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Summary: We consider a "non-relativistic" limit in a model of radiative flow where matter is described by a viscous heat-conducting system coupled to radiation through the radiative transfer equation. We prove the convergence of this coupled system toward a compressible Navier-Stokes-Fourier system coupled to a stationary radiative transfer equation.

1 Introduction

We revisit a model of radiation hydrodynamics introduced in [8] where various singular regimes were studied (low Mach number limit, equilibrium and non-equilibrium diffusion). In the present paper we consider the "non-relativistic" limit for the radiation where the velocity of light c goes formally to infinity. The motion of the matter is described by compressible fluid mechanics giving the evolution of the mass density $\varrho = \varrho(t,x)$, the velocity field $\vec{u} = \vec{u}(t,x)$, and the temperature $\vartheta = \vartheta(t,x)$ as functions of the time t and the spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation is incorporated in a unique function: the radiative intensity $I = I(t,x,\vec{\omega},\nu)$, depending on the direction $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$. The evolution of I is described by the radiative transfer equation: a linear transport equation with a (non linear) source term and the fluid-radiation interaction appears through radiative sources in the momentum and energy equations.The system under study reads as follows:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathbb{S} - \vec{S}_F \quad \text{in } (0,T)\times\Omega, \tag{1.2}$$

$$\partial_t\left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta)\right) + \mathrm{div}_x\left(\left(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e(\varrho,\vartheta) + p\right)\vec{u} + \vec{q} - \mathbb{S}\vec{u}\right) = -S_E \quad \text{in } (0,T)\times\Omega, \tag{1.3}$$

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \quad \text{in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.4}$$

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The symbol $p = p(\varrho, \vartheta)$ denotes the thermodynamic pressure and $e = e(\varrho, \vartheta)$ is the specific internal energy, related through Maxwell's equation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho, \vartheta) - \vartheta \frac{\partial p}{\partial \vartheta}\right). \tag{1.5}$$

In (1.2) $\mathbb{S}$ is the viscous stress tensor given by $\mathbb{S} = \mu\left(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3}\mathrm{div}_x \vec{u}\right) + \eta\, \mathrm{div}_x \vec{u}\, \mathbb{I}$, where the viscosity coefficients $\mu = \mu(\vartheta) > 0$ and $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature. Similarly in (1.3) $\vec{q}$ is the heat flux given by Fourier's law $\vec{q} = -\kappa \nabla_x \vartheta$, with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$.

We suppose that S is given by

$$\begin{aligned} S &= \sigma_a \Big[B(\nu, \vec{\omega}, \vec{u}, \vartheta) - I(t, x, \nu, \vec{\omega})\Big] \\ &\quad + \sigma_s \left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I(t, x, \nu, \vec{\omega}')\, d\vec{\omega}' - I(t, x, \nu, \vec{\omega})\right) =: S_{a,e} + S_s. \end{aligned} \tag{1.6}$$

In the right-hand side the first term is the emission-absorption contribution where $\sigma_a > 0$ is the absorption coefficient and B is a perturbation of the equilibrium Planck's function introduced in [8] and given by

$$B(\nu, \vec{\omega}, \vec{u}, \vartheta) = \frac{2h}{c^2} \frac{\nu^3}{e^{\frac{h\nu}{k\vartheta}\left(1 - \alpha \frac{\vec{\omega}\cdot\vec{u}}{c}\right)} - 1}, \tag{1.7}$$

where h is the Planck's constant, k is the Boltzmann's constant and $0 \leq \alpha(\vartheta) \leq 1$ is a smooth function pecised below.

The second term in S is the scattering contribution where $\sigma_s > 0$ is the scattering coefficient and in the right-hand sides of (1.2) and (1.3) appear the coupling sources.

$$\vec{S}_F(t, x) = \frac{1}{c}\int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} S \,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu, \quad S_E(t, x) = \int_0^\infty \int_{\mathcal{S}^2} S \,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu.$$

We first suppose that the transport coefficients are smooth functions satisfying $\sigma_a(\vartheta, \vec{u}) = \chi(|\vec{u}|)\tilde{\sigma}_a(\vartheta) \geq 0$ and $\sigma_s(\vartheta) \geq 0$ and that both depend neither on angular variable (1.1 - 1.4) (isotropy of radiation), nor on frequency (the so called "grey" hypothesis), and the function χ appearing in the emission-absorption coefficient is a C^∞ cut-off satisfying

$$\chi(s) = \begin{cases} 1 & \text{if } s \leq c, \\ 0 & \text{if } s \geq c + \beta, \end{cases}$$

for an arbitrary $\beta > 0$, taking into account the singularity of B. More restrictions on properties of these constitutive quantities will be imposed in Section 2.1 below.

Finally system (1.1 - 1.4) is supplemented with the boundary conditions:

$$\vec{u}|_{\partial\Omega} = 0, \ \vec{q}\cdot\vec{n}|_{\partial\Omega} = 0, \tag{1.8}$$

$$I(t, x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega}\cdot\vec{n} \leq 0, \tag{1.9}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$, and initial conditions

$$(\varrho(t,x),\ \vec{u}(t,x),\ \vartheta(t,x),\ I(t,x,\omega,\nu))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x),\ I^0(x,\vec{\omega},\nu)\right), \tag{1.10}$$

for any $x \in \Omega$, $\vec{\omega} \in \mathcal{S}^2$, $\nu \in \mathbb{R}_+$.

The fully relativistic version of system (1.1 - 1.9) has been introduced by Pomraning [14] and Mihalas and Weibel-Mihalas [13] and investigated by Lowrie, Morel and Hittinger [12] and Buet and Desprès [3]. A global existence result was also proved in [4] under some cut-off hypotheses on transport coefficients in the time-dependent case and in [11] in the stationary case and various singular limits of this system have been investigated in [8]. In the following, we are interested in the limit $c \to \infty$ leading to a compressible limit. As emphasized in [8] compressible singular limits can be dealt with by using relative entropy inequalities. Let us mention that, as shown in [6] and [7], this difficulty disappears in simplified cases (see the model of Teleaga, Seaïd, Gasser, Klar and Struckmeier [15]), where the radiative momentum source is absent in the right hand side of (1.9).

The paper is organized as follows. In Section 2.1, we list the principal hypotheses imposed on constitutive relations and state the existence result for our model. In Section 3 we study the "non-relativistic" limit.

2 Hypotheses and existence result

We suppose that the pressure satisfies

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ a > 0, \tag{2.1}$$

where $P : [0,\infty) \to [0,\infty)$ is a given function with the following properties:

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(Z) > 0, \text{ for all } Z \geq 0, \tag{2.2}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.3}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.4}$$

After Maxwell's equation (1.5), the specific internal energy e is

$$e(\varrho,\vartheta) = \frac{3}{2}\left(\frac{\vartheta^{5/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{2.5}$$

and the specific fluid entropy reads

$$s(\varrho,\vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho} \text{ with } M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0. \tag{2.6}$$

Coefficients μ, η, and κ are continuously differentiable functions of temperature such that

$$0 < c_1(1+\vartheta) \leq \mu(\vartheta),\ \mu'(\vartheta) < c_2,\ 0 \leq \eta(\vartheta) \leq c(1+\vartheta), \tag{2.7}$$

$$0 < c_1(1+\vartheta^3) \leq \kappa(\vartheta) \leq c_2(1+\vartheta^3) \tag{2.8}$$

for any $\vartheta \geq 0$. We also assume that σ_a and σ_s are smooth functions such that

$$0 \leq \sigma_a(\vartheta, \vec{u}),\ \sigma_s(\vartheta) \leq c_1,\ \sigma_a(\vartheta, \vec{u}) B(\nu, \vec{\omega}, \vec{u}, \vartheta) \leq c_2, \tag{2.9}$$

$$\sigma_a(\vartheta, \vec{u}) B(\nu, \vec{\omega}, \vec{u}, \vartheta) \leq h(\nu),\ h \in L^1(0, \infty), \tag{2.10}$$

where $c_{1,2,3}$ are positive constants.

In the weak formulation of the Navier-Stokes-Fourier system the equation of continuity (1.1) is replaced by its renormalized version

$$\begin{aligned}\int_0^T \int_\Omega ((\varrho + b(\varrho))\partial_t \varphi + (\varrho + b(\varrho))\vec{u} \cdot \nabla_x \varphi + (b(\varrho) - b'(\varrho)\varrho) \mathrm{div}_x \vec{u} \varphi) \, \mathrm{d}x \, \mathrm{d}t \\ = - \int_\Omega (\varrho_0 + b(\varrho_0))\varphi(0, \cdot) \, \mathrm{d}x\end{aligned} \tag{2.11}$$

satisfied for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, and any $b \in C^\infty[0, \infty)$, $b' \in C_c^\infty[0, \infty)$. Similarly, the momentum equation (1.2) is replaced by

$$\begin{aligned}&\int_0^T \int_\Omega (\varrho \vec{u} \cdot \partial_t \vec{\varphi} + \varrho \vec{u} \otimes \vec{u} : \nabla_x \vec{\varphi} + p \mathrm{div}_x \vec{\varphi}) \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^T \int_\Omega \mathbb{S} : \nabla_x \vec{\varphi} \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \vec{S}_F \vec{\varphi} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\varrho \vec{u})_0 \cdot \vec{\varphi}(0, \cdot) \, \mathrm{d}x,\end{aligned} \tag{2.12}$$

for any $\vec{\varphi} \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$. As usual we require that $\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3))$ which contains the no-slip boundary condition (1.8). As usual [9] we replace (1.3) by the internal energy equation

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \vec{u}) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \mathrm{div}_x \vec{u} - S_E + \vec{S}_F \cdot \vec{u}, \tag{2.13}$$

and dividing (2.13) by ϑ we rewrite (2.13) as an entropy equation

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) - \frac{S_E}{\vartheta} + \frac{\vec{S}_F \cdot \vec{u}}{\vartheta} =: \varsigma, \tag{2.14}$$

where the first term of the right hand side $\varsigma_m := \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$ is the (positive) matter entropy production. In order to identify the second term in the right hand side of (2.14), let us recall [1] the formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] d\vec{\omega} d\nu, \tag{2.15}$$

where $n = n(I) = \frac{c^2 I}{2h\alpha^3\nu^3}$ is the occupation number. Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] \vec{\omega} \, d\vec{\omega} d\nu, \tag{2.16}$$

and using the radiative transfer equation, we get the equation

$$\partial_t s^R + \mathrm{div}_x \vec{q}^R = -\frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1} \, S \, d\vec{\omega} d\nu =: \varsigma^R. \tag{2.17}$$

