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Leading terms of velocity and its gradient of the stationary rotational viscous incompressible flows with nonzero velocity at infinity

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Abstract

We consider the Navier-Stokes system with Oseen and rotational terms describing the stationary flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating at a constant angular velocity. In a previous paper, we prove a representation formula for weak solutions of the system. Here the representation formula is used to get an asymptotic expansion of respectively velocity and its gradient, and to establish pointwise decay estimates of remainder terms. Our results are based on a fundamental solution proposed by Guenther and Thomann [31]. We thus present a different approach to this result, besides the one, given by Kyed [43].

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Key words. exterior domain, viscous incompressible flow, rotating body, fundamental solution, asymptotic expansion, Navier-Stokes system.

1   Introduction

The aim of this paper is to find the asymptotic structure, particularly the leading terms, of the velocity part of the solution to the system

$$-\mu\Delta u(z) - (U + \omega \times z)\cdot\nabla u(z) + \omega \times u(z) + u\cdot\nabla u(z) + \nabla\pi(z) = f(z), \atop \mathrm{div}\ u(z) = 0, \biggr\} \qquad (1.1)$$

$$u(x) \to 0 \ \text{ for } |x| \to \infty. \qquad (1.2)$$

This system describes the stationary flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating at a constant angular velocity. We refer to [21] for more details on the physical background of (1.1). Here we only indicate that $\mathfrak{D} \subset \mathbb{R}^3$ is an open bounded set describing the rigid body, the vector $U \in \mathbb{R}^3\backslash\{0\}$ represents the constant translational velocity of this body, the vector $\omega \in \mathbb{R}^3\backslash\{0\}$ stands

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for its constant angular velocity, and μ denotes the constant kinematic viscosity of the fluid. The given function $f : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ describes a body force, and the unknowns $u : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}^3$ and $\pi : \mathbb{R}^3 \backslash \overline{\mathfrak{D}} \mapsto \mathbb{R}$ correspond respectively to the velocity and pressure field of the fluid. We assume that $U \cdot \omega \neq 0$. Then, according to [23], without loss of generality we may replace (1.1) by the normalized system

$$L(u) + \tau (u \cdot \nabla)u + \nabla \pi = f, \ \ \mathrm{div}\, u = 0 \quad \mathrm{in} \ \ \mathbb{R}^3 \backslash \overline{\mathfrak{D}}, \tag{1.3}$$

where the differential operator L is defined by

$$L(u)(z) := -\Delta u(z) + \tau \, \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z)$$
$$\mathrm{for} \ \ u \in W^{2,1}_{loc}(U)^3, \ z \in U, \ U \subset \mathbb{R}^3 \ \mathrm{open},$$

with $\tau \in (0,\infty)$ (Reynolds number) and $\omega = \varrho(1,0,0)$ for some $\varrho \in \mathbb{R} \setminus \{0\}$ (Taylor number).

Suppose that $f \in L^{p_0}(\mathbb{R}^3)^3$ for some $p_0 \in (1,\infty)$ and f has compact support. Further suppose there is a pair of functions (u, π) with $u \in L^6(\overline{\mathfrak{D}}^c)^3$, $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$ and $\pi \in L^2_{loc}(\overline{\mathfrak{D}}^c)$ satisfying (1.1) in the distributional sense ("Leray solution"). Such a solution exists under suitable assumptions on $\partial \mathfrak{D}$, $u|\partial \mathfrak{D}$ and p_0 ([25, Theorem XI.3.1]). Note that the condition $u \in L^6(\overline{\mathfrak{D}}^c)^3$, $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$ means in particular that (1.2) holds in a weak sense; compare [26, Theorem II.5.1]. In this situation, it was shown by Galdi and Kyed [23] that

$$|\partial^\alpha u(x)| = O\big[\, \big(|x| \, s_\tau(x)\big)^{-1-|\alpha|/2}\,\big] \quad (|x| \to \infty), \tag{1.4}$$

where $\alpha \in \mathbb{N}_0^3$ with $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 \leq 1$ (decay of u and ∇u). The term $s_\tau(x)$ in (1.4) is defined by

$$s_\tau(x) := 1 + \tau \, (|x| - x_1) \quad (x \in \mathbb{R}^3). \tag{1.5}$$

Its presence in (1.4) may be considered as a mathematical manifestation of the wake extending downstream behind the rigid body. Even in the linear nonrotational case, that is, in the case of solutions to the Oseen system

$$-\Delta u + \tau \, \partial_1 u + \nabla \pi = f, \ \ \mathrm{div}\, u = 0, \tag{1.6}$$

the velocity cannot be expected to decay more rapidly than $\big(|x| \, s_\tau(x)\big)^{-1}$ for $|x| \to \infty$, nor its gradient more rapidly than $\big(|x| \, s_\tau(x)\big)^{-3/2}$ ([40]). Therefore the decay rate in (1.4) should be best possible in the present case, too. By Kyed [43] it was shown that

$$u(x) = \mathfrak{O}(x) \cdot \alpha + R(x), \quad \nabla u(x) = \nabla \mathfrak{O}(x) \cdot \alpha + S(x), \tag{1.7}$$

where $\mathfrak{O}$ is the fundamental solution of the stationary Oseen system, α represents the force $\mathfrak{F}$ exerted by the liquid on the body, and R and S are some remainder terms decaying faster than $\mathfrak{O}$ and $\nabla \mathfrak{O}$, respectively, as $|x| \to \infty$.

In the work at hand, we also derive an asymptotic expansion of respectively u and ∇u. These expansions – stated in Theorem 3.1 below – differ in two respects from those presented in [43] and indicated in (1.7). Firstly, our leading term is less explicit than the term

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$\mathfrak{O} \cdot \alpha$ in (1.7). Instead of the fundamental solution $\mathfrak{O}$ of the <u>stationary</u> Oseen system, we use the time integral of the fundamental solution of the <u>evolutionary</u> Oseen system multiplied by a rotation depending on time. Secondly, and this is an aspect which goes beyond the theory in [43], we establish pointwise decay estimates of our remainder terms (see (3.2)), whereas in [43], it is only shown that the function R in (1.7) belongs to $L^p(\mathbb{R}^3 \backslash B_S)^3$ for $p \in (4/3, \infty)$, and S to $L^p(\mathbb{R}^3 \backslash B_S)^9$ for $p \in (1, \infty)$, where B_S is an open ball with sufficiently large radius $S > 0$. Interestingly, by integrating the decay rates in (3.2) and using Lemma 2.1, we find that our remainder terms belong to the same L^p-spaces.

We further indicate that our results are derived by an approach different from the one in [43]: whereas the theory in [43] reduces (1.7) to estimates of solutions to the time-periodic Oseen system in the whole space $\mathbb{R}^3$, our results are based on a representation formula of solutions to (1.3) (see Theorem 2.15). As a consequence of our approach, our remainder terms are expressed explicitly in terms of u, π and f. In particular, sharpening (1.7), we obtain that $S = \nabla R$.

Our access is made difficult by the structure of the Guenther-Thomann fundamental solution. In fact, as was already pointed out in [17] for the case $\tau = 0$, a fundamental solution $\mathfrak{Z}(x, y)$ to (1.3) cannot be bounded by $c \, |x - y|^{-1}$ uniformly in $x, y \in \mathbb{R}^3$ with $|x|$ and $|y|$ large, contrary to what may be expected in view of the situation in the Stokes and Oseen case. Actually it seems that no uniform bound $c \, |x - y|^{-\epsilon}$ exists, for whatever $\epsilon \in (0, \infty)$.

In [3] – [6], we proved a representation formula, a decay estimate as in (1.4), and asymptotic expansions for weak solutions of the linearized problem

$$L(u) + \nabla \pi = f, \quad \operatorname{div} u = 0, \quad u(x) \to \infty \ \ (|x| \to \infty),$$

as well as a representation formula for weak solutions of (1.3). In the context of these papers, a weak solution (u, π) of (1.3) is characterized by the assumptions that u is L^6-integrable outside a ball containing $\overline{\mathfrak{D}}$, and ∇u and π are L^2-integrable outside such a ball. In [8], we extended the results from [3] – [6] from weak solutions to Leray solutions.

In [7], we considered the nonlinear problem (1.3), deriving optimal rates of decay as in (1.4) for the velocity and its gradient, on the basis of the representation formula proved in [4] and [8] and restated below as Theorem 2.15.

The asymptotic behavior of purely rotating case was studied by Farwig, Hishida; see [15], [14] for the linear case and [16, 11] in the nonlinear one.

Concerning further articles related to the work at hand, we mention [1], [10], [12], [13], [18] – [20], [22], [24], [28], [30], [32] – [39], [41] – [45], [47], [48].

Let us briefly indicate how we will proceed in the following. In Section 2 we will present various auxiliary results. Section 3 deals with the main theorem - leading term for the velocity field and its gradient.

2 Notation and preliminaries

The open bounded set $\mathfrak{D} \subset \mathbb{R}^3$ introduced in Section 1 will be kept fixed throughout. We assume its boundary $\partial \mathfrak{D}$ to be of class C^2, and we denote its outward unit normal by

$n^{(\mathfrak{D})}$. The numbers τ and ϱ and the vector ω also introduced in Section 1 will be kept fixed, too. Define the matrix $\Omega \in \mathbb{R}^{3\times 3}$ by

$$\Omega := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

so that $\omega \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$.

Let us denote $s(x) := s_1(x) = 1 + (|x| - x_1)$. We recall that the function s_τ was defined in Section 1, as was the notation $|\alpha|$ for the length of a multi-index $\alpha \in \mathbb{N}_0^3$. If $A \subset \mathbb{R}^3$, we write A^c for the complement $\mathbb{R}^3 \backslash A$ of A. The open ball centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. Put $e_1 := (1,0,0)$. Let $x \times y$ denote the usual vector product of $x, y \in \mathbb{R}^3$. For $T \in (0,\infty)$, set $\mathfrak{D}_T := B_T \backslash \overline{\mathfrak{D}}$ ("truncated exterior domain"). By the symbol $\mathfrak{C}$, we denote constants only depending on $\mathfrak{D}$, τ or ω. We write $\mathfrak{C}(\beta_1, ..., \beta_n)$ for positive constants that additionally depend on parameters $\beta_1, ..., \beta_n \in \mathbb{R}$, for some $n \in \mathbb{N}$. As usual, $C(\gamma_1, \ldots, \gamma_n)$ means a positive constant only depending on $\gamma_1, \ldots, \gamma_n$.

We will further use the ensuing estimate, which was proved in [9].

