_s(0), \gamma_s(0)) - Q_V(p_s(0), \gamma_s(0)). \end{aligned} \tag{3.29}$$

We intend to adapt some procedures known from the theory of linear hyperbolic systems to the problem defined by (3.20), or in more concise equivalent form (3.21), with initial conditions (3.26) and boundary condition (3.28). To this end we employ the left eigenvectors

$$\ell^i(x) = (\ell_1^i(x), \ell_2^i(x), \ell_3^i(x)), \quad i = 1,2,3, \tag{3.30}$$

corresponding to the eigenvalues $\lambda^i, i = 1,2,3$ of the matrix $A(x)$. These eigenvectors are computed from the equations

$$(\ell_1^i(x), \ell_2^i(x), \ell_3^i(x))(\lambda_i(x)I - A(x)) = 0. \tag{3.31}$$

Elementary algebra leads to the result

$$\begin{aligned} \ell_1(x) = (c(p_s(x),\gamma_s(x)),\rho_0^{-1},0), \quad \text{if } w_s(x) \neq c(p_s(x),\gamma_s(x)), \\ \ell_2(x) = (-c(p_s(x),\gamma_s(x)),\rho_0^{-1},0) \quad \text{if } w_s(x) \neq -c(p_s(x),\gamma_s(x)), \\ \ell_3(x) = (0,0,1) \quad \text{if } w_s(x) \neq \pm c(p_s(x),\gamma_s(x)). \end{aligned} \tag{3.32}$$

Denote for short

$$c_s = c_s(x) = c_s(p_s(x),\gamma_s(x)). \tag{3.33}$$

Multiply the system (3.21) by $\ell_i, i = 1,2$, respectively. Then we get

$$\ell_i u_t + \lambda_i \ell_i \cdot u_x + \ell_i \cdot (B \cdot u) = \ell \cdot G. \quad i = 1,2,3. \tag{3.34}$$

Substituting

$$v = L \cdot u, \tag{3.35}$$

where the matrix L is given by

$$L = (\ell_{ij})_{i,j=1}^3, \quad \text{with } \ell_{ij} = \ell_i^j, \tag{3.36}$$

we obtain the system (3.34) in the form

$$(v_i)_t + \lambda_i (v_i)_x + \ell_i \cdot ((I+B)\cdot(L^{-1}v)) = \ell_i \cdot G, \quad i = 1,2,3, \tag{3.37}$$

which in vector notation reads

$$v_t + \Lambda v_x + L\cdot(I+B)\cdot(L^{-1}v)) = L \cdot G, \tag{3.38}$$

denoting by Λ the diagonal matrix given by

$$\Lambda = (\lambda_{ij})_{i,j=1}^3, \quad \text{where } \lambda_{ij} = \lambda_i\delta_{ij},\ i,j = 1,2,3. \tag{3.39}$$

Denote

$$C = L\cdot(I+B)\cdot L^{-1} \quad \text{and} \quad F = L\cdot G. \tag{3.40}$$

Then system (3.38) reads

$$v_t + \Lambda v_x + Cv = F. \tag{3.41}$$

Now we are in position to use comfortably the method of characteristics. Define functions $\xi_i = \xi_i(x,t;\tau)$ as solutions of the problems

$$\frac{d\xi_i(x,t;\tau)}{d\tau} = \lambda_i(\xi_i(x,t;\tau)), \quad x \in [0,\ell),\ t \geq 0,\ \tau \geq 0, \quad \xi_i(x,t;t) = x. \tag{3.42}$$

Clearly, by our assumptions and the theory od ODE's the problem (3.41) has a unique solution for each (x,t).

Then the functions

$$z_i(x,t;\tau) = v_i(x + \lambda_i(\xi_i(x,t;\tau))(\tau - t), \tau), \quad i = 1,2,3, \tag{3.43}$$

satisfy the relations

$$\begin{aligned} \frac{dz_i(x,t;\tau)}{d\tau} &= \frac{d}{d\tau} v_i(x + \lambda_i(\xi_i(x,t;\tau))(\tau - t), \tau) \\ = \frac{\partial v_i}{\partial \tau}(x + \lambda_i(\xi_i(x,t;\tau))(\tau - t), \tau) &+ \frac{\partial v_i}{\partial y}(x + \lambda_i(\xi_i(x,t;\tau))(\tau - t), \tau)) \\ \times(\lambda_i(\xi_i(x,t;\tau)) &+ (\tau - t)\frac{d\lambda_i}{dy}(\xi_i(x,t;\tau)\lambda_i(\xi_i(x,t;\tau))) \\ &= F_i(\xi_i(x,t;\tau)) - c_{ij}(\xi_i(x,t;\tau)) \cdot z_j(\xi_i(x,t;\tau), \tau)) \end{aligned} \tag{3.44}$$

accepting the Einstein summation convention.
Finally, we arrive at the system

$$\begin{aligned} \frac{dz_i(x,t;\tau)}{d\tau} &= F_i(\xi_i(x,t;\tau)) - c_{ij}(\xi_i(x,t;\tau)) \cdot z_j(\xi_i(x,t;\tau)), \\ \frac{d\xi_i(x,t;\tau)}{d\tau} &= \lambda_i(\xi_i(x,t;\tau)), \\ \xi_i(x,t;t) &= x, \end{aligned} \tag{3.45}$$

with

$$\begin{aligned} \lambda_{1,2} = \pm c(p_s(x), \gamma_s(x)) &= \pm \frac{c_1 p_s(x)^2}{c_2 p_s(x)^2 + \gamma_s(x) + c_3}, \\ \lambda_3(x) &= w_s(x). \end{aligned} \tag{3.46}$$

This implies that the explicit form of the system (3.42) is

$$\begin{aligned} \frac{d\xi_i^{1,2}}{d\tau}(x,t;\tau) &= \pm \frac{c_1 p_s(\xi_i^{1,2}(x,t;\tau)^2}{c_2 p_s(\xi_i^{1,2}(x,t;\tau))^2 + \gamma_s(\xi_i^{1,2}(x,t;\tau)) + c_3,} \\ \frac{d\xi_i^3}{d\tau}(x,t;\tau) &= w(\xi_i(x,t;\tau)). \end{aligned} \tag{3.47}$$

Since $w_s \equiv w_0 = \text{const}$, we find

$$\xi_i^3(x,t;\tau) = w_0(\tau - t) + x. \tag{3.48}$$

As far as the solutions of equations (3.47) are concerned, we know, that the solutions $\xi_i^{1,2}$ exist. We can decompose the set $\{\tau \in (0,\infty)\}$ into components where g is given by one prescription and by the other on the rest. On these intervals the solution is given by one respective formula which inserted into (3.47) yields equation given by one analytical expression.

Let us express the equations (3.47) in a more explicit form. To simplify notation, write ξ instead of $\xi^{1,2}$, and K instead of K_u or K_H, respectively. Then in the respective interval γ satisfies the equation

$$\gamma' = g(\gamma, p) = g(\gamma, p_0\xi + p_1) = K(\frac{\overline{\gamma} - \gamma}{K_H} - p_0\xi - p_1). \tag{3.49}$$

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