Checking the identity $\log \frac{n(B)}{n(B)+1} = -\frac{h\nu}{k\vartheta}\left(1 - \alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\right)$ where B is the Planck's function, using the definition of S and taking into account that $\vec{S}_F = (\sigma_a+\sigma_s)\int_0^\infty \int_{\mathcal{S}^2} \omega I \, d\vec{\omega} I \, d\nu$, the right-hand side of (2.17) rewrites

$$\begin{aligned}
\varsigma^R = -\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\Bigg\{ &\left[\log \frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I) \\
&+ \left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I}-I)\Bigg\} \, d\vec{\omega} d\nu \\
&+ \frac{1}{\vartheta}\int_0^\infty \int_{\mathcal{S}^2}\left(1 - \alpha\frac{\vec{\omega}\cdot\vec{u}}{c}\right) S \, d\vec{\omega} d\nu - \frac{\alpha\sigma_s}{\sigma_a+\sigma_s}\frac{\vec{S}_F\cdot\vec{u}}{\vartheta}.
\end{aligned}$$

Choosing now

$$\alpha = \frac{\sigma_a+\sigma_s}{\sigma_a+2\sigma_s}, \tag{2.18}$$

we get

$$\begin{aligned}
\varsigma^R = &-\frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I) \, d\vec{\omega} d\nu \\
&- \frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I}-I) \, d\vec{\omega} d\nu + \frac{1}{\vartheta} S_E - \frac{\vec{S}_F\cdot\vec{u}}{\vartheta}.
\end{aligned} \tag{2.19}$$

From (2.14), (2.17) and (2.19) we obtain finally

$$\begin{aligned}
&\partial_t\left(\varrho s + s^R\right) + \mathrm{div}_x\left(\varrho s\vec{u} + \vec{q}^R\right) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) \\
&= \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right) - \frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I) \, d\vec{\omega} d\nu \\
&- \frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I}-I) \, d\vec{\omega} d\nu,
\end{aligned} \tag{2.20}$$

and equation (2.14) is replaced in the weak formulation by the inequality

$$\begin{aligned}
&\int_0^T \int_\Omega \left([\varrho s + s^R]\partial_t\varphi + \varrho s\vec{u}\cdot\nabla_x\varphi + [\frac{\vec{q}}{\vartheta} + \vec{q}^R]\cdot\nabla_x\varphi\right) \mathrm{d}x \, \mathrm{d}t \\
&\quad \le -\int_\Omega (\varrho s + s^R)_0\varphi(0,\cdot) \, \mathrm{d}x - \int_0^T \int_\Omega \frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\varphi \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \frac{k}{h}\int_0^T \int_\Omega \Bigg[\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I) \, d\vec{\omega} d\nu \, \mathrm{d}x \\
&\quad + \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I}-I) \, d\vec{\omega} d\nu\Bigg]\varphi \mathrm{d}x \, \mathrm{d}t
\end{aligned} \tag{2.21}$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$, where the sign of all the terms in the right hand side may be controlled.

Following [9] (2.11), (2.12), (2.21) is supplemented with the total energy balance

$$\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) + E^R \right)(\tau, \cdot) \, \mathrm{d}x + \int_0^\tau \int_{\Gamma_+} \int_0^\infty \vec{\omega} \cdot \vec{n} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega} \, \mathrm{d}S_x \, \mathrm{d}t,$$
$$= \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho \vec{u})_0|^2 + (\varrho e)_0 + E_{R,0} \right) \, \mathrm{d}x, \tag{2.22}$$

where $\Gamma_+ = \{(x, \vec{\omega}) \in \partial\Omega \times \mathcal{S}^2 \, : \, \vec{\omega} \cdot \vec{n}_x > 0\}$.

Finally for later purposes, we define the radiative energy

$$E^R(t,x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu, \tag{2.23}$$

with $E_{R,0} = \int_{\mathcal{S}^2} \int_0^\infty I_0(\cdot, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu$, the radiative momentum

$$\vec{F}^R(t,x) = \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu, \tag{2.24}$$

and the radiative tensor

$$\mathbb{P}^R(t,x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} \otimes \vec{\omega} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu. \tag{2.25}$$

Definition 2.1 *We say that $\varrho, \vec{u}, \vartheta, I$ is a weak solution of problem (1.1 - 1.9) if*

$$\varrho \geq 0, \ \vartheta > 0 \textit{ for a.a. } (t,x) \times \Omega, \ I \geq 0 \textit{ a.a. in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0, \infty),$$

$$\varrho \in L^\infty(0,T; L^{5/3}(\Omega)), \ \vartheta \in L^\infty(0,T; L^4(\Omega)),$$

$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \ \vartheta \in L^2(0,T; W^{1,2}(\Omega)),$$

$$I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)), \ I \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty)),$$

and if ϱ, $\vec{u}$, ϑ, I satisfy the integral identities (2.11), (2.12), (2.21), (2.22), together with the transport equation (1.4).

The existence result reads now (see [8] for a proof)

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1-2.6), that B satisfies (1.7) and (2.18) and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (2.7 - 2.10). Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon > 0}$ be a family of weak solutions to problem (1.1 - 1.9) in the sense of*

Definition 2.1.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega), \tag{2.26}$$

$$\int_\Omega (\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon) + E_{R,\varepsilon})(0,\cdot)\,\mathrm{d}x$$
$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}}|(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon}\right)\,\mathrm{d}x \le E_0, \tag{2.27}$$
$$\int_\Omega [\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) + s^R(I_\varepsilon)](0,\cdot)\,\mathrm{d}x \equiv \int_\Omega (\varrho s + s^R)_{0,\varepsilon}\,\mathrm{d}x \ge S_0,$$

and

$$0 \le I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \le I_0,\ |I_{0,\varepsilon}(\cdot,\nu)| \le h(\nu) \text{ for a certain } h \in L^1(0,\infty).$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\text{weak}}([0,T];L^{5/3}(\Omega)),$$
$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),$$
$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

at least for suitable subsequences, where $\{\varrho,\vec{u},\vartheta,I\}$ *is a weak solution of problem (1.1 - 1.9).*

3 The non-relativistic limit

In order to get a non dimensional system we perform a scaling, denoting by

$$L_{ref},\ T_{ref},\ U_{ref}, \rho_{ref},\ \vartheta_{ref},\ p_{ref},\ e_{ref},\ \mu_{ref},\ \kappa_{ref},$$

the reference hydrodynamical quantities (length, time, velocity, density, temperature, pressure, energy, viscosity, conductivity) and by $I_{ref},\ \nu_{ref},\ \sigma_{a,ref},\ \sigma_{s,ref}$, the reference radiative quantities. We also assume the compatibility conditions $p_{ref} \equiv \rho_{ref}e_{ref},\ \nu_{ref} = \frac{k_B\vartheta_{ref}}{h},\ I_{ref} = \frac{2h\nu_{ref}^3}{c^2}$ and we denote by $Sr := \frac{L_{ref}}{T_{ref}U_{ref}},\ Ma := \frac{U_{ref}}{\sqrt{\rho_{ref}p_{ref}}},\ Re := \frac{U_{ref}\rho_{ref}L_{ref}}{\mu_{ref}},\ Pe := \frac{U_{ref}p_{ref}L_{ref}}{\vartheta_{ref}\kappa_{ref}},\ \mathcal{C} := \frac{c}{U_{ref}}$, the Strouhal, Mach, Reynolds, Péclet (dimensionless) and "infrarelativistic" numbers corresponding to hydrodynamics, and by $\mathcal{L} := L_{ref}\sigma_{a,ref},\ \mathcal{L}_s := \frac{\sigma_{s,ref}}{\sigma_{a,ref}},\ \mathcal{P} := \frac{2k_B^4\vartheta_{ref}^4}{h^3c^3\ \rho_{ref}e_{ref}}$, various dimensionless numbers corresponding to radiation. Using carets to symbolize renormalized variables we get

$$\hat{S} = \mathcal{L}\hat{\sigma}_a\left(B(\hat{\nu},\vec{\omega},\hat{\vec{u}},\hat{\vartheta}) - \hat{I}\right) + \mathcal{L}\mathcal{L}_s\hat{\sigma}_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2}\hat{I}(\cdot,\vec{\omega})\,\mathrm{d}\vec{\omega} - \hat{I}\right).$$

We have also for the non dimensional α

$$\alpha = \frac{\widehat{\sigma}_a + \mathcal{L}_s\widehat{\sigma}_s}{\widehat{\sigma}_a + 2\mathcal{L}_s\widehat{\sigma}_s}. \tag{3.1}$$

Omitting the carets in the following, we get first the scaled equation for I, in the region $(0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2$

$$\frac{Sr}{\mathcal{C}}\,\partial_t I+\vec{\omega}\cdot\nabla_x I = s = \mathcal{L}\sigma_a\,(B-I)+\mathcal{L}\mathcal{L}_s\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\,\mathrm{d}\vec{\omega}-I\right), \tag{3.2}$$

where we used the same notation B for the dimensionless Planck function $B(\nu,\vec{\omega},\vec{u},\vartheta)=\frac{\nu^3}{e^{\frac{\nu}{\vartheta}(1-\alpha\frac{\vec{\omega}\cdot\vec{u}}{\mathcal{C}})}-1}$. We denote also by $E^R=\int_{\mathcal{S}^2}\int_0^\infty I\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}$, the renormalized radiative energy, by $\vec{F}^R=\int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}I\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}$, the renormalized radiative momentum, by $\mathbb{P}^R=\int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}\otimes\vec{\omega}I\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}$, the renormalized radiative tensor, by $s_E=\int_{\mathcal{S}^2}\int_0^\infty s\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}$, the renormalized radiative energy source, by $\vec{s}_F=\int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}I\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}s$, the renormalized radiative momentum source, by $\vec{s}^R=-\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n-(n+1)\log(n+1)\right]\,d\vec{\omega}d\nu$, the renormalized radiative entropy with $n=n(I)=\frac{I}{\alpha^3\nu^3}$, by $\vec{q}^R=-\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n-(n+1)\log(n+1)\right]\vec{\omega}\,d\vec{\omega}d\nu$, the renormalized radiative entropy flux.

In order to analyze the large c regime we suppose that $\mathcal{C}=O(\varepsilon^{-1})$ and that a small amount of radiation is present so $\mathcal{P}=\varepsilon$. Finally we put $Ma=1$, $Sr=1$, $Pe=1$, $Re=1$, $\mathcal{L}=\mathcal{L}_s=1$ in the previous system.