Lemma 2.1. *Let $\beta \in (1, \infty)$. Then $\int_{\partial B_r} s_\tau(x)^{-\beta}\, do_x \le \mathfrak{C}(\beta)\, r$ for $r \in (0, \infty)$.*

We begin by introducing the fundamental solutions used in what follows. We set

$K(x,t) = (4\pi t)^{-3/2} e^{-\frac{|x|^2}{4t}}, \ \ x \in \mathbb{R}^3, \ t \in (0,\infty),$

$N_{jk}(x) = x_j x_k |x|^{-2}, \ x \in \mathbb{R}^3 \setminus \{0\},$

$\Lambda_{jk}(x,t) = K(x,t)\left(\delta_{jk} - N_{jk}(x) - {}_1F_1\left(1, 5/2, \frac{|x|^2}{4t} \right) (\delta_{jk}/3 - N_{jk}(x)) \right),$

$x \in \mathbb{R}^3 \setminus \{0\}, \ t \in (0,\infty), \ j,k \in \{1,2,3\},$

${}_1F_1(1,c,u) := \sum_{n=0}^{\infty} \big(\Gamma(c)/\Gamma(n+c) \big) \cdot u^n \ \ \text{for} \ \ u \in \mathbb{R}, \ c \in (0,\infty),$

where Γ denotes the usual Gamma function. In the following, the letter Γ will stand for the matrix-valued function defined by

$(\Gamma_{jk}(y,z,t))_{1 \le j,k \le 3} := (\Lambda_{rs}(y - \tau\, t\, e_1 - e^{-t\Omega} \cdot z, t))_{1 \le r,s \le 3} \cdot e^{-t\Omega},$

$y, z \in \mathbb{R}^3, \ t \in (0,\infty)$ with $y - \tau\, t\, e_1 - e^{-t\Omega} \cdot z \ne 0.$

$E_{4j}(x) := (4\pi)^{-1} x_j |x|^{-3}, \quad 1 \le j \le 3, \ x \in \mathbb{R}^3 \backslash \{0\}.$

Our following lemma restates [3, Corollary 3.1]:

Lemma 2.2. *The function Γ may be continuously extended to a function from $C^\infty\big(\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty) \big).$*

According to [3, Theorem 3.1], we have

Lemma 2.3. $\int_0^\infty |\Gamma_{jk}(y,z,t)|dt < \infty$ for $y, z \in \mathbb{R}^3$ with $y \ne z$, $1 \le j, k \le 3$.

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Thus we may define

$$\mathfrak{Z}_{jk}(y,z) := \int_0^\infty \Gamma_{jk}(y,z,t)dt$$

for $y, z \in \mathbb{R}^3$ with $y \neq z$, $1 \leq j, k \leq 3$.

The matrix-valued function $\mathfrak{Z}$ constitutes the velocity part of the fundamental solution introduced by Guenther, Thomann [31] for the system (1.3).

We will use the following technical lemmas:

Lemma 2.4. *Let $\delta > 0$. Assuming $z \in B_\delta(x)$, we have*

$$|z| \geq |x|/2, \quad for \ |x| \geq 2\delta, \tag{2.1}$$

$$s_\tau(z)^{-1} \leq \mathfrak{C}\,(1 + |x - z|)\, s_\tau(x)^{-1} \leq \mathfrak{C}(\delta)\, s_\tau(x)^{-1}. \tag{2.2}$$

Proof: For $|x| \geq 2\delta$ we have $|z| \geq |x| - |x - z| \geq |x| - \delta \geq |x|/2$, i.e. the relation (2.1) is satisfied. For the proof of (2.2) see [2, Lemma 4.8].

Lemma 2.5 ([5, Corollary 3.1]). *Let $j, k \in \{1, 2, 3\}$, $\alpha, \beta \in \mathbb{N}_0^3$ with $|\alpha + \beta| \leq 2$, $y, z \in \mathbb{R}^3$, $t \in (0, \infty)$. Then*

$$|\partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y,z,t)| \leq \mathfrak{C}\,(|y - \tau\,t\,e_1 - e^{-t\cdot\Omega} \cdot z|^2 + t)^{-3/2 - |\alpha + \beta|/2}.$$

Lemma 2.6 ([4, Theorem 2.19]). *Let $S_1, S \in (0, \infty)$ with $S_1 < S$, $\nu \in (1, \infty)$. Then*

$$\int_0^\infty (|y - \tau\,t\,e_1 - e^{-t\cdot\Omega} \cdot z|^2 + t)^{-\nu}\, dt \leq \mathfrak{C}(S_1, S, \nu)\,\big(|y| \cdot s_\tau(y)\big)^{-\nu + 1/2} \tag{2.3}$$

for $y \in B_S^c$, $z \in \overline{B_{S_1}}$.

Lemma 2.7 ([5, Lemma 3.2]). *Let $j, k \in \{1, 2, 3\}$. For $\alpha, \beta \in \mathbb{N}_0^3$ with $|\alpha + \beta| \leq 2$, $y, z \in \mathbb{R}^3$ with $y \neq z$, the function $(0, \infty) \ni t \mapsto \partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y, z, t) \in \mathbb{R}$ is integrable, the derivative $\partial_y^\alpha \partial_z^\beta \mathfrak{Z}_{jk}(y, z)$ exists, and*

$$\partial_y^\alpha \partial_z^\beta \mathfrak{Z}_{jk}(y,z) = \int_0^\infty \partial_y^\alpha \partial_z^\beta \Gamma_{jk}(y,z,t)\, dt. \tag{2.4}$$

Moreover, for α, β as before, the derivative $\partial_y^\alpha \partial_z^\beta \mathfrak{Z}_{jk}(y, z)$ is a continuous function of $y, z \in \mathbb{R}^3$ with $y \neq z$.

Lemma 2.8. *Let $S_1, S \in (0, \infty)$ with $S_1 < S$, $\alpha, \beta \in \mathbb{N}_0^3$ with $|\alpha + \beta| \leq 2$, $1 \leq j, k \leq 3$. Then*

$$|\partial_y^\alpha \partial_z^\beta \mathfrak{Z}_{jk}(y,z)| \leq \mathfrak{C}(S_1, S)\,\big(|y| \cdot s_\tau(y)\big)^{-1 - |\alpha + \beta|/2} \quad for \ y \in B_S^c, \ z \in \overline{B_{S_1}},$$

$$|\partial_y^\alpha \partial_z^\beta \mathfrak{Z}_{jk}(y,z)| \leq \mathfrak{C}(S_1, S)\,\big(|z| \cdot s_\tau(z)\big)^{-1 - |\alpha + \beta|/2} \quad for \ z \in B_S^c, \ y \in \overline{B_{S_1}}.$$

Proof: Lemma 2.6 - 2.7.

Lemma 2.9 ([3, Theorem 3.1]). *Let* $k \in \{0,1\}$, $R \in (0,\infty)$, $y, z \in B_R$ *with* $y \neq z$. *Then*

$$\int_0^\infty \left(|y - \tau \, t \, e_1 - e^{-t \, \Omega} \cdot z|^2 + t \right)^{-3/2-k/2} dt \leq \mathfrak{C}(R) \, |y - z|^{-1-k}.$$

Due to Lemma 2.5, this means for y, z *as above, and for* $j, k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \leq 1$ *that*

$$\left| \partial_y^\alpha \mathfrak{Z}(y,z) \right| + \left| \partial_z^\alpha \mathfrak{Z}(y,z) \right| \leq \mathfrak{C}(R) \, |y - z|^{-1-|\alpha|}.$$

Lemma 2.10 ([5, Lemma 4.1]). *Let* $j, k \in \{1, 2, 3\}$, $g \in L^1(\partial \mathfrak{D})$, *and put*

$$F(y) := \int_{\partial \mathfrak{D}} \mathfrak{Z}_{jk}(y,z) \, g(z) \, do_z \quad for \ y \in \overline{\mathfrak{D}}^c.$$

Then $F \in C^1(\overline{\mathfrak{D}}^c)$ *and*

$$\partial_m F(y) = \int_{\partial \mathfrak{D}} \partial_{y_m} \mathfrak{Z}_{jk}(y,z) \, g(z) \, do_z \quad for \ 1 \leq m \leq 3, \ y \in \overline{\mathfrak{D}}^c. \tag{2.5}$$

Lemma 2.11. *([5, Lemma 4.2]) Let* $j, k, l \in \{1, 2, 3\}$, $R > 0$, $g \in L^1(B_R)$, *and put*

$$F(y) := \int_{B_R} \partial_{z_l} \mathfrak{Z}_{jk}(y,z) \, g(z) \, dz \quad for \ y \in \overline{B_R}^c.$$

Then $F \in C^1(\overline{B_R}^c)$ *and*

$$\partial_m F(y) = \int_{B_R} \partial_{y_m} \partial_{z_l} \mathfrak{Z}_{jk}(y,z) \, g(z) \, dz \quad for \ y \in \overline{B_R}^c, \ 1 \leq m \leq 3.$$

Lemma 2.12. *Let* $\gamma \in (1/4, \infty)$. *Then there is a constant* $C(\gamma) > 0$ *such that for all* $x \in \mathbb{R}^3$:

$$\int_{\mathbb{R}^3} \left[(1 + |x - y|) \, s(x - y) \right]^{-3/2} \left[(1 + |y|) s(y) \right]^{-\gamma} dy \leq C(\gamma) \, (1 + |x|)^{-c} \, s(x)^{-d} \ln^k(2 + |x|),$$

where

$$c := \begin{cases} \gamma - 1/2 & if \ \gamma \in (1/4, 2] \\ 3/2 & if \ \gamma \in (2, +\infty) \end{cases} \quad d := \begin{cases} \gamma & if \ \gamma \in (1/4, 3/2] \\ 3/2 & if \ \gamma \in (3/2, +\infty) \end{cases} \quad k := \begin{cases} 0 & if \ \gamma \neq 2 \\ 1 & if \ \gamma = 2. \end{cases}$$

Proof: See the proof of [40, Theorem 3.2].

Lemma 2.13. *There exist a constant* $C > 0$ *such that for all* $x \in \mathbb{R}^3$:

$$\int_{\mathbb{R}^3} \left[(1 + |x - y|) \, s(x - y) \right]^{-2} \left[(1 + |y|) \, s(y) \right]^{-2} dy \leq C \left[(1 + |x|) \, s(x) \right]^{-2} \ln(2 + |x|)$$

Proof.