Taking the first moment of (3.2) with respect to $\vec{\omega}$, we get first equations for E^R and $\vec{S}_F$

$$\varepsilon\partial_t E^R+\mathrm{div}_x\vec{F}^R=s_E, \tag{3.3}$$

$$\varepsilon\partial_t\vec{F}_R+\mathrm{div}_x\mathbb{P}^R=\vec{s}_F, \tag{3.4}$$

then the scaled system reads

$$\varepsilon\,\partial_t I+\vec{\omega}\cdot\nabla_x I=\sigma_a\,(B-I)+\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\,\mathrm{d}\vec{\omega}-I\right), \tag{3.5}$$

$$\partial_t\varrho+\mathrm{div}_x(\varrho\vec{u})=0, \tag{3.6}$$

$$\partial_t\left(\varrho\vec{u}+\varepsilon\vec{F}^R\right)+\mathrm{div}_x\left(\varrho\vec{u}\otimes\vec{u}+\mathbb{P}^R\right)+\nabla_x p-\ \mathrm{div}_x\mathbb{S}=0. \tag{3.7}$$

$$\partial_t\Big(\frac{1}{2}\varrho|\vec{u}|^2+\varrho e+\varepsilon E^R\Big)+\mathrm{div}_x\Big(\Big(\frac{1}{2}\varrho|\vec{u}|^2+\varrho e+p\Big)\vec{u}+\vec{F}^R+\vec{q}-\mathbb{S}\vec{u}\Big)=0, \tag{3.8}$$

$$\begin{aligned}\partial_t\left(\varrho s+\varepsilon s_R\right)+\mathrm{div}_x\left(\varrho\vec{u}s+\vec{q}_R\right)+\ \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right)\geq\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\vec{u}-\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\\
+\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\,d\vec{\omega}d\nu\\
+\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\,d\vec{\omega}d\nu.\end{aligned} \tag{3.9}$$

with

$$\frac{d}{dt}\int_\Omega\left(\varrho\mathcal{E}+\varepsilon E_R\right)\,dx+\int_0^\infty\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}\,I\,d\Gamma_+d\nu=0, \tag{3.10}$$

where $\mathcal{E} = \frac{1}{2}\,|\vec{u}|^2 + e$. In order to compute the limit system, we use the formal expansions

$$\begin{cases} I = I_0 + \varepsilon I_1 + O(\varepsilon^2), \\ \varrho = \varrho_0 + \varepsilon \varrho_1 + O(\varepsilon^2), \\ \vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1 + O(\varepsilon^2), \\ \vartheta = \vartheta_0 + \varepsilon \vartheta_1 + O(\varepsilon^2). \end{cases} \tag{3.11}$$

Introducing the renormalized unperturbed Planck's function $B(\nu, \vartheta_0) = \frac{\nu^3}{e^{\frac{\nu}{\vartheta_0}} - 1}$ and computing in (2.18) the expansion $\alpha = \frac{\sigma_a(\vartheta_0) + \sigma_s(\vartheta_0)}{\sigma_a(\vartheta_0) + 2\sigma_s(\vartheta_0)} = 1 + O(\varepsilon)$, we have in (1.7)

$$B(\nu, \vec{\omega}, \vec{u}, \vartheta) = B(\nu, \vartheta_0) + (\vec{\omega} \cdot \vec{u}_0 \vartheta_0 + \vartheta_1)\, \partial_\vartheta B(\nu, \vartheta_0)\varepsilon + O(\varepsilon^2).$$

The related radiative quantities are

$$\begin{cases} E^R = E_0^R + \varepsilon E_1^R + O(\varepsilon^2), \\ \vec{F}^R = \vec{F}_0^R + \varepsilon \vec{F}_1^R + O(\varepsilon^2), \\ \mathbb{P}^R = \mathbb{P}_0^R + \varepsilon \mathbb{P}_1^R + O(\varepsilon^2). \end{cases} \tag{3.12}$$

Passing formally to the limit, we finally obtain the limit system in $(0, T) \times \Omega$

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_{a0}\,(B_0 - I_0) + \sigma_{s0} \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I_0 \,\mathrm{d}\vec{\omega} - I_0 \right), \tag{3.13}$$

$$\partial_t \varrho_0 + \mathrm{div}_x(\varrho_0 \vec{u}_0) = 0, \tag{3.14}$$

$$\partial_t(\varrho_0 \vec{u}_0) + \mathrm{div}_x(\varrho_0 \vec{u}_0 \otimes \vec{u}_0 + \mathbb{P}_0) + \nabla_x p_0 = \mathrm{div}_x \mathbb{S}_0, \tag{3.15}$$

$$\partial_t \Big(\frac{1}{2} \varrho_0 |\vec{u}_0|^2 + \varrho_0 \mathbf{e}_0 \Big) + \mathrm{div}_x \Big(\Big(\frac{1}{2} \varrho_0 |\vec{u}_0|^2 + \varrho_0 e_0 + p_0 \Big) \vec{u}_0 + \vec{q}_0 - \mathbb{S}_0 \vec{u}_0 \Big) = 0, \tag{3.16}$$

$$\partial_t\,(\varrho_0 s_0) + \mathrm{div}_x\,(\varrho s_0 \vec{u}_0) + \mathrm{div}_x \left(\frac{\vec{q}_0}{\vartheta_0} \right) = \frac{1}{\vartheta_0} \left(\mathbb{S}_0 : \nabla_x \vec{u}_0 - \frac{\vec{q} \cdot \nabla_x \vartheta_0}{\vartheta_0} \right), \tag{3.17}$$

where $p_0 = p(\varrho_0, \vartheta_0)$, $e_0 = e(\varrho_0, \vartheta_0)$, $s_0 = s(\varrho_0, \vartheta_0)$, $\sigma_{a0} = \sigma_a(\vartheta_0)$ and $\sigma_{s0} = \sigma_s(\vartheta_0)$.

We also get boundary conditions

$$\vec{u}_0|_{\partial\Omega} = 0,\ \nabla \vartheta_0 \cdot \vec{n}|_{\partial\Omega} = 0,\ I_0(t, x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega} \cdot \vec{n} \le 0, \tag{3.18}$$

and initial conditions

$$(\varrho_0(x,t),\ \vec{u}_0(x,t),\ \vartheta_0(x,t))|_{t=0} = \left(\varrho^0(x),\ \vec{u}^0(x),\ \vartheta^0(x) \right), \tag{3.19}$$

for any $x \in \Omega$ and with the compatibility conditions

$$\vec{u}_0|_{\partial\Omega, t=0} = 0,\ \nabla \vartheta_0 \cdot \vec{n}|_{\partial\Omega, t=0} = 0,\ I_0(t, x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ t = 0,\ \vec{\omega} \cdot \vec{n} \le 0, \tag{3.20}$$

As expected, this system corresponds to the compressible Navier-Stokes system coupled to the stationary radiative transfer equation. Adapting the time-dependent result of [5] one shows the following existence result for this system.

Theorem 3.1 *Suppose that* $\|\varrho^0\|_{H^2} + \|\vec{u}^0\|_{H^2} + \|\vartheta^0\|_{H^2} < \infty$ *and that* $\inf_\Omega \varrho_0 > 0$ *and* $\inf_\Omega \vartheta_0 > 0$. *Then there exists a positive constant* T_* *such that* $(\varrho, \vec{u}, \vartheta, I)$ *is the unique classical solution to the problem (3.14)-(3.16) with boundary conditions (3.18), initial conditions (3.19) and with the compatibility conditions (3.20) in* $(0,T) \times \Omega$ *for any* $T < T_*$ *such that*

$$\varrho \in H^1(0,T;H^2(\Omega)),\ \inf_\Omega \varrho > 0,$$

$$\vec{u} \in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)),$$

$$\vartheta \in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)),\ \inf_\Omega \vartheta > 0,$$

$$I \in L^\infty(0,T \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+),\ E^R \in L^2(0,T;H^1(\Omega)),$$

$$\partial_t \varrho \in L^\infty(0,T;H^1(\Omega)).$$

3.1 Relative entropy inequality

We first rephrase the existence result of Theorem 2.2 in the rescaled context

Proposition 3.2 *Suppose that the conditions of Theorem 2.2 are satisfied. Then for any* $\varepsilon > 0$ *small enough there exists a weak solution* $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ *to the radiative Navier-Stokes systems (1.1-1.4) for* $(t, x, \vec{\omega}, \nu) \in [0,T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, *with boundary conditions (1.8 - 1.9) and initial conditions* $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$. *More precisely we have*

$$\int_\Omega \varrho_\varepsilon(\tau,\cdot)\phi(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\phi(0,\cdot)\ dx = \int_0^\tau \int_\Omega \varrho_\varepsilon \left(\partial_t \phi + \vec{u}_\varepsilon \cdot \nabla_x \phi\right)\ dx\ dt \quad (3.21)$$

for any $\phi \in C^1([0,T) \times \overline{\Omega})$, *and any* $\tau \in [0,T]$,

$$\int_\Omega \varrho_\varepsilon \vec{u}_\varepsilon(\tau,\cdot) \cdot \vec{\phi}(\tau,\cdot)\ dx - \int_\Omega \varrho_{0,\varepsilon}\vec{u}_{0,\varepsilon} \cdot \vec{\phi}(0,\cdot)\ dx$$

$$= \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon \vec{u}_\varepsilon + \varepsilon \vec{F}^R_\varepsilon\right) \cdot \partial_t \vec{\phi} + \left(\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon + \mathbb{P}^R_\varepsilon\right) : \nabla_x \vec{\phi} + p_\varepsilon \operatorname{div}_x \vec{\phi} - \mathbb{S}_\varepsilon : \nabla_x \vec{\phi} \right) dx\ dt = 0, \quad (3.22)$$

for any $\vec{\phi} \in C^1([0,T) \times \overline{\Omega}; \mathbb{R}^3)$, *and any* $\tau \in [0,T]$, *such that* $\phi \cdot n|_{\partial\Omega} = 0$, *with* $p_\varepsilon = p(\varrho_\varepsilon, \vartheta_\varepsilon)$ *and* $\mathbb{S}_\varepsilon = \mathbb{S}(\vec{u}_\varepsilon, \vartheta_\varepsilon)$,

$$\int_0^\tau \int_\Omega \left(\frac{1}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon\right)\ dx\ dt + \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$
$$= \int_\Omega \left(\frac{1}{2}\ \varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon}\right)\ dx =: \mathcal{E}_0, \quad (3.23)$$

for a.a. $t \in [0,T]$ *with* $\Gamma_+ = \{(x,\vec{\omega}) \in \partial\Omega \times \mathcal{S}^2 \ : \ \vec{\omega}\cdot\vec{n}_x \geq 0\}$ *and with* $e_\varepsilon = e(\varrho_\varepsilon,\vartheta_\varepsilon)$ *and* $E^R_\varepsilon(t,x) = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu)\, d\vec{\omega}\, d\nu$

$$\begin{aligned}\int_0^\tau \int_\Omega &\left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right)\partial_t\varphi + \left(\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon + \vec{q}^R_\varepsilon\right)\cdot\nabla_x\varphi\right)\, dx\, dt \\ &+ \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\, dx\, dt + \left\langle \varsigma^m_\varepsilon + \varsigma^R_\varepsilon; \phi\right\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} \\ &\leq -\int_\Omega \left(\varrho s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon}\right)\varphi(0,\cdot)\, dx + \int_\Omega \left(\varrho s_\varepsilon + \varepsilon s^R_\varepsilon\right))(\tau,\cdot)\varphi(\tau,\cdot)\, dx,\end{aligned} \tag{3.24}$$

where

$$\varsigma^m_\varepsilon = \frac{1}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x\vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\right), \tag{3.25}$$

and

$$\begin{aligned}\varsigma^R_\varepsilon \geq &\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right]\sigma_{a\varepsilon}(B_\varepsilon - I_\varepsilon)\, d\vec{\omega} d\nu \\ &+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right]\sigma_{s\varepsilon}(\tilde{I}_\varepsilon - I_\varepsilon)\, d\vec{\omega} d\nu,\end{aligned} \tag{3.26}$$

for $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$ *and any* $\tau \in [0,T]$, *with* $\varsigma^m_\varepsilon \in \mathcal{M}^+([0,T)\times\overline{\Omega})$ *and* $\varsigma^R_\varepsilon \in \mathcal{M}^+([0,T)\times\overline{\Omega})$, *where* $\mathcal{M}(X)$ *is the set of signed Borel measures on* X *and* $\mathcal{M}^+(X)$ *is the cone of non-negative elements of* $\mathcal{M}(X)$.

Denoting $B_\varepsilon = B(\nu,\vec{\omega},\vec{u}_\varepsilon,\vartheta_\varepsilon)$, $\vec{q}_\varepsilon = \kappa(\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon$, $s_\varepsilon = s(\varrho_\varepsilon, \ \vartheta_\varepsilon)$, $s^R_\varepsilon = s^R(I_\varepsilon)$, $\vec{q}^R_\varepsilon = \vec{q}^R(I_\varepsilon)$, *and* $\tilde{I}_\varepsilon := \frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(t,x,\nu,\vec{\omega})\,\mathrm{d}\vec{\omega}$, *we have finally*

$$\begin{aligned}\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} &\left(\varepsilon\partial_t\psi + \vec{\omega}\cdot\nabla_x\psi\right) I_\varepsilon\, d\vec{\omega}\, d\nu\, dx\, dt \\ &+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[\sigma_{a\varepsilon}\left(B_\varepsilon - I_\varepsilon\right) + \sigma_{s\varepsilon}\left(\tilde{I}_\varepsilon - I_\varepsilon\right)\right]\psi\, d\vec{\omega}\, d\nu\, dx\, dt, \\ &= \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_{0,\varepsilon}\psi(0,x,\vec{\omega},\nu)\, d\vec{\omega}\, d\nu\, dx - \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_\varepsilon\psi(\tau,x,\vec{\omega},\nu)\, d\vec{\omega}\, d\nu\, dx \\ &+ \int_0^\tau \int_{\Gamma_+} \int_0^\infty \vec{\omega}\cdot\vec{n}_x I_\varepsilon\psi\, d\Gamma\, d\nu\, dt,\end{aligned} \tag{3.27}$$

for any $\psi \in C^1([0,T)\times\overline{\Omega}\times\mathcal{S}^2\times\mathbb{R}_+)$ *and any* $\tau \in [0,T]$.

Following now the lines of [8] we give just main lines of proof to introduce a relative entropy inequality satisfied by any weak solution $(\varrho,\vec{u},\vartheta,I)$ of the radiative Navier-Stokes system. Let us consider a set $\{r,\Theta,\vec{U}\}$ of arbitrary smooth functions such that r and Θ are bounded below away from zero and $\vec{U}\Big|_{\partial\Omega} = 0$. We call *ballistic free energy* the thermodynamical potential given by $H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta)$, and *radiative* *ballistic free energy* the potential $H_\Theta^R(I) = E^R(I) - \Theta s^R(I)$. The *relative entropy* is then defined by

$$\mathcal{E}(\varrho,\vartheta|r,\Theta) := H_\Theta(\varrho,\vartheta) - \partial_\rho H_\Theta(r,\Theta)(\varrho - r) - H_\Theta(r,\Theta).$$

One observes that, after thermodynamical stability, $\rho \to H_\Theta(\rho,\Theta)$ is strictly convex and $\theta \to H_\Theta(\rho,\theta)$ attains its global minimum at $\theta = \Theta$.

Testing equation (3.21) with $\phi = \frac{1}{2}\,|\vec{U}|^2$, equation (3.22) with $\phi = \vec{U}$, and equation (3.24) by $\varphi = \Theta$ and combining the resulting identities, we get

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\,\varrho_\varepsilon|\vec{u}_\varepsilon - \vec{U}|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon - (\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon)\Theta\right)(\tau,\cdot)\;dx + \int_0^T \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\;d\Gamma\;d\nu\;dt \\
&+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x\vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon}\right)\mathrm{d}x\;dt \\
&+ \int_0^\tau \int_\Omega \Theta\left\{\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1}\right]\sigma_{a\varepsilon}(B_\varepsilon - I_\varepsilon)\;d\vec{\omega}\;d\nu\right. \\
&+ \left.\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1}\right]\sigma_{s\varepsilon}(\tilde{I}_\varepsilon - I_\varepsilon)\;d\vec{\omega}\;d\nu\right\}\;dx\;dt \\
&\leq \int_\Omega \left(\frac{1}{2}\,\varrho_{0,\varepsilon}|\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \varrho_{0,\varepsilon}e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} - (\varrho_0 s_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon})\Theta(0,\cdot)\right)\;dx \\
&+ \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon\partial_t\vec{U} + \varrho_\varepsilon\vec{u}_\varepsilon\cdot\nabla_x\vec{U}\right)\cdot\left(\vec{U} - \vec{u}_\varepsilon\right) - p_\varepsilon\,\mathrm{div}_x\vec{U} + \mathbb{S}_\varepsilon : \nabla_x\vec{U} - \varepsilon\vec{F}^R_\varepsilon\cdot\partial_t\vec{U} - \mathbb{P}_\varepsilon : \nabla_x\vec{U}\right)\;dx\;dt \\
&- \int_0^\tau \int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\right)\partial_t\Theta + \left(\varrho_\varepsilon s_\varepsilon\vec{u}_\varepsilon + \vec{q}^R_\varepsilon\right)\cdot\nabla_x\Theta\right)\;dx\;dt - \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\Theta\;dx\;dt.
\end{aligned} \tag{3.28}$$

Testing equation (3.21) with $\phi = \partial_\rho H_\Theta(r,\Theta)$, and using (3.5) we get now

$$\varepsilon\;\partial_t I_\varepsilon + \vec{\omega}\cdot\nabla_x(I_\varepsilon - I) = S_\varepsilon - S,$$

where $S = \sigma_{a\varepsilon}\,(B_\varepsilon - I_\varepsilon) + \sigma_{s\varepsilon}\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon\;\mathrm{d}\vec{\omega} - I_\varepsilon\right)$ and $S = \sigma_a\,(B - I) + \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\;\mathrm{d}\vec{\omega} - I\right)$. Multiplying by $I_\varepsilon - I$ and integrating by parts it follows

$$\begin{aligned}
&\frac{1}{2}\,\varepsilon\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}(I_\varepsilon - I)^2(\tau,\cdot)\;\mathrm{d}\vec{\omega}\;d\nu\;dx + \frac{1}{2}\int_0^T\int_0^\infty\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}_x(I_\varepsilon - I)^2(t,x,\vec{\omega},\nu)\;d\Gamma\;d\nu\;dt \\
&= \frac{1}{2}\,\varepsilon\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}(I_\varepsilon - I)^2(0,\cdot)\;d\vec{\omega}\;d\nu\;dx + \int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}(S_\varepsilon - S - \varepsilon\partial_t I)\,(I_\varepsilon - I)\;d\vec{\omega}\;d\nu\;dx\;dt,
\end{aligned} \tag{3.29}$$

where $I_0 := I(0,\cdot)$ is the unique solution of the problem

$$\vec{\omega}\cdot\nabla_x I_0 = S(\Theta(0,\cdot),\vec{U}(0,\cdot)),$$

$$I_0|_{\Gamma_-} = 0.$$

Using previous calculations together with (3.29) we get

$$
\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + H_\Theta(\varrho_\varepsilon, \vartheta_\varepsilon) - H_\Theta(r,\Theta) - \partial_\rho H_\Theta(r,\Theta)(\varrho_\varepsilon - r) \\
&+ \frac{1}{2}\, \varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_\varepsilon - I)^2 \,\mathrm{d}\vec{\omega}\, d\nu + \varepsilon H^R_\Theta(I_\varepsilon) \Big)(\tau,\cdot)\, dx \\
&+ \int_0^T \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x \left[I_\varepsilon + \frac{1}{2}\,\varepsilon (I_\varepsilon - I)^2 \right](t,x,\vec{\omega},\nu)\, d\Gamma\, d\nu\, dt \\
&+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right) \mathrm{d}x\, dt \\
&+ \int_0^\tau \int_\Omega \Theta \Big\{ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right] \sigma_{a\varepsilon}(B_\varepsilon - I_\varepsilon)\, d\vec{\omega}\, d\nu \\
&+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s\varepsilon}(\tilde{I}_\varepsilon - I_\varepsilon)\, d\vec{\omega}\, d\nu \Big\}\, dx\, dt \\
&\le \int_\Omega \frac{1}{2}\, \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2\, dx \\
&+ \int_\Omega \big(H_{\Theta(0,\cdot)}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot)) - \partial_{\varrho_{0,\varepsilon}} H_{\Theta(0,\cdot)}(r(0,\cdot),\Theta(0,\cdot))(\varrho_{0,\varepsilon} - r(0,\cdot)) \\
&+ \varepsilon H^R_{\Theta(0,\cdot)}(I_{0,\varepsilon}) + \frac{1}{2}\, \varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_{0,\varepsilon} - I(0,\cdot))^2 \,\mathrm{d}\vec{\omega}\, d\nu \Big)\, dx \\
&+ \int_0^\tau \int_\Omega \Big(\Big(\varrho_\varepsilon \partial_t \vec{U} + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \vec{U} \Big) \cdot \Big(\vec{U} - \vec{u}_\varepsilon \Big) - p_\varepsilon \,\mathrm{div}_x \vec{U} + \mathbb{S}_\varepsilon : \nabla_x \vec{U} - \varepsilon \vec{F}^R_\varepsilon \cdot \partial_t \vec{U} - \mathbb{P}^R_\varepsilon : \nabla_x \vec{U} \Big)\, dx\, dt \\
&- \int_0^\tau \int_\Omega \left(\varrho_\varepsilon s_\varepsilon \partial_t \Theta + \varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \Theta + \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta \right) dx\, dt \\
&- \int_0^\tau \int_\Omega \big(\varepsilon s^R_\varepsilon \partial_t \Theta + \vec{q}^R_\varepsilon \cdot \nabla_x \Theta \big)\, dx\, dt \\
&- \int_0^\tau \int_\Omega \Big(\varrho_\varepsilon \partial_t \Big(\partial_{\varrho_\varepsilon} H_\Theta(r,\Theta) \Big) + \varrho_\varepsilon \vec{u}_\varepsilon \cdot \nabla_x \Big(\partial_\varrho H_\Theta(r,\Theta) \Big) \Big)\, dx\, dt \\
&+ \int_0^\tau \int_\Omega \partial_t \Big(r \partial_\varrho H_\Theta(r,\Theta) - H_\Theta(r,\Theta) \Big)\, dx\, dt.
\end{aligned}
\tag{3.30}
$$