Let us denote $r^* := \min(1, r)$, $r \in \mathbb{R}$, $\eta_\beta^\alpha(x) := (1 + |x - y|)^\alpha s(x - y)^\beta$, $x, y \in \mathbb{R}^N$, $N \in \mathbb{N}$, $N \geq 2$, $\alpha, \beta \in \mathbb{R}$. In [40] the inequalities of the type $\eta_{-b}^{-a} * \eta_{-d}^{-c} \leq c \, \eta_{-f}^{-e}$ are studied in $\mathbb{R}^N$. If $|x| \leq 1$, the conditions $a + b^* + c + d > N$ and $a + b + c + d^* > N$ ensure that

the convolution $\eta_{-b}^{-a} * \eta_{-d}^{-c}$ is bounded by $C(a,b,c,d)$, and thus by $C(a,b,c,d)\,\eta_{-f}^{-e}$, ([40, p. 73]). So, we may consider the case $|x| \geq 1$. In that case the whole space $\mathbb{R}^N$ is divided into sixteen regions Ω_i, $0 \leq i \leq 15$, and the optimal choice of e_i, f_i in the inequality

$$\eta_{-b}^{-a} * \eta_{-d}^{-c} \leq C_i\,\eta_{-f_i}^{-e_i} \quad \text{in}\ \ \Omega_i, \ \ 0 \leq i \leq 15,$$

for given a,b,c,d is stated in [40, Tab. 1, 2] included in this paper as an appendix. Using the expressions of e_i and f_i, we have to find

$$e = \min_{i=0,1,\ldots 15} e_i, \quad e + f = \min_{i=0,1,\ldots 15}(e_i + f_i).$$

Let us mention that $\eta_\beta^\alpha(x) \leq 2^{\gamma-\alpha}\,\eta_\delta^\gamma(x)$, for $x \in \mathbb{R}^N$ if $\alpha \leq \gamma$ and $\alpha + \beta \leq \gamma + \delta$, $\alpha,\beta,\gamma,\delta \in \mathbb{R}$. Using expressions of e_i from [40, Tab. 1, 2], where we put $N = 3$, we define e as the minimum of the following values:

$$c + \tfrac{1}{2}\min(0,\, a + b^* - 3), \qquad a + \tfrac{1}{2}\min(0,\, c + d^* - 3),$$
$$a + c - 2 + \tfrac{1}{2}\min(0, 1 + b^* - a, 1 + d - c), \quad a + c - 2 + \tfrac{1}{2}\min(0, 1 + b - a, 1 + d^* - c),$$
$$a + c + d - 2 + \tfrac{1}{2}\min(0, 1 - c - d), \qquad a + c + b - 2 + \tfrac{1}{2}\min(0, 1 - a - b),$$
$$a + b + c + d - 3 + \tfrac{1}{2}\min(0, 3 - 2b - c - d), \quad a + b + c + d - 3 + \tfrac{1}{2}\min(0, 3 - a - b - 2d),$$
$$a + b^* + c + d - 3, \qquad a + b + c + d^* - 3.$$

Substituting $a = b = c = d = 2$, we get $e = 2$. Analogously, using expressions of e_i, f_i from [40, Tab. 1, 2] we define $e + f$ as the minimum of the following values

$$c + d + \min(0,\, a + b^* - 3), \quad a + b + \min(0,\, c + d^* - 3),$$
$$a + b^* + c + d - 3, \qquad a + b + c + d^* - 3.$$

So, we get $e + f = 4$, hence $f = 2$.

The logarithmic factor of the type $\ln(2 + |x|)$ appears on Ω_0 because $a + b^* = 3$, and on Ω_1 because $c + d^* = 3$, see [40, Tab. 1, 2]. Regions Ω_2, Ω_3 and Ω_4 also contribute logarithmic factors, they are covered by the logarithmic factor of the mentioned type $\ln(2 + |x|)$. $\quad\square$

Starting point of our considerations will be the following theorem about the integrability and pointwise decays of the velocity and its gradient, where the velocity is a solution of the rotational Navier-Stokes equations:

Theorem 2.14 ([7, Theorem 1.1]). *Let $\tau \in (0,\infty)$, $\omega \in \mathbb{R}^3\backslash\{0\}$, $\mathfrak{D} \subset \mathbb{R}^3$ open and bounded. Take γ, $S_1 \in (0,\infty)$, $p_0 \in (1,\infty)$, $A \in (2,\infty)$, $B \in [0,3/2]$, $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ measurable with $\overline{\mathfrak{D}} \subset B_{S_1}$, $A + \min\{B, 1\} > 3$, $A + B \geq 7/2$, $f|B_{S_1} \in L^{p_0}(B_{S_1})^3$,*

$$|f(y)| \leq \gamma\,|y|^{-A}\,s_\tau(y)^{-B} \quad \text{for}\ \ y \in B_{S_1}^c. \tag{2.6}$$

Let $u \in L^6(\overline{\mathfrak{D}}^c)^3 \cap W_{loc}^{1,1}(\overline{\mathfrak{D}}^c)^3$, $\pi \in L_{loc}^2(\overline{\mathfrak{D}}^c)$, $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$, and

$$\fint_{\overline{\mathfrak{D}}^c}\Big[\nabla u \cdot \nabla\varphi \ + \ \big(\tau\,\partial_1 u + \tau\,(u \cdot \nabla)u$$
$$-(\omega \times z) \cdot \nabla u + \omega \times u\big) \cdot \varphi - \pi\,div\,\varphi - f \cdot \varphi\Big]\,dz = 0, \quad div\,u = 0 \tag{2.7}$$

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for $\varphi \in C_0^\infty(\overline{\mathfrak{D}}^c)^3$. Let $S \in (S_1, \infty)$. Then

$$|\partial^\alpha u(y)| \le D\left(|y|\, s_\tau(y)\right)^{-1-|\alpha|/2} \quad for \ \ x \in B_S^c, \ \ \alpha \in \mathbb{N}_0^3, \ \ |\alpha| \le 1, \tag{2.8}$$

with the constant D depending on $\tau, \rho, \gamma, S_1, p_0, A, B, \|f|B_{S_1}\|_1, u, \pi, S$, and on an arbitrary but fixed number $S_0 \in (0, S_1)$ with $\overline{\mathfrak{D}} \subset B_{S_0}$.

Let $p \in (1, \infty)$, $q \in (1, 2)$, $f \in L_{loc}^p(\mathbb{R}^3)^3$ with $f|B_S^c \in L^q(B_S^c)^3$ for some $S \in (0, \infty)$. For $y \in \mathbb{R}^3$, $j \in \{1, 2, 3\}$, we set

$$\mathfrak{R}_j(f)(y) := \int_{\mathbb{R}^3} \sum_{k=1}^3 \mathfrak{Z}_{jk}(y, z)\, f_k(z)\, dz.$$

According to [4, Lemma 3.1], the integral appearing in the definition of $\mathfrak{R}_j(f)$ is well defined at least for almost every $y \in \mathbb{R}^3$. If f is a function on $\overline{\mathfrak{D}}^c$, the function f in the previous definition is to be replaced by the extension of f by zero to $\mathbb{R}^3$.

In order to derive the leading terms of the velocity and its gradient we are going to use the representation formula of a solution of the rotational Navier-Stokes equation:

Theorem 2.15. *Let* $u \in W_{loc}^{1,1}(\overline{\mathfrak{D}}^c)^3 \cap L^6(\overline{\mathfrak{D}}^c)^3$ *with* $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$. *Let* $p \in (1, \infty)$, $q \in (1, 2)$, $f : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}^3$ *a function with* $f|\mathfrak{D}_T \in L^p(\mathfrak{D}_T)^3$ *for* $T \in (0, \infty)$ *with* $\overline{\mathfrak{D}} \subset B_T$, $f|B_S^c \in L^q(B_S^c)^3$ *for some* $S \in (0, \infty)$ *with* $\overline{\mathfrak{D}} \subset B_S$. *Further assume that* $u|\partial\mathfrak{D} \in W^{2-1/p,p}(\partial\mathfrak{D})^3$ *and* $\pi : \overline{\mathfrak{D}}^c \mapsto \mathbb{R}$ *is a function with* $\pi|\mathfrak{D}_T \in L^p(\mathfrak{D}_T)$ *for* T *as above.*

Suppose that the pair (u, π) *is a weak solution of the Navier-Stokes system with Oseen and rotational terms, and with right-hand side* f *in the sense of (2.7). Then* $u \in W^{2,\min\{p,3/2\}}(\mathfrak{D}_T)^3$, $\pi \in W^{1,\min\{p,3/2\}}(\mathfrak{D}_T)$ *for any* $T \in (0, \infty)$ *with* $\overline{\mathfrak{D}} \subset B_T$,

$$u_j(y) = \mathfrak{R}_j\left(f - \tau\,(u \cdot \nabla)u\right)(y) + \mathfrak{B}_j(u, \pi)(y) \quad for \ j \in \{1, 2, 3\}, \ a.e. \ y \in \overline{\mathfrak{D}}^c, \tag{2.9}$$

where $\mathfrak{B}_j(u, \pi)$ *is defined by*

$$\mathfrak{B}_j(u, \pi)(y) \tag{2.10}$$

$$:= \int_{\partial\mathfrak{D}} \sum_{k=1}^3 \Big[\sum_{l=1}^3 \Big(\mathfrak{Z}_{jk}(y, z)\,\big(-\partial_l u_k(z) + \delta_{kl}\,\pi(z) + u_k(z)\,(\tau\,e_1 - \omega \times z)_l\big)$$

$$+ \partial_{z_l}\mathfrak{Z}_{jk}(y, z)\,u_k(z)\Big)\,n_l^{(\mathfrak{D})}(z) \ + \ E_{4j}(y - z)\,u_k(z)\,n_k^{(\mathfrak{D})}(z) \Big]\,do_z$$

for $y \in \overline{\mathfrak{D}}^c$.