Using the identity $D\partial_\varrho H_\Theta = -sD\Theta - r\partial_\varrho s D\Theta + \partial^2_{\varrho,\varrho} H_\Theta D\varrho + \partial^2_{\varrho,\vartheta} H_\Theta D\vartheta$, valid for $D = \partial_t$ or $D = \nabla_x$, and the thermodynamical relations: $\partial^2_{\varrho,\varrho} H_\Theta = \frac{1}{r}\, \partial_\varrho p(r,\Theta)$, $r\partial_\varrho s = -\frac{1}{r}\, \partial_\vartheta p$, $\partial^2_{\varrho,\vartheta} H_\Theta = \partial_\varrho \Big(\varrho(\vartheta - \Theta)\partial_\vartheta s \Big) = (\vartheta - \Theta)\partial_\vartheta \Big(\varrho \partial_\vartheta s(\varrho,\vartheta) \Big) = 0$, equation (3.30) rewrites after some algebraic rearrangements (see [10] for details)

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) + \frac{1}{2}\, \varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_\varepsilon - I)^2 d\vec{\omega}\, d\nu \right)(\tau, \cdot)\, dx \\
&+ \int_0^\tau \int_{\Gamma_+} \vec{\omega}\cdot \vec{n}_x \left[I_\varepsilon + \frac{1}{2}\, \varepsilon (I_\varepsilon - I)^2 \right](t, x, \vec{\omega}, \nu)\, d\Gamma\, d\nu\, dt \\
&+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \right) dx\, dt \\
&+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right] \sigma_{a\varepsilon}(B_\varepsilon - I_\varepsilon)\, d\vec{\omega}\, d\nu\, dx\, dt \\
&+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log \frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log \frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s\varepsilon}(\tilde{I}_\varepsilon - I_\varepsilon)\, d\vec{\omega}\, d\nu\, dx\, dt, \\
&\le \int_\Omega \frac{1}{2} \bigg(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon}) \\
&+ \frac{1}{2}\, \varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_{0,\varepsilon} - I(0,\cdot))^2 d\vec{\omega}\, d\nu \bigg)\, dx \\
&+ \int_0^\tau \int_\Omega \varrho_\varepsilon (\vec{u}_\varepsilon - \vec{U}) \cdot \nabla_x \vec{U} \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) dx\, dt \\
&+ \int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \left(\vec{U} - \vec{u}_\varepsilon \right) \cdot \nabla_x \Theta\, dx\, dt \\
&+ \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(\partial_t \vec{U} + \vec{U} \cdot \nabla_x \vec{U} \right) \cdot \left(\vec{U} - \vec{u}_\varepsilon \right) \right) dx\, dt \\
&- \int_0^\tau \int_\Omega \left(p_\varepsilon\, \mathrm{div}_x \vec{U} - \mathbb{S}_\varepsilon : \nabla_x \vec{U} \right) dx\, dt - \int_0^\tau \int_\Omega \left(\varepsilon s_\varepsilon^R \partial_t \Theta + \vec{q}_\varepsilon^R \cdot \nabla_x \Theta \right) dx\, dt \\
&- \int_0^\tau \int_\Omega \left(\varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \partial_t \Theta \right) dx\, dt - \int_0^\tau \int_\Omega \varrho_\varepsilon \left(s_\varepsilon - s(r,\Theta) \right) \vec{U} \cdot \nabla_x \Theta\, dx\, dt \\
&- \int_0^\tau \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x \Theta\, dx\, dt + \int_0^\tau \int_\Omega \left(\left(1 - \frac{\varrho_\varepsilon}{r} \right) \partial_t p(r,\Theta) - \frac{\varrho_\varepsilon}{r}\, \vec{u}_\varepsilon \nabla_x p(r,\Theta) \right) dx\, dt \\
&- \int_0^\tau \int_\Omega \left(\varepsilon \vec{F}_\varepsilon^R \cdot \partial_t \vec{U} + \mathbb{P}_\varepsilon^R : \nabla_x \vec{U} \right) dx\, dt \\
&+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left(S_\varepsilon - S - \varepsilon \partial_t I \right) \left(I_\varepsilon - I \right) d\vec{\omega}\, d\nu\, dx\, dt \\
&=: K_0 + \sum_{j=1}^{11} \int_0^\tau K_j(t)\, dt.
\end{aligned}
\tag{3.31}
$$

It will be the goal of the next Section to provide a bound for the right-hand side of (3.31).

3.2 Uniform estimates

We choose positive numbers $(\underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta}, \underline{E}, \overline{E})$ such that

$$0 < \underline{\varrho} \leq \frac{1}{2} \min_{(t,x)\in[0,T]\times\overline{\Omega}} r(t,x) \leq 2 \max_{(t,x)\in[0,T]\times\overline{\Omega}} r(t,x) \leq \overline{\varrho},$$

$$0 < \underline{\vartheta} \leq \frac{1}{2} \min_{(t,x)\in[0,T]\times\overline{\Omega}} \Theta(t,x) \leq 2 \max_{(t,x)\in[0,T]\times\overline{\Omega}} \Theta(t,x) \leq \overline{\vartheta},$$

$$0 < \underline{E} \leq \frac{1}{2} \min_{(t,x)\in[0,T]\times\overline{\Omega}} E^R(t,x) \leq 2 \max_{(t,x)\in[0,T]\times\overline{\Omega}} E^R(t,x) \leq \overline{E},$$

and we split any measurable function h as $h = h_{ess} + h_{res}$, where $h_{ess}(t,x) = h(t,x)$ if $(\varrho, \vartheta, E^R) \in [\underline{\varrho}, \overline{\varrho}] \times [\underline{\vartheta}, \overline{\vartheta}] \times [\underline{E}, \overline{E}]$ and $h_{ess}(t,x) = 0$ otherwise.

Then we see [9] that there exist positive constants C_j for $j = 1, \cdots, 6$ such that

$$\begin{aligned} C_1 \left(|\varrho_\varepsilon - r|^2 + |\vartheta_\varepsilon - \Theta|^2 \right) &\leq H_\Theta(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - r)\partial_\Theta H_\Theta(r,\Theta) - H_\Theta(r,\Theta) \\ &\leq C_2 \left(|\varrho_\varepsilon - r|^2 + |\vartheta_\varepsilon - \Theta|^2 \right), \end{aligned} \tag{3.32}$$

for all $(\varrho_\varepsilon, \vartheta_\varepsilon) \in [\underline{\varrho}, \overline{\varrho}] \times [\underline{\vartheta}, \overline{\vartheta}]$,

$$H_\Theta(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - r)\partial_\Theta H_\Theta(r,\Theta) - H_\Theta(r,\Theta) \geq C_3 \left(1 + \varrho e(\varrho_\varepsilon, \vartheta_\varepsilon) + \varrho |s(\varrho_\varepsilon, \vartheta_\varepsilon)| \right), \tag{3.33}$$

otherwise. In the same stroke we have

$$C_4 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu, \vec{\omega}, \vec{U}, \Theta)|^2 d\vec{\omega}\, d\nu \leq H^R(I_\varepsilon) \leq C_5 \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - B(\nu, \vec{\omega}, \vec{U}, \Theta)|^2 d\vec{\omega}\, d\nu, \tag{3.34}$$

for all $E^R_\varepsilon \in [\underline{E}, \overline{E}]$,

$$H^R(I_\varepsilon) \geq C_6 (1 + E^R_\varepsilon + |s^R_\varepsilon|), \tag{3.35}$$

otherwise.

We have now the crucial inequality (see the Appendix for a proof)

Lemma 3.3 *Let $(r, \vec{U}, \Theta, I)$ be the solution of problem (3.14-3.19) satisfying the conditions of Theorem 3.1.*

One has the following relative entropy inequality

$$\begin{aligned} &\int_\Omega \left[\frac{1}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + H^R(I_\varepsilon) + \int_0^\infty \int_{\mathcal{S}^2} |I_\varepsilon - I|^2 d\vec{\omega}\, d\nu \right](t,\cdot)\, dx \\ &\leq \frac{1}{\varepsilon} \Big\{ \mathcal{C} e_0 + \int_\Omega \Big[\frac{1}{2} \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)) + H^R(I_{0,\varepsilon}) \\ &+ \int_0^\infty \int_{\mathcal{S}^2} |I_{0,\varepsilon} - I(0,\cdot)|^2 d\vec{\omega}\, d\nu \Big] dx \Big\} e^{\frac{\mathcal{C}'}{\varepsilon} t}, \end{aligned} \tag{3.36}$$

where $\mathcal{C}$ and $\mathcal{C}'$ are positive constants depending on $(r, \vec{U}, \Theta, I)$ and e_0 is the same as in Theorem 3.1.

The following result is a straightforward consequence of (3.36) and we omit its proof

Lemma 3.4 *Assume ("well prepared" data) that* $e_0 = O(\varepsilon)^2$ *and suppose that*

$$\|\varrho_{0,\varepsilon} - \varrho_0\|_{L^2(\Omega)} \le C\varepsilon, \|\vartheta_{0,\varepsilon} - \vartheta_0\|_{L^2(\Omega)} \le C\varepsilon, \|\sqrt{\varrho_{0,\varepsilon}}\,(\vec{u}_{0,\varepsilon} - \vec{u})\|_{L^2(\Omega;\mathbb{R}^3)} \le C\varepsilon.$$

Then the following estimates hold

$$\left(\varsigma_\varepsilon^m + \varsigma_\varepsilon^R\right)\left[[0,T]\times\overline{\Omega}\right] \le C\varepsilon, \tag{3.37}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \le C\varepsilon, \tag{3.38}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\varepsilon - \varrho]_{ess}(t)\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}, \tag{3.39}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\vartheta_\varepsilon - \vartheta]_{ess}(t)\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}, \tag{3.40}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\,(\vec{u}_\varepsilon(t) - \vec{u}(t))\,\|_{L^2(\Omega;\mathbb{R}^3)} \le C\sqrt{\varepsilon}, \tag{3.41}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[E^R_\varepsilon - E^R(I)]_{ess}(t)\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}, \tag{3.42}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\sqrt{\varepsilon}, \tag{3.43}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\sqrt{\varepsilon}, \tag{3.44}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[E^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\sqrt{\varepsilon}, \tag{3.45}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|[s^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \le C\sqrt{\varepsilon}. \tag{3.46}$$

Let us finally quote the following result which is a straightforward application of Proposition 5.2 of [9] (the proof is omitted)

Proposition 3.5 *Let* $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\vartheta_\varepsilon\}_{\varepsilon>0}\{I_\varepsilon\}_{\varepsilon>0}$ *three sequences of non-negative measurable functions such that*

$$\left[\varrho_\varepsilon^{(1)}\right]_{ess} \to \varrho^{(1)}\ weakly-(*)\ in\ L^\infty(0,T;L^2(\Omega)),$$

$$\left[\vartheta_\varepsilon^{(1)}\right]_{ess} \to \vartheta^{(1)}\ weakly-(*)\ in\ L^\infty(0,T;L^2(\Omega)),$$

$$\left[I_\varepsilon^{(1)}\right]_{ess} \to I^{(1)}\ weakly-(*)\ in\ L^\infty(0,T;L^2(\Omega)),\ a.e.\ in\ \mathcal{S}^2\times\mathbb{R}_+,$$

where

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \varrho}{\varepsilon},\ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon - \vartheta}{\varepsilon},\ I_\varepsilon^{(1)} = \frac{I_\varepsilon - I}{\varepsilon}.$$

Suppose that

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \le C\varepsilon^2. \tag{3.47}$$

Let $G, G^R \in C^1(\overline{\mathcal{O}_{ess}})$ *be given functions. Then*

$$\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess} - G(\varrho,\vartheta)}{\varepsilon} \to \frac{\partial G(\varrho,\vartheta)}{\partial\varrho}\ \varrho^{(1)} + \frac{\partial G(\varrho,\vartheta)}{\partial\vartheta}\ \vartheta^{(1)},$$

$weakly-(*)\ in\ L^\infty(0,T;L^2(\Omega))$, *and if we note*

$$\left[G^R(I_\varepsilon)\right]_{ess} := \left[G^R(I_\varepsilon(\cdot,\cdot,\vec{\omega},\nu))\right]_{ess} = G^R(I_\varepsilon)\cdot\mathbb{I}_{\mathcal{M}^\varepsilon_{ess}},\ for\ a.a.\ (\vec{\omega},\nu)\in\mathcal{S}^2\times\mathbb{R}_+,$$

we have

$$\frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(I)}{\varepsilon} \to \frac{\partial G(I)}{\partial I}\ I^{(1)},$$

$weakly-(*)\ in\ L^\infty(0,T;L^2(\Omega)),\ a.e.\ in\ \mathcal{S}^2\times\mathbb{R}_+.$

Moreover if $G, G^R \in C^2(\overline{\mathcal{O}_{ess}})$ *then*

$$\left\|\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess} - G(\varrho,\vartheta)}{\varepsilon} - \frac{\partial G(\varrho,\vartheta)}{\partial\varrho}\ \left[\varrho^{(1)}\right]_{ess} - \frac{\partial G(\varrho,\vartheta)}{\partial\vartheta}\ \left[\vartheta^{(1)}\right]_{ess}\right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

and

$$\left\|\frac{\left[G^R(I_\varepsilon)\right]_{ess} - G^R(I)}{\varepsilon} - \frac{\partial G(I)}{\partial I}\ \left[I^{(1)}\right]_{ess}\right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

$for\ a.a.\ (\vec{\omega},\nu)\in\mathcal{S}^2\times\mathbb{R}_+.$

3.3 Convergence toward the target system

We are now in position to prove that the *non relativistic target system* (3.14)-(3.16) is the limit in a suitable sense, of the primitive system (3.5)-(3.10) when $\varepsilon \to 0$.

Theorem 3.6 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain of class* $C^{2,\nu}$*. Assume that the thermodynamic functions* p*,* e*,* s *satisfy hypotheses (2.1 - 2.6) and that the coefficients* μ*,* η*,* κ*,* σ_a*,* σ_s *and* B *comply with (2.7) - (2.10).*

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon)$ *be a weak solution to the scaled radiative Navier-Stokes system (3.5 - 3.10) for* $(t,x,\vec{\omega},\nu)\in[0,T]\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+$*, supplemented with boundary conditions (1.8 - 1.9), the compatibility conditions (3.20) and initial conditions* $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon})$ *such that*

$$\varrho_\varepsilon(0,\cdot) = \varrho_0 + \sqrt{\varepsilon}\varrho^{(1)}_{0,\varepsilon},\quad \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon},\quad \vartheta_\varepsilon(0,\cdot) = \vartheta_0 + \sqrt{\varepsilon}\vartheta^{(1)}_{0,\varepsilon},$$

where $(\varrho_0, \vec{u}, \vartheta_0) \in H^3(\Omega)$ *are smooth functions such that* (ϱ_0, ϑ_0) *belong to the set* $\mathcal{O}^H_{ess}$ *where* $\overline{\varrho} > 0$*,* $\overline{\vartheta} > 0$*, are two constants and* $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\,dx = 0,\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\,dx = 0.$

Suppose also that

$$\vec{u}_{0,\varepsilon} \to \vec{u}_0\ \ strongly\ in\ L^2(\Omega;\mathbb{R}^3),$$

$$\varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0\ strongly\ in\ L^2(\Omega),$$

$$\vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \ strongly \ in \ L^2(\Omega).$$

Then up to subsequences

$$\varrho_\varepsilon \to \varrho \ \ strongly \ in \ L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \ strongly \ in \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \ strongly \ in \ L^\infty(0,T;L^4(\Omega)),$$

$$I_\varepsilon \to I \ \ strongly \ in \ L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

where $(\varrho,\vec{u},\vartheta,I)$ *is the smooth solution of (3.14)-(3.16) on* $[0,T]\times\Omega$*, with initial data* $(\varrho_0,\vec{u}_0,\vartheta_0)$*.*

Proof: Let us observe that after Theorem 2.2, bounds (2.9) and (2.10) and relative entropy inequality (3.36), the temperature ϑ_ε is bounded in $L^2(0,T;W^{1,2}(\Omega))$ then after extraction of a subsequence

$$\vartheta_\varepsilon \to \vartheta \ \ in \ L^2([0,T]\times\Omega). \tag{3.48}$$

1. For the continuity equation, Lemma 3.4 implies: $\int_0^T \left\| \nabla_x \vec{u}_\varepsilon + \nabla_x^T \vec{u}_\varepsilon - \frac{2}{3}\,\mathrm{div}_x \vec{u}_\varepsilon \mathbb{I} \right\|_{L^2(\Omega;\mathbb{R}^3)} \, dt \le C$. Using this fact together with bounds in Lemma 3.4, we see that

$$\int_0^T \|\vec{u}_\varepsilon(t) - \vec{u}(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \, dt \le C,$$

so, passing to the limit after possible extraction of a subsequence, we have $\vec{u}_\varepsilon \to \vec{u}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$. In the same stroke $\varrho_\varepsilon \to \varrho$, weakly in $L^\infty(0,T;L^{5/3}(\Omega;\mathbb{R}^3))$. So we can pass to the limit in the weak continuity equation (3.21) which rewrites as (3.14), together with the boundary condition $\vec{u}\cdot n_x|_{\partial\Omega} = 0$, provided $\partial\Omega$ is regular.

2. For the radiative transfer equation we have shown in the previous sections, using the result of Bardos, Golse, Perthame and Sentis [2] that $I_\varepsilon \to I$, weakly in $L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$, and that $\vartheta_\varepsilon \to \vartheta$, weakly in $L^2(0,T;W^{1,2}(\Omega))$. As the equation is linear in I, we can pass to the limit in the weak formulation of radiative transfer equation which gives

$$\int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left\{ \vec{\omega}\cdot\nabla_x\psi \, I + \left[\sigma_a \left(B - I\right) + \sigma_s \left(\tilde{I} - I\right) \right] \psi \right\} d\vec{\omega}\, d\nu\, dx\, dt$$
$$= \int_0^\tau \int_{\Gamma_+} \int_0^\infty \vec{\omega}\cdot\vec{n}_x I\psi \, d\Gamma\, d\nu\, dt,$$

for any test function $\psi \in C_c^\infty((0,T)\times\overline{\Omega}\times\mathcal{S}^2\times\mathbb{R}_+)$ which is the weak formulation of equation (3.5).

3. For the momentum equation, one knows after the analysis of [9] (see [4]) that one can pass to the limit in the convective term and obtain

$$\int_0^T \int_\Omega \overline{\varrho_\epsilon \, \vec{u}_\epsilon \otimes \vec{u}_\epsilon} : \nabla_x\phi \, dx\, dt \to \int_0^T \int_\Omega \varrho \, \vec{u}\otimes\vec{u} : \nabla_x\phi \, dx\, dt.$$

Moreover after the hypotheses on pressure, ϑ_ε is bounded in $L^\infty((0,T);L^4(\Omega))\cap L^2(0,T;L^6(\Omega))$, which implies that $\mathbb{S}_\varepsilon \to \mu(\vartheta)(\nabla_x\vec{u}+\nabla_x^t\vec{u})$, weakly in $L^q(0,T;L^q(\Omega;\mathbb{R}^3))$ for a $q>1$.

Using (3.39) and (3.40)

$$\text{ess}\sup_{t\in(0,T)} \|p_\varepsilon - p]_{ess}(t)\|_{L^2(\Omega)} \leq C\varepsilon,$$

then $\nabla_x p_\varepsilon \to \nabla_x p$ in $\mathcal{D}'$.