Proof: See [8, Theorem 4.1], and its proof, as well as [4, Theorem 4.4]. $\qquad\square$

In comparison with the linear case we will need some additional lemma:

Lemma 2.16. *Let* $\phi \in W_{loc}^{1,1}(U)$ *for some open set* $U \subset \mathbb{R}^3$ *and* $A \in \mathbb{R}^{3\times3}$ *such that* $A^{-1} = A^T$. *Then:*

$$A\nabla_z\left(\phi(Az)\right) = \nabla\phi(Az)$$

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Proof: Indeed:

$$\frac{\partial}{\partial z_l}\left(\phi(Az)\right) = \sum_{k=1}^{3} \partial_k \phi(Az) \frac{\partial (Az)_k}{\partial z_l} = \sum_{k=1}^{3} \partial_k \phi(Az) A_{k\,l} = \sum_{k=1}^{3} A_{l\,k}^T \, \partial_k \phi(Az)$$

i.e. $\nabla_z \left(\phi(Az)\right) = A^T \nabla \phi(Az)$, which gives the mentioned formula. $\qquad\square$

Corollary 2.17. *In the situation of Theorem 2.14, we get for $z \in \overline{B_{S_1}}^c$ that*

$$\sum_{l=1}^{3} \left(u_l \, \partial_l u\right)\left(e^{t\Omega} z\right) = \sum_{l=1}^{3} \partial_l \left(u_l u\right)\left(e^{t\Omega} z\right)$$

$$= \sum_{l=1}^{3} \sum_{k=1}^{3} \left(e^{t\Omega}\right)_{lk} \frac{\partial}{\partial z_k}\left[\left(u_l u\right)\left(e^{t\Omega} z\right)\right] = \sum_{l=1}^{3} \left(e^{t\Omega} \nabla_z\right)_l \left[\left(u_l u\right)\left(e^{t\Omega} z\right)\right].$$

Lemma 2.18. *In the situation of Theorem 2.14, we have*

$$\int_{\overline{\mathfrak{D}}^c} |\partial_x^\alpha \mathfrak{Z}(x,y)\left[(u\cdot\nabla)u\right](y)| \, dy < \infty \quad \textit{for } x \in \overline{B_{S_1}}^c, \; \alpha \in \mathbb{N}_0^3 \textit{ with } |\alpha| \le 1. \quad (2.11)$$

Moreover the function $\mathfrak{V}(x) := \int_{\overline{\mathfrak{D}}^c} \mathfrak{Z}(x,y)\left[(u\cdot\nabla)u\right](y) \, dy \quad (x \in \overline{B_{S_1}}^c)$ belongs to $C^1(\overline{B_{S_1}}^c)^3$, with

$$\partial^\alpha \mathfrak{V}(x) = \int_{\overline{\mathfrak{D}}^c} \partial_x^\alpha \mathfrak{Z}(x,y)\left[(u\cdot\nabla)u(y)\right] \, dy \quad \textit{for } x, \, \alpha \textit{ as in (2.11).} \qquad (2.12)$$

Proof: Let $U \subset \mathbb{R}^3$ be open and bounded, with $\overline{U} \subset \overline{B_{S_1}}^c$. It is enough to show that (2.11) holds for $x \in U$, that $\mathfrak{V}|U \in C^1(U)^3$, and (2.12) is valid for $x \in U$.

Due to our assumptions on U, we may choose $R, S \in (S_1, \infty)$ such that $\overline{B_S} \cap \overline{U} = \emptyset$ and $\overline{U} \subset B_R$. In particular we have $\text{dist}(B_S, U) > 0$ and $\text{dist}(U, B_R^c) > 0$. This observation and Lemma 2.8 imply that $|\partial_x^\alpha \mathfrak{Z}(x,y)| \le C_0$ for $x \in U$, $y \in B_S \backslash \overline{\mathfrak{D}}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, where C_0 is independent of x and y. We further observe that $(u\cdot\nabla)u \in L^{3/2}(\overline{\mathfrak{D}}^c)^3$, hence $(u\cdot\nabla)u|B_S\backslash\overline{\mathfrak{D}} \in L^1(B_S\backslash\overline{\mathfrak{D}})^3$. Lemma 2.8 and (2.8) yield that $|\partial_x^\alpha \mathfrak{Z}(x,y)\left[(u\cdot\nabla)u\right](y)| \le C_1 \cdot |y|^{-7/2-|\alpha|/2}$ for $x \in U$, $y \in B_R^c$, with C_1 again being independent of x and y. In view of the last statement of Lemma 2.7, we may thus conclude by Lebesgue's theorem that the function

$$y \mapsto \partial_x^\alpha \mathfrak{Z}(x,y)\left[(u\cdot\nabla)u\right](y), \; y \in A := (B_S\backslash\overline{\mathfrak{D}}) \cup B_R^c,$$

is integrable for $x \in U$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, that the function

$$\mathfrak{V}^{(I)}(x) := \int_A \mathfrak{Z}(x,y)\left[(u\cdot\nabla)u\right](y) \, dy, \;\; x \in U,$$

belongs to $C^1(U)^3$, and that $\partial^\alpha \mathfrak{V}^{(I)}(x) = \int_A \partial_x^\alpha \mathfrak{Z}(x,y)\left[(u\cdot\nabla)u\right](y) \, dy$ for x, α as before.

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ with $0 \le \varphi \le 1$, $\varphi|B_{1/2} = 0$, $\varphi|B_1^c = 1$, and define $\varphi_\delta(x) := \varphi(\delta^{-1}x)$ for $x \in \mathbb{R}^3$, $\delta > 0$. Then $\varphi_\delta \in C_0^\infty(\mathbb{R}^3)$, $0 \le \varphi_\delta \le 1$, $\varphi_\delta|B_{\delta/2} = 0$, $\varphi_\delta|B_\delta^c = 1$ and $|\nabla\varphi_\delta(x)| \le C\delta^{-1}$ for $x \in \mathbb{R}^3$, $\delta > 0$.

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Using Lemma 2.9 and Theorem 2.14, we see there are constants C_2, C_3 with

$$|\partial_x^\alpha\big(\,\mathfrak{Z}(x,y)\,\varphi_\delta(x-y)\,\big)\,[(u\cdot\nabla)u](y)|$$
$$\leq C_2(|x-y|^{-2}+\delta^{-1}|x-y|^{-1})\,\chi_{(\delta/2,\infty)}(|x-y|)\leq C_3\delta^{-2} \qquad (2.13)$$

for $x \in U$, $y \in B_R\backslash B_S$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. In addition, if $y \in B_R\backslash B_S$, the function $x \mapsto \mathfrak{Z}(x,y)\,\varphi_\delta(x-y)\,[(u\cdot\nabla)u](y)$, $x \in U$, is continuously differentiable, as follows from Lemma 2.6. Now we may conclude from Lebesgue's theorem that the function $y \mapsto \partial_x^\alpha\big(\,\mathfrak{Z}(x,y)\,\varphi_\delta(x-y)\,\big)\,[(u\cdot\nabla)u](y)$, $y \in B_R\backslash B_S$, is integrable for any $\delta > 0$, $x \in U$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, the function

$$\mathfrak{B}_\delta(x) := \int_{B_R\backslash B_S} \mathfrak{Z}(x,y)\,\varphi_\delta(x-y)\,[(u\cdot\nabla)u](y)\,dy, \quad x\in U,$$

belongs to $C^1(U)^3$ for any $\delta > 0$, and

$$\partial^\alpha\mathfrak{B}_\delta(x) = \int_{B_R\backslash B_S} \partial_x^\alpha\big(\,\mathfrak{Z}(x,y)\,\varphi_\delta(x-y)\,\big)\,[(u\cdot\nabla)u](y)\,dy$$

for δ, x, α as before. Proceeding as in (2.13), we further obtain

$$\int_{B_R\backslash B_S} |\partial_x^\alpha\big(\,\mathfrak{Z}(x,y)\,\varphi_\delta(x-y)-\mathfrak{Z}(x,y)\,\big)\,[(u\cdot\nabla)u](y)|\,dy$$
$$\leq C_4 \int_{B_R\backslash B_S} \chi_{B_\delta}(x-y)\,(|x-y|^{-2}+\delta^{-1}|x-y|^{-1})\,dy \leq C_5\delta$$

for $x \in U$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, with the C_4, C_5 denoting constants independent of δ and x. Therefore, by an argument involving uniform convergence of $\mathfrak{B}_\delta$ and $\nabla\mathfrak{B}_\delta$ for $\delta \downarrow 0$, we may conclude that the function

$$\mathfrak{V}^{(II)}(x) := \int_{B_R\backslash B_S} \mathfrak{Z}(x,y)\,[(u\cdot\nabla)u](y)\,dy, \quad x\in U,$$

belongs to $C^1(U)^3$, and

$$\partial^\alpha\mathfrak{V}^{(II)}(x) = \int_{B_R\backslash B_S} \partial_x^\alpha\mathfrak{Z}(x,y)\,[(u\cdot\nabla)u](y)\,dy \quad \text{for } x\in U, \ \alpha\in\mathbb{N}_0^3 \text{ with } |\alpha|\leq 1.$$

Since $\mathfrak{V}(x) = \mathfrak{V}^{(I)}(x)+\mathfrak{V}^{(II)}(x)$ for $x\in U$, the proof of the lemma is complete. $\qquad\square$

3   Leading term of the velocity and of its gradient

The aim of this part is to find the leading term of the velocity and its gradient for the Navier-Stokes problem with rotation. Let us recall that the quantities τ, ω and the set $\mathfrak{D}$ were fixed in Section 2. We study the case f has a compact support in $\overline{\mathfrak{D}}^c$. The result we will prove in the work at hand may be stated as:

Theorem 3.1. *Let $S_1 \in (0, \infty)$ with $\overline{\mathfrak{D}} \subset B_{S_1}$, $p \in (1, \infty)$, $f \in L^p(\overline{\mathfrak{D}}^c)^3$ with $\operatorname{supp}(f) \subset B_{S_1}$, $u \in L^6(\overline{\mathfrak{D}}^c)^3 \cap W_{loc}^{1,1}(\overline{\mathfrak{D}}^c)^3$ with $\nabla u \in L^2(\overline{\mathfrak{D}}^c)^9$ and $u|\partial\mathfrak{D} \in W^{2-1/p,\,p}(\partial\mathfrak{D})^3$, $\pi \in L_{loc}^2(\overline{\mathfrak{D}}^c)$ with $\pi|\mathfrak{D}_{S_1} \in L^p(\mathfrak{D}_{S_1})$. Suppose that the pair (u, π) is a weak solution of the Navier-Stokes system with Oseen and rotational terms, and with right-hand side f in the sense of (2.7). Then there are coefficients $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and functions $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \in C^1(\overline{B_{S_1}}^c)$ such that for $j \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$, $x \in \overline{B_{S_1}}^c$,*

$$\partial^\alpha u_j(x) = \left\{ \sum_{k=1}^3 \beta_k\,\partial^\alpha \mathfrak{Z}_{jk}(x,0) + \left(\int_{\partial\mathfrak{D}} u \cdot n^{(\mathfrak{D})}\,do_z \right) \partial^\alpha E_{4j}(x) \right\} + \partial^\alpha \mathfrak{F}_j(x), \qquad (3.1)$$

and if $S \in (S_1, \infty)$, $x \in B_S^c$,

$$|\partial^\alpha \mathfrak{F}(x)| \le \mathfrak{C}\,(|x|\,s_\tau(x))^{-3/2 - |\alpha|/2}\ln(2 + |x|), \qquad (3.2)$$

where $\mathfrak{C}$ depends on τ, ω, p, S_1, S, certain norms of u, π and f, and on the constant D from (2.8).

In Theorem 3.1, the estimate presented in [8, Theorem 3.14] for the linear case is extended to the nonlinear one. Note that by [31, (3.9)], the function $\mathfrak{Z}(x,0)$ in the leading term on the right-hand side of (3.1) corresponds to the time integral of a fundamental solution of the evolutionary Oseen system multiplied by a rotation depending on time.