Finally as we know that $I_\varepsilon \to I$, weakly in $L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$ then $\mathbb{P}_\varepsilon^R \to \mathbb{P}^R$ and $\varepsilon\vec{F}_\varepsilon^R \to 0$, so we can pass to the limit in all the terms of the momentum equation (3.22) and obtain (3.15).

4. For the entropy balance we rewrite equation (3.26) in the form

$$\begin{aligned}
&\int_0^\tau\int_\Omega\Big((\varrho_\varepsilon s_\varepsilon + s_\varepsilon^R)\partial_t\varphi + \varrho_\varepsilon s_\varepsilon\vec{u}_\varepsilon\cdot\nabla_x\varphi + \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\Big)\,dx\,dt + \frac{1}{\varepsilon}\int_0^\tau\int_\Omega \frac{\vec{q}_\varepsilon^R}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\,dx\,dt \\
&\quad + \langle\varsigma^m;\phi\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} + \left\langle\varsigma_\varepsilon^R;\phi\right\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} - \int_\Omega \left(\varrho s + s^R\right))(0,\cdot)\varphi(0,\cdot)\,dx \\
&\quad \leq \int_\Omega \left[\left(\varrho_{0,\varepsilon}s_{0,\varepsilon} + s_{0,\varepsilon}^R\right) - \left(\varrho_0 s_0 + s_0^R\right)\right]\varphi(0,\cdot)\,dx \\
&\quad - \int_0^\tau\int_\Omega \left\{\varrho_\varepsilon(s_\varepsilon - s) + (\varrho_\varepsilon - \varrho)s + s_\varepsilon^R - s^R)\right\}\partial_t\varphi\,dx\,dt \\
&\quad - \int_0^\tau\int_\Omega \left\{\varrho_\varepsilon(s_\varepsilon - s)\vec{u}_\varepsilon + (\varrho_\varepsilon\vec{u}_\varepsilon - \varrho\vec{u})s\right\}\cdot\nabla_x\varphi\,dx\,dt \\
&\quad - \int_0^\tau\int_\Omega \left[\frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon} - \frac{\vec{q}}{\vartheta}\right]\cdot\nabla_x\varphi\,dx\,dt + \langle\varsigma_\varepsilon^m - \varsigma^m;\phi\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})}.
\end{aligned}$$

for any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$.

Using Proposition 3.5, one computes first

$$\frac{1}{\varepsilon}\int_0^\tau\int_\Omega \frac{\vec{q}_\varepsilon^R}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\,dx\,dt \to \int_0^\tau\int_\Omega \frac{\vec{F}_1^R}{\vartheta}\cdot\nabla_x\varphi\,dx\,dt,$$

as $\varepsilon \to 0$. In the same stroke, we find

$$\left\langle\varsigma_\varepsilon^R;\phi\right\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} \to \int_0^\tau\int_\Omega \frac{\vec{F}_1^R\cdot\nabla_x\vartheta}{\vartheta^2}\,\varphi\,dx\,dt.$$

as $\varepsilon \to 0$, by using once more Proposition 3.5.

After the conditions on the data and the estimates in Lemma 3.4 and using verbatim the techniques of [9](Chap. 5) one concludes that all of the integrals in the right hand side converge to zero as $\varepsilon \to 0$, which proves that the limit entropy inequality (3.17) is obtained □

Appendix: Proof of Lemma 3.3

After estimates (3.32)-(3.35) all of the terms in the left-hand side of (3.31) are positive. Proceeding as in [8] we get

$$\int_\Omega \left(\frac{1}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) + \frac{1}{2}\,\varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_\varepsilon - I)^2 d\vec{\omega}\, d\nu \right)(\tau, \cdot)\, dx$$
$$+ \int_0^\tau \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\, d\Gamma\, d\nu\, dt$$
$$+ \int_0^\tau \int_\Omega \Big(\frac{\vartheta}{\vartheta_\varepsilon} \mathbb{S}(\varrho_\varepsilon, \nabla_x \vec{u}_\varepsilon) : \nabla_x \vec{u}_\varepsilon - \mathbb{S}(r, \vec{U}) : (\nabla_x \vec{u}_\varepsilon - \nabla_x \vec{U}) - \mathbb{S}(\varrho_\varepsilon, \nabla_x \vec{u}_\varepsilon) : \nabla_x \vec{U} \Big)\, dx\, dt$$
$$+ \int_0^\tau \int_\Omega \Big(\frac{\vec{q}(\varrho_\varepsilon, \vartheta_\varepsilon)\cdot \nabla_x \Theta}{\vartheta_\varepsilon} - \frac{\Theta}{\vartheta_\varepsilon}\, \frac{\vec{q}(\varrho_\varepsilon, \vartheta_\varepsilon)\cdot\nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \Big)\, dx\, dt$$
$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right] \sigma_{a\varepsilon}(B_\varepsilon - I_\varepsilon)\, d\vec{\omega}\, d\nu\, dx\, dt$$
$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{\Theta}{\nu} \left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s\varepsilon}(\tilde{I}_\varepsilon - I_\varepsilon)\, d\vec{\omega}\, d\nu\, dx\, dt,$$
$$\le \int_\Omega \frac{1}{2}\bigg(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)\right) + H^R(I_{0,\varepsilon})$$
$$+ \frac{1}{2}\,\varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_{0,\varepsilon} - I_0)^2 d\vec{\omega}\, d\nu \bigg)\, dx$$
$$+ \int_0^\tau \left[\delta \left\| \vec{U} - \vec{u}_\varepsilon \right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}(\delta; r, \vec{U}, \Theta) \int_\Omega \Big(\frac{1}{2}\, \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) \Big)\, dx \right] dt + \mathcal{A},$$

where

$$\mathcal{A} := \int_\Omega \Big(p(r,\Theta) - p(\varrho_\varepsilon, \vartheta_\varepsilon) \Big) \mathrm{div}_x \vec{U}\, dx + \int_\Omega \Big(\Big(1 - \frac{\varrho_\varepsilon}{r}\Big) \Big(\partial_t p(r,\Theta) + \vec{u}\cdot\nabla_x p(r,\Theta)\Big)\, dx$$
$$- \int_\Omega \varrho \Big(\partial_r s(r,\Theta)(\varrho_\varepsilon - r) - \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta) \Big) \Big(\partial_t \Theta + \vec{U}\cdot\nabla_x \Theta \Big)\, dx$$
$$- \int_0^\tau \int_\Omega \left(\varepsilon s^R_\varepsilon \partial_t \Theta + \vec{q}^R_\varepsilon \cdot \nabla_x \Theta \right)\, dx\, dt - \int_0^\tau \int_\Omega (\varepsilon \vec{F}^R_\varepsilon \cdot \partial_t \vec{U} + \mathbb{P}_\varepsilon : \nabla_x \vec{U})\, dx\, dt$$
$$+ \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left(S_\varepsilon - S - \varepsilon \partial_t I\right)(I_\varepsilon - I)\, d\vec{\omega}\, d\nu\, dx\, dt =: \sum_{k=1}^6 . \tag{3.49}$$

In order to estimate the last terms in the right-hand side, we begin with the three last integrals. We get first

$$\left| \int_0^\tau \int_\Omega \left(\varepsilon s^R_\varepsilon \partial_t \Theta \right)\, dx\, dt \right| \le \int_0^\tau \int_\Omega \varepsilon H^R_\varepsilon |\partial_t \log\Theta|\, dx\, dt + \int_0^\tau \int_\Omega \varepsilon E^R_\varepsilon |\partial_t \log\Theta|\, dx\, dt$$
$$\le \|\partial_t \log\Theta\|_{L^\infty(\Omega)} \left(\int_0^\tau \int_\Omega \varepsilon H^R_\varepsilon\, dx\, dt + \epsilon e_0 \right),$$

and also

$$\left|\int_0^\tau\int_\Omega \vec{q}_\varepsilon^R\cdot\nabla_x\Theta\,dx\,dt\right| \le \int_0^\tau\int_\Omega \Theta|s_\varepsilon^R|\|\nabla_x\log\Theta\|\,dx\,dt$$
$$\le C\|\nabla_x\Theta\|_{L^\infty(\Omega)}\left(e_0+\int_0^\tau\int_\Omega H_\varepsilon^R\,dx\,dt\right),$$

then

$$|A_4| = \left|\int_0^\tau\int_\Omega \left(\varepsilon s_\varepsilon^R\partial_t\Theta+\vec{q}_\varepsilon^R\cdot\nabla_x\Theta\right)\,dx\,dt\right| \le C\left(e_0+\int_0^\tau\int_\Omega H_\varepsilon^R\,dx\,dt\right), \tag{3.50}$$

where we took into account (3.34) and (3.35). In the same way

$$\left|\int_0^\tau\int_\Omega \varepsilon\vec{F}_\varepsilon^R\partial_t\vec{U}\,dx\,dt\right| = \varepsilon\left|\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}\left(I_\varepsilon-B(\nu,\Theta)\right)\cdot\partial_t\vec{U}\,d\vec{\omega}\,d\nu\,dx\,dt\right|$$
$$=\varepsilon\left|\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}\left([I_\varepsilon-B(\nu,\Theta)]_{ess}+[I_\varepsilon-B(\nu,\Theta)]_{res}\right)\cdot\partial_t\vec{U}\,d\vec{\omega}\,d\nu\,dx\,dt\right|$$
$$\le C\varepsilon\|\partial_t\vec{U}\|_{L^\infty}\left(\int_0^\tau\int_\Omega H_\varepsilon^R\,dx\,dt\right),$$

and in the second part

$$\left|\int_0^\tau\int_\Omega \mathbb{P}_\varepsilon^R\cdot\nabla_x\vec{U}\,dx\,dt\right| \le \left|\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}\otimes\vec{\omega}\left(I_\varepsilon-B(\nu,\Theta)\right)\cdot\nabla_x\vec{U}\,d\vec{\omega}\,d\nu\,dx\,dt\right|$$
$$+\left|\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}\otimes\vec{\omega}B(\nu,\Theta)\cdot\nabla_x\vec{U}\,d\vec{\omega}\,d\nu\,dx\,dt\right|$$
$$\le\left|\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}\otimes\vec{\omega}([I_\varepsilon-B(\nu,\Theta)]_{ess}\right.$$
$$\left.+[I_\varepsilon-B(\nu,\Theta)]_{res})\cdot\nabla_x\Theta\,d\vec{\omega}\,d\nu\,dx\,dt\right|+\frac{1}{3}\left|\int_0^\tau\int_\Omega\Theta^4\mathrm{div}_x\vec{U}\,dx\,dt\right|$$
$$\le C\|\nabla_x\vec{U}\|_{L^\infty}\left(\int_0^\tau\int_\Omega H_\varepsilon^R\,dx\,dt+e_0\right).$$