Proof of Theorem 3.1 The term of (3.1) contained in braces $\{\dots\}$ we will call "the leading term", term $\mathfrak{F}$ we will call "the remainder". From Theorem 2.15 we have

$$u_j(x) = \mathfrak{R}_j\big(f - \tau\,(u \cdot \nabla)u\big)(x) + \mathfrak{B}_j(u, \pi)(x), \ j \in \{1, 2, 3\}, \ \text{for a.e. } x \in \overline{\mathfrak{D}}^c, \qquad (3.3)$$

where $\mathfrak{B}_j(u, \pi)$ was defined in (2.10).

We put

$$\beta_k := \beta_k^{(I)} - \tau \beta_k^{(II)}$$

$$\beta_k^{(I)} := \int_{B_{S_1}} f_k(y)\,dy$$

$$\qquad + \int_{\partial\mathfrak{D}} \sum_{l=1}^3 \big(-\partial_l u_k(y) + \delta_{kl}\,\pi(y) + u_k(y)\,(\tau\,e_1 - \omega \times y)_l \big)\, n_l^{(\mathfrak{D})}(y)\,do_y$$

$$\beta_k^{(II)} := \int_{\partial\overline{\mathfrak{D}}} \sum_{m=1}^3 (n_m u_m\,u_k)(y)\,\mathrm{d}o_y$$

for $1 \le k \le 3$. By the definition of β_k the leading term in formula (3.1) is determined. Because (3.1) is in fact rearrangement of formula (3.3), we now define the value $\mathfrak{F}_j$ as the difference of the right-hand side of the representation formula (3.3) minus the leading term. We will distinguish $\mathfrak{F}^{(I)}$ coming from the linear terms and $\mathfrak{F}^{(II)}$ arising from the

non-linear part, i.e. from $\mathfrak{R}_j\big((u\cdot\nabla)u\big)$:

$$\mathfrak{F}_j(x) := \mathfrak{F}^{(I)}{}_j(x) - \tau\,\mathfrak{F}_j^{(II)}(x),$$

$$\mathfrak{F}^{(I)}{}_j(x) := \int_{B_{S_1}}\left(\sum_{k=1}^{3}\big[\,(\,\mathfrak{Z}_{jk}(x,y) - \mathfrak{Z}_{jk}(x,0)\,)\,f_k(y)\,\big]\right)dy$$

$$+ \int_{\partial\mathfrak{D}}\sum_{k=1}^{3}\Big(\big(\,\mathfrak{Z}_{jk}(x,y) - \mathfrak{Z}_{jk}(x,0)\,\big)$$

$$\cdot\sum_{l=1}^{3}\big(-\partial_l u_k(y) + \delta_{kl}\,\pi(y) + u_k(y)\,(\tau\,e_1 - \omega\times y)_l\,\big)\,n_l^{(\mathfrak{D})}(y)$$

$$+\big(\,E_{4j}(x-y) - E_{4j}(x)\,\big)\,u_k(y)\,n_k^{(\mathfrak{D})}(y)\Big)\,do_y + \int_{\partial\mathfrak{D}}\sum_{k,l=1}^{3}\partial_{y_l}\,\mathfrak{Z}_{jk}(x,y)\,u_k(y)\,n_l^{(\mathfrak{D})}(y)\,do_y\,,$$

$$\mathfrak{F}^{(II)}{}_j(x) := \int_{\overline{\mathfrak{D}}^c}\sum_{k,l=1}^{3}\mathfrak{Z}_{jk}(x,y)\,(u_l\,\partial_l u_k)\,(y)\mathrm{d}y - \int_{\partial\mathfrak{D}}\sum_{k,l=1}^{3}\mathfrak{Z}_{jk}(x,0)(n_l u_l\,u_k)(y)\,do_y \quad (3.4)$$

for $x\in\overline{B_{S_1}}^{\,c}$, $1\le j\le 3$. Then by (3.3) we get (3.1).

The assertion of the theorem will be proved in four steps:

1. Estimates and continuity of $\partial^\alpha\mathfrak{F}^{(I)}$, where $|\alpha|=0$ or $|\alpha|=1$.

By exactly the same proof as given in [5, p. 473-474] for [5, Theorem 1.1], we obtain that $\mathfrak{F}^{(I)}\in C^1(\overline{B_{S_1}}^{\,c})^3$ and

$$|\partial^\alpha\mathfrak{F}^{(I)}(x)| \le \mathfrak{C}(\|f\|_1 + \|\nabla u|\partial\mathfrak{D}\|_1 + \|\pi|\partial\mathfrak{D}\|_1 + \|u|\partial\mathfrak{D}\|_1)(|x|\,s_\tau(x))^{-3/2-|\alpha|/2}$$

for $S\in(S_1,\infty)$, $x\in\overline{B_S}^{\,c}$, $\alpha\in\mathbb{N}_0^3$ with $|\alpha|\le 1$, with $\mathfrak{C}$ depending on τ, ω, p, S_1 and S.

2. C^1-continuity of $\mathfrak{F}^{II}$.

By Lemma 2.18, the function $\mathfrak{F}^{(II)}\in C^1(\overline{B_{S_1}}^{\,c})^3$, and first-order derivatives may be moved into the volume integral appearing on the right-hand side of (3.4).

3. Estimates of $\partial^\alpha\mathfrak{F}^{(II)}$: first steps.

Let $x\in B_S^c$. Recalling that

$$\mathfrak{F}^{(II)}(x) = \int_{\overline{\mathfrak{D}}^c}\sum_{l=1}^{3}\mathfrak{Z}(x,y)\,(u_l\,\partial_l u)\,(y)\mathrm{d}y \;-\int_{\partial\mathfrak{D}}\sum_{l=1}^{3}\mathfrak{Z}(x,0)(n_l u_l\,u)(y)\,do_y, \qquad (3.5)$$

we apply firstly the integration by parts and and then split the resulting volume integral in an integral $\mathfrak{B}_R$ on the bounded domain $B_R\setminus\overline{\mathfrak{D}}$ and integral $\mathcal{E}_R$ on the exterior domain $(B_R)^c$, where $R=(S_1+S)/2$. Thus $\mathfrak{F}^{II}(x)$ becomes

$$\int_{\partial\mathfrak{D}}\sum_{l=1}^{3}[\,\mathfrak{Z}(x,y) - \mathfrak{Z}(x,0)\,]\,(n_l u_l\,u)(y)\,do_y - \left\{\int_{B_R\setminus\overline{\mathfrak{D}}} + \int_{(B_R)^c}\right\}\sum_{l=1}^{3}\partial_{y_l}\,\mathfrak{Z}(x,y)(u_l\,u)(y)\,\mathrm{d}y$$

$$= \mathcal{S}_{\partial\mathfrak{D}} - \mathcal{B}_R - \mathcal{E}_R. \tag{3.6}$$

Of course, here and in similar situations in the following, a partial integration has to be performed first on a bounded domain, where $B_T \backslash (\overline{B_R} \cup B_\epsilon(x))$ with $T > \max\{2R, 2|x|\}$, $0 < \epsilon$ is a good choice for such a domain. In the next step we let ϵ tend to zero. This passage to the limit may be handled by referring to Lemma 2.9 and (2.8). Finally we let T tend to infinity. The surface integral on ∂B_T which came up in the partial integration then vanishes, as follows from Lemma 2.8 and (2.8). The same references imply that all the volume integrals involved tend to integrals on B_R^c when $T \to \infty$.

In volume integral $\mathcal{E}_R$ over the exterior domain $(B_R)^c$ we use firstly the definition of $\mathfrak{Z}$, (2.4) and the Fubini's theorem, and then the domain invariant transformation $y = e^{t\Omega}z$ for fixed $t > 0$. The reason why we use the mentioned transformation is that we would like to avoid a periodic term in the right-hand side of (3.11):

$$\mathcal{E}_R = \int_{(B_R)^c} \sum_{l=1}^3 \partial_{y_l}\, \mathfrak{Z}(x,y)(u_l\, u)(y)\, \mathrm{d}y = \int_0^\infty \int_{(B_R)^c} \sum_{l=1}^3 \partial_{y_l}\, \Gamma(x,y,t)[u_l\, u](y)\, \mathrm{d}y\, \mathrm{d}t$$

$$= \int_0^\infty \int_{(B_R)^c} \sum_{l=1}^3 \partial_{y_l}\, \Gamma(x,y,t)_{|y=e^{t\Omega}z}[u_l\, u](e^{t\Omega}z)\, \mathrm{d}z\, \mathrm{d}t$$

Finally we split the the exterior domain of integration B_R^c on two domains: Let δ be a sufficiently small positive number comparing to $1, R$ and $S - S_1$, f.e. $\delta := \min\{1, (S - S_1)/2, R/2\} = \min\{1, S - R, R/2\}$. Note that $\overline{B_\delta(x)} \subset B_R^c$. We obtain:

$$\mathcal{E}_R = \left\{ \int_0^\infty \int_{B_\delta(x)} + \int_0^\infty \int_{(B_R)^c \backslash B_\delta(x)} \right\} \sum_{l=1}^3 \partial_{y_l}\, \Gamma(x,y,t)_{|y=e^{t\Omega}z}[(u_l\, u)(e^{t\Omega}z)\, \mathrm{d}z\, \mathrm{d}t$$

$$= \mathcal{V}_\delta + \mathcal{V}_{R,\delta}$$

Substituting the expression of $\mathcal{E}_R$ into (3.6) we get:

$$\mathfrak{F}^{II} = \mathcal{S}_{\partial\mathfrak{D}} - \mathcal{B}_R - \mathcal{V}_\delta - \mathcal{V}_{R,\delta} \tag{3.7}$$

$\partial_x^\alpha \mathcal{B}_R,\ \partial_x^\alpha \mathcal{S}_{\partial\mathfrak{D}}$: Estimating the behavior of the first two terms and their derivatives ∂_x^α for $|\alpha| = 0, 1$, we get the following estimate:

$$|\partial_x^\alpha \mathcal{B}_R| + |\partial_x^\alpha \mathcal{S}_{\partial\mathfrak{D}}| \le \mathfrak{C}(S_1, S)\, (|x| s_\tau(x))^{-3/2 - |\alpha|/2},\ x \in B_S^c \tag{3.8}$$

Indeed, from Lemma 2.8 for $y \in B_R$, $x \in B_S^c$:

$$|\partial_x^\alpha \partial_{y_l}\, \mathfrak{Z}(x,y)| \le \mathfrak{C}(S_1, S)\, (|x| s_\tau(x))^{-3/2 - |\alpha|/2}, \tag{3.9}$$

$$\left|\partial_x^\alpha\big(\mathfrak{Z}(x,y) - \mathfrak{Z}(x,0)\big)\right| = \left|\sum_{k=1}^3 \partial_x^\alpha \partial_{y_k}\, \mathfrak{Z}(x,\theta y)\, y_k\right| \le \mathfrak{C}(S_1, S)\, (|x| s_\tau(x))^{-3/2 - |\alpha|/2} \tag{3.10}$$

for some $0 \le \theta \le 1$. So, with Lemma 2.11 and (3.9)

$$|\partial_x^\alpha \mathcal{B}_R| \le \left|\int_{B_R \backslash \overline{\mathfrak{D}}} \sum_{l=1}^3 \partial_x^\alpha \partial_{y_l}\, \mathfrak{Z}(x,y)(u_j\, u)(y)\, \mathrm{d}y\right| \le$$

$$\leq \mathfrak{C}(S_1, S)(|x|s_\tau(x))^{-3/2-|\alpha|/2} \int_{B_R\setminus\overline{\mathfrak{D}}} \sum_{l=1}^{3} \big|(u_l\,u)(y)\big| \mathrm{d}y \leq \mathfrak{C}(S_1, S)\,(|x|s_\tau(x))^{-3/2-|\alpha|/2},$$

because $|u|^2$ is L^1-integrable on bounded domain $B_R \setminus \overline{\mathfrak{D}}$. Similarly, we have with (2.5) and (3.10):

$$|\partial_x^\alpha \mathcal{S}_{\partial\mathfrak{D}}| \leq \mathfrak{C}(S_1, R)\,(|x|s_\tau(x))^{-3/2-|\alpha|/2} \int_{\partial\mathfrak{D}} \sum_{l=1}^{3} \big|(n_l u_l\,u)(y)\,\big| \mathrm{do}_y$$

$$\leq \mathfrak{C}(S_1, R)\,(|x|s_\tau(x))^{-3/2-|\alpha|/2}.$$

4. Estimates of $\partial^\alpha \mathfrak{F}^{(II)}$ for $\alpha = 0$.

$\mathcal{V}_\delta$: For the estimation of this term we use Lemma 2.5 for the first order derivatives of Γ : We have (for $x \neq \mathrm{e}^{t\Omega}z$)

$$\left|\partial_{y_j}\Gamma(x,y,t)_{|y=e^{t\Omega}z}\right| \leq \mathfrak{C}\left(|x-\tau t\mathrm{e}_1-z|^2+t\right)^{-2}. \tag{3.11}$$

From Theorem 2.14 we have for $y \in B_R^c$

$$|u(y)|^2 \leq \mathfrak{C}(R)\,(|y|\,s_\tau(y))^{-2}.$$

If $z \in (B_R)^c$ then $\mathrm{e}^{t\Omega}z \in (B_R)^c$, we get:

$$|u(\mathrm{e}^{t\Omega}z)|^2 \leq \mathfrak{C}(R)\left(|\mathrm{e}^{t\Omega}z|\,s_\tau(\mathrm{e}^{t\Omega}z)\right)^{-2} = \mathfrak{C}(R)\,(|z|\,s_\tau(z))^{-2} \tag{3.12}$$

Since $\overline{B_\delta(x)} \subset B_R^c$, we thus get due to (2.1), (2.2)

$$|u(e^{t\Omega}z)| \leq \mathfrak{C}(R,\delta)\,(|x|\,s_\tau(x))^{-2} \quad \text{for } z \in \overline{B_\delta(x)}. \tag{3.13}$$

So, we have

$$|\mathcal{V}_\delta| = \left|\int_0^\infty \int_{B_\delta(x)} \sum_{j=1}^{3} \partial_{y_j}\Gamma(x,y,t)_{|y=e^{t\Omega}z}\left[(u_j u)\,(e^{t\Omega}z)\right] \mathrm{d}z\,\mathrm{d}t\right|$$

$$\leq \mathfrak{C}(R)\int_{B_\delta(x)} \int_0^\infty \left(|x-\tau t\mathrm{e}_1-z|^2+t\right)^{-2}(|x|\,s_\tau(x))^{-2}\,\mathrm{d}t\,\mathrm{d}z$$

$$\leq \mathfrak{C}(R,\delta)\,(|x|s_\tau(x))^{-2}\int_{B_\delta(x)} |x-z|^{-2}\,\mathrm{d}z \leq \mathfrak{C}(R,\delta)\,(|x|s_\tau(x))^{-2},$$

where the integral with respect to variable t is estimated using Lemma 2.9, choosing in its application $y := x-z,\ z := 0$.

$\mathcal{V}_{R,\delta}$: Similarly as in the previous case using (3.11) and (3.12):

$$|\mathcal{V}_{R,\delta}| \leq \left|\int_0^\infty \int_{B_R^c\setminus B_\delta(x)} \sum_{j=1}^{3} \partial_{y_j}\Gamma(x,y,t)_{|y=e^{t\Omega}z}\left[(u_j u)\,(e^{t\Omega}z)\right] \mathrm{d}z\,\mathrm{d}t\right|$$

$$\leq \ \mathfrak{C}(R) \int_0^\infty \int_{B_R^c \setminus B_\delta(x)} \left(|x - \tau t e_1 - z|^2 + t \right)^{-2} (|z| \ s_\tau(z))^{-2} \, \mathrm{d}z \, \mathrm{d}t$$

Now, the integral with respect to t can be estimated using Lemma 2.6, $y := x - z, \ z := 0$:

$$\int_0^\infty \left(|x - \tau t e_1 - z|^2 + t \right)^{-2} \mathrm{d}t \leq \mathfrak{C}(S_1, S) \, (|x - z| s_\tau(x - z))^{-3/2} \ , \ z \in B_R^c \setminus B_\delta(x)$$

$$|\mathcal{V}_{R,\delta}| \leq \mathfrak{C}(S_1, S) \int_{B_R^c \setminus B_\delta(x)} (|x - z| s_\tau(x - z))^{-3/2} \, (|z| \, |s_\tau(z)|)^{-2} \, \mathrm{d}z$$

$$\leq \mathfrak{C}(S_1, S) \ln(2 + |x|) \, (|x| |s_\tau(x)|)^{-3/2} \ .$$

The last inequality follows from Lemma 2.12 ($\gamma = 2$).

5. Estimates of $\partial^\alpha \mathfrak{F}^{(II)}$ for $|\alpha| = 1$.

Let us mention that S, S_1, R, δ are the same as in the previous section, so $\overline{B_\delta(x)} \subset B_R^c$. The aim of this part is to find the leading term of the gradient of velocity for the Navier-Stokes problem with rotation: The difference with the previous case is that we cannot apply the integration by parts over the whole domain $\overline{\mathfrak{D}}^c$ because we have to protect the neighbourhood $B_\delta(x)$ due to singularities of the second order derivatives of $\mathfrak{Z}$. On the other hand, to avoid some technical difficulties, we are able to handle the integrals with respect to t only in domains invariant with respect to the transformation $y = e^{t\Omega} z, \ t > 0$. These facts causes some additional computations. So, we use Lemma 2.18, split the domain of integration into the bounded part $B_R \setminus \overline{\mathfrak{D}}^c$ and the exterior domain $(B_R)^c$, and we apply the integration by parts firstly only on the bounded domain:

$$\partial_x^\alpha \mathfrak{F}_j^{(II)}(x) = \int_{\overline{\mathfrak{D}}^c} \sum_{k,l=1}^3 \partial_x^\alpha \mathfrak{Z}_{jk}(x, y) \, (u_l \, \partial_l u_k) \, (y) \mathrm{d}y - \int_{\partial \mathfrak{D}} \sum_{k,l=1}^3 \partial_x^\alpha \mathfrak{Z}_{jk}(x, 0)(n_l u_l \, u_k)(y) \, \mathrm{d}o_y$$

$$= \left\{ \int_{B_R \setminus \overline{\mathfrak{D}}} + \int_{(B_R)^c} \right\} \sum_{k,l=1}^3 \partial_x^\alpha \mathfrak{Z}_{jk}(x, y) \, (u_l \, \partial_l u_k) \, (y) \mathrm{d}y$$

$$- \int_{\partial \mathfrak{D}} \sum_{k,l=1}^3 \partial_x^\alpha \mathfrak{Z}_{jk}(x, 0)(n_l u_l \, u_k)(y) \, \mathrm{d}o_y$$

$$= \int_{\partial \overline{\mathfrak{D}}} \sum_{k,l=1}^3 [\partial_x^\alpha \mathfrak{Z}_{jk}(x, y) - \partial_x^\alpha \mathfrak{Z}_{jk}(x, 0)] \, (n_l u_l \, u_k)(y) \, \mathrm{d}o_y$$

$$- \int_{B_R \setminus \overline{\mathfrak{D}}} \sum_{k,l=1}^3 \partial y_l \, \partial_x^\alpha \mathfrak{Z}_{jk}(x, y)(u_l \, u_k)(y) \, \mathrm{d}y + \int_{\partial B_R} \sum_{k,l=1}^3 \partial_x^\alpha \mathfrak{Z}_{jk}(x, y)(u_l \, u_k)(y) \, \frac{y_l}{R} \mathrm{d}o_y$$

$$+ \int_{(B_R)^c} \sum_{k,l=1}^3 \partial_x^\alpha \mathfrak{Z}_{jk}(x, y) \, (u_l \, \partial_l u_k) \, (y) \mathrm{d}y$$

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So, we get:
$$\partial_x^\alpha \mathfrak{F}^{(II)}(x) = \partial^\alpha \mathcal{S}_{\partial \mathfrak{D}} - \partial^\alpha \mathcal{B}_R + \mathcal{S}_R' + \mathcal{E}_R' \tag{3.14}$$

Evaluation of the last term in (3.14) with (2.4):

$$\mathcal{E}_R'(x)_j = \int_{(B_R)^c} \sum_{k,l=1}^3 \partial_x^\alpha \mathfrak{Z}_{jk}(x,y) \, (u_l \, \partial_l u_k) \, (y) \mathrm{d}y$$

$$= \int_{(B_R)^c} \int_0^{+\infty} \sum_{k,l=1}^3 \partial_x^\alpha \Gamma_{jk}(x,y,t) \, (u_l \, \partial_l u_k) \, (y) \mathrm{d}y \, \mathrm{d}t$$

The domain of integration of $\mathcal{E}_R'$ is $(B_R)^c$. This exterior domain is invariant with respect to the transformation $y = e^{t\Omega} z$, $t > 0$. We use the same transformation to avoid periodic terms as in the case $|\alpha| = 0$:

$$\mathcal{E}_R'(x)_j = \int_0^{+\infty} \int_{(B_R)^c} \sum_{k,l=1}^3 \partial_x^\alpha \Gamma_{jk}(x, e^{\tau\Omega} z, t) \, (u_l \, \partial_l u_k) \, (e^{\tau\Omega} z) \mathrm{d}z \, \mathrm{d}t$$