Then finally

$$|A_5| = \left|\int_0^\tau\int_\Omega(\varepsilon\vec{F}_\varepsilon^R\cdot\partial_t\vec{U}+\mathbb{P}_\varepsilon:\nabla_x\vec{U})\,dx\,dt\right| \le C(e_0+\int_0^\tau\int_\Omega H_\varepsilon^R\,dx\,dt). \tag{3.51}$$

Using a Taylor argument we have also

$$|A_6| = \left|\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\left(S_\varepsilon-S-\varepsilon\partial_t I\right)\left(I_\varepsilon-I\right)d\vec{\omega}\,d\nu\,dx\,dt\right|$$
$$\le C\left(\int_0^\tau\int_\Omega\mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon|r,\Theta\right)\,dx\,dt+\int_0^\tau\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}(I_\varepsilon-I)^2\,d\vec{\omega}\,d\nu\,dx\,dt\right).$$

Now using the previous thermodynamical identities for H_Θ and the continuity equation for the target system, we get rid of the remaining integrals in the right-hand side of (3.49) (see [10]) by observing that

$$
\begin{aligned}
A_1 + A_2 + A_3 &= \int_\Omega \Big(p(r,\Theta) - p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}_x \vec{U}\, dx \\
&+ \int_\Omega \Big(\Big(1 - \frac{\varrho_\varepsilon}{r}\Big) \Big(\partial_t p(r,\Theta) + \vec{u}\cdot\nabla_x p(r,\Theta)\Big)\, dx \\
&- \int_\Omega \varrho \Big(\partial_r s(r,\Theta)(\varrho_\varepsilon - r) - \partial_\Theta s(r,\Theta)(\vartheta_\varepsilon - \Theta)\Big)\Big(\partial_t\Theta + \vec{U}\cdot\nabla_x\Theta\Big)\, dx \\
&= \int_\Omega \Big(p(r,\Theta) - p(\varrho_\varepsilon,\vartheta_\varepsilon)\Big)\mathrm{div}_x\vec{u}\, dx + \int_\Omega \varrho\,(\Theta - \vartheta_\varepsilon)\,\partial_\Theta s(r,\Theta)\Big(\partial_t\Theta + \vec{U}\cdot\nabla_x\Theta\Big)\, dx \\
&- \int_\Omega \Big(r - \varrho_\varepsilon\Big)\partial_r p(r,\Theta)\mathrm{div}_x\vec{U}\, dx.
\end{aligned}
$$

As the second term in the right-hand side rewrites as follows

$$
\begin{aligned}
&\int_\Omega \varrho\,(\Theta - \vartheta_\varepsilon)\,\partial_\Theta s(r,\Theta)\Big(\partial_t\Theta + \vec{U}\cdot\nabla_x\Theta\Big)\, dx \\
&\quad= \int_\Omega r\,(\Theta - \vartheta_\varepsilon)\Big[\partial_t s(r,\Theta) + \vec{U}\cdot\nabla_x s(r,\Theta)\Big]\, dx - \int_\Omega \Big(\Theta - \vartheta_\varepsilon\Big)\partial_\Theta p(r,\Theta)\mathrm{div}_x\vec{U}\, dx \\
&\quad= \int_\Omega (\vartheta - \vartheta_\varepsilon)\Big[\frac{1}{\Theta}\Big(\mathbb{S}(r,\vec{U}) : \nabla_x\vec{U} - \frac{\vec{q}(\Theta,\nabla_x\Theta)\cdot\nabla_x\Theta}{\Theta}\Big) - \mathrm{div}_x\Big(\frac{\vec{q}(\Theta,\nabla_x\Theta)}{\Theta}\Big)\Big]\, dx \\
&\quad- \int_\Omega \Big(\Theta - \vartheta_\varepsilon\Big)\partial_\Theta p(r,\Theta)\mathrm{div}_x\vec{U}\, dx,
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
A_1 + A_2 + A_3 &= \int_\Omega \Big(p(r,\Theta) - p(\varrho_\varepsilon,\vartheta_\varepsilon) - \partial_\varrho p(r,\Theta)(\varrho_\varepsilon - r) - \partial_\vartheta p(r,\Theta)(\vartheta_\varepsilon - \Theta)\Big)\mathrm{div}_x\vec{U}\, dx. \\
&+ \int_\Omega (\Theta - \vartheta_\varepsilon)\Big[\frac{1}{\Theta}\Big(\mathbb{S}(r,\vec{U}) : \nabla_x\vec{U} - \frac{\vec{q}(\Theta,\nabla_x\Theta)\cdot\nabla_x\Theta}{\Theta}\Big) - \mathrm{div}_x\Big(\frac{\vec{q}(\Theta,\nabla_x\Theta)}{\Theta}\Big)\Big]\, dx,
\end{aligned}
$$

where we observe that

$$
\begin{aligned}
&\left|\int_\Omega \Big(p(r,\Theta) - p(\varrho_\varepsilon,\vartheta_\varepsilon) - \partial_r p(r,\Theta)(\varrho_\varepsilon - r) - \partial_\Theta p(r,\Theta)(\vartheta_\varepsilon - \Theta)\Big)\mathrm{div}_x\vec{U}\, dx\right| \\
&\qquad\le C\left\|\mathrm{div}_x\vec{U}\right\|_{L^\infty(\Omega)} \int_\Omega \mathcal{E}\left(\varrho_\varepsilon,\vartheta_\varepsilon | r,\Theta\right)\, dx.
\end{aligned}
$$

Plugging all of these estimates into (3.49) we see that it reduces finally to

$$\int_\Omega \left(\frac{1}{2} \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + \varepsilon H^R(I_\varepsilon) + \frac{1}{2}\,\varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_\varepsilon - I)^2 d\vec{\omega}\; d\nu \right)(\tau, \cdot)\; dx$$
$$+ \int_0^\tau \int_\Omega \Big(\frac{\Theta}{\vartheta_\varepsilon} \mathbb{S}(\varrho_\varepsilon, \nabla_x \vec{u}_\varepsilon) : \nabla_x \vec{u}_\varepsilon - \mathbb{S}(r, \vec{U}) : (\nabla_x \vec{u}_\varepsilon - \nabla_x \vec{U}) - \mathbb{S}(\varrho_\varepsilon, \nabla_x \vec{u}_\varepsilon) : \nabla_x \vec{U} \Big)\; dx\; dt$$
$$+ \int_0^\tau \int_\Omega \Big(\frac{\tilde{\vartheta}_\varepsilon - \Theta}{\tilde{\vartheta}_\varepsilon} \mathbb{S}(\varrho_\varepsilon, \nabla_x \vec{u}_\varepsilon) : \nabla_x \vec{U} \Big)\; dx\; dt +$$
$$+ \int_0^\tau \int_\Omega \Big(\frac{\vec{q}(\vartheta_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \Theta}{\vartheta_\varepsilon} - \frac{\Theta}{\vartheta_\varepsilon}\; \frac{\vec{q}(\varrho_\varepsilon, \nabla_x \vartheta_\varepsilon) \cdot \nabla_x \vartheta_\varepsilon}{\vartheta_\varepsilon} \Big)\; dx\; dt$$
$$+ \int_0^\tau \int_\Omega \Big((\Theta - \vartheta_\varepsilon) \frac{\vec{q}(\Theta, \nabla_x \Theta) \cdot \nabla_x \Theta}{\Theta^2} + \frac{\vec{q}(\Theta, \nabla_x \Theta) \cdot \nabla_x (\vartheta_\varepsilon - \Theta)}{\Theta} \Big)\; dx\; dt$$
$$\leq \int_\Omega \frac{1}{2} \bigg(\varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon})$$
$$+ \frac{1}{2}\,\varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_{0,\varepsilon} - I_0)^2 d\vec{\omega}\; d\nu \bigg)\; dx$$
$$+ \int_0^\tau \left[\delta \left\| \vec{U} - \vec{u}_\varepsilon \right\|^2_{W^{1,2}(\Omega;\mathbf{R}^3)} + \mathcal{C}'(\delta; r, \vec{U}, \Theta) \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right)\; dx \right] dt$$
$$+ \mathcal{C}''(\delta; r, \vec{U}, \Theta)(e_0 + \int_0^\tau \int_\Omega H^R_\varepsilon\; dx\; dt)$$
$$+ \mathcal{C}'''(\delta; r, \vec{U}, \Theta) \left(\int_0^\tau \int_\Omega \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right)\; dx\; dt + \int_0^\tau \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} (I_\varepsilon - I)^2\; d\vec{\omega}\; d\nu\; dx\; dt \right). \tag{3.52}$$

Finally we can control the dissipative terms (the three last integrals in the left-hand side), by using verbatim the computations in ([10]) which leads to the final inequality

$$\varepsilon \int_\Omega \left(\frac{1}{2}\; \varrho_\varepsilon |\vec{u}_\varepsilon - \vec{U}|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + H^R(I_\varepsilon) + \frac{1}{2} \int_0^\infty \int_{\mathcal{S}^2} (I_\varepsilon - I)^2 d\vec{\omega}\; d\nu \right)(\tau, \cdot)\; dx$$
$$\leq k_1 e_0 + \int_\Omega \left(\frac{1}{2} \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon} - \vec{U}(0,\cdot)|^2 + \mathcal{E}\left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon} | r(0,\cdot), \Theta(0,\cdot)\right) + \varepsilon H^R(I_{0,\varepsilon}) \right.$$
$$\left. + \frac{1}{2}\,\varepsilon \int_0^\infty \int_{\mathcal{S}^2} (I_{0,\varepsilon} - I_0)^2 d\vec{\omega}\; d\nu \right) dx$$
$$+ k_2 \int_0^\tau \int_\Omega \Big(\frac{1}{2}\; \varrho_\varepsilon \left| \vec{u}_\varepsilon - \vec{U} \right|^2 + \mathcal{E}\left(\varrho_\varepsilon, \vartheta_\varepsilon | r, \Theta\right) + H^R(I_\varepsilon) + \frac{1}{2} \int_0^\infty \int_{\mathcal{S}^2} (I_\varepsilon - I)^2 d\vec{\omega}\; d\nu \Big)\; dx\; dt, \tag{3.53}$$

where the positive constants k_j depend on $(r, \vec{U}, \Theta)$ through the norms involved in Theorem 3.1. Using Gronwall's lemma we get finally the requested inequality (3.36).

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