Unlike the case $|\alpha| = 0$, the mentioned transformation is used *before* the integration by parts. We split the domain of integration into two domains $B_\delta(x)$ and $(B_R)^c \setminus B_\delta(x)$. In the integral over the unbounded domain we apply the identity from Corollary 2.17 and integrate by parts:

$$\mathcal{E}_R'(x)_j = \int_0^{+\infty} \int_{B_\delta(x)} \sum_{k,l=1}^3 \partial_x^\alpha \Gamma_{jk}(x, e^{\tau\Omega} z, t) \, (u_l \, \partial_l u_k) \, (e^{\tau\Omega} z) \mathrm{d}z \, \mathrm{d}t$$

$$+ \int_0^\infty \int_{\partial B_\delta(x)} \sum_{k,l=1}^3 \partial_x^\alpha \Gamma_{jk}(x, e^{t\Omega} z, t) \left[(u_l u_k) \, (e^{t\Omega} z) \right] \left(e^{t\Omega} \, (x-z)/\delta \right)_l \mathrm{d}o_z \, \mathrm{d}t$$

$$+ \int_0^\infty \int_{\partial B_R} \sum_{k,l=1}^3 \partial_x^\alpha \Gamma_{jk}(x, e^{t\Omega} z, t) \left[(u_l u_k) \, (e^{t\Omega} z) \right] \left(e^{t\Omega} \, (-z)/R \right)_l \mathrm{d}o_z \, \mathrm{d}t$$

$$- \int_0^\infty \int_{B_R^c \setminus B_\delta(x)} \sum_{k,l=1}^3 \left(e^{t\Omega} \, \nabla_z \right)_l \partial_x^\alpha \Gamma_{jk}(x, e^{t\Omega} z, t) \left[(u_l u_k) \, (e^{t\Omega} z) \right] \mathrm{d}z \, \mathrm{d}t$$

$$= \left(\mathcal{U}_\delta \right)_j + \left(\mathcal{S}_\delta' \right)_j + \left(-\mathcal{S}_R' \right)_j + \left(\mathcal{U}_{R,\delta} \right)_j$$

Substituting the expression of $\mathcal{E}_R'(x)$ into (3.14) and using (2.4), we get finally:

$$\partial_x^\alpha \mathfrak{F}^{(II)}(x) = \partial^\alpha \mathcal{S}_{\partial \mathfrak{D}} - \partial^\alpha \mathcal{B}_R + \mathcal{U}_\delta + \mathcal{S}_\delta' + \mathcal{U}_{\mathcal{R},\delta} \tag{3.15}$$

Now we will estimate all terms of (3.15) for $|\alpha| = 1$:

$\partial^\alpha \mathcal{S}_{\partial \mathfrak{D}}$, $\partial^\alpha \mathcal{B}_{B_R}$: From (3.8) we know that $|\partial_x^\alpha \mathcal{S}_{\partial \mathfrak{D}}| + |\partial_x^\alpha \mathcal{B}_R| \leq \mathfrak{C}_1(S_1, S) \, (|x| \, s_\tau(x))^{-2}$.

$\mathcal{U}_\delta$: Estimates of this term are completely analogous to the evaluation of $\mathcal{V}_\delta$ in the case $|\alpha| = 0$. Only difference is that from Theorem 2.14: $|u(y)||\nabla u(y)| \leq \mathfrak{C}(S)\,(|y|\,s_\tau(y))^{-5/2}$ for $y \in (B_R)^c$: We get

$$|\mathcal{U}_\delta| \leq \mathfrak{C}(S_1, S)\,(|x||s_\tau(x)|)^{-5/2}\,.$$

$\mathcal{S}'_\delta$: From Lemma 2.5 for the first order derivatives of Γ, $x \neq e^{t\Omega}z$:

$$\left|\partial_x^\alpha \Gamma(x, e^{t\Omega}z, t)\right| \leq \mathfrak{C}\left(|x - \tau t e_1 - z|^2 + t\right)^{-2}\,.$$

By (3.13)

$$|u(e^{t\Omega}z)|^2 \leq \mathfrak{C}(R, \delta)\,(|x|\,s_\tau(x))^{-2} \quad \text{for}\ z \in \partial B_\delta(x).$$

It is also clear that $\left|e^{t\Omega}\,(x - z)/\delta\right| = 1$ for $z \in B_\delta(x)$.

So, we have

$$\left|\mathcal{S}'_\delta\right| \leq \mathfrak{C}(R) \int_{\partial B_\delta(x)} \int_0^\infty \left(|x - \tau t e_1 - z|^2 + t\right)^{-2} (|x|\,s_\tau(x))^{-2}\,\mathrm{d}t\,\mathrm{d}o_z$$

$$\leq \mathfrak{C}(R, \delta)\,(|x|\,s_\tau(x))^{-2} \int_{\partial B_\delta(x)} |x - z|^{-2}\,\mathrm{d}o_z \leq \mathfrak{C}(R, \delta)\,(|x|\,s_\tau(x))^{-2}$$

where the integral with respect to variable t is estimated using Lemma 2.9, $y := x - z$, $z := 0$. So, the integral $\mathcal{S}'_\delta$ belongs to the remainder.

$\mathcal{U}_{R,\delta}$: We shall use Lemma 2.5, for the evaluation of the second order derivatives of the function Γ:

$$\left|\left(e^{t\Omega}\,\nabla_z\right)_j \partial_x^\alpha \Gamma(x, e^{t\Omega}z, t)\right| \leq \mathfrak{C}\left(|x - \tau t e_1 - z|^2 + t\right)^{-5/2}$$

The integral with respect to t of the right-hand side can be estimated using Lemma 2.6 choosing y, z from the lemma by the following way: $y := x - z$, $z := 0$. Hence:

$$\int_0^\infty \left(|y - \tau t e_1 - z|^2 + t\right)^{-5/2}\,\mathrm{d}t \leq \mathfrak{C}(S_1, S)\,(|x - z|\,s_\tau(x - z))^{-2}$$

Using (3.12), we find

$$|\mathcal{U}_{R,\delta}| \leq \left|\int_0^\infty \int_{B_R^c \setminus B_\delta(x)} \sum_{j=1}^3 \left(e^{t\Omega}\,\nabla_z\right)_j \partial x_m \Gamma(x, e^{t\Omega}z, t) \left[(u_j u)\,(e^{t\Omega}z)\right]\,\mathrm{d}z\,\mathrm{d}t\right|$$

$$\leq \mathfrak{C}(S_1, S) \int_{B_R^c \setminus B_\delta(x)} (|x - z|s_\tau(x - z))^{-2}\,(|z||s_\tau(z)|)^{-2}\,\mathrm{d}z$$

$$\leq \mathfrak{C}(S_1, S)\,(|x|\,s_\tau(x))^{-2} \ln(2 + |x|).$$

The last inequality we get by Lemma 2.13. $\qquad\square$

Remark: So, finally we get that the leading term of $\partial^\alpha u_j$ is expressed as in the linear case

$$\sum_{k=1}^3 \beta_k \, \partial_x^\alpha \mathfrak{Z}_{jk}(x,0),$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ contains additionally the term $\int_{\partial \mathfrak{D}} \sum_{j=1}^3 (n_j u_j \, u)(y) \, \mathrm{d}o_y$.

Corollary 3.2. *Let $U \subset \mathbb{R}^3$ be open and bounded, $S_1 \in (0, \infty)$ with $\overline{U} \subset B_{S_1}$, $p \in (1, \infty)$, $f \in L^p(U)^3$ with $supp(f) \subset B_{S_1}$. Let $u \in W^{1,1}_{loc}(\overline{U}^c)^3$ with $u \in L^6(\overline{U}^c)^3$, $\nabla u \in L^2(\overline{U}^c)^9$, $\pi \in L^2_{loc}(\overline{U}^c)$, and suppose that u, π satisfy (2.7) (weak form of rotational Navier-Stokes system, as in Theorem 2.14) with U in the place of $\mathfrak{D}$. Let $S_0 \in (0, S_1)$ with $\overline{U} \subset B_{S_0}$. Then the conclusions of Theorem 3.1 hold, with $\mathfrak{D}$ replaced by B_{S_0}.*

Proof: Obviously $(u \cdot \nabla)u \in L^{3/2}(\overline{U}^c)^3$ so that $f - (u \cdot \nabla)u \in L^{\min\{p,3/2\}}_{loc}(\overline{U}^c)^3$. Therefore, by interior regularity of the Stokes system, as stated in [26, Theorem IV.4.1], we have $u \in W^{2, \min\{p, 3/2\}}_{loc}(\overline{U}^c)^3$ and $\pi \in W^{1, \min\{p, 3/2\}}_{loc}(\overline{U}^c)$; also see the proof of [4, Theorem 5.5]. Now the corollary follows from Theorem 3.1, with $p_0 = \min(p, 3/2)$ and B_{S_0} in the place of $\mathfrak{D}$. $\qquad\square$

Acknowledgments

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Tab.1, see [40] $N = 3$

Dom.	t	ϱ	r	$\tilde{t}$	$\tilde{\varrho}$	$\tilde{r}$	$\eta_{-b}^{-a}(\mathbf{y})$	$\eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})$	e	f	log. factors
Ω_0	$(-\tfrac{1}{8}R^v; \tfrac{1}{8}R^v)$	$\sim \lvert\mathbf{x}'\rvert$ $(0; \tfrac{1}{8}R^v)$	$\sim \lvert\mathbf{x}\rvert$ $(0; \tfrac{1}{8}R^v)$	$\sim x_1$ R	$\sim \lvert\mathbf{x}'\rvert$ R^v	$\sim \lvert\mathbf{x}\rvert$ R	$r^{-a}(1+s(\mathbf{y}))^{-b}$	$R^{-c-2\sigma d}$	$c+\tfrac{1}{2}\min(0,a+b^*-3)$	$d+\tfrac{1}{2}\min(0,a+b^*-3)$	$\ln R\,(b=1\wedge a<2)\vee$ $\vee(b\neq1\wedge a+b^*=3)\vee$ $\ln^2 R\,(b=1\wedge a=2)$
Ω_1	$\sim x_1$ R	$\sim \lvert\mathbf{x}'\rvert$ R^v	$\sim \lvert\mathbf{x}\rvert$ R	$(-\tfrac{1}{8}R^v; \tfrac{1}{8}R^v)$	$(0; \tfrac{1}{8}R^v)$	$(0; \tfrac{1}{8}R^v)$	$R^{-a-2\sigma b}$	$\tilde{r}^{-c}(1+s(\mathbf{x}-\mathbf{y}))^{-d}$	$a+\tfrac{1}{2}\min(0,c+d^*-3)$	$b+\tfrac{1}{2}\min(0,c+d^*-3)$	$\ln R\,(d=1\wedge c<2)\vee$ $\vee(d\neq1\wedge c+d^*=3)\vee$ $\ln^2 R\,(c=2\wedge d=1)$
Ω_2	$\sim r$ $(R^v; R)$	$(0; \tfrac{1}{8}R^v)$	$\sim t$ $R+R^v-\tilde{r}\ (R^v; R)$	$(-\tfrac{1}{8}R^v;$ $R-R^v)$	$\sim \lvert\mathbf{x}'\rvert$ R^v	$(R^v; R)$	$r^{-a}\quad \varrho<\sqrt{t}$ $r^{-a+b}\varrho^{-2b}\quad \varrho>\sqrt{t}$	$\tilde{r}^{-c+d}R^{-2dv}$	$a+c-2+\tfrac{1}{2}\min$ $(0,1+b^*-a,1+d-c)$	$b^*+d-1-\tfrac{1}{2}\min$ $(0,1+b^*-a,1+d-c)$	$\ln\tfrac{R}{1+s}\,(\min(1+b^*-a,\ 1+d-c)=0\wedge b\neq1)\wedge$ $\ln_+ s\cdot\ln\tfrac{R}{1+s}$ $(b=1\wedge 1+d-c=0)\,$ $(\ln_+ s\ a<2)(\ln R\ a>2)$ $(\ln R\ \ln\tfrac{R}{1+s}\ a=2)\wedge b=1$
Ω_3	$R-R^v$	R^v	$R+R^v-\tilde{r}$ $(R^v; R)$	$\sim \tilde{r}$ $(R^v; R)$	$(0; \tfrac{1}{8}R^v)$	$\sim \tilde{t}$ $(0; \tfrac{1}{8}R^v)$	$r^{-a+b}R^{-2bv}$	$\tilde{r}^{-c}\quad \tilde{\varrho}<\sqrt{\tau}$ $\tilde{r}^{-c+d}\tilde{\varrho}^{-2d}\quad \tilde{\varrho}>\sqrt{\tau}$	$a+c-2+\tfrac{1}{2}\min$ $(0,1+b-a,1+d^*-c)$	$b+d^*-1-\tfrac{1}{2}\min$ $(0,1+b-a,1+d^*-c)$	$\ln\tfrac{R}{1+s}\,(\min(1+b-a,\ 1+d^*-c)=0\wedge d\neq1)\wedge$ $\ln_+ s\cdot\ln\tfrac{R}{1+s}\,(d=1\wedge$ $\wedge 1+b-a=0)$ $(\ln_+ s\,c<2)(\ln R\,c>2)$ $(\ln R\ \tfrac{R}{1+s}\,c=2)\wedge d=1$
Ω_4									see Ω_2, Ω_3	see Ω_2, Ω_3	see Ω_2, Ω_3
Ω_5	$\sim r$ R	$(0; R^v)$	$\sim t$ $(R^v; R)$	$\sim -\tilde{r}$ $(-R; -R^v)$	$(0; R^v)$	$\sim \lvert\tilde{t}\rvert$ $(R^v; R)$	$R^{-a}\quad \varrho<\sqrt{t}$ $R^{-a+b}\varrho^{-2b}\quad \varrho>\sqrt{t}$	$\lvert\tilde{t}\rvert^{-c-d}$	$a+c+d-2+\tfrac{1}{2}\min$ $(0,1-c-d)$	$b^*-1-\tfrac{1}{2}\min$ $(0,1-c-d)$	$(\ln_+ s\quad b=1)\cdot$ $\cdot(\ln\tfrac{R}{1+s}\quad c+d=1)$
Ω_6	$\sim -r$ $(-R; -R^v)$	$(0; R^v)$	$\sim \lvert t\rvert$ $(R^v; R)$	$\sim \tilde{r}$ R	$(0; R^v)$	$\sim \tilde{t}$ R	$\lvert t\rvert^{-a-b}$	$R^{-c}\quad \tilde{\varrho}<\sqrt{\tilde{t}}$ $R^{-c+d}\tilde{\varrho}^{-2d}\quad \tilde{\varrho}>\sqrt{\tilde{t}}$	$a+b+c-2+\tfrac{1}{2}\min$ $(0,1-a-b)$	$d^*-1-\tfrac{1}{2}\min$ $(0,1-a-b)$	$(\ln_+ s\quad d=1)\cdot$ $\cdot(\ln\tfrac{R}{1+s}\quad a+b=1)$
Ω_7	R	$\sim \tilde{\varrho}$ $(\lvert\tilde{t}\rvert; R)$	$\sim \varrho$ $(\lvert t\rvert; R)$	R	$\sim \varrho, \tilde{r}$ $(\lvert\tilde{t}\rvert; R)$	$\sim \tilde{\varrho}$ $(\lvert\tilde{t}\rvert; R)$	$R^{-a+b}\varrho^{-2b}$	ϱ^{-c-d}	$a+b+c+d-3+$ $\tfrac{1}{2}\min(0,3-2b-c-d)$	$-\tfrac{1}{2}\min(0,3-2b-$ $-c-d)$	$\ln\tfrac{R}{1+s}\quad 2b+c+d=3$
Ω_8	$(-R; -R^v)$	$\sim \tilde{\varrho}, r$ $(\lvert t\rvert; R)$	$\sim \varrho$ $(\lvert t\rvert; R)$	R	$\sim \varrho$ $(\lvert t\rvert; R)$	R	$\tilde{\varrho}^{-a-b}$	$R^{-c+d}\tilde{\varrho}^{-2d}$	$a+b+c+d-3+$ $\tfrac{1}{2}\min(0,3-a-b-2d)$	$-\tfrac{1}{2}\min(0,3-a-b-2d)$	$\ln\tfrac{R}{1+s}\quad a+b+2d=3$

Tab.2, see [40] $N = 3$

Dom.	t	ϱ	r	$\tilde{t}$	$\tilde{\varrho}$	$\tilde{r}$	$\eta_{-b}^{-a}(\mathbf{y})$	$\eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})$	e	f	log. factors
Ω_9	$\sim -r$ / $(-R;-R^v)$	$\sim \tilde{\varrho}$ / $(R^v;|\tilde{t}|)$	$\sim \tilde{t}$ / R	$\sim -\tilde{r}$ / $(-R;-R^v)$	$\sim \varrho$ / $(R^v;|t|)$	$\sim |\tilde{t}|$ / $(R^v;R)$	$R^{-a+b}\,\varrho^{-2b}$	$|\tilde{t}|^{-c-d}$	$b>1$ see Ω_5; $b<1$ see Ω_7; $b=1$ see Ω_5		$\left(\ln\frac{R}{1+s}\ c+d<1\right)$; $\left(\ln^2\frac{R}{1+s}\ c+d=1\right)\wedge b=1$
Ω_{10}	$\sim -r$ / $(-R;-R^v)$	$\sim \tilde{\varrho}$ / $(R^v;|t|)$	$\sim |t|$ / $(R^v;R)$	$\sim \tilde{r}$ / R	$\sim \varrho$ / $(R^v;|t|)$	$\sim \tilde{t}$ / R	$|t|^{-a-b}$	$R^{-c+d}\,\tilde{\varrho}^{-2d}$	$d>1$ see Ω_6; $d<1$ see Ω_8; $d=1$ see Ω_6		$\left(\ln\frac{R}{1+s}\ a+b<1\right)$; $\left(\ln^2\frac{R}{1+s}\ a+b=1\right)\wedge d=1$
Ω_{11}	$(R;\infty)$	$(0;\infty)$	$\sim \tilde{r}$ / $(R;\infty)$	$(-\infty;-R^v)$	$(0;\infty)$	$\sim r$ / $(R;\infty)$	$(1+r)^{-a}(1+s(\mathbf{y}))^{-b}$	r^{-c-d}	$a+b^*+c+d-3>0$	0	$\ln R$ $b=1$
Ω_{12}	$(-\infty;-R^v)$	$(0;\infty)$	$\sim \tilde{r}$ / $(R;\infty)$	$(R;\infty)$	$(0;\infty)$	$\sim r$ / $(R;\infty)$	$\tilde{r}^{-a-b}$	$(1+\tilde{r})^{-c}(1+s(\mathbf{x}-\mathbf{y}))^{-d}$	$a+b+c=d^*-3>0$	0	$\ln R$ $d=1$
Ω_{13}	$\sim r,\tilde{r}$ / $(R;\infty)$	$\sim \tilde{\varrho}$ / $(R;\infty)$	$\sim \tilde{r}$ / $(R;\infty)$		$\sim \varrho$ / $(R;\infty)$	$\sim r$ / $(R;\infty)$			see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}
Ω_{14}	$\sim r$ / R	$\sim \tilde{\varrho}$ / $(R^v;R)$	$\sim t$ / R	$(-\tfrac{1}{8}R^v;\tfrac{R}{2})$	$\sim \varrho$ / $(R^v;R)$	$\sim |\tilde{t}|+\tilde{\varrho}$	$\sim |\tilde{t}|+\tilde{\varrho}\ R^{-a+b}\,\varrho^{-2b}$	$(\tilde{t}+\tilde{\varrho})^{-c+d}\,\tilde{\varrho}^{-2d}\ \tilde{t}>0$; $\tilde{\varrho}^{-c-d}\ \tilde{t}<0$	see Ω_2,Ω_3; Ω_7; Ω_2; $a+b+c+d-3$	$b+d>1\wedge1+d-c>0$; $1+d-c<0$; $1+d-c<0\wedge b+d>1$; 0 otherwise	$\ln\frac{R}{1+s}\ b+d=1\wedge1+d-c>0$; $\ln^2\frac{R}{1+s}\ b+d=1\wedge1+d-c=0$
Ω_{15}	$(-R^v;\tfrac{R}{2})$	$\sim \tilde{\varrho}$ / $(R^v;R)$	$\sim |t|+\varrho$	$\sim r$ / R	$\sim \varrho$ / $(R^v;R)$	$\sim t$ / R	$(t+\varrho)^{-a+b}\,\varrho^{-2b}\ t>0$; $\varrho^{-a-b}\ t<0$	$R^{-c+d}\,\tilde{\varrho}^{-2d}$	see Ω_2,Ω_3; Ω_8; Ω_3; $a+b+c+d-3$	$b+d>1\wedge1+b-a>0$; $1+b-a<0$; $1+b-a=0\wedge b+d>1$; 0 otherwise	$\ln\frac{R}{1+s}\ b+d=1\wedge1+b-a>0$; $\ln^2\frac{R}{1+s}\ b+d=1\wedge1+b-a=